On the Distribution and Applications of Ramanujan sums

by

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PhD19306

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Submitted
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

to the

Indraprastha Institute of Information Technology
Delhi
May 2024

Dedicated to my parents

Certificate

This is to certify that the thesis entitled **"On the Distribution and Applications of Ramanujan sums"** being submitted by **"Ms. Shivani Goel"** to the **Indraprastha Institute of Information Technology Delhi**, for the award of the Degree of **Doctor of Philosophy** is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
May 2024

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Acknowledgements

I extend my heartfelt gratitude to my Ph.D. supervisor, Dr. Sneha Chaubey, for her unwavering inspiration and support throughout my mathematical journey. Her encouragement, patience, and enthusiastic involvement have been crucial to my growth and understanding of number theory. I am deeply grateful for her availability to discuss my doubts, her advice and guidance, and her efforts in thoroughly reviewing and improving the manuscripts.

I express my profound thanks to Professor M. Ram Murty for his expertise, persistent encouragement, and support that have shaped this research endeavor. I am ever thankful to him for taking the time for our meetings and the engaging math sessions. Beyond mathematics, his philosophy lectures during my visit to Queen's University were truly motivating.

I acknowledge all the faculty members of the Department of Mathematics at IIIT Delhi for their continuous support and cooperation. My gratitude extends to the supporting staff whose contributions, direct or indirect, facilitated the smooth completion of my work. I am thankful to the University Grants Commission (UGC) and IIIT Delhi for the financial support of my PhD studies.

My special thanks to Dr. Debika Banerjee (IIIT, Delhi), Dr. Atul Dixit (IIT, Gandhinagar), and Dr. Shanta Laisharam (ISI, Delhi) for agreeing to be part of my committee.

I also thank all my collaborators who contributed to the manuscripts included in this thesis and to the other projects I have been involved with over the years.

I would like to thank my friends and research fellows, Dipa, Divya Aggarwal, Atul, Divya Khurana, Sayantan, Tanisha, Archana, Ishani, Madhu, and Bittu at

IIIT-Delhi.

Most importantly, I would like to thank my family members, especially my parents, Mr. Pramod Kumar Goel and Mrs. Savita Goel. Their support, love, guidance, and faith in me have been my pillars of strength. I am incredibly grateful to my siblings Anu, Rajni, and Akshat, and my brother-in-law Akash for their constant support, inspiration, and love. Their encouragement has been crucial in helping me overcome the challenges of this journey. Lastly, I thank my dear niece Nitya and nephew Advait for their joyful presence.

New Delhi Shivani Goel

May 2024

Abstract

While studying the trigonometric series expansion of certain arithmetic functions, Ramanujan, in 1918, defined a sum of the nth power of the primitive qth roots of unity and denoted it as $c_q(n)$. These sums are now known as Ramanujan sums. Since then, Ramanujan sums have been widely used and studied in mathematics and other areas. Most importantly, it is used in the proof of Vinogradov's theorem that every sufficiently large odd number is the sum of three primes. It is also used to simplify the computations of Arithmetic Fourier Transform (AFT), Discrete Fourier Transform (DFT), and Discrete Cosine Transform (DCT) coefficients for a special type of signal.

We study Ramanujan sums in the context of the k-tuple prime conjecture. A twin prime is a prime number that is either two less or two more than another prime number. It is conjectured that there are infinitely many twin primes. Hardy and Littlewood generalized the twin prime conjecture and gave the k-tuple conjecture. Let $d_1, \cdots, d_k$ be distinct integers, and $b(p)$ is the number of distinct residue classes (mod p) represented by d_i. If $b(p) < p$ for every prime p, the k-tuple conjecture gives an asymptotic formula for the number of $n \leq x$ such that all the k numbers $n + d_i$ are primes. We study a heuristic proof of the k-tuple conjecture using the convolution of Ramanujan sums.

Additionally, we study questions on the distribution of Ramanujan sums. One way to study distribution is via moments of averages. Chan and Kumchev studied the first and second moments of Ramanujan sums. In this thesis, we estimate the higher moment of their averages using the theory of functions of several variables initiated by Vaidyanathaswamy.

Ramanujan sums can also be generalized over number fields. A number field is an extension field $\mathbb{K}$ of the field of rational numbers $\mathbb{Q}$ such that the field extension $\mathbb{K}/\mathbb{Q}$ has a finite degree. Nowak first studied the first moment for Ramanujan sums over quadratic number fields, and later, it was estimated for the higher degree number fields as well. For a general number field, assuming generalized Lindelöf Hypothesis, we improve the first moment result and also study the second moment. Furthermore, unconditionally, we estimate asymptotic formulas for the second moment for quadratic, cubic, and cyclotomic number fields. Our primary tool for these results is a Perron-type formula. Finally, we obtain the second moment result for certain integral domains called Prüfer domains.

Contents

List of Symbols

Symbol	**Meaning**
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{K}$	A number field
(a, b) and $[a, b]$	gcd and lcm of a and b, respectively
$(a_1, \cdots, a_k)$ and $[a_1, \cdots, a_k]$	gcd and lcm of $a_1, \cdots, a_k$, respectively
μ	The Möbius function
ϕ	The Euler totient function
Λ	The von Mangoldt function
$\omega(n)$	The number of distinct prime divisors of n
$\mathit{1}(n) = 1$	A constant function
$f \star g(n) = \sum_{d\mid n} f(d) g\left(\frac{n}{d}\right)$	The Dirichlet convolution
$r(n)$	The number of ways of writing n as a sum of two squares in $\mathbb{Z}^2$
$d(n)$	The divisor function
B_k	The k^{th} Bernoulli number
$\ll$	Vinogradov asymptotic notation
$\mathrm{O}(\cdot)$ and $\mathrm{o}(\cdot)$	Big oh and little oh asymptotic notation
$\zeta(s)$	The Riemann zeta function

$L(s, \chi)$	The Dirichlet L-function where χ is a Dirichlet character
$\zeta_{\mathbb{K}}(s)$	The Dedekind zeta function corresponding to a number field $\mathbb{K}$

Research Publications

Publications Related to the Dissertation

1. Sneha Chaubey, **Shivani Goel**, and M Ram Murty. On the Hardy-Littlewood prime tuples conjecture and higher convolutions of Ramanujan sums. *Functiones et Approximatio Commentarii Mathematici, 69(2)*, 2023: 161-175.

2. Sneha Chaubey and **Shivani Goel**. On the distribution of Ramanujan sums over number fields. *Ramanujan Journal, 61(3)*, 2023: 813-837.

3. Sneha Chaubey and **Shivani Goel**. Moments of Averages of Ramanujan Sums over Number Fields *Functiones et Approximatio Commentarii Mathematici (accepted)* .

4. **Shivani Goel** and M Ram Murty. Higher convolutions of Ramanujan sums (communicated).

5. **Shivani Goel** and M Ram Murty. On the moments of averages of Ramanujan sums (communicated), preprint at http://arxiv.org/abs/2401.07321.

Other Publications

1. Bittu Chahal, Sneha Chaubey, and **Shivani Goel**. On the distribution of index of Farey sequences. *Research in Number Theory 10(2)*, 2024.

2. Sneha Chaubey and **Shivani Goel**. Pair correlation of real-valued vector sequences. *Monatshefte fur Mathematik 204*, 2024: 217-239.

3. Sneha Chaubey and **Shivani Goel**. Mean value estimates of gcd and lcm-sums. *International Journal of Number Theory, 18(3)*, 2022: 629-654.

1

Introduction

In 1918, Ramanujan [52] identified the significance of the exponential sum defined as

$$c_q(n) := \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e^{\frac{2\pi i n a}{q}}. \tag{1.0.1}$$

This function is now known as Ramanujan sums. Here, q and n are positive integers. Ramanujan sums can also be interpreted as a sum of the n^{th} power of q^{th} primitive roots of unity. Furthermore, let $\zeta_q^n = e^{2\pi i n/q}$ be a q^{th} root of unity. Each of its power, $\zeta_q^n, \cdots, \zeta_q^{(q-1)n}$, $\zeta_q^0 = \zeta_q^{qn} = 1$ is also a root of unity. If

$$\eta_q(n) = \sum_{a=1}^{q} e^{2\pi i a n/q}$$

is the sum of n^{th} power of all roots, then

$$\eta_q(n) = \sum_{d|q} c_d(n).$$

Next, we use the Möbius inversion formula, which says that if two arithmetical functions f and g are related as $f = 1 \star g$, then we can write $g = \mu \star f$. Thus, by the Möbius inversion formula, we have

$$c_q(n) = \sum_{d|q} \eta_d(n)\mu\left(\frac{q}{d}\right).$$

Therefore, we can write an explicit form for $c_q(n)$ using Möbius function as

$$c_q(n) = \sum_{\substack{d|n \\ d|q}} d\mu\left(\frac{q}{d}\right). \tag{1.0.2}$$

We can derive various interesting properties of Ramanujan sums from (1.0.2), such as it is an integer-valued function and a q-periodic function of n. In other words, $c_q(n+q) = c_q(n)$. Additionally, for a fix n, it is a multiplicative function of q. That is, if $(q_1, q_2) = 1$, then we have

$$c_{q_1 q_2}(n) = c_{q_1}(n)c_{q_2}(n).$$

While for a fixed q, it is a multiplicative function of n if and only if $\mu(n) = 1$. This also implies $c_q(1) = \mu(q)$ and $c_q(q) = \phi(q)$. For a prime p,

$$c_p(n) = \begin{cases} -1 & \text{if } p \nmid n, \\ \phi(p) & \text{if } p|n. \end{cases}$$

Also, for a prime p and any positive integer k, we have

$$c_{p^k}(n) = \begin{cases} 0 & \text{if } p^{k-1} \nmid n, \\ -p^{k-1} & \text{if } p^{k-1}|n \text{ and } p^k \nmid n, \\ \phi(p^k) & \text{if } p^k|n. \end{cases}$$

The above result and the multiplicative property give another explicit formula of Ramanujan sums:

$$c_q(n) = \frac{\phi(q)}{\phi(q/(q,n))}\mu(q/(q,n)). \tag{1.0.3}$$

The right side of the above is called von Sterneck's arithmetic function, and this equivalence was derived by Hölder [32] in 1936. In the following sections, we discuss the literature on Ramanujan expansion and the orthogonality property of Ramanujan sums. Finally, we study the results of the distribution of Ramanujan sums.

1.1 Ramanujan expansion

For a periodic function, we can write the series expansion into trigonometric functions called Fourier series. Joseph Fourier first used the Fourier series to find solutions to the heat equation. Fourier series can also be applied to functions that are not necessarily periodic. Motivated by this, one can ask for a series expansion of arithmetical functions using Ramanujan sums. In fact, it is possible that it was this that led Ramanujan to define (1.0.1).

In [52], Ramanujan used these sums to obtain pointwise convergent series representations of the normalized arithmetic function $f(n)$ of the following form:

$$f(n) = \sum_{q=1}^{\infty} \hat{f}(q) c_q(n), \tag{1.1.1}$$

with coefficients $\hat{f}(q)$. The representation of $f(n)$ in (1.1.1) is called Ramanujan expansion or Ramanujan Fourier series, or simply Ramanujan series, and $\hat{f}(q)$ is called the q-th Ramanujan coefficient of f. Ramanujan derives the series expansion of some well-known functions written below:

$$d(n) = -\sum_{q=1}^{\infty} \frac{\log q}{q} c_q(n), \quad \frac{\phi(n)}{n} = \frac{6}{\pi^2} \sum_{q=1}^{\infty} \left(\frac{\mu(q)}{q^2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{-1} \right) c_q(n),$$

$$\Lambda(n) = -\sum_{q=1}^{\infty} \frac{c_q(n)}{n}, \quad r(n) = \pi \sum_{q=1}^{\infty} \frac{(-1)^{q-1}}{2q-1} c_{2q-1}(n).$$

Hardy obtained a Ramanujan Fourier series of the von Mangoldt function given by

$$\frac{\phi(n)\Lambda(n)}{n} = \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)} c_q(n). \tag{1.1.2}$$

Ramanujan further discovered that the prime number theorem is equivalent to showing

$$\sum_{q=1}^{\infty} \frac{1}{q} c_q(n) = 0. \tag{1.1.3}$$

More precisely, from (1.0.2), the above series evaluation is equivalent to $\sum_{q=1}^{\infty} \frac{\mu(q)}{q} = 0$, which is a well-known form of prime number theorem. We can see that (1.1.3) is an expansion of the zero function. Therefore, it also shows that the Ramanujan Fourier series need not be unique.

A natural question arises: is there Ramanujan expansion for every arithmetical function? This question was answered by Spilker [64]. He proved that every bounded function has a Ramanujan expansion. Later, Hilderbrand [31] gave a simple proof of Spilker's result and also removed the boundedness condition. A comprehensive review paper by Lucht [41] discusses the Ramanujan expansion of arithmetical functions. Moreover, notable monographs in this direction include the works of [59] and [63]. It is non-trivial exercise to determine the Ramanujan coefficients for several arithmetical functions. Therefore, another question arises: how can we determine the corresponding Ramanujan coefficients? Carmichael first encountered this question using the orthogonality principle of Ramanujan sums.

1.2 An orthogonality principle

In 1932, Carmichael [10] discovered the orthogonality principle of Ramanujan sums. The Ramanujan coefficient of an arithmetical function can be predicted using Carmichael's orthogonality property for Ramanujan sums.

Theorem 1.2.1 (Carmichael, 1932)**.** *We have*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} c_r(n) c_s(n+h) = \begin{cases} c_r(h) & \text{if } r = s, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From (1.0.1), we have

$$\sum_{n\le x} c_r(n) c_s(n+h) = \sum_{(a,r)=1} \sum_{(b,s)=1} e^{2\pi i h b/s} \sum_{n\le x} e^{2\pi i n(a/r+b/s)}.$$

The innermost sum is bounded unless $a/r + b/s$ is an integer m (say). This implies

$$as + br = mrs$$

forces $r = s$ because $(a, r) = (b, s) = 1$. The result is now immediate. $\square$

Let suppose $f(n)$ be an arithmetical function, and $M(f)$ denotes its mean value. In particular,

$$M(f) = \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} f(n),$$

and $f(n)$ has a convergent Ramanujan expansion given by

$$f(n) = \sum_{q=1}^{\infty} \hat{f}(q) c_q(n). \tag{1.2.1}$$

From the orthogonality principle, we have

$$\hat{f}(r) = \frac{1}{\phi(r)} M(fc_r), \tag{1.2.2}$$

provided that, $M(fc_r)$ exists. This gives the Ramanujan coefficients for the functions f, with a mean value $M(fc_r)$. Ramanujan and Carmichael's work set the stage for a general theory of Ramanujan expansions.

Later on, in 1943, Wintner [69] obtained the Ramanujan coefficients for a large class of functions. He supposed two arithmetical functions f and g such that

$$f(n) = \sum_{d|n} g(d), \tag{1.2.3}$$

and

$$\sum_{n=1}^{\infty} \frac{|g(n)|}{n} < \infty. \tag{1.2.4}$$

Then, the q^{th} Ramanujan coefficient of f is equal to

$$\sum_{n=1}^{\infty} \frac{g(nq)}{nq}. \tag{1.2.5}$$

Delange [21] improved Wintner's result with a hypothesis

$$\sum_{n=1}^{\infty} 2^{\omega(n)} \frac{|g(n)|}{n} < \infty$$

instead of (1.2.4) with the same Ramanujan coefficient given in (1.2.5). Therefore, the work of Wintner and Delange allows us to determine a large number of Ramanujan expansions. Today, we have a mature theory on the pointwise convergence of Ramanujan expansions with Fourier analysis on almost even arithmetic functions (see [22, 36]). For more recent developments, see the work Coppola [17], Schwarz [58], Lucht and Reifenrath [42], and the monograph of Schwarz and Spilker [59].

Furthermore, Gadiyar and Padma [25] used the orthogonality property of Ramanujan sums to derive a simple heuristic of the Hardy-Littlewood k-tuple conjecture for the case $k = 2$. Let $d_1, \cdots, d_k$ be distinct integers; the Hardy-Littlewood k-tuple conjecture gives an asymptotic formula for the number of $n \leq x$ such that all the k numbers $n + d_i$ are primes.

Stimulated by heuristic results in [25], the authors [24] employed the Ramanujan expansion to deduce asymptotic formulas for the convolution sums of distinct arithmetical functions. Murty and Saha [47] adopted the method in [24] to derive an asymptotic formula with explicit error terms for the shifted 2-convolution sums of arithmetical functions with absolutely convergent Ramanujan expansion under bounded conditions on Ramanujan coefficients. Subsequently, Coppola, Murty, and Saha [18, 19, 56] extended the results with a weaker hypothesis.

We develop the theory of the triple convolution of Ramanujan sums. We use this theory to derive a heuristic derivation of the Hardy-Littlewood 3-tuple conjecture. This derivation is inspired by the method of Gadiyar and Padma [25] for 2-tuple conjecture. In the estimation of the triple convolution of Ramanujan sums, we encounter an interesting exponential sum defined as:

$$\mathscr{K}_r(h, j) := \sum_{\substack{(b,r)=(c,r)=1 \\ (b+c,r)=1}} e^{2\pi i(hb+jc)/r}. \tag{1.2.6}$$

We refer to it as a two-variable variant of the Ramanujan sums. We prove that it is a multiplicative function of r. Similar to the usual Ramanujan sums, the two-variable variant of Ramanujan sums also satisfies the orthogonality property. Moreover, we can express an arithmetical function as a series expansion involving the two-variable variant of Ramanujan sums leveraging the orthogonality property.

The method used to develop the theory of triple convolution when applied to higher convolution, leads to exponential sums of several variables, which is cumbersome to solve. Therefore, we adopt a different approach for higher convolution. This naturally leads to the study of arithmetical functions of several variables initiated by Vaidyanathswamy [67] in 1927. We derive the Hardy-Littlewood k-tuple conjecture for $k > 3$ heuristically using the higher convolution of Ramanujan sums. Chapter 2 of the thesis consists of the above study. This Chapter also appeared in [15] and [27].

1.3 Distribution of Ramanujan sums

Understanding Ramanujan sums and their distribution is an important topic of study in number theory, with profound connections to problems in arithmetic such as in the proof of Vinogradov's theorem [48, Chapter 8], Waring type formulas [37], distribution of rational numbers in short intervals [35], equipartition modulo odd integers [7], large sieve inequality [53], as well as other areas of mathematics.

For any positive integer r, Alkan [3] studied the weighted average of Ramanujan sums given by

$$\frac{1}{q^{r+1}}\sum_{n=1}^{q} n^r c_q(n) = \frac{\phi(q)}{2q} + \frac{1}{r+1}\sum_{d|q}\mu(d)\sum_{m=1}^{[r/2]}\binom{r+1}{2m}\frac{B_{2m}}{d^{2m}}. \tag{1.3.1}$$

In [5], Alkan showed that the average of Ramanujan sums in (1.3.1) is positive for all $r \geq 1$, and as $r \to \infty$, the second sum of right side in (1.3.1) vanished completely. From this, he concluded that Ramanujan sums is not orthogonal to vectors with powers of integers but close to being orthogonal to these vectors. He also proved that any number in $[0, 1]$ is a limit point of the set of values of weighted average (1.3.1) for a semigroup of positive integers.

In [1, 2], Alkan used (1.3.1) to find the special values of the L-function and to calculate the maximum value of the partial sum of Ramanujan sums. In [4], he also gave a relation between the singularity of the Burgess zeta function and Ramanujan sums with the help of (1.3.1).

Note that the sum in (1.3.1) only deals with the average over the variable n. The question on the average order over both variables q and n of $c_q(n)$ was first considered by Chan and Kumchev [12] motivated by applications to problems on Diophantine approximations of reals by sums of rational numbers. More specifically, they consider

$$S_k(x,y) := \sum_{n\le y}\left(\sum_{q\le x} c_q(n)\right)^k \tag{1.3.2}$$

for $k = 1, 2$. From the elementary arguments, we have

$$S_1(x,y) = y + \mathrm{O}\left(x^2\right).$$

Chan and Kumchev refined the result to obtain broader support than $y > x^2$. A pair of real numbers (r, l) such that $0 \le r \le 1/2 \le l \le 1$ is called an exponential pair if the average of exponential sums can be bounded by a certain bound with exponents r and l. Chan and Kumchev decompose $S_1(x, y)$ into subsums and use the exponential pair $(1/2, 1/2)$ from van der Corput's method [29] for the same. Similarly, from the elementary methods, it can be shown

$$S_2(x,y) = \frac{yx^2}{2\zeta(2)} + \mathrm{O}\left(x^4 + xy\log x\right).$$

Chan and Kumchev [12] also sharpened the support by means of an analytic argument. More precisely, they used Perron's formula and double complex integrals.

In [16], Cohen generalized the Ramanujan sums in the following way:

$$c_q^\beta(n) := \sum_{\substack{1\le j\le q^\beta\\ (j,q^\beta)_\beta=1}} e\left(\frac{jn}{q^\beta}\right) = \sum_{\substack{d|q\\ d^\beta|n}} d^\beta \mu\left(\frac{q}{d}\right). \tag{1.3.3}$$

We refer to these sums as Cohen-Ramanujan sums. Here,

$$(j, q^\beta)_\beta = \max\{l^\beta : l^\beta | j \text{ and } l|q\}.$$

For $\beta = 1$, it is the usual Ramanujan sums. In [55], Robles and Roy estimated the average of the first and second moments of Cohen-Ramanujan sums using Perron's formula and double complex integrals. They also gave a result for higher moments (Proposition 1.1), but their result for higher moments is incorrect, which we show in Chapter 3.

In Chapter 3, we estimate higher moments for usual, and Cohen Ramanujan sums using the Brèteche Tauberian theorem and the theory of functions of several variables. The contents of Chapter 3 appeared in [28].

In the final part of the thesis, we investigate Ramanujan sums for number fields. It is defined as follows: let $\mathbb{K}$ be a number field. If $\mathcal{J}$ and $\mathcal{I}$ are non-zero integral ideal in its ring of integers $\mathcal{O}_{\mathbb{K}}$, then Ramanujan sums over $\mathbb{K}$ are defined as

$$C_{\mathcal{J}}(\mathcal{I}) := \sum_{\substack{\mathcal{I}_1 | \mathcal{J} \\ \mathcal{I}_1 | \mathcal{I}}} \mathcal{N}(\mathcal{I}_1) \mu \left(\frac{\mathcal{J}}{\mathcal{I}_1} \right). \tag{1.3.4}$$

Here, $\mathcal{N}(\mathcal{I}_1)$ is the norm of $\mathcal{I}_1$ and $\mu(\mathcal{I})$ is the generalization of classical Mobius function such that

$$\mu(\mathcal{I}) := \begin{cases} (-1)^r & \text{if } \mathcal{I} \text{ is a product of } r \text{ distinct prime ideals,} \\ 0 & \text{if there exists a prime ideal } \mathcal{P} \text{ of } \mathcal{O}_{\mathbb{K}} \text{ such that } \mathcal{P}^2 | \mathcal{I}. \end{cases} \tag{1.3.5}$$

Note that for $\mathbb{K} = \mathbb{Q}$, it is the usual Ramanujan sum $c_q(n)$ in (1.0.1).

The average of the first moment of Ramanujan sums over number fields is studied by Nowak [50], Zhai [70], and Fujisawa [23]. Therefore, our study focuses on the second moment of Ramanujan sums over number fields. In Chapter 4 we examine the second moment for quadratic, cubic, and cyclotomic number fields and Prüfer domains. More generally, by pushing our arguments a little further, we can deal with arbitrary number fields. We prove the first and second moment results for them under the assumption of the generalized Lindelöf hypothesis. Part of the contents of Chapter 4 appeared in [13] and [14].

In the last chapter, we conclude the thesis by pointing out some directions for future research related to two variable variants of Ramanujan sums defined in (2.3.2) and Ramanujan expansion.

2

Ramanujan sums and the Hardy-Littlewood Prime Tuples Conjecture

In this chapter, we will derive a generalization of a limit theorem of Carmichael (Theorem 1.2.1) involving Ramanujan sums adopting a different method, which naturally leads to the study of certain arithmetical functions of several variables. We apply the general theory to give a heuristic derivation of the Hardy-Littlewood prime k-tuple conjecture which Hardy and Littlewood formulated using the more complicated circle method. We also estimate the triple convolution of the Jordan totient function using the convolution of Ramanujan sums.

2.1 The Hardy-Littlewood Prime Tuples Conjecture

A prime number p is called a twin prime if $p+2$ is also a prime. For example, 3, 5, 11, 17, 41 are twin primes. It is conjectured that there are infinitely many twin primes. In 1922, Hardy and Littlewood [30] generalized the celebrated twin prime conjecture and formulated what is now called the prime k-tuple conjecture, which is the following: Suppose that $d_1, ..., d_k$ are distinct integers, and let $b(p)$ be the number of distinct residue classes (mod p) represented by d_i. If $b(p) < p$ for every prime p, the prime k-tuple conjecture asserts that the number of $n \leq x$ such that **all** the k numbers $n + d_i$ are prime for $1 \leq i \leq k$ is asymptotic to

$$\mathfrak{S}(d_1, ..., d_k) \frac{x}{(\log x)^k},$$

where

$$\mathfrak{S}(d_1, ..., d_k) = \prod_p \left(1 - \frac{b(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-k}, \tag{2.1.1}$$

and the product is over all primes p.

Hardy and Littlewood formulated their conjecture using the intuition provided by the circle method and essentially ignoring the contribution from the so-called minor arcs emanating from the technique and focusing only on the major arcs. Though the idea is simple, the analysis of the major arcs could be more complex and delicate.

In 1999, Gadiyar and Padma [25] discovered a simple heuristic to derive the case $k = 2$ (or the generalized twin prime conjecture) using a simple orthogonality principle for Ramanujan sums originally discovered by Carmichael [10]. The key idea of their proof is first to write the conjecture as a convolution of the von Mangoldt function, which is possible as the von Mangoldt function is the best approximation function for primes. Consequently, they use the Ramanujan expansion of the von Mangoldt function (see (1.1.2)) and convert the problem into an estimation of the convolution of Ramanujan sums. Though their approach does not lead to a solution to the twin prime problem, it does open up other lines of productive investigations related to the theory of Ramanujan sums.

Building upon the beautiful idea of Gadiyar and Padma, we obtain a heuristic proof of the Hardy-Littlewood conjecture for $k > 2$. This is not straightforward, and we encounter a multivariable generalization of Ramanujan sums, which in itself demands further investigation. We begin with generalizing Carmichael's theorem on the convolution of Ramanujan sums. Our aim is to generalize Gadiyar and Padma's method. Therefore, we first need to find the convolution of Ramanujan sums.

2.2 Carmichael's theorem revisited

It will be useful to re-derive Carmichael's theorem through this new optic. We consider

$$\sum_{n\leq x} c_r(n+h)c_s(n+j) = \sum_{(a,r)=1}\sum_{(b,s)=1} e^{2\pi iha/r}e^{2\pi ibj/s}\sum_{n\leq x} e^{2\pi in(a/r+b/s)}.$$

Therefore,

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leq x} c_r(n+h)c_s(n+j) = \sum_{\substack{(a,r)=1,(b,s)=1\\ \frac{a}{r}+\frac{b}{s}\in\mathbb{Z}}} e^{2\pi iha/r}e^{2\pi ibj/s}$$

Using our theory of fractions, it is now clear that the condition

$$\frac{a}{r}+\frac{b}{s}\in\mathbb{Z}$$

can hold if and only if $r = s$ and $a = -b$ which gives the limit to be $c_r(h-j)$. We state this for future reference.

Lemma 2.2.1.

$$\sum_{\substack{(a,r)=1,(b,s)=1\\ \frac{a}{r}+\frac{b}{s}\in\mathbb{Z}}} e^{2\pi iha/r}e^{2\pi ibj/s} = c_r(h-j)\delta_{r,s}$$

where $\delta_{r,s}$ is the Kronecker delta function.

2.3 Triple convolution of Ramanujan sums

We will now consider "triple convolutions." More precisely, let us consider

$$\sum_{n\le x} c_r(n)c_s(n+h)c_t(n+j) = \sum_{(a,r)=1}\sum_{(b,s)=1} e^{2\pi i hb/s}\sum_{(c,t)=1} e^{2\pi i jc/t}\sum_{n\le x} e^{2\pi i n(a/r+b/s+c/t)}. \tag{2.3.1}$$

Let us first look at the case $r = s = t$. In here, the innermost sum is bounded unless $a+b+c = 0 \pmod r$ **and** $b+c$ is coprime to r. Therefore, in this case, we have

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x} c_r(n)c_r(n+h)c_r(n+j) = \sum_{\substack{(b,r)=(c,r)=1\\(b+c,r)=1}} e^{2\pi i(hb+jc)/r}.$$

This motivates the study of a two-variable variant of the Ramanujan sum.

Definition 2.3.1. For positive intergers r, h, and j the two-variable variant of the Ramanujan sum is defined as

$$\mathscr{K}_r(h,j) := \sum_{\substack{(b,r)=(c,r)=1\\(b+c,r)=1}} e^{2\pi i(hb+jc)/r}. \tag{2.3.2}$$

This function is worthy of independent study. For the case when $r = p$ is prime, the condition $b+c$ coprime to p is always satisfied unless $c = p-b$, so we have

$$\mathscr{K}_p(h,j) = c_p(h)c_p(j) - c_p(h-j).$$

Since $c_p(n) = c_p(-n)$, this is a symmetric function of h and j as it should be. It should be possible to derive similar formulas in the general case.

Let us consider the case when r, s, t are not all equal. The innermost sum in (2.3.1) is bounded unless $a/r + b/s + c/t$ is an integer. This means that

$$\frac{a}{r} + \frac{bt+cs}{st}$$

is an integer. We will first study the case that the three numbers r, s, t are mutually coprime. Then

$$\frac{a}{r} + \frac{bt+cs}{st}$$

being an integer implies that $r = st$ from the earlier discussion because then $bt+cs$ is coprime to st. Similarly, $s = rt$ and $t = rs$ from which we conclude $r = s = t = 1$. Thus, we have:

Theorem 2.3.1. *If r, s, t are mutually coprime, then*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} c_r(n)c_s(n+h)c_t(n+j) = 0$$

unless $r = s = t = 1$, in which case the limit is 1.

This result holds in greater generality provided the p-adic valuations of r, s, t are distinct for some prime p. We examine this in the later sections.

2.3.1 Multiplicativity of $\mathscr{K}_r(h, j)$

In this section, we will prove that $\mathscr{K}_r(h, j)$ is a multiplicative function of r.

Lemma 2.3.2. *Let $(m, n) = 1$. Then $(b, m) = (c, n) = 1$, if and only if $(bn + cm, mn) = 1$.*

Proof. Let $n = \prod_p p^{v_p(n)}$ and $m = \prod_q q^{v_q(m)}$ be the prime factorization of n and m, where p's and q's are distinct since $(m, n) = 1$. Suppose that $(b, m) = (c, n) = 1$, and $(bn+cm, mn) = d$. Then, we have $d = \prod_{p,q} p^{v_p(d)}q^{v_q(d)}$ where $0 \leq v_p(d) \leq v_p(n)$ and $0 \leq v_q(d) \leq v_q(m)$. For some p, one can write $p^{v_p(d)}|bn + cm$ and so $p^{v_p(d)}|c$ which implies $p^{v_p(d)}|(c, n)$. It is possible only if $v_p(d) = 0$. Similarly, we can show that $v_q(d) = 0$, that is $d = 1$. Conversely, assume that $(bn + cm, mn) = 1$ and $(b, m) = d$, then $d|b$ and $d|m$. This implies $d|bn+cm$ and $d|mn$. Therefore, we have $(bn + cm, mn) = d$ which gives $d = 1$. Similarly, one can prove $(c, n) = 1$. □

Remark 2.3.1. Let $(m, n) = 1$ and $(b, m) = (c, n) = 1$. If $1 \leq b \leq m$ and $1 \leq c \leq n$, then we have $\phi(mn)$ choices for $bn + cm$. The lemma shows that each of these numbers will be coprime to mn.

Remark 2.3.2. For $(m, n) = 1$, one can represent every coprime residue class of mn as $bn + cm$ where b and c belong to a coprime residue class of m and n, respectively.

Theorem 2.3.3. *If $(m, n) = 1$, then*

$$\mathscr{K}_{mn}(h, j) = \mathscr{K}_m(h, j)\mathscr{K}_n(h, j). \tag{2.3.3}$$

That is, $\mathscr{K}_r(h, j)$ is a multiplicative function of r.

Proof. From (2.3.2), we have

$$\mathscr{K}_{mn}(h, j) = \sum_{\substack{(b,mn)=(c,mn)=1 \\ (b+c,mn)=1}} e^{2\pi i(hb+jc)/mn},$$

Next, from Remark 2.3.1, Remark 2.3.2 and Lemma 2.3.2, we see that for every b and c, we can find b_1, b_2, c_1, c_2 such that $b = b_1 n + b_2 m$, $c = c_1 n + c_2 m$, and $(b_1, m) = (c_1, m) = (b_2, n) = (c_2, n) = 1$. This implies

$$\begin{aligned}\mathscr{K}_{mn}(h, j) &= \sum_{\substack{(b_1,m)=(c_1,m)=1 \\ (b_1+c_1,m)=1}} e^{2\pi i(hb_1+jc_1)/m} \sum_{\substack{(b_2,n)=(c_2,n)=1 \\ (b_2+c_2,n)=1}} e^{2\pi i(hb_2+jc_2)/n} \\ &= \mathscr{K}_m(h, j)\mathscr{K}_n(h, j).\end{aligned}$$

□

2.3.2 Orthogonality property of $\mathscr{K}_r(h, j)$

In this section, we show the principle of orthogonality for $\mathscr{K}_r(h, j)$.

Lemma 2.3.4. *We have*

$$\lim_{\substack{x\to\infty \\ y\to\infty}} \frac{1}{xy} \sum_{\substack{h\le x \\ j\le y}} \mathscr{K}_r(h, j)\overline{K_s(h, j)} = f(r)\delta_{r,s},$$

where $\delta_{r,s}$ is the Kronecker delta function and

$$f(r) = \sum_{\substack{(b,r)=(c,r)=1 \\ (b+c,r)=1}} 1. \tag{2.3.4}$$

Proof. From (2.3.2), we have

$$\begin{aligned}\sum_{\substack{h\le x\\ j\le y}}\mathscr{K}_r(h,j)\overline{K_s(h,j)} &= \sum_{\substack{h\le x\\ j\le y}}\sum_{\substack{(b,r)=(c,r)=1\\ (b+c,r)=1}}\sum_{\substack{(b',s)=(c',s)=1\\ (b'+c',s)=1}} e^{2\pi i\left(\frac{hb+jc}{r}-\frac{hb'+jc'}{s}\right)}\\ &= \sum_{\substack{h\le x\\ j\le y}}\sum_{\substack{(b,r)=(c,r)=1\\ (b+c,r)=1}}\sum_{\substack{(b',s)=(c',s)=1\\ (b'+c',s)=1}} e^{2\pi i(h(b/r-b'/s)+j(c/r-c'/s))}\\ &= \sum_{\substack{(b,r)=(c,r)=1\\ (b+c,r)=1}}\sum_{\substack{(b',s)=(c',s)=1\\ (b'+c',s)=1}}\sum_{h\le x} e^{2\pi i(h(b/r-b'/s)}\sum_{j\le y} e^{2\pi i(j(c/r-c'/s)}.\end{aligned} \tag{2.3.5}$$

The innermost sum is unbounded unless $(b/r-b'/s),(c/r-c'/s)\in\mathbb{Z}$. This implies $r=s$, $b\equiv b' \bmod r$ and, $c\equiv c' \bmod r$. Therefore, we have

$$\sum_{h\le x} e^{2\pi i(h(b/r-b'/s)} = x+O(1)$$

and

$$\sum_{j\le y} e^{2\pi i(j(c/r-c'/s)} = y+O(1),$$

otherwise it is a bounded by $O(f(r)f(s))$. Compiling all results and substituting in (2.3.5) yields,

$$\sum_{\substack{h\le x\\ j\le y}}\mathscr{K}_r(h,j)\overline{K_s(h,j)} = xyf(r)+O(f(r)f(s)).$$

Dividing the above result by xy and taking $x\to\infty$ and $y\to\infty$ gives the required Lemma 2.3.4. □

2.3.3 Fractions revisited

Since the innermost sum of (2.3.1) is bounded unless $a/r+b/s+c/t$ is an integer. Therefore, to simplify this condition for other cases, it may be useful to think of a fraction as a rational number with "poles" at the prime divisors of the denominator. This motivates the following "partial fraction expansion" of a rational number.

Lemma 2.3.5. *Let*

$$q = \prod_p p^{v_p(q)}$$

be the unique factorization of q as a product of prime powers. If $1 \le a < q$, we can write

$$\frac{a}{q} = \sum_{p|q} m_p p^{-v_p(q)} \pmod 1,$$

where $0 \le m_p < p^{v_p(q)}$ and this representation is unique.

Proof. We write $Q_p = q/p^{v_p(q)}$. Since the greatest common divisor of all the Q_p's is 1, we have, by the Euclidean algorithm, integers x_p such that

$$a = \sum_{p|q} x_p Q_p$$

and dividing by q gives

$$\frac{a}{q} = \sum_{p|q} x_p p^{-v_p(q)}.$$

Now we choose m_p satisfying $0 \le m_p < p^{v_p(q)}$ such that $m_p \equiv x_p \pmod{p^{v_p(q)}}$ which completes the proof. □

This lemma also explains why we could prove Theorem 2.3.1 when r, s, t are mutually coprime relatively quickly. In that case, the "poles" of each fraction do not interfere with each other, and the only way the "poles" can all disappear is if $r = s = t = 1$. That is, there are no "poles" to start with.

Since we will be working $\pmod 1$, we can drop the condition $p|q$ in the summation of the lemma. Thus, we can write any fraction as

$$\frac{a}{q} = \sum_p x_p(a) p^{-v_p(q)} \pmod 1$$

and speak of the "order of the pole" at p as $v_p(q)$. Now, the condition of the innermost sum of (2.3.1) is that for every prime p,

$$x_p(a)p^{-v_p(r)} + x_p(b)p^{-v_p(s)} + x_p(c)p^{-v_p(t)} = 0 \pmod 1. \tag{2.3.6}$$

If $v_p(r), v_p(s), v_p(t)$ are all distinct for some prime p, this condition is never satisfied and so we get:

Theorem 2.3.6. *If for some prime p, $v_p(r), v_p(s), v_p(t)$ are all distinct, then*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} c_r(n)c_s(n+h)c_t(n+j) = 0.$$

In other words, the only way we can have a possible non-zero limit is when, for every prime p, at least two of $v_p(r), v_p(s), v_p(t)$ are equal.

2.3.4 The squarefree case

When r, s, t are all squarefree, then the set of values of $v_p(r), v_p(s), v_p(t)$ can only be 0 or 1. Viewing this as a triple, $(v_p(r), v_p(s), v_p(t))$ for which (2.3.6) holds, the only possibilities are $(1,1,1),(1,1,0),(1,0,1),(0,1,1)$. In any case, we can define

$$\mathscr{K}_{r,s,t}(k,h,j) := \sum_{\substack{(a,r)=1,(b,s)=1,(c,t)=1 \\ \frac{a}{r}+\frac{b}{s}+\frac{c}{t}\in\mathbb{Z}}} \exp\left(2\pi i \left(\frac{ak}{r} + \frac{bh}{s} + \frac{cj}{t}\right)\right), \tag{2.3.7}$$

and determine the conditions when this function is non-zero. This determination should be sufficient to extend the heuristic mentioned in the introduction to the Hardy-Littlewood 3-triple conjecture in view of Hardy's formula

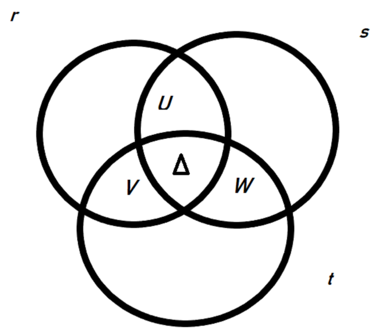
$$\frac{\phi(n)\Lambda(n)}{n} = \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)} c_q(n), \tag{2.3.8}$$

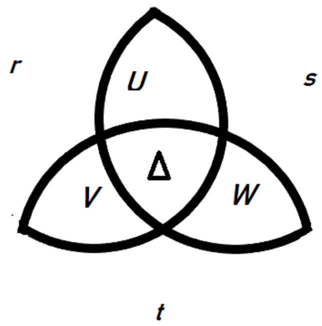
where the summation is over squarefree q. In this way, we seem to be getting multivariable generalizations of the classical Ramanujan sums that are new and worthy of independent study.

2.3.5 The evaluation of $\mathscr{K}_{r,s,t}(k,h,j)$

We will confine our attention to the case that r, s, t are all squarefree. As noted above, the integrality condition forces either all $v_p(r), v_p(s), v_p(t)$ are equal to 1 or exactly two are equal to 1 for every prime divisor p of rst. This suggests we write

$\Delta = (r, s, t)$ and $U = (r, s)/\Delta$, $V = (r, t)/\Delta$ and $W = (s, t)/\Delta$. The Venn diagrams below may help the reader visualize the situation. The circles represent the set of prime divisors of r, s, t, respectively. Applying our theory of fractions, Figure 1 shows that r, s, t cannot have prime factors outside of Δ, U, V, W, and upon close analysis (Theorem 2.3.7), our diagram shrinks to Figure 2.





Indeed, using our theory of fractions (along with the Chinese remainder theorem), we can separate the sum (2.3.7) into parts corresponding to Δ, U, V and W. We then find:

Theorem 2.3.7. *Let r, s, t be squarefree with $(a, r) = (b, s) = (c, t) = 1$. Then*

$$\frac{a}{r} + \frac{b}{s} + \frac{c}{t} = 0 \pmod 1 \tag{2.3.9}$$

implies $r = \Delta UV$, $s = \Delta UW$, and $t = \Delta VW$ with Δ, U, V, W all mutually coprime.

Proof. As noted earlier, using our theory of fractions, we see that if r has a prime divisor not dividing ΔUV, it gives rise to a "pole" on the left-hand side of (2.3.9). The same argument applies to s and t. □

Theorem 2.3.8. *Let r, s, t be squarefree with $(a, r) = (b, s) = (c, t) = 1$. Suppose (2.3.9) holds. Then,*

$$\mathscr{K}_{r,s,t}(k, h, j) = \mathscr{K}_\Delta(h - k, j - k)c_U(h - j)c_V(j - k)c_W(h - k),$$

where the c_U, c_V, c_W are Ramanujan sums and $\mathscr{K}_r$ is given by (2.3.2).

Proof. We have already noted that

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}c_r(n+k)c_s(n+h)c_t(n+j)=\mathscr{K}_{r,s,t}(k,h,j).$$

This limit is zero unless the conditions of Theorem 2.3.7 are met. Since r is squarefree and equal to ΔUV by the previous theorem, we see by the multiplicativity of the Ramanujan sum that

$$c_r(n)=c_\Delta(n)c_U(n)c_V(n).$$

Similarly, $c_s(n)=c_\Delta(n)c_U(n)c_W(n)$ and $c_t(n)=c_\Delta(n)c_V(n)c_W(n)$. The integrality condition, along with our earlier results, completes the proof. □

We can now supply the heuristic argument for the Hardy-Littlewood prime 3-tuple conjecture, which we do in the next section.

2.4 A heuristic derivation of the Hardy-Littlewood 3-tuple conjecture

By partial summation, the Hardy-Littlewood 3-tuple conjecture is equivalent to

$$\sum_{n\le x}\Lambda(n)\Lambda(n+h)\Lambda(n+j)\sim x\prod_p\left(1-\frac{b(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-3}, \tag{2.4.1}$$

where $b(p)$ is the size of the image of $\{0,h,j\} \pmod p$. The product on the right-hand side of (2.4.1) can be rewritten as

$$\prod_p\frac{p^2(p-b(p))}{(p-1)^3}.$$

Our objective is to present a heuristic proof of (2.4.1) by employing the convolution of Ramanujan sums. First, we observe that

$$\Upsilon:=\sum_{n\le x}\Lambda(n)\Lambda(n+h)\Lambda(n+j)\sim\sum_{n\le x}\frac{\phi(n)}{n}\Lambda(n)\frac{\phi(n+h)}{n+h}\Lambda(n+h)\frac{\phi(n+j)}{n+j}\Lambda(n+j).$$

To see this, we need only note that the sum on the left hand side is negligible if $n \leq x^{1-\epsilon}$ for any $\epsilon > 0$. Thus, the sum on the right hand side can also be restricted to $x^{1-\epsilon} \leq n \leq x$ and in this interval, we have

$$\frac{\phi(n+a_i)}{n+a_i}\Lambda(n+a_i) \sim \Lambda(n+a_i).$$

where the summation is over squarefree q. Next, inserting the Ramanujan-Fourier series of Hardy from(2.3.8), we, therefore, expect using Theorem 2.3.7,

$$\frac{\Upsilon}{x} \sim \sum_{r,s,t} \frac{\mu(r)\mu(s)\mu(t)}{\phi(r)\phi(s)\phi(t)} \mathscr{K}_{r,s,t}(0,h,j).$$

Using Theorems 2.3.8 and 2.3.7, the right hand side can be written as

$$\sum_{\Delta,U,V,W} \mu^2(\Delta UVW)\frac{\mu(\Delta)^3}{\phi(\Delta)^3}\frac{\mu^2(U)\mu^2(V)\mu^2(W)}{\phi(U)^2\phi(V)^2\phi(W)^2}\mathscr{K}_\Delta(h,j)c_U(h-j)c_V(j)c_W(h),$$

where the term $\mu^2(\Delta UVW)$ ensures that Δ, U, V, W are all mutually coprime as required by Theorem 2.3.7. Writing

$$f_d(h,j); = \sum_{UVW=d} c_U(h-j)c_V(j)c_W(h), \tag{2.4.2}$$

we can rewrite our sum in a simpler way as

$$\sum_{m=1}^{\infty} \frac{\mu(m)}{\phi(m)^2} \sum_{\Delta d = m} \frac{\mu(d)}{\phi(\Delta)} \mathscr{K}_\Delta(h,j) f_d(h,j).$$

The inner sum is a multiplicative function of m. For $m = p$ a prime, we have that the inner sum is

$$\frac{\mathscr{K}_p(h,j)}{p-1} - \mathscr{K}_1(h,j)f_p(h,j).$$

Thus the sum can be written as a product:

$$\prod_p \left(1 - \frac{1}{(p-1)^2}\left\{\frac{c_p(h)c_p(j) - c_p(h-j)}{p-1} - (c_p(h-j) + c_p(h) + c_p(j))\right\}\right).$$

Let us first consider the case $b(p) = 3$, that is, when $0, h, j$ are all distinct mod p. In particular, h, j and $h - j$ are all coprime to p. Using our formulas for each of these terms, we find the Euler factor is

$$\frac{p^2(p-3)}{(p-1)^3}$$

as predicted. If $b(p) = 1$, then p divides both h and j. Again, the Euler factor turns out to be

$$\frac{p^2}{(p-1)^2}$$

as predicted by the conjecture. Finally, when $b(p) = 2$, then possible cases are p divides h and j is coprime to p or p divides j and h is coprime to p or both h and j are coprime to p and p divides $h - j$. In all cases, the Euler factor is

$$\frac{p^2(p-2)}{(p-1)^3}.$$

Thus, in all cases, the Euler factor is consistent with that predicted by the Hardy-Littlewood conjecture.

Remark 2.4.1. This method involves leveraging exponential sums to estimate the triple convolution of Ramanujan sums. Unfortunately, the generalization of this approach for the case when $k > 3$ leads to a calculation of exponential sums of several variables, which is not easy to solve. Therefore, when $k > 3$, we adopt a different method, which leads to the study of certain arithmetical functions of several variables.

2.5 A synoptic view of arithmetical functions of several variables

The theory of arithmetical functions of several variables was initiated by Vaidyanathswamy [67] in 1927. Apart from sporadic and isolated results, no formal theory has emerged and it seems timely to delineate such a theory. Several expositions will assist us in developing the theory such as the one by Tòth [66].

An arithmetical function of several variables is a map $f : \mathbb{N}^k \to \mathbb{C}$. We will use vector notation as much as possible. Thus $\underline{n}$ will denote the k-tuple $(n_1, \cdots, n_k)$. Following [67], we will say the vector $\underline{d}$ divides $\underline{n}$ (and write $\underline{d}|\underline{n}$) if $d_i|n_i$ for $1 \le i \le k$. The constant function $\underline{1}$ is simply the function that assigns the value 1 for every k-tuple. We will write $\underline{n}/\underline{d}$ to mean the vector $(n_1/d_1, ..., n_k/d_k)$.

We define the Möbius function $\underline{\mu}$ by

$$\underline{\mu}(\underline{n}) := \mu(n_1) \cdots \mu(n_k),$$

where μ is the classical Möbius function. We then have the generalization of the Möbius inversion formula:

$$f(\underline{n}) = \sum_{\underline{d}|\underline{n}} g(\underline{d}) \iff g(\underline{n}) = \sum_{\underline{d}|\underline{n}} \underline{\mu}(\underline{d}) f(\underline{n}/\underline{d}).$$

There are several ways to generalize the notion of a multiplicative function of a single variable to the several variable context. In 1931, Vaidyanathaswamy [67] was the first to give the definition that is suitable for our purposes. Selberg [60] seems to have rediscovered this definition much later in 1977 in his paper dealing with extensions of the large sieve.

We say a function f is multiplicative if

$$f(m_1, ..., m_k) f(n_1, ..., n_k) = f(m_1 n_1, ..., m_k n_k)$$

provided $(m_1 \cdots m_k, n_1 \cdots n_k) = 1$. With this definition, it is **not true** that if we fix one component, n_1 (say), then $f(n_1, ..., n_k)$ is a multiplicative function in the remaining variables $n_2, ..., n_k$. (Selberg says otherwise on pages 233-234 in his paper [60] and as he does not use this, the results of his paper are unaffected.) For instance, the Ramanujan sum $c_q(n)$ is a multiplicative function of q for fixed n but is not a multiplicative function of n for fixed q. However, $c_q(n)$ is a multiplicative function of two variables q, n as we have defined it above using Vaidyanathaswamy's definition. In particular, $\underline{\mu}$ is multiplicative and generally, a multiplicative function f is completely determined by its values $f(p^{v_1}, ..., p^{v_k})$ for every prime p and every tuple $(v_1, ..., v_k) \in \mathbb{N}^k$.

It is not hard to see that if f and g are multiplicative, then so is their Dirichlet convolution $f \star g$ defined as

$$(f \star g)(n) = \sum_{\underline{d}|\underline{n}} f(\underline{d})g(\underline{n}/\underline{d}).$$

For multiplicative functions f, we can introduce a formal Dirichlet series of several variables along with an Euler product:

$$\sum_{\underline{n}=\underline{1}}^{\infty} \frac{f(n_1, ..., n_k)}{n_1^{s_1} \cdots n_k^{s_k}} = \prod_p \left(\sum_{v_1,...,v_k=0}^{\infty} \frac{f(p^{v_1}, ..., p^{v_k})}{p^{v_1 s_1} \cdots p^{v_k s_k}} \right).$$

2.6 Generalized Chinese Remainder Theorem

We will use the following variant of the classical Chinese remainder theorem. The familiar version is often stated when the $d_1, ..., d_k$ are pairwise coprime. It is a simple exercise to derive this general version from the classical version (see for example, the inductive proof on page 155 of [61]).

Lemma 2.6.1. *For a fixed set $T = \{a_1, \cdots, a_k\}$ and $d_1, \cdots, d_k \in \mathbb{Z}$, the system*

$$\begin{aligned} x &\equiv a_1 \mod d_1 \\ &\vdots \\ x &\equiv a_k \mod d_k \end{aligned} \tag{2.6.1}$$

has a solution if and only if $(d_i, d_j)|(a_i - a_j)$ for all $1 \leq i, j \leq k$. When the solution exists, it is unique modulo $[d_1, \cdots, d_k]$.

Proof. Our proof is direct and more conceptual than the one in [61]. For a prime p, let $v_p(n)$ be the largest power of p dividing n. Then, the system of congruences (2.6.1) is equivalent to $x \equiv a_i \pmod{p^{v_p(d_i)}}$ for $1 \leq i \leq k$ and all primes p dividing the lcm $[d_1, ..., d_k]$. Therefore, it suffices to prove the theorem when all the d_i are the powers of a single prime p. The result is now self-evident since the existence of a solution implies that $(d_i, d_j)|(a_i - a_j)$ for all $1 \leq i, j \leq k$. For the converse, the condition that $(d_i, d_j)|(a_i - a_j)$ for all $1 \leq i, j \leq k$ implies the compatability of the a_i. That is, if $v_p(d_i) \le v_p(d_j)$, then a_j is indeed a "lift" $\pmod{d_j}$ of a_i as required. □

From now on, we will fix T and define a function

$$g(d_1, \cdots, d_k) := \begin{cases} 1 & \text{if (2.6.1) has a solution,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.6.2}$$

2.7 Higher convolutions of Ramanujan sums

We will generalize the Carmichael's orthogonality limit Theorem 1.2.1 in the following way. Let $T = \{a_1, a_2, ..., a_k\}$ be a given multiset of integers. Then, the limit

$$f(q_1, ..., q_k) := \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} c_{q_1}(n+a_1)\cdots c_{q_k}(n+a_k).$$

exists and can be evaluated as follows. From (1.0.2), we have

$$f(q_1, ..., q_k) = \lim_{x\to\infty} \frac{1}{x} \sum_{d_1|q_1,...,d_k|q_k} d_1\mu\left(\frac{q_1}{d_1}\right)\cdots d_k\mu\left(\frac{q_k}{d_k}\right) \sum_{\substack{n\le x \\ d_1|a_1+n,\cdots,d_k|a_k+n}} 1.$$

Therefore, from (2.6.2) we have

$$f(q_1, ..., q_k) := \sum_{d_1|q_1,...,d_k|q_k} d_1\mu\left(\frac{q_1}{d_1}\right)\cdots d_k\mu\left(\frac{q_k}{d_k}\right) \frac{g(d_1, ..., d_k)}{[d_1, ..., d_k]}, \tag{2.7.1}$$

Since $g(d_1, ..., d_k)$ is multiplicative, we see that $f(n_1, ..., n_k)$ is multiplicative. This proves the following generalization of Carmichael's theorem.

Theorem 2.7.1. *For fixed integers $a_1, \cdots, a_k$ and $q_1, \cdots, q_k$, we have*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} c_{q_1}(n+a_1)\cdots c_{q_k}(n+a_k) = \sum_{d_1|q_1,...,d_k|q_k} d_1\mu\left(\frac{q_1}{d_1}\right)\cdots d_k\mu\left(\frac{q_k}{d_k}\right) \frac{g(d_1, ..., d_k)}{[d_1, ..., d_k]}.$$

Since the function on the right-hand side of the above theorem is a multiplicative function, it suffices to determine the values $f(p^{v_1}, ..., p^{v_k})$ for a fixed prime p. For our application, we need this when all the v_i are less than or equal to 1. We will derive the required formula in the next section.

It is worth highlighting that in the case $k = 2$, our Theorem 2.7.1 agrees with Carmichael's theorem. Indeed, to verify this, we must compute explicitly $f(q_1, q_2)$ and ascertain its identity with Carmichael's limit. That is, we must check $f(q_1, q_2) = 0$ if $q_1 \neq q_2$ and $c_q(h)$ when $q_1 = q_2 = q$. By multiplicativity, it suffices to determine $f(p^a, p^b)$ for a fixed prime p. Without any loss of generality, we may suppose that $a \leq b$. The sum (2.7.1) has only four terms corresponding to $d_1 = p^a$ or p^{a-1} and $d_2 = p^b$ or p^{b-1}. In the case $a \leq b-1$, the summation is easily checked to be zero. In the case $a = b$, the summation is $p^a - p^{a-1} = c_{p^a}(h)$ since $p^a | h$ by the compatibility condition of Lemma 2.6.1 to ensure a solution.

2.7.1 Explicit evaluation of $f(p^{v_1}, ..., p^{v_k})$

We define an equivalence relation on $\{1, 2, ..., k\}$ using T. We say $i \sim j$ if and only if $a_i \equiv a_j \pmod{p}$. This partitions T into equivalence classes C_i. Let $b(p)$ be the number of equivalence classes. Note that this induces an equivalence relation on any subset S of T and the corresponding equivalence classes for S are simply $S \cap C_i$ (some of which can be empty).

Lemma 2.7.2. *For $0 \leq v_i \leq 1$ for $1 \leq i \leq k$, we have*

$$f(p^{v_1}, ..., p^{v_k}) = (-1)^{|S|} + \frac{(-1)^{|S|}}{p} \sum_{C_i} [(1-p)^{|C_i \cap S|} - 1]$$

where $S = \{i : v_i = 1\}$.

Proof. As remarked earlier, the equivalence relation on T induces an equivalence relation on S. From (2.7.1), we see that in the sum for $f(p^{v_1}, ..., p^{v_k})$, the contribution from $d_1 = d_2 = \cdots = d_k = 1$ is $(-1)^{|S|}$. For the remaining tuples of divisors $(d_1, ..., d_k)$, we must have $d_i = p$ for some $i \in S$. If $d_j = p$ for some other $j \neq i$, then we must have j equivalent to i by the definition of our equivalence relation. In other words, the remaining sum can be re-written as

$$\sum_{C_i}\sum_{j=1}^{|C_i\cap S|}\binom{|C_i\cap S|}{j}p^{j-1}(-1)^{|S|-j}=\frac{(-1)^{|S|}}{p}\sum_{C_i}[(1-p)^{|C_i\cap S|}-1]$$

which completes the proof. □

Theorem 2.7.3. *Let $T \pmod p$ have size $b(p)$. Then,*

$$\sum_{v_1,\dots,v_k\geq 0}\frac{\mu(p^{v_1})\cdots\mu(p^{v_k})}{\phi(p^{v_1})\cdots\phi(p^{v_k})}f(p^{v_1},...,p^{v_k})=\left(1-\frac{b(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-k}$$

Proof. To evaluate the sum on the left hand side, we need only consider the terms with $v_i \leq 1$ for all $1 \leq i \leq k$ because the Möbius function vanishes otherwise. We insert our formula for $f(p^{v_1},...,p^{v_k})$ from Lemma 1 into the sum to get

$$\sum_{v_1,\dots,v_k\geq 0}\frac{\mu(p^{v_1})\cdots\mu(p^{v_k})}{\phi(p^{v_1})\cdots\phi(p^{v_k})}\left\{(-1)^{|S|}+\frac{(-1)^{|S|}}{p}\sum_{C_i}[(1-p)^{|C_i\cap S|}-1]\right\},$$

where $S = \{i : v_i = 1\}$ (as before). The first part of the sum is easily evaluated:

$$\sum_{v_1,\dots,v_k\geq 0}\frac{\mu(p^{v_1})\cdots\mu(p^{v_k})}{\phi(p^{v_1})\cdots\phi(p^{v_k})}(-1)^{|S|}=\sum_{j=0}^{k}\binom{k}{j}\frac{1}{(p-1)^j}=\left(1+\frac{1}{p-1}\right)^k. \qquad (2.7.2)$$

The second part of the sum is a bit more delicate. Let $[k]$ denote the set $\{1,2,...,k\}$. Since the product of the Möbius functions is $(-1)^{|S|}$ and the product of the ϕ functions is $(p-1)^{|S|}$, we get

$$\frac{1}{p}\sum_{\emptyset\neq S\subseteq[k]}\frac{1}{(p-1)^{|S|}}\sum_{C_i}[(1-p)^{|C_i\cap S|}-1].$$

We interchange the sums to get

$$\frac{1}{p}\sum_{C_i}\sum_{\emptyset\neq S\subseteq[k]}\frac{1}{(p-1)^{|S|}}[(1-p)^{|C_i\cap S|}-1].$$

We examine the inner sum. Writing $A = C_i \cap S$ we see that $S = A \sqcup B$ (where $\sqcup$

denotes disjoint union) and $B \subseteq [k] \backslash C_i$. Since $|S| = |A| + |B|$, the sum becomes

$$\frac{1}{p} \sum_{C_i} \sum_{A \subseteq C_i} \frac{(1-p)^{|A|} - 1}{(p-1)^{|A|}} \sum_{B \subseteq [k] \backslash C_i} \frac{1}{(p-1)^{|B|}}.$$

The innermost sum is equal to

$$\left(1 + \frac{1}{p-1}\right)^{k-|C_i|}.$$

Now

$$\sum_{A \subseteq C_i} \frac{(1-p)^{|A|} - 1}{(p-1)^{|A|}} = -\left(1 + \frac{1}{p-1}\right)^{|C_i|},$$

because

$$\sum_{A \subseteq C_i} \frac{(1-p)^{|A|}}{(p-1)^{|A|}} = \sum_{A \subseteq C_i} (-1)^{|A|} = 0.$$

Putting everything together gives

$$-\frac{1}{p} \sum_{C_i} \left(1 + \frac{1}{p-1}\right)^{|C_i|} \left(1 + \frac{1}{p-1}\right)^{k-|C_i|} = -\frac{b(p)}{p} \left(1 + \frac{1}{p-1}\right)^{k}.$$

Combining this with the first part (2.7.2) gives the desired result:

$$\left(1 + \frac{1}{p-1}\right)^{k} \left(1 - \frac{b(p)}{p}\right) = \left(1 - \frac{b(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-k}.$$

□

2.8 A heuristic derivation of the Hardy-Littlewood k-tuple conjecture

We can now combine the above discussion and give the promised heuristic derivation of the Hardy-Littlewood prime k-tuple conjecture. By partial summation, the conjecture is easily seen to be equivalent to

$$\sum_{n\leq x}\Lambda(n+a_1)\cdots\Lambda(n+a_k)\sim x\prod_p\left(1-\frac{b(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-k}, \tag{2.8.1}$$

where $b(p)$ is the size of the image of $T \bmod p$.

Our objective is to present a heuristic proof of (2.8.1) by employing the convolution of Ramanujan sums. First, we observe that

$$\Upsilon := \sum_{n\leq x}\Lambda(n+a_1)\cdots\Lambda(n+a_k)\sim\sum_{n\leq x}\frac{\phi(n+a_1)}{n+a_1}\cdots\frac{\phi(n+a_k)}{n+a_k}\Lambda(n+a_1)\cdots\Lambda(n+a_k).$$

This modification enables us to use Hardy's formula

$$\frac{\phi(n)\Lambda(n)}{n}=\sum_{q=1}^{\infty}\frac{\mu(q)}{\phi(q)}c_q(n), \tag{2.8.2}$$

where the summation is over squarefree q. Next, inserting the Ramanujan-Fourier series of Hardy from equation (2.8.2) and ignoring issues of convergence, we expect upon using Theorem 2.7.1,

$$\frac{\Upsilon}{x}\sim\sum_{q_1,\cdots,q_k=1}^{\infty}\frac{\mu(q_1)\cdots\mu(q_k)}{\phi(q_1)\cdots\phi(q_k)}f(q_1,\cdots,q_k).$$

Therefore, the Hardy-Littlewood constant is expected to be equal to

$$\sum_{q_1,\dots,q_k=1}^{\infty}\frac{\mu(q_1)\cdots\mu(q_k)}{\phi(q_1)\cdots\phi(q_k)}f(q_1,\dots,q_k). \tag{2.8.3}$$

We want to show that this agrees with the classical evaluation of this constant as

$$\prod_p\left(1-\frac{b(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-k},$$

where $b(p)$ is the size of the image of $T \pmod p$.

By multiplicativity, the series in (2.8.3) can be written as the Euler product:

$$\prod_p \left(\sum_{v_1,\dots,v_k \geq 0} \frac{\mu(p^{v_1})\cdots\mu(p^{v_k})}{\phi(p^{v_1})\cdots\phi(p^{v_k})} f(p^{v_1},\dots,p^{v_k}) \right).$$

We now examine the p-Euler factor and evaluate explicitly $f(p^{v_1},\dots,p^{v_k})$ for $0 \leq v_i \leq 1$ for $1 \leq i \leq k$. Let us henceforth fix p, then from Theorem 2.7.3, we obtain the required result.

2.9 Triple convolution of the Jordan totient function

We can obtain the convolution of other arithmetic functions using the results of the convolution of Ramanujan sums. To illustrate, we estimate the triple convolution of the Jordan totient function.

For a positive integer α, the Jordan totient function is defined as

$$\phi_\alpha(n) := n^\alpha \prod_{p|n} \left(1 - \frac{1}{p^\alpha}\right).$$

When $\alpha = 1$, it coincides with the Euler totient function. In [24], the authors obtained the shifted 2-convolution of the Euler totient function employing the Ramanujan-Fourier series and the orthogonality property of Ramanujan sums. Later, Balasubramanian and Giri [8] derived an asymptotic formula for the weighted shifted 2-convolution of a class of functions sufficiently close to the constant function $f(x) = 1$ for all x using the information about the average value of weight function in arithmetic progressions. Consequently, they also obtained the shifted 2-convolution of the Euler and Jordan totient functions. Alternatively, one can obtain the shifted 2-convolution of the Jordan totient function using the orthogonality property of Ramanujan sums. In this section, we estimate the triple convolution of the Jordan totient function.

Theorem 2.9.1. *For positive integers α, h, and j, we have as x tends to infinity,*

$$\sum_{n\leq x}\frac{\phi_\alpha(n)}{n^\alpha}\frac{\phi_\alpha(n+h)}{(n+h)^\alpha}\frac{\phi_\alpha(n+j)}{(n+j)^\alpha}$$
$$\sim x\prod_{p}\left(1-\frac{3}{p^{\alpha+1}}\right)\prod_{\substack{p\\ b(p)=2}}\left(1+\frac{1}{p^{2\alpha+1}-3p^\alpha}\right)\prod_{\substack{p\\ b(p)=3}}\left(1+\frac{3p^\alpha-1}{p^{3\alpha+1}-3p^{2\alpha}}\right),$$

where $b(p)$ be the number of distinct residue classes (mod p) represented by 0, h and j.

When $\alpha=1$, Theorem 2.9.1 provides the convolution formula for the Euler function. Mirsky has also explored the convolution of the Euler function [44] through a distinct methodology. Mirsky examined a specific class of functions and employed their properties to express the k-convolution of a function as a simultaneous solution to k congruence equations. It is worth noting that Mirsky's method can also be employed to derive Theorem 2.9.1.

Proof. Ramanujan [52] obtained the series expansion of Jordan totient function in terms of Ramanujan sums given by:

$$\frac{\phi_\alpha(n)}{n^\alpha}=\frac{1}{\zeta(\alpha+1)}\sum_{r=1}^{\infty}\frac{\mu(r)}{\phi_{\alpha+1}(r)}c_r(n). \tag{2.9.1}$$

The above Ramanujan-Fourier series is absolutely convergent. This implies

$$\sum_{n\leq x}\frac{\phi_\alpha(n)}{n^\alpha}\frac{\phi_\alpha(n+h)}{(n+h)^\alpha}\frac{\phi_\alpha(n+j)}{(n+j)^\alpha}$$
$$=\frac{1}{\zeta^3(\alpha+1)}\sum_{r,s,t=1}^{\infty}\frac{\mu(r)\mu(s)\mu(t)}{\phi_{\alpha+1}(r)\phi_{\alpha+1}(s)\phi_{\alpha+1}(t)}\sum_{n\leq x}c_r(n)c_s(n+h)c_t(n+j).$$

We define

$$\Gamma:=\sum_{n\leq x}\frac{\phi_\alpha(n)}{n^\alpha}\frac{\phi_\alpha(n+h)}{(n+h)^\alpha}\frac{\phi_\alpha(n+j)}{(n+j)^\alpha}.$$

Next, using Theorem 2.3.7, we have

$$\frac{\Gamma}{x} \sim \frac{1}{\zeta^3(\alpha+1)} \sum_{r,s,t=1}^{\infty} \frac{\mu(r)\mu(s)\mu(t)}{\phi_{\alpha+1}(r)\phi_{\alpha+1}(s)\phi_{\alpha+1}(t)} \mathscr{K}_{r,s,t}(0,h,j).$$

Therefore, using Theorems 2.3.8 and 2.3.7, the right hand side can be written as

$$\frac{1}{\zeta^3(\alpha+1)} \sum_{\Delta,U,V,W} \mu^2(\Delta UVW) \frac{\mu^3(\Delta)\mu^2(U)\mu^2(V)\mu^2(W)}{\phi^3_{\alpha+1}(\Delta)\phi^2_{\alpha+1}(U)\phi^2_{\alpha+1}(V)\phi^2_{\alpha+1}(W)} \mathscr{K}_\Delta(h,j)c_U(h-j)c_V(j)c_W(h),$$

where the term $\mu^2(\Delta UVW)$ ensures that Δ, U, V, W are all mutually coprime as required by Theorem 2.3.7. From (2.4.2), we can rewrite our sum in a simpler way as

$$\frac{1}{\zeta^3(\alpha+1)} \sum_{m=1}^{\infty} \frac{\mu(m)}{\phi^2_{\alpha+1}(m)} \sum_{\Delta d=m} \frac{\mu(d)}{\phi_{\alpha+1}(\Delta)} \mathscr{K}_\Delta(h,j) f_d(h,j).$$

The inner sum is a multiplicative function of m. For $m = p$ a prime, we have that the inner sum is

$$\frac{\mathscr{K}_p(h,j)}{p^{\alpha+1}-1} - \mathscr{K}_1(h,j) f_p(h,j).$$

Thus the sum can be written as a product:

$$\frac{1}{\zeta^3(\alpha+1)} \prod_p \left(1 - \frac{1}{(p^{\alpha+1}-1)^2}\left\{\frac{c_p(h)c_p(j) - c_p(h-j)}{p^{\alpha+1}-1} - (c_p(h-j) + c_p(h) + c_p(j))\right\}\right).$$

Let us first consider the case $b(p) = 3$, that is, when 0, h, j are all distinct mod p. In particular, h, j and $h - j$ are all coprime to p. Using our formulas for each of these terms, we find the Euler factor is

$$\frac{1}{\zeta^3(\alpha+1)} \prod_p \left(1 - \frac{1}{(p^{\alpha+1}-1)^2}\left\{\frac{2}{p^{\alpha+1}-1} + 3\right\}\right).$$

If $b(p) = 1$, then p divides both h and j. Again, the Euler factor turns out to be

$$\frac{1}{\zeta^3(\alpha+1)} \prod_p \left(1 - \frac{1}{(p^{\alpha+1}-1)^2}\left\{\frac{(p-1)(p-2)}{p^{\alpha+1}-1} - 3(p-1)\right\}\right).$$

Finally, when $b(p) = 2$, then possible cases are p divides h and j is coprime to p

or p divides j and h is coprime to p or both h and j are coprime to p and p divides $h-j$. In all cases, the Euler factor is

$$\frac{1}{\zeta^3(\alpha+1)}\prod_p\left(1-\frac{1}{(p^{\alpha+1}-1)^2}\left\{\frac{(2-p)}{p^{\alpha+1}-1}-(p-3)\right\}\right).$$

By combining the above cases, we have

$$\Gamma\sim x\prod_p\left(1-\frac{3}{p^{\alpha+1}}\right)\prod_{\substack{p\\ b(p)=2}}\left(1+\frac{1}{p^{2\alpha+1}-3p^{\alpha}}\right)\prod_{\substack{p\\ b(p)=3}}\left(1+\frac{3p^{\alpha}-1}{p^{3\alpha+1}-3p^{2\alpha}}\right).$$

□

Using Theorem 2.9.1, we conclude that

Corollary 1.

$$\sum_{n\le x}\phi_\alpha(n)\phi_\alpha(n+h)\phi_\alpha(n+j)$$
$$\sim\frac{x^{3\alpha+1}}{3\alpha+1}\prod_p\left(1-\frac{3}{p^{\alpha+1}}\right)\prod_{\substack{p\\ b(p)=2}}\left(1+\frac{1}{p^{2\alpha+1}-3p^{\alpha}}\right)\prod_{\substack{p\\ b(p)=3}}\left(1+\frac{3p^{\alpha}-1}{p^{3\alpha+1}-3p^{2\alpha}}\right),$$

where $b(p)$ is the number of distinct residue classes (mod p) represented by 0, h and j.

Proof. We can write

$$\sum_{n\le x}\phi_\alpha(n)\phi_\alpha(n+h)\phi_\alpha(n+j)=\sum_{n\le x}\frac{\phi_\alpha(n)}{n^\alpha}\frac{\phi_\alpha(n+h)}{(n+h)^\alpha}\frac{\phi_\alpha(n+j)}{(n+j)^\alpha}n^\alpha(n+h)^\alpha(n+j)^\alpha. \tag{2.9.2}$$

From Theorem 2.9.1, we have

$$A(x):=\sum_{n\le x}\frac{\phi_\alpha(n)}{n^\alpha}\frac{\phi_\alpha(n+h)}{(n+h)^\alpha}\frac{\phi_\alpha(n+j)}{(n+j)^\alpha}\sim xC,$$

where

$$C = \prod_{p}\left(1 - \frac{3}{p^{\alpha+1}}\right) \prod_{\substack{p \\ b(p)=2}}\left(1 + \frac{1}{p^{2\alpha+1} - 3p^{\alpha}}\right) \prod_{\substack{p \\ b(p)=3}}\left(1 + \frac{3p^{\alpha} - 1}{p^{3\alpha+1} - 3p^{2\alpha}}\right).$$

Therefore, applying partial summation formula to right side of (2.9.2) yields

$$\begin{aligned}
&\sum_{n\le x}\phi_\alpha(n)\phi_\alpha(n+h)\phi_\alpha(n+j) \\
&= x^\alpha(x+h)^\alpha(x+j)^\alpha A(x) - \int_1^x A(t)\frac{d}{dt}(t^\alpha(t+h)^\alpha(t+j)^\alpha)dt \\
&\sim \left(1 - \frac{3\alpha}{3\alpha+1}\right)x^{3\alpha+1}C = \frac{x^{3\alpha+1}}{3\alpha+1}C.
\end{aligned}$$

□

3

Moments of Ramanujan sums over $\mathbb{Q}$

In this chapter, we estimate the higher moments of averages of Ramanujan sums and Cohen-Ramanujan sums using the Brèteche Tauberian theorem. The question on the average order of moments of Ramanujan sums was first considered by Chan and Kumchev [12] motivated by applications to problems on Diophantine approximations of reals by sums of rational numbers. More precisely, they estimated asymptotic formulas for

$$S_k(x,y) := \sum_{n \le y} \left(\sum_{q \le x} c_q(n) \right)^k \tag{3.0.1}$$

for $k = 1, 2$ using both elementary and analytic techniques. They proved that for $y \ge x$,

$$S_1(x,y) = y - \frac{x^2}{4\zeta(2)} + \mathrm{O}\left(xy^{1/3}\log x + x^3y^{-1}\right). \tag{3.0.2}$$

Also, for $y \geq x^2(\log x)^B$ and for $B > 0$,

$$S_2(x, y) = \frac{yx^2}{2\zeta(2)} + \mathrm{O}\left(x^4 + xy\log x\right),$$

and for $x \leq y \leq x^2(\log x)^B$

$$S_2(x, y) = \frac{yx^2}{2\zeta(2)}(1 + 2\kappa(u)) + \mathrm{O}\left(yx^2 \log^{10} x(x^{-1/2} + (y/x)^{-1/2})\right), \qquad (3.0.3)$$

where $u = \log(yx^{-2})$, and $\kappa(u)$ is defined as

$$\kappa(u) := \frac{1}{2\pi}\int_{-\infty}^{\infty} f(it)e^{-iut}dt, \text{where } f(s) := \frac{\zeta(1-s)}{(1+s)^2(1-s)\zeta(1+s)}.$$

For any $\epsilon > 0$, it satisfies

$$\kappa(u) > -0.4, \quad \kappa(u) \ll \exp -|u|^{3/5-\epsilon}.$$

In particular, $\kappa(u) = o(1)$ as $|u| \to \infty$.

3.1 Main results

The behavior of (3.0.1) for $k \geq 3$ is an interesting question for several reasons. First, since this problem prompts an exploration of the theory of the arithmetical functions of several variables, a study initiated by Vaidyanathaswamy [67] in 1931 (see details in Section 2.5). This theory is still in evolution and several recent papers [11, 20, 57] highlight the importance of developing such a theory. Second, the earlier known result on higher moments of Ramanujan sums is determined to be incorrect. We derive the asymptotic behavior of the moments of Ramanujan sums (3.0.1) for $k \geq 3$. This result is an important step in developing the theory of the arithmetical functions of several variables. More precisely, we prove:

Theorem 3.1.1. *For $k \geq 3$ and $y > x^k$, as $x \to \infty$, we have*

$$S_k(x, y) = yx^kQ(\log x) + \mathrm{O}\left(yx^{k-\theta}\right),$$

where $Q \in \mathbb{R}[X]$ *is a polynomial of exact degree* $2^k - 2k - 1$ *and* $0 \leq \theta \leq 1$.

Ramanujan sums have applications in the study of arithmetic functions and Fourier analysis. Generalizing these sums can provide new tools and methods for solving problems in these areas. Therefore, in [16], Cohen generalized the Ramanujan sums defined as:

$$c_q^\beta(n) := \sum_{\substack{1 \leq j \leq q^\beta \\ (j, q^\beta)_\beta = 1}} e\left(\frac{jn}{q^\beta}\right) = \sum_{\substack{d|q \\ d^\beta | n}} d^\beta \mu\left(\frac{q}{d}\right), \tag{3.1.1}$$

where

$$(j, q^\beta)_\beta = \max\{l^\beta : l^\beta | j \text{ and } l|q\}.$$

We refer to these sums as Cohen Ramanujan sums. He obtained that these sums have several interesting properties of these sums similar to the usual Ramanujan sums. Robles and Roy [55] computed the moments for Cohen Ramanujan sums. However, in their study presented in Proposition 1.1, it is implied for $k > 1$ and $\beta = 1$ that

$$S_k(x, y) = \frac{3yx^2}{\pi^2} + O(yx \log x + x^{2k} \log^k x), \tag{3.1.2}$$

for $y > x^{2k} \log^{k+1} x$. This is correct for $k = 2$, but for $k = 4$, the theorem contradicts itself, as can be seen by a simple application of the Cauchy-Schwarz inequality:

$$S_2(x, y) \leq y^{1/2} S_4(x, y)^{1/2}.$$

If we denote the moments of Cohen-Ramanujan sums by $S_{k,\beta}(x, y) = \sum_{n \leq y} \left(\sum_{q \leq x} c_q^\beta(n)\right)^k$, then, applying our method to obtain the higher moments of these sums, we get:

Theorem 3.1.2. *For* $k \geq 3$ *and* $y > x^{k(\beta+1)/2}$*, as* $x \to \infty$*, we have*

$$S_{k,\beta}(x, y) = yx^{k(\beta+1)/2} Q(\log x) + \mathrm{O}\left(yx^{k(\beta+1)/2 - \theta}\right),$$

where $Q(\log x)$ *is a polynomial of exact degree* $2^k - 2k - 1$ *and* $0 \leq \theta \leq 1$.

3.2 The Brèteche Tauberian theorem

The study of (3.0.1) inevitably leads us into the theory of arithmetical functions of several variables. In the one variable case, the classical Tauberian theorems provide us with asymptotic behaviours of the summatory function of the non-negative arithmetical function of a single variable by relating it with the analytic properties of the associated Dirichlet series. In the multivariable case, a similar theorem exists, but it does not seem to be well-known. The extension of Cauchy's residue theorem for functions of several variables seems to have first been addressed by Leray [39] in 1959 using the language of sheaf theory. Later, in the 1980's, Cassou-Nogues [11] and Sargos [57] derived more precise results that could be applied to counting problems involving arithmetical functions of several variables. We should also mention the work of Lichtin [40] in this regard. In the early part of the 21st century, Brèteche [20] derived a multi-variable version of the Tauberian theorem using classical methods of analytic number theory, and it is this version that we apply to our situation. His theorems in this context are as follows.

Theorem 3.2.1. *Let $f : \mathbb{N}^k \to \mathbb{R}$ be a non-negative function and F the associated Dirichlet series of f defined by*

$$F(\boldsymbol{s}) = F(s_1, \cdots, s_k) = \sum_{n_1,\cdots,n_k=1}^{\infty} \frac{f(n_1, \cdots, n_k)}{n_1^{s_1} \cdots n_k^{s_k}}.$$

Denote by $\mathcal{LR}_k^+(\mathbb{C})$ the set of non-negative $\mathbb{C}$ linear forms from $\mathbb{C}^k$ to $\mathbb{C}$ on $\mathbb{R}_+^k$. Moreover, assume that there exists $(c_1, \cdots, c_k) \in \mathbb{R}_+^k$ such that:

1. *For $\boldsymbol{s} \in \mathbb{C}^k$, $F(s_1, \cdots, s_k)$ is absolutely convergent for $\mathrm{Re}(s_i) > c_i$ for all $1 \le i \le k$.*

2. *There exist a finite family $\mathcal{L} = (l^{(i)})_{1\le i\le q}$ of non-zero elements of $\mathcal{LR}_k^+(\mathbb{C})$, a finite family $(h^{(i)})_{1\le i\le q'}$ of elements of $\mathcal{LR}_k^+(\mathbb{C})$ and $\delta_1, \delta_2, \delta_3 > 0$ such that the function H defined by*

$$H(\boldsymbol{s}) = F(\boldsymbol{s} + \boldsymbol{c}) \prod_{i=1}^{q} l^{(i)}(\boldsymbol{s})$$

has a holomorphic continuation to the domain

$$D(\delta_1, \delta_3) = \big\{ \boldsymbol{s} \in \mathbb{C}^k : \operatorname{Re} l^{(i)}(\boldsymbol{s}) > -\delta_1 \ \textit{ for all } i = 1, \cdots, q \textit{ and } \\ \operatorname{Re} h^{(i)}(\boldsymbol{s}) > -\delta_3 \textit{ for all } i = 1, \cdots, q' \big\},$$

and verifies the estimate: for $\epsilon, \epsilon' > 0$ *we have uniformly in* $\boldsymbol{s} \in D(\delta_1 - \epsilon, \delta_3 - \epsilon')$

$$H(\boldsymbol{s}) \ll \prod_{i=1}^{q} \left(|\operatorname{Im} l^{(i)}(\boldsymbol{s})| + 1\right)^{1-\delta_2 \min\left(0, \operatorname{Re} l^{(i)}(\boldsymbol{s})\right)} (1 + (\operatorname{Im} s_1 + \cdots + \operatorname{Im} s_k)^{\epsilon}).$$

Set $J = J(\mathbb{C}) = \{j \in \{1, \cdots, k\} : c_j = 0\}$. *Denote* w *to be the cardinality of* J *and by* $j_1 < \cdots < j_w$ *its elements in increasing order. Define the* w *linear forms* $l^{(q+i)}$ $(1 \le i \le w)$ *by* $l^{(q+i)}(\boldsymbol{s}) = e^*_{j_i}(\boldsymbol{s}) = s_{j_i}$.

Then, for any $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_k) \in (0, \infty)^k$, *there exist a polynomial* $Q_\beta \in \mathbb{R}[X]$ *of degree at most* $q + w - \mathit{Rank}\{l^{(1)}, \cdots, l^{(q)}\}$ *and* $\theta > 0$ *such that as* $x \to \infty$

$$\sum_{n_1 \le x^{\beta_1}} \cdots \sum_{n_k \le x^{\beta_k}} f(n_1, \cdots, n_k) = x^{<\boldsymbol{c}, \boldsymbol{\beta}>} Q_{\boldsymbol{\beta}}(\log x) + \mathrm{O}\left(x^{<\boldsymbol{c}, \boldsymbol{\beta}> - \theta}\right).$$

Here, $< \cdot, \cdot >$ *denotes the usual dot product in* $\mathbb{R}^k$.

The next theorem gives a determination of the precise degree of the polynomial $Q_{\boldsymbol{\beta}}$ appearing in the previous theorem. Denoting by $\mathbb{R}_*^+$ the set of strictly positive real numbers, the notation $\mathrm{con}^*(\{l^{(1)}, \cdots, l^{(q)}\})$ means $\mathbb{R}_*^+ l^{(1)} + \cdots + \mathbb{R}_*^+ l^{(q)}$.

Theorem 3.2.2. *Let* $f : \mathbb{N}^k \to \mathbb{R}$ *be a non-negative function satisfying the assumptions of Theorem 3.2.1. Let* $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_k) \in (0, \infty)^k$ *and set* $\mathcal{B} = \sum_{i=1}^k \beta_i e_i^* \in \mathcal{LR}_k^+(\mathbb{C})$. *Then, if* $\mathit{Rank}\{l^{(1)}, \cdots, l^{(q)}\} = n$, $H(\boldsymbol{0}) \neq 0$, *and* $\mathcal{B} \in \mathit{con}^*(\{l^{(1)}, \cdots, l^{(q)}\})$, *then* $\deg(Q_{\boldsymbol{\beta}}) = q + w - n$.

3.3 Higher moments of Ramanujan sums

To obtain the higher moments of Ramanujan sums, we use (1.0.2) and substitute in (3.0.1). After changing the order of summation, we arrive at the function

$$f(n_1, ..., n_k) := \sum_{d_1|n_1,\, d_2|n_2,\, ..., d_k|n_k} \mu(n_1/d_1) \cdots \mu(n_k/d_k) g(d_1, ..., d_k) \tag{3.3.1}$$

where

$$g(n_1, ..., n_k) := \frac{n_1 \cdots n_k}{[n_1, ..., n_k]}.$$

Since g is multiplicative, we see that f is multiplicative by our remarks in Section 2.5. We will show that $f(n_1, ..., n_k)$ satisfies all the hypotheses of Theorems 3.2.1 and 3.2.2 and apply these theorems to obtain the average order of the above function.

3.3.1 Dirichlet series of $f(n_1, ..., n_k)$

Theorem 3.3.1. *For the function $f(n_1, \cdots, n_k)$, we have*

$$\sum_{n_1,\cdots,n_k=1}^{\infty} \frac{f(n_1, \cdots, n_k)}{n_1^{s_1} \cdots n_k^{s_k}} = \left(\prod_{\substack{I \subseteq [k] \\ |I| \geq 2}} \zeta(s_I - |I| + 1) \right) E(s_1, \cdots, s_k), \tag{3.3.2}$$

where $[k] := \{1, \cdots, k\}$ and for any subset $I = \{l_1, \cdots, l_r\}$ of $[k]$, we have $s_I := s_{l_1} + \cdots + s_{l_r}$ and $E(s_1, ..., s_k)$ is a Dirichlet series absolutely convergent for $\mathrm{Re}(s_i) > 1 - 1/k$.

Proof. It is easy to prove that a factorization of the form (3.3.2) exists as follows. We first note that $f(d_1, \cdots, d_k)$ is a convolution of multiplicative functions. Therefore, from (3.3.1), we have

$$\begin{aligned} \sum_{n_1,\cdots,n_k=1}^{\infty} \frac{f(n_1, \cdots, n_k)}{n_1^{s_1} \cdots n_k^{s_k}} &= \sum_{d_1,\cdots,d_k=1}^{\infty} \frac{g(d_1, \cdots, d_k)}{d_1^{s_1} \cdots d_k^{s_k}} \sum_{e_1,\cdots,e_k=1}^{\infty} \frac{\mu(e_1) \cdots \mu(e_k)}{e_1^{s_1} \cdots e_k^{s_k}} \\ &= \frac{1}{\zeta(s_1) \cdots \zeta(s_k)} \sum_{d_1,\cdots,d_k=1}^{\infty} \frac{g(d_1, \cdots, d_k)}{d_1^{s_1} \cdots d_k^{s_k}} \end{aligned} \tag{3.3.3}$$

since

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

We examine the series on the right of (3.3.3) as follows.

$$\sum_{d_1,\cdots,d_k=1}^{\infty} \frac{g(d_1, \cdots, d_k)}{d_1^{s_1} \cdots d_k^{s_k}} = \prod_p \left(\sum_{v_1,\cdots,v_k=0}^{\infty} \frac{p^{v_1+\cdots+v_k-\max(v_1,\cdots,v_k)}}{p^{s_1 v_1+\cdots+s_k v_k}} \right)$$

$$= \prod_p \left(\sum_{n=0}^{\infty} p^{-n} \sum_{\substack{v_1,\cdots,v_k=0 \\ \max(v_1,\cdots,v_k)=n}}^{\infty} \frac{p^{v_1+\cdots+v_k}}{p^{s_1v_1+\cdots+s_kv_k}} \right). \tag{3.3.4}$$

The Euler factor can be written as

$$1 + p^{-1} \sum_{\substack{v_1,\cdots,v_k=0 \\ \max(v_1,\cdots,v_k)=1}}^{\infty} \frac{p^{v_1+\cdots+v_k}}{p^{s_1v_1+\cdots+s_kv_k}} + \sum_{n=2}^{\infty} p^{-n} \sum_{\substack{v_1,\cdots,v_k=0 \\ \max(v_1,\cdots,v_k)=n}}^{\infty} \frac{p^{v_1+\cdots+v_k}}{p^{s_1v_1+\cdots+s_kv_k}}. \tag{3.3.5}$$

The inner sum in the second summation is actually a finite sum with at most $(n+1)^k$ terms and with $\sigma = \mathrm{Re}(s_i)$, it is easily estimated to be

$$\ll (n+1)^k p^{kn(1-\sigma)}.$$

This means that

$$\sum_p \sum_{n=2}^{\infty} p^{-n} \sum_{\substack{v_1,\cdots,v_k=0 \\ \max(v_1,\cdots,v_k)=n}}^{\infty} \frac{p^{v_1+\cdots+v_k}}{p^{s_1v_1+\cdots+s_kv_k}}$$

converges absolutely for $\mathrm{Re}(s_i) > 1-1/k$. We can therefore factor the Euler product in (3.3.4) to get

$$\sum_{d_1,\cdots,d_k=1}^{\infty} \frac{g(d_1,\cdots,d_k)}{d_1^{s_1}\cdots d_k^{s_k}} = \left(\prod_p \left(1 + p^{-1} \sum_{\substack{v_1,\cdots,v_k=0 \\ \max(v_1,\cdots,v_k)=1}}^{\infty} \frac{p^{v_1+\cdots+v_k}}{p^{s_1v_1+\cdots+s_kv_k}} \right) \right) E^*(s_1,...,s_k),$$

where $E^*(s_1,...,s_k)$ is a Dirichlet series absolutely convergent in $\mathrm{Re}(s_i) > 1-\frac{1}{k}$. The Euler product above can be analyzed as follows. The p-Euler factor can be written as

$$1 + \frac{1}{p} \sum_{\emptyset \neq I \subseteq [k]} \prod_{i \in I} pT_i^{v_i}$$

where $T_i = p^{-s_i}$. This observation allows us to further factor the Euler product as

$$\left(\prod_{\emptyset \neq I \subseteq [k]} \zeta(s_I - |I| + 1) \right) E^{**}(s_1,...,s_k)$$

where $E^{**}(s_1, ..., s_k)$ is a Dirichlet series absolutely convergent for $\mathrm{Re}(s_i) > 1/2$. Combining all these observations and setting

$$E(s_1, ..., s_k) = E^*(s_1, ..., s_k)E^{**}(s_1, ..., s_k),$$

we obtain

$$\sum_{d_1,\cdots,d_k=1}^{\infty} \frac{g(d_1, \cdots, d_k)}{d_1^{s_1} \cdots d_k^{s_k}} = \left(\prod_{\emptyset \neq I \subseteq [k]} \zeta(s_I - |I| + 1) \right) E(s_1, ..., s_k).$$

Taking into account (3.3.3) and noting that the singleton sets are removed from our product of zeta functions, we obtain (3.3.2), as claimed. □

Remark 3.3.1. Though it is not needed for our purposes, we can determine $E(s_1, ..., s_k)$ very explicitly:

$$E(s_1, \cdots, s_k) = \prod_p \frac{\sum_{\emptyset \neq I \subseteq [k]} (-1)^{|I|} (p^{|I| - s_I} - 1) \prod_{\substack{\emptyset \neq J \subseteq [k] \\ I \neq J}} (1 - p^{|J| - s_J - 1})}{1 + \sum_{\emptyset \neq I \subseteq [k]} (-1)^{|I|} p^{|I| - s_I}}$$

To see this, let $p^{-s_1} = T_1, \cdots, p^{-s_k} = T_k$ as before. Then,

$$\begin{aligned}
\sum_{\substack{v_1,\cdots,v_k=0 \\ \max(v_1,\cdots,v_k)=n}}^{\infty} \frac{p^{v_1+\cdots+v_k}}{p^{s_1 v_1 + \cdots + s_k v_k}} &= \sum_{\substack{v_1,\cdots,v_k=0 \\ \max(v_1,\cdots,v_k)\leq n}}^{\infty} p^{v_1+\cdots+v_k} T_1^{v_1} \cdots T_k^{v_k} \\
&\quad - \sum_{\substack{v_1,\cdots,v_k=0 \\ \max(v_1,\cdots,v_k)\leq n-1}}^{\infty} p^{v_1+\cdots+v_k} T_1^{v_1} \cdots T_k^{v_k} \\
&= \prod_{i=1}^{k} \left(\sum_{v_i \leq n} p^{v_i} T_i^{v_i} \right) - \prod_{i=1}^{k} \left(\sum_{v_i \leq n-1} p^{v_i} T_i^{v_i} \right) \\
&= \prod_{i=1}^{k} \left(\frac{1 - (pT_i)^{n+1}}{1 - pT_i} \right) - \prod_{i=1}^{k} \left(\frac{1 - (pT_i)^{n}}{1 - pT_i} \right). \qquad (3.3.6)
\end{aligned}$$

Now, multiplying this by p^{-n} and summing from $n = 0$ to ∞ gives, (using the abbreviation $T_I = p^{-s_I}$),

$$\frac{1}{\prod_{i=1}^k(1-pT_i)}\sum_{n=0}^{\infty}\sum_{\emptyset\neq I\subseteq[k]}(-1)^{|I|}\left(p^{n(|I|-1)+|I|}T_I^{n+1}-p^{n(|I|-1)}T_I^n\right)$$

$$=\frac{1}{\prod_{i=1}^k(1-pT_i)}\sum_{\emptyset\neq I\subset[k]}(-1)^{|I|}\left(\frac{p^{|I|}T_I}{1-p^{|I|-1}T_I}-\frac{1}{1-p^{|I|-1}T_I}\right)$$

$$=\frac{1}{\prod_{i=1}^k(1-pT_i)}\sum_{\emptyset\neq I\subseteq[k]}(-1)^{|I|}\left(\frac{p^{|I|}T_I-1}{1-p^{|I|-1}T_I}\right)$$

We therefore have from (3.3.4) and (3.3.6), and the above calculation that

$$\sum_{d_1,\cdots,d_k=1}^{\infty}\frac{g(d_1,\cdots,d_k)}{d_1^{s_1}\cdots d_k^{s_k}}=\prod_p\frac{1}{\prod_{i=1}^k(1-pT_i)}\sum_{\emptyset\neq I\subseteq[k]}(-1)^{|I|}\left(\frac{p^{|I|}T_I-1}{1-p^{|I|-1}T_I}\right)$$
$$=\left(\prod_{\emptyset\neq I\subseteq[k]}\zeta(s_I-|I|+1)\right)E(s_1,\cdots,s_k). \tag{3.3.7}$$

Here,

$$E(s_1,\cdots,s_k)=\prod_p\frac{\sum_{\emptyset\neq I\subset[k]}(-1)^{|I|}(p^{|I|}T_I-1)\prod_{\substack{\emptyset\neq J\subset[k]\\ I\neq J}}(1-p^{|J|-1}T_J)}{\prod_{i=1}^k(1-pT_i)}$$
$$=\prod_p\frac{\sum_{\emptyset\neq I\subseteq[k]}(-1)^{|I|}(p^{|I|}T_I-1)\prod_{\substack{\emptyset\neq J\subseteq[k]\\ I\neq J}}(1-p^{|J|-1}T_J)}{1+\sum_{\emptyset\neq I\subseteq[k]}(-1)^{|I|}p^{|I|}T_I}.$$

From this explicit expression, the region of absolute convergence for $E(s_1,...,s_k)$ is not immediately clear and thus, we have opted for the more expedient method in the proof of our theorem.

3.3.2 Non-negativity of $f(n_1,...,n_k)$

In this section, we will show that $f(n_1,...,n_k)$ is a non-negative function, thus paving our way for an application of the Brèteche Tauberian theorem.

Theorem 3.3.2. $f(n_1,...,n_k)\geq 0$ *for all* $(n_1,...,n_k)\in\mathbb{N}^k$.

Proof. Since f is a convolution of two multiplicative functions, it is also multiplicative. Therefore, to prove the lemma, it suffices to show for each prime p,

$$f(p^{v_1}, p^{v_2}, ..., p^{v_k}) \geq 0, \qquad v_1, ..., v_k \geq 0.$$

Since f is symmetric, we can suppose without any loss of generality that $v_1 \geq v_2 \geq \cdots \geq v_k$. We proceed by induction on k. For $k = 1$, the result is clear. We may also suppose that all $v_i \geq 1$ for otherwise, we are again done by induction. If $v_1 > v_2$, then noting that

$$f(p^{v_1}, p^{v_2}, ..., p^{v_k}) = \sum_{d_2|p^{v_2}, ..., d_k|p^{v_k}} \mu(d_2) \cdots \mu(d_k) \left\{ g\left(p^{v_1}, \frac{p^{v_2}}{d_2}, ..., \frac{p^{v_k}}{d_k}\right) - g\left(p^{v_1 - 1}, \frac{p^{v_2}}{d_2}, ..., \frac{p^{v_k}}{d_k}\right) \right\}, \tag{3.3.8}$$

we have

$$g\left(p^{v_1}, \frac{p^{v_2}}{d_2}, ..., \frac{p^{v_k}}{d_k}\right) = g\left(1, \frac{p^{v_2}}{d_2}, ..., \frac{p^{v_k}}{d_k}\right)\left(p^{v_1}, \left[\frac{p^{v_2}}{d_2}, ..., \frac{p^{v_k}}{d_k}\right]\right)$$

and

$$g\left(p^{v_1 - 1}, \frac{p^{v_2}}{d_2}, ..., \frac{p^{v_k}}{d_k}\right) = g\left(1, \frac{p^{v_2}}{d_2}, ..., \frac{p^{v_k}}{d_k}\right)\left(p^{v_1 - 1}, \left[\frac{p^{v_2}}{d_2}, ..., \frac{p^{v_k}}{d_k}\right]\right).$$

We see in this case that the gcd in both cases is the same and so the term in braces in (3.3.8) above is zero. Now suppose that $v_1 = v_2 = \cdots v_\ell > v_{\ell+1} \geq \cdots \geq v_k$. We have

$$f(p^{v_1}, ..., p^{v_k}) = \sum_{d_1|p, ..., d_k|p} \mu(d_1) \cdots \mu(d_k) g(p^{v_1}/d_1, ..., p^{v_k}/d_k).$$

Noting that in the sum over divisors that each d_i can only be 1 or p, we arrange the sum as follows. We write $d_i = p^{e_i}$ where $e_i = 0$ or 1. We can then identify each tuple $(d_1, ..., d_k)$ with a subset $I \subseteq [k]$ where $I = \{i : e_i = 1\}$. Our sum becomes

$$f(p^{v_1}, ..., p^{v_k}) = \sum_{I \subseteq [k]} (-1)^{|I|} p^{s - |I| - \max(v_i - e_i : 1 \leq i \leq k)},$$

where

$$s = v_1 + \cdots + v_k.$$

We let $I_0 = \{1, 2, ..., \ell\}$ and set $s' = v_2 + \cdots + v_k$. We split the sum on the right into three parts:

$$\sum_{I:I\cap I_0=\emptyset} \quad + \quad \sum_{I:\emptyset\neq I\cap I_0\neq I_0} \quad + \quad \sum_{I:I\supseteq I_0} .$$

Letting $J = \{\ell+1, \cdots, k\}$, the first part is equal to

$$\sum_{I\subseteq J}(-1)^{|I|}p^{s'-|I|} = p^{s'}\left(1-\frac{1}{p}\right)^{k-\ell},$$

because in this case $\max(v_i - e_i : 1 \leq i \leq k) = v_1$. In the second part, we again have $\max(v_i - e_i : 1 \leq i \leq k) = v_1$ so that the second part is equal to

$$\sum_{I:\emptyset\neq I\cap I_0\neq I_0}(-1)^{|I|}p^{s'-|I|} = p^{s'}\sum_{j=1}^{\ell-1}\binom{\ell}{j}(-1)^j p^{-j}\left(1-\frac{1}{p}\right)^{k-\ell}.$$

Finally, in the third part, $\max(v_i - e_i : 1 \leq i \leq k) = v_1 - 1$ so that the third part equals

$$\sum_{I:I\supseteq I_0}(-1)^{|I|}p^{s'+1-|I|} = p^{s'+1-\ell}(-1)^{\ell}\left(1-\frac{1}{p}\right)^{k-\ell}.$$

Notice that the first part and the second part combine to give

$$p^{s'}\sum_{j=0}^{\ell-1}\binom{\ell}{j}(-1)^j p^{-j}\left(1-\frac{1}{p}\right)^{k-\ell},$$

since the term corresponding to $j = 0$ is the contribution from the first part. Putting everything together gives

$$p^{s'}\sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^j p^{-j}\left(1-\frac{1}{p}\right)^{k-\ell} - (-1)^{\ell}p^{s'-\ell}\left(1-\frac{1}{p}\right)^{k-\ell} + p^{s'+1-\ell}(-1)^{\ell}\left(1-\frac{1}{p}\right)^{k-\ell}.$$

This simplifies to

$$p^{s'}\left(1-\frac{1}{p}\right)^{k} + (-1)^{\ell}\left(1-\frac{1}{p}\right)^{k-\ell}\left(p^{s'+1-\ell} - p^{s'-\ell}\right).$$

Noting that $\ell \geq 2$, we see that this is certainly positive if ℓ is even. If ℓ is odd, the term is equal to

$$p^{s'}\left(1-\frac{1}{p}\right)^{k}-\left(1-\frac{1}{p}\right)^{k-\ell}\left(p^{s'+1-\ell}-p^{s'-\ell}\right).$$

We easily see that this reduces to checking that

$$p^{s'}(p-1)^{\ell}\geq p^{\ell}\left(p^{s'+1-\ell}-p^{s'-\ell}\right)=p^{s'}(p-1),$$

which is evidently true. This completes the proof of non-negativity. □

3.3.3 Average order of $f(n_1,...,n_k)$

The average order of the function $f(n_1,...,n_k)$ is the most essential step in the proof of Theorem 3.1.1. In this section, we estimate an average of $f(n_1,...,n_k)$ as an application of Theorems 3.2.1 and 3.2.2.

Theorem 3.3.3. *For $0<\theta<1$, we have as $x\to\infty$*

$$\sum_{n_1,\cdots,n_k\leq x} f(n_1,...,n_k)=x^kQ(\log x)+\mathrm{O}\left(x^{k-\theta}\right),$$

where $Q\in\mathbb{R}[X]$ is a polynomial of exact degree 2^k-2k-1.

Proof. In Theorem 3.3.2, we proved that $f(n_1,...,n_k)$ is non-negative and in Theorem 3.3.1, we proved $f(n_1,...,n_k)$ has an absolutely convergent series $F(\boldsymbol{s})$ for $\mathrm{Re}(s_i)>1$ for all $1\leq i\leq k$. This shows $f(n_1,...,n_k)$ satisfies (1) of Theorem 3.2.1. Next, we show that $f(n_1,...,n_k)$ also satisfies (2) of Theorem 3.2.1. Write $\mathbf{1}=(1,\cdots,1)$ then, $F(\boldsymbol{s}+\mathbf{1})$ is an absolutely convergent series for $\mathrm{Re}(s_i)>0$. Therefore, for the linear forms $\prod_{\substack{I\subseteq[k]\\|I|\geq 2}} s_I$, define the function

$$H(\boldsymbol{s}):=F(\boldsymbol{s}+\mathbf{1})\prod_{\substack{I\subseteq[k]\\|I|\geq 2}} s_I.$$

Since $c_i=1$ for all $1\leq i\leq k$, we take $q'=0$ in the notation of Theorem 3.2.1. Furthermore, for any $\zeta(s_1+\cdots+s_\ell+1)$ in $F(\boldsymbol{s}+\mathbf{1})$, there is a linear form $(s_1+\cdots+s_\ell)$ such that

$$(s_1+\cdots+s_\ell)\zeta(s_1+\cdots+s_\ell+1)$$

has analytic continuation on the plane $\mathrm{Re}\,(s_1 + \cdots + s_\ell) > -\epsilon$, where $\epsilon > 0$ and for $I \subseteq K$, we have $|I| = \ell \geq 2$. Therefore, $H(\boldsymbol{s})$ also has analytic continuation on the plane $\mathrm{Re}\,(s_1 + \cdots + s_\ell) > -\epsilon$. Consider $h^i(\boldsymbol{s}) = s_i$, set $\delta_1 = \delta_3 = \epsilon$. Moreover, from Lemma 3.3.1, $E(\boldsymbol{s} + \mathbf{1})$ has analytic continuation on the plane $\mathrm{Re}\,(s_1 + \cdots + s_\ell) > -\epsilon$. We know that for $\mathrm{Re}\, s_i > -1$ and for all $\epsilon_0 > 0$

$$s_I \zeta(1 + s_I) \ll_{\epsilon_0} (1 + |s_I|)^{1 - \frac{1}{2}\min(0, \mathrm{Re}\, s_I) + \epsilon_0}.$$

The above argument shows that $H(\boldsymbol{s})$ satisfies (2) of Theorem 3.2.1 with $\delta_2 = 1/2$. Therefore, we have as $x \to \infty$,

$$\sum_{n_1, \cdots, n_k \leq x} f(n_1, ..., n_k) = x^k Q(\log x) + \mathrm{O}\left(x^{k-\theta}\right),$$

where $Q(\log x)$ is a polynomial of degree at most $2^k - 2k - 1$. Next, $c_i > 0$ for all $1 \leq i \leq k$, this implies $w = 0$. Again, it is easy to see that the rank of the collection of linear forms s_I is k and the interior of the cone generated by linear forms is the set $\mathcal{B} = \sum_{i=1}^{k} \beta_i \mathbf{e}_i^*$ for $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_k) \in (0, \infty)^k$ and $\mathbf{e}_i^*(\boldsymbol{s}) = s_i$. Also, as $s_I \to 0$, $s_I \zeta(s_I + 1) \to 1$ and hence $H(\mathbf{0}) \neq 0$ Thus, from Theorem 3.2.2, $\deg(Q) = 2^k - 2k - 1$. This gives the required result. □

3.3.4 Proof of Theorem 3.1.1

In this section, we provide a proof of Theorem 3.1.1 using the average order of $f(n_1, ..., n_k)$ obtained in the previous section.

Proof of Theorem 3.1.1. From the definition of Ramanujan sums, we have

$$\begin{aligned} S_k(x, y) &= \sum_{n \leq y} \left(\sum_{q \leq x} c_q(n) \right)^k = \sum_{n \leq y} \sum_{q_1, \cdots, q_k \leq x} \sum_{\substack{d_1 | q_1 \\ d_1 | n}} d_1 \mu\left(\frac{q_1}{d_1}\right) \cdots \sum_{\substack{d_k | q_k \\ d_k | n}} d_k \mu\left(\frac{q_k}{d_k}\right) \\ &= \sum_{q_1, \cdots, q_k \leq x} \sum_{d_1 | q_1, \cdots, d_k | q_k} d_1 \cdots d_k \mu\left(\frac{q_1}{d_1}\right) \cdots \mu\left(\frac{q_k}{d_k}\right) \sum_{\substack{n \leq y \\ [d_1, \cdots, d_k] | n}} 1 \\ &= \sum_{q_1, \cdots, q_k \leq x} \sum_{d_1 | q_1, \cdots, d_k | q_k} d_1 \cdots d_k \mu\left(\frac{q_1}{d_1}\right) \cdots \mu\left(\frac{q_k}{d_k}\right) \left(\frac{y}{[d_1, \cdots, d_k]} + \mathrm{O}(1)\right) \end{aligned}$$

$$= y \sum_{q_1,\cdots,q_k \le x} \sum_{d_1|q_1,\cdots,d_k|q_k} \frac{d_1 \cdots d_k}{[d_1,\cdots,d_k]} \mu\left(\frac{q_1}{d_1}\right) \cdots \mu\left(\frac{q_k}{d_k}\right) + \mathrm{O}\left(x^{2k}\right).$$

The inner sum is precisely $f(q_1,...,q_k)$ in our notation by virtue of (3.3.1). Thus, from Theorem 3.3.3, for $0<\theta<1$, we have as $x \to \infty$

$$S_k(x,y) = yx^k Q(\log x) + \mathrm{O}\left(yx^{k-\theta} + x^{2k}\right),$$

where $Q(\log x)$ is a polynomial of degree $2^k - 2k - 1$. This completes the proof. □

3.4 Moments of Cohen Ramanujan sums

We prove Theorem 3.1.2 by modifying the proof of Theorem 3.1.1, taking into account the differences in the definitions between Ramanujan sums and the Cohen-Ramanujan sums. From (1.3.3), in this context, the function we will study is

$$f_\beta(n_1,...,n_k) := \sum_{d_1|n_1,\, d_2|n_2,\, ...,d_k|n_k} \mu(n_1/d_1)\cdots\mu(n_k/d_k) g_\beta(d_1,...,d_k) \tag{3.4.1}$$

where

$$g_\beta(n_1,...,n_k) := \frac{n_1^\beta \cdots n_k^\beta}{[n_1^\beta,...,n_k^\beta]}.$$

We will follow the same steps used in the proof of Theorem 3.1.1.

3.4.1 Dirichlet series of $f_\beta(n_1,...,n_k)$

Theorem 3.4.1. *For the function $f_\beta(n_1,\cdots,n_k)$, we have*

$$\sum_{n_1,\cdots,n_k=1}^{\infty} \frac{f_\beta(n_1,\cdots,n_k)}{n_1^{s_1}\cdots n_k^{s_k}} = \prod_{\substack{I\subseteq[k]\\ |I|\ge 2}} \zeta(s_I - \beta|I| + \beta) E_\beta(s_1,\cdots,s_k),$$

where $[k]=\{1,\cdots,k\}$ and for any subset $I=\{l_1,\cdots,l_r\}$ of $[k]$, $s_I = s_{l_1}+\cdots+s_{l_r}$ and

$$E_\beta(s_1,\cdots,s_k) = \prod_p \frac{\sum_{\emptyset\ne I\subset[k]} (-1)^{-|I|}(p^{\beta|I|-s_I}-1)\prod_{\substack{\emptyset\ne J\subset[k]\\ I\ne J}}(1-p^{\beta(|J|-1)-s_J})}{1+\sum_{\emptyset\ne I\subset[k]}(-1)^{-|I|}p^{\beta|I|-s_I}}.$$

which is convergent for $\mathrm{Re}(s_i) > 1 - 1/k$.

Proof. Note that $f_\beta(d_1, \cdots, d_k)$ is a convolution of multiplicative functions. Therefore, from (3.3.1) we have

$$\sum_{n_1,\cdots,n_k=1}^{\infty} \frac{f_\beta(n_1,\cdots,n_k)}{n_1^{s_1}\cdots n_k^{s_k}} = \sum_{d_1,\cdots,d_k=1}^{\infty} \frac{g_\beta(d_1,\cdots,d_k)}{d_1^{s_1}\cdots d_k^{s_k}} \sum_{n_1,\cdots,n_k=1}^{\infty} \frac{\mu(n_1)\cdots\mu(n_k)}{n_1^{s_1}\cdots n_k^{s_k}}$$
$$= \frac{1}{\zeta(s_1)\cdots\zeta(s_k)} \sum_{d_1,\cdots,d_k=1}^{\infty} \frac{g_\beta(d_1,\cdots,d_k)}{d_1^{s_1}\cdots d_k^{s_k}} \tag{3.4.2}$$

Therefore, we estimate the series

$$\sum_{d_1,\cdots,d_k=1}^{\infty} \frac{g_\beta(d_1,\cdots,d_k)}{d_1^{s_1}\cdots d_k^{s_k}} = \prod_p \left(\sum_{v_1,\cdots,v_k=0}^{\infty} \frac{p^{\beta(v_1+\cdots+v_k-\max(v_1,\cdots,v_k))}}{p^{s_1v_1+\cdots+s_kv_k}} \right)$$
$$= \prod_p \left(\sum_{n=0}^{\infty} p^{-\beta n} \sum_{\substack{v_1,\cdots,v_k=0 \\ \max(v_1,\cdots,v_k)=n}}^{\infty} \frac{p^{\beta(v_1+\cdots+v_k)}}{p^{s_1v_1+\cdots+s_kv_k}} \right). \tag{3.4.3}$$

Let $p^{-s_1} = T_1, \cdots, p^{-s_k} = T_k$ The innermost sum of the above can be written as

$$\sum_{\substack{v_1,\cdots,v_k=0 \\ \max(v_1,\cdots,v_k)=n}}^{\infty} \frac{p^{\beta(v_1+\cdots+v_k)}}{p^{s_1v_1+\cdots+s_kv_k}} = \prod_{i=1}^{k} \left(\frac{1-(p^\beta T_i)^{n+1}}{1-p^\beta T_i} \right) - \prod_{i=1}^{k} \left(\frac{1-(p^\beta T_i)^{n}}{1-p^\beta T_i} \right). \tag{3.4.4}$$

Denote $T_I = p^{-s_I}$, therefore from (3.4.3) and (3.4.4), we have

$$\sum_{d_1,\cdots,d_k=1}^{\infty} \frac{g_\beta(d_1,\cdots,d_k)}{d_1^{s_1}\cdots d_k^{s_k}} = \prod_{\emptyset\neq I\subset[k]} \zeta(s_I - \beta|I| + \beta) E_\beta(s_1,\cdots,s_k). \tag{3.4.5}$$

We have the explicit value of $E_\beta(s_1,\cdots,s_k)$ given by

$$E_\beta(s_1,\cdots,s_k) = \prod_p \frac{\sum_{\emptyset\neq I\subset[k]} (-1)^{-|I|} (p^{\beta|I|}T_I - 1) \prod_{\substack{\emptyset\neq J\subset[k] \\ I\neq J}} (1-p^{\beta(|J|-1)}T_J)}{\prod_{i=1}^{k}(1-p^\beta T_i)}$$
$$= \prod_p \frac{\sum_{\emptyset\neq I\subset[k]} (-1)^{-|I|} (p^{\beta|I|}T_I - 1) \prod_{\substack{\emptyset\neq J\subset[k] \\ I\neq J}} (1-p^{\beta(|J|-1)}T_J)}{1+\sum_{\emptyset\neq I\subset[k]} (-1)^{-|I|} p^{\beta|I|} T_I}.$$

From (3.4.2) and (3.4.5), we obtain the required result. □

3.4.2 Non-negativity of $f_\beta(n_1, ..., n_k)$

In this section, we prove that $f_\beta(n_1, ..., n_k)$ is non-negative for all $(n_1, ..., n_k) \in \mathbb{N}^k$.

Theorem 3.4.2. *$f_\beta(n_1, ..., n_k) \geq 0$ for all $(n_1, ..., n_k) \in \mathbb{N}^k$.*

Proof. Since f_β is a convolution of two multiplicative functions, it is also multiplicative. Therefore, to prove the lemma, it suffices to show for each prime p,

$$f_\beta(p^{v_1}, p^{v_2}, ..., p^{v_k}) \geq 0, \qquad v_1, ..., v_k \geq 0.$$

Since f_β is symmetric, we can suppose without any loss of generality that $v_1 \geq v_2 \geq \cdots \geq v_k$. We proceed by induction on k. For $k = 1$, the result is clear. We may also suppose that all $v_i \geq 1$ for otherwise, we are again done by induction. If $v_1 > v_2$, then noting that

$$f_\beta(p^{v_1}, p^{v_2}, ..., p^{v_k}) = \tag{3.4.6}$$
$$\sum_{d_2|p^{v_2},...,d_k|p^{v_k}} \mu(d_2)\cdots\mu(d_k)\left\{g_\beta\left(p^{v_1}, \frac{p^{v_2}}{d_2}, ..., \frac{p^{v_k}}{d_k}\right) - g_\beta\left(p^{v_1-1}, \frac{p^{v_2}}{d_2}, ..., \frac{p^{v_k}}{d_k}\right)\right\},$$

we have

$$g_\beta\left(p^{v_1}, \frac{p^{v_2}}{d_2}, ..., \frac{p^{v_k}}{d_k}\right) = g_\beta\left(1, \frac{p^{v_2}}{d_2}, ..., \frac{p^{v_k}}{d_k}\right)\left(p^{\beta v_1}, \left[\frac{p^{\beta v_2}}{d_2^\beta}, ..., \frac{p^{\beta v_k}}{d_k^\beta}\right]\right)$$

and

$$g_\beta\left(p^{v_1-1}, \frac{p^{v_2}}{d_2}, ..., \frac{p^{v_k}}{d_k}\right) = g_\beta\left(1, \frac{p^{v_2}}{d_2}, ..., \frac{p^{v_k}}{d_k}\right)\left(p^{\beta(v_1-1)}, \left[\frac{p^{\beta v_2}}{d_2^\beta}, ..., \frac{p^{\beta v_k}}{d_k^\beta}\right]\right).$$

We see in this case that the gcd in both the cases is the same, so the term in braces in (3.4.6) above is zero. Now suppose that $v_1 = v_2 = \cdots = v_\ell > v_{\ell+1} \geq \cdots \geq v_k$. We have

$$f_\beta(p^{v_1}, ..., p^{v_k}) = \sum_{d_1|p,...,d_k|p} \mu(d_1)\cdots\mu(d_k) g_\beta(p^{v_1}/d_1, ..., p^{v_k}/d_k).$$

Noting that in the sum over divisors that each d_i can only be 1 or p, we arrange the sum as follows. We write $d_i = p^{e_i}$ where $e_i = 0$ or 1. We can then identify each tuple $(d_1, ..., d_k)$ with a subset $I \subseteq [k]$ where $I = \{i : e_i = 1\}$. Our sum becomes

$$f_\beta(p^{v_1}, ..., p^{v_k}) = \sum_{I \subseteq [k]} (-1)^{|I|} p^{\beta s - \beta |I| - \max(\beta(v_i - e_i):1 \le i \le k)},$$

where

$$s = v_1 + \cdots + v_k.$$

We let $I_0 = \{1, 2, ..., \ell\}$ and set $s' = v_2 + \cdots + v_k$. We split the sum on the right into three parts:

$$\sum_{I: I \cap I_0 = \emptyset} + \sum_{I: \emptyset \neq I \cap I_0 \neq I_0} + \sum_{I: I \supseteq I_0}.$$

Letting $J = \{\ell + 1, \cdots, k\}$, the first part is equal to

$$\sum_{I \subseteq J} (-1)^{|I|} p^{\beta s' - \beta |I|} = p^{\beta s'} \left(1 - \frac{1}{p^\beta}\right)^{k-\ell},$$

because in this case $\max(\beta(v_i - e_i) : 1 \le i \le k) = \beta v_1$. In the second part, we again have $\max(\beta(v_i - e_i) : 1 \le i \le k) = \beta v_1$ so that the second part is equal to

$$\sum_{I: \emptyset \neq I \cap I_0 \neq I_0} (-1)^{|I|} p^{\beta s' - \beta |I|} = p^{\beta s'} \sum_{j=1}^{\ell - 1} \binom{\ell}{j} (-1)^j p^{-j\beta} \left(1 - \frac{1}{p^\beta}\right)^{k-\ell}.$$

Finally, in the third part, $\max(\beta(v_i - e_i) : 1 \le i \le k) = \beta(v_1 - 1)$ so that the third part equals

$$\sum_{I: I \supseteq I_0} (-1)^{|I|} p^{\beta(s' + 1 - |I|)} = p^{\beta(s' + 1 - \ell)} (-1)^\ell \left(1 - \frac{1}{p^\beta}\right)^{k-\ell}.$$

Notice that the first part and the second part combine to give

$$p^{\beta s'} \sum_{j=0}^{\ell - 1} \binom{\ell}{j} (-1)^j p^{-j\beta} \left(1 - \frac{1}{p^\beta}\right)^{k-\ell},$$

since the term corresponding to $j = 0$ is the contribution from the first part. Putting everything together and simplify it gives

$$p^{\beta s'}\left(1-\frac{1}{p^\beta}\right)^k+(-1)^\ell\left(1-\frac{1}{p^\beta}\right)^{k-\ell}\left(p^{\beta(s'+1-\ell)}-p^{\beta(s'-\ell)}\right).$$

Noting that $\ell \geq 2$, we see that this is certainly positive if ℓ is even. If ℓ is odd, the term is equal to

$$p^{\beta s'}\left(1-\frac{1}{p^\beta}\right)^k-\left(1-\frac{1}{p^\beta}\right)^{k-\ell}\left(p^{\beta(s'+1-\ell)}-p^{\beta(s'-\ell)}\right).$$

We see that this reduces to checking that

$$p^{\beta s'}(p^\beta-1)^\ell \geq p^{\beta\ell}\left(p^{\beta(s'+1-\ell)}-p^{\beta(s'-\ell)}\right)=p^{\beta s'}(p^\beta-1),$$

which is evidently true. □

3.4.3 Average order of $f_\beta(n_1,...,n_k)$

Theorem 3.4.3. *For $0<\theta<1$, we have as $x\to\infty$*

$$\sum_{n_1,\cdots,n_k\leq x} f_\beta(n_1,...,n_k)=x^{k(1+\beta)/2}Q(\log x)+\mathrm{O}\left(x^{k-\theta}\right),$$

where $Q(\log x)$ is a polynomial of degree 2^k-2k-1.

Proof. To prove the Theorem, we show that $f_\beta(n_1,...,n_k)$ satisfies all assumptions of Theorems 3.2.1 and 3.2.2. In Theorem 3.4.2, we prove that $f_\beta(n_1,...,n_k)$ is non-negative and in Theorem 3.4.1, we show $f_\beta(n_1,...,n_k)$ has an absolutely convergent series $F_\beta(\boldsymbol{s})$ for $\mathrm{Re}(s_i)>(1+\beta)/2$ for all $1\leq i\leq k$. Moreover, this implies for $\mathbf{c}=((1+\beta)/2,\cdots,(1+\beta)/2)$, $F(\boldsymbol{s}+\mathbf{c})$ is absolutely convergent series for $\mathrm{Re}(s_i)>((1-\beta)/2)$. Consider the linear forms $\prod_{\substack{I\subseteq[k]\\|I|=2}} s_I\prod_{\substack{I\subseteq[k]\\|I|\geq 3}}\left(s_I+|I|(1-\beta)/2+\beta-1\right)$. Define the function

$$H(\boldsymbol{s}):=F(\boldsymbol{s}+\mathbf{c})\prod_{\substack{I\subseteq[k]\\|I|\geq 2}}\left(s_I+\frac{|I|(1-\beta)}{2}+\beta-1\right).$$

Since $c_i=(1+\beta)/2$ for all $1\leq i\leq k$, therefore $q'=0$. Furthermore, for every zeta function in $F(\boldsymbol{s}+\boldsymbol{c})$, there is a linear form such that the product of linear form and zeta function has analytic continuation on the plane $\mathrm{Re}\,(s_1 + \cdots + s_\ell) > -\epsilon$, where $\epsilon > 0$ and for $I \subseteq K$, we have $|I| = \ell \geq 2$. Therefore, $H(\boldsymbol{s})$ also has analytic continuation on the plane $\mathrm{Re}\,(s_1 + \cdots + s_\ell) > -\epsilon$. Consider $h^i(\boldsymbol{s}) = s_i$, set $\delta_1 = \delta_3 = \epsilon$. Moreover, from Lemma 3.3.1, $E(\boldsymbol{s} + \boldsymbol{c})$ is has analytic continuation on the plane $\mathrm{Re}\,(s_1 + \cdots + s_\ell) > -\epsilon$. We know that for $\mathrm{Re}\, s_i > -1$ and for all $\epsilon_0 > 0$

$$s_I \zeta(1 + s_I) \ll_{\epsilon_0} (1 + |s_I|)^{1 - \frac{1}{2} \min(0, \mathrm{Re}\, s_I) + \epsilon_0}.$$

The above argument shows that $H(\boldsymbol{s})$ satisfies (2) of Theorem 3.2.1 with $\delta_2 = 1/2$. Therefore, we have

$$\sum_{n_1, \cdots, n_k \leq x} f_\beta(n_1, ..., n_k) = x^{k(1+\beta)/2} Q(\log x) + \mathrm{O}\left(x^{k-\theta}\right),$$

where $Q(\log x)$ is a polynomial of degree at most $2^k - 2k - 1$ and $x \to \infty$. Moreover, $c_i > 0$ for all $1 \leq i \leq k$, this implies $w = 0$. Again, it is easy to see that the rank of linear forms is k and the interior of the cone generated by linear forms is the set $\mathcal{B} = \sum_{i=1}^{k} \beta_i \mathbf{e}_i^*$ for $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_k) \in (0, \infty)^k$ and $\mathbf{e}_i^*(\boldsymbol{s}) = s_i$. Also, as $s_I \to 0$, $s_I \zeta(s_I + 1) \to 1$ and hence $H(\mathbf{0}) \neq 0$ Thus, from Theorem 3.2.2, $\deg(Q) = 2^k - 2k - 1$. This gives the required result. □

3.4.4 Proof of Theorem 3.1.2

Proof. From the definition of Cohen Ramanujan sums (1.3.3), we have

$$\begin{aligned} S_{k,\beta}(x, y) &= \sum_{n \leq y} \sum_{q_1, \cdots, q_k \leq x} \sum_{\substack{d_1 | q_1 \\ d_1^\beta | n}} d_1^\beta \mu\left(\frac{q_1}{d_1}\right) \cdots \sum_{\substack{d_k | q_k \\ d_k^\beta | n}} d_k^\beta \mu\left(\frac{q_k}{d_k}\right) \\ &= \sum_{q_1, \cdots, q_k \leq x} \sum_{d_1 | q_1, \cdots, d_k | q_k} d_1^\beta \cdots d_k^\beta \mu\left(\frac{q_1}{d_1}\right) \cdots \mu\left(\frac{q_k}{d_k}\right) \sum_{\substack{n \leq y \\ [d_1^\beta, \cdots, d_k^\beta] | n}} 1 \\ &= \sum_{q_1, \cdots, q_k \leq x} \sum_{d_1 | q_1, \cdots, d_k | q_k} d_1^\beta \cdots d_k^\beta \mu\left(\frac{q_1}{d_1}\right) \cdots \mu\left(\frac{q_k}{d_k}\right) \\ &\quad \times \left(\frac{y}{[d_1^\beta, \cdots, d_k^\beta]} + \mathrm{O}\,(1)\right) \end{aligned}$$

$$= y \sum_{q_1,\cdots,q_k \le x} \sum_{d_1|q_1,\cdots,d_k|q_k} \frac{d_1^\beta \cdots d_k^\beta}{[d_1^\beta,\cdots,d_k^\beta]} \mu\left(\frac{q_1}{d_1}\right) \cdots \mu\left(\frac{q_k}{d_k}\right) + \mathrm{O}\left(x^{(\beta+1)k}\right).$$

Next, form Theorem 3.4.3, for $0 < \theta < 1$, we have as $x \to \infty$

$$S_{k,\beta}(x,y) = yx^{k(1+\beta)/2} Q(\log x) + \mathrm{O}\left(yx^{k-\theta} + x^{(\beta+1)k}\right),$$

where $Q(\log x)$ is a polynomial of degree $2^k - 2k - 1$. □

4

Moments of Ramanujan sums over Number Fields

In this Chapter, we study moments of Ramanujan sums over number fields. Assuming the generalized Lindelöf hypothesis, we provide asymptotic formulas for the mean values of the first and second moments of Ramanujan sums over any number field. Additionally, unconditionally, we extend this investigation by establishing asymptotic formulas for the second moment of averages of Ramanujan sums over quadratic, cubic number fields, and cyclotomic number fields. Using a special property of certain integral domains, we obtain second-moment results for Ramanujan sums over some other number fields.

4.1 Preliminaries from algebraic number theory

In this section, we briefly recall some basic definitions and results from the algebraic number theory. This can be found in [45] and [49].

Definition 4.1.1. An algebraic number field $\mathbb{K}$ is a finite degree extension of $\mathbb{Q}$. The ring of integers of $\mathbb{K}$, denoted by $\mathcal{O}_{\mathbb{K}}$, is the integral closure of $\mathbb{Z}$ in $\mathbb{K}$.

A number field is a simple extension, that is, for an algebraic integer α, $\mathbb{K} =$

$\mathbb{Q}(\alpha)$. Therefore, a number field is a finite-dimensional vector space over $\mathbb{Q}$. A two-degree extension of $\mathbb{Q}$ is called a quadratic number field. For a square-free integer d, $\mathbb{K} = \mathbb{Q}(\sqrt{d})$ is a quadratic number field and its ring of integer is given as

$$\mathcal{O}_{\mathbb{K}} = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}\left[\dfrac{1+\sqrt{d}}{2}\right] & \text{if } d \equiv 1 \bmod 4, \\ \mathbb{Z} \oplus \mathbb{Z}\left[\sqrt{d}\right] & \text{if } d \not\equiv 1 \bmod 4. \end{cases}$$

Similarly, a cubic number field is a number field with degree 3. If ω is a cube root of unity, then $\mathbb{Q}(\omega)$ is a cubic number field. Also, for a n^{th} primitive root of unity ζ_n, $\mathbb{Q}(\zeta_n)$ is called a cyclotomic number field. This field contain all complex n^{th} roots of unity and dimension of $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$ is $\phi(n)$. For a prime p, if $\mathbb{K} = \mathbb{Q}(\zeta_p)$, then

$$\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\zeta_p] \oplus \mathbb{Z}[\zeta_p^2] \oplus \cdots \oplus \mathbb{Z}[\zeta_p^{p-1}].$$

Definition 4.1.2. For a number field $\mathbb{K}$, we associate a zeta function defined as

$$\zeta_{\mathbb{K}}(s) := \sum_{\mathcal{I} \subseteq \mathcal{O}_{\mathbb{K}}} \frac{1}{\mathcal{N}(\mathcal{I})^s},$$

where sum is over all non-zero integral ideal of $\mathcal{O}_{\mathbb{K}}$ and $\mathcal{N}(\mathcal{I})$ is the norm of ideal $\mathcal{I}$ given as:

$$\mathcal{N}(\mathcal{I}) = [\mathcal{O}_{\mathbb{K}} : \mathcal{I}] = |\mathcal{O}_{\mathbb{K}}/\mathcal{I}|.$$

The function $\zeta_{\mathbb{K}}(s)$ is called Dedekind zeta function associated to $\mathbb{K}$. For $\mathbb{K} = \mathbb{Q}$, we have $\zeta_{\mathbb{K}}(s) = \zeta(s)$. The Euler product of $\zeta_{\mathbb{K}}(s)$ is given as follows:

$$\zeta_{\mathbb{K}}(s) = \prod_{\mathcal{P} \subseteq \mathcal{O}_{\mathbb{K}}} \left(1 - \frac{1}{\mathcal{N}(\mathcal{P})^s}\right)^{-1},$$

where sum is over all non-zero prime integral ideal of $\mathcal{O}_{\mathbb{K}}$.

Remark 4.1.1. For any $s \in \mathbb{C}$, $\zeta_{\mathbb{K}}(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$ and has an analytic continuation to $\mathbb{C}\backslash\{1\}$ with a simple pole at $s = 1$.

In [54], Riemann conjectured that all non-trivial zeroes of the Riemann zeta function lie on $\mathrm{Re}(s) = 1/2$. The Riemann hypothesis is generalized for number fields. The generalized Riemann hypothesis states that all non-trivial zeroes of the Dedekind zeta function lie on $\mathrm{Re}(s) = 1/2$. The other important conjecture on the rate of growth of the Riemann zeta function is Lindelöf hypothesis. This conjecture is also generalized over number fields as follows:

Conjecture 4.1.1 (Generalized Lindelöf hypothesis)**.** Let $\zeta_{\mathbb{K}}(s)$ be the Dedekind zeta function over the number field $\mathbb{K}$, then for any $\epsilon > 0$, we have

$$\left|\zeta_{\mathbb{K}}\left(\frac{1}{2}+it\right)\right| \ll |t|^{\epsilon}.$$

The GLH follows from the GRH. On the other hand, Backlund [6] proved that LH is equivalent to the following statement about the zeros of the zeta function: for every $\epsilon > 0$, the number of zeros with real part at least $1/2+\epsilon$ and imaginary part between T and $T+1$ is $o(\log T)$ as T tends to infinity.

4.2 Main results

First, we recall the definition of Ramanujan sums over number fields. Let $\mathbb{K}$ be a number field. If $\mathcal{J}$ and $\mathcal{I}$ are non-zero integral ideal in its ring of integers $\mathcal{O}_{\mathbb{K}}$, then Ramanujan sums over $\mathbb{K}$ are defined as

$$C_{\mathcal{J}}(\mathcal{I}) := \sum_{\substack{\mathcal{I}_1|\mathcal{J}\\ \mathcal{I}_1|\mathcal{I}}} \mathcal{N}(\mathcal{I}_1)\mu\left(\frac{\mathcal{J}}{\mathcal{I}_1}\right). \tag{4.2.1}$$

Here, $\mathcal{N}(\mathcal{I}_1)$ is the norm of $\mathcal{I}_1$ and $\mu(\mathcal{I})$ is the generalization of classical Mobius function defined in (1.3.5). Concerning the number field analogue of Ramanujan sums, Nowak [50] showed that if $\mathbb{K}$ is a fixed quadratic number field, and $y > x^{\delta}$ where $\delta > \frac{1973}{820} = 2.40609\cdots$, then

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq y}\ \sum_{0<\mathcal{N}(\mathcal{J})\leq x} C_{\mathcal{J}}(\mathcal{I}) \sim \rho_{\mathbb{K}} y, \tag{4.2.2}$$

More precisely, for $y > x^{\delta}$ and arbitrary $\epsilon > 0$,

$$\begin{aligned}\sum_{0<\mathcal{N}(\mathcal{I})\leq y}\ \sum_{0<\mathcal{N}(\mathcal{J})\leq x} C_{\mathcal{J}}(\mathcal{I}) =& \rho_{\mathbb{K}} y + \mathrm{O}\left(x^{\frac{1973}{1358}} y^{\frac{269}{679}+\epsilon} + x^{\frac{1234823}{737394}} y^{\frac{205}{679}+\epsilon} + x^{\frac{23917}{21728}} y^{\frac{8675}{16296}+\epsilon}\right)\\ &+ \mathrm{O}\left(x^2 y^{\epsilon}\right),\end{aligned}$$

where $\rho_{\mathbb{K}}$ is the residue of $\zeta_{\mathbb{K}}(s)$ at $s = 1$. To be precise,

$$\rho_{\mathbb{K}} = \lim_{t\to\infty} \frac{1}{t}\#\{\text{integral ideals } \mathcal{I} \text{ in } O_{\mathbb{K}} : 0 < N(\mathcal{I}) \le t\}. \tag{4.2.3}$$

With regard to other degree two extensions, for example, for the field of Gaussian integers, Nowak [51] proved (4.2.2) with uniform error terms with $\delta > \frac{29}{12} = 2.416\cdots$. His result was later improved Zhai in [70] for $\delta > 2.3235\cdots$.

For the cubic case, a result on the first moment is derived in [43], where the authors obtain an asymptotic formula (4.2.2) with condition $y > x^{11/4}$,

$$\sum_{0<\mathcal{N}(\mathcal{I})\le y}\ \sum_{0<\mathcal{N}(\mathcal{J})\le x} C_{\mathcal{J}}(\mathcal{I}) = \rho_{\mathbb{K}} y + \mathrm{O}\left(x^{\frac{8}{5}} y^{\frac{2}{5}+\epsilon} + x^{\frac{11}{8}} y^{\frac{1}{2}+\epsilon}\right).$$

For mean values of Ramanujan sums over general number fields, the only known result is due to Fujisawa [23] who proved that if $\mathbb{K}$ is any number field, then for some $c > 0$, and for any $\delta > \frac{2-\alpha}{1-\alpha}$ where $\alpha \in [0, 1)$, with condition $y \gg x^{\delta}$, and $y \to \infty$

$$\sum_{0<\mathcal{N}(\mathcal{I})\le y}\ \sum_{0<\mathcal{N}(\mathcal{J})\le x} C_{\mathcal{J}}(\mathcal{I}) = \rho_{\mathbb{K}} y + o(y). \tag{4.2.4}$$

It is evident that Fujisawa's results exhibits limitations and does not hold for small values of y. Consequently, we estimate the first moment for general number fields in cases where $y > x^2$, under the Generalized Lindelöf hypothesis (GLH). This improves Fujisawa's result. Our result also improves Nowak and Zhai's results for quadratic and cubic number fields, respectively. Following this, we proceed to present our results.

Theorem 4.2.1. *Let $\mathbb{K}$ be a number field, then under GLH for $\zeta_{\mathbb{K}}(s)$ and for any $\epsilon > 0$ if $y > x^2$, we have*

$$\sum_{0<\mathcal{N}(\mathcal{I})\le y}\ \sum_{0<\mathcal{N}(\mathcal{J})\le x} C_{\mathcal{J}}(\mathcal{I}) = \rho_{\mathbb{K}} y + \mathrm{O}_{\mathfrak{q}}\left(x y^{1/2+\epsilon} \log x\right).$$

We also estimate the sum

$$\sum_{0<\mathcal{N}(\mathcal{J})\le x} C_{\mathcal{J}}(\mathcal{I})$$

in average over ideals $\mathcal{I}$ such that $\mathcal{N}(\mathcal{I}) \in \{1, \cdots, y\}$ via the second moment. There is no result for the second moment over number fields in the literature. For $\mathbb{K} = \mathbb{Q}$, the second moment has been derived in [12] and [55]. We have studied the result on moments over $\mathbb{Q}$ in Chapter 3. Under GLH, the second moment of Ramanujan sums over the general number fields is estimated by the following theorem:

Theorem 4.2.2. *For a number field $\mathbb{K}$, and $\epsilon > 0$, under GLH for $\zeta_{\mathbb{K}}(s)$, we have:*

For $y < x^{5/2}$,

$$\sum_{0<\mathcal{N}(\mathcal{I})\le y}\left(\sum_{0<\mathcal{N}(\mathcal{J})\le x} C_{\mathcal{J}}(\mathcal{I})\right)^2 = \frac{\rho_{\mathbb{K}}^2}{\zeta_{\mathbb{K}}(2)} yx^2 + \frac{\rho_{\mathbb{K}}^2\zeta_{\mathbb{K}}(0)}{4\zeta_{\mathbb{K}}(2)^2}x^4 + \mathrm{O}_{\mathfrak{q}}\left(yx^{3/2+\epsilon}\right),$$

and for $y \ge x^{5/2}$, we have

$$\sum_{0<\mathcal{N}(\mathcal{I})\le y}\left(\sum_{0<\mathcal{N}(\mathcal{J})\le x} C_{\mathcal{J}}(\mathcal{I})\right)^2 = \frac{\rho_{\mathbb{K}}^2}{\zeta_{\mathbb{K}}(2)} yx^2 + \mathrm{O}_{\mathfrak{q}}\left(yx^{3/2+\epsilon}\right).$$

Next, we derive asymptotic formulas for the second moment of Ramanujan sums over specific quadratic, cubic, and cyclotomic number fields and Prüfer domains unconditionally. We have the following result for a quadratic number field $\mathbb{K}$.

Theorem 4.2.3. *Let $\mathbb{K}$ be a quadratic number field, $\epsilon > 0$ be any arbitrary small real number, then for $y \le x^{11/9-\epsilon}$, we have*

$$\sum_{0<\mathcal{N}(\mathcal{I})\le y}\left(\sum_{0<\mathcal{N}(\mathcal{J})\le x} C_{\mathcal{J}}(\mathcal{I})\right)^2 = \frac{\rho_{\mathbb{K}}^2\zeta_{\mathbb{K}}(0)}{4\zeta_{\mathbb{K}}(2)^2}x^4 + \mathrm{O}_{\mathfrak{q}}\left(x^{47/18+\epsilon}y^{1/2}\log^{12} x\right),$$

for $x^{11/9-\epsilon} \le y < x^{36/17-\epsilon}$,

$$\sum_{0<\mathcal{N}(\mathcal{I})\le y}\left(\sum_{0<\mathcal{N}(\mathcal{J})\le x} C_{\mathcal{J}}(\mathcal{I})\right)^2 = \frac{\rho_{\mathbb{K}}^2}{2\zeta_{\mathbb{K}}(2)} yx^2 + \frac{\rho_{\mathbb{K}}^2\zeta_{\mathbb{K}}(0)}{4\zeta_{\mathbb{K}}(2)^2}x^4 + \mathrm{O}_{\mathfrak{q}}\left(x^2y^{17/18}\log^{24} x\right) + \mathrm{O}_{\mathfrak{q}}\left(x^{47/18+\epsilon}y^{1/2}\log^{12} x\right),$$

and for $y \ge x^{36/17-\epsilon}$,

$$\sum_{0<\mathcal{N}(\mathcal{I})\le y}\left(\sum_{0<\mathcal{N}(\mathcal{J})\le x} C_{\mathcal{J}}(\mathcal{I})\right)^2 = \frac{\rho_{\mathbb{K}}^2}{2\zeta_{\mathbb{K}}(2)} yx^2 + \mathrm{O}_{\mathfrak{q}}\left(yx^{\frac{5}{3}+\epsilon}\log^7 x + x^2y^{\frac{17}{18}}\log^{24} x\right).$$

We obtain estimates on the second moment for a cubic number field in the following theorem.

Theorem 4.2.4. *Let $\mathbb{K}$ be a cubic number field, $\epsilon > 0$ be any arbitrary small real number, then for $y \leq x^{237/196-\epsilon}$,*

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq y}\left(\sum_{0<\mathcal{N}(\mathcal{J})\leq x} C_{\mathcal{J}}(\mathcal{I})\right)^2 = \frac{\rho_{\mathbb{K}}^2\zeta_{\mathbb{K}}(0)}{4\zeta_{\mathbb{K}}(2)^2}x^4 + \mathrm{O}_{\mathfrak{q}}\left(x^{1021/392+\epsilon}y^{1/2}\log^{20}x\right),$$

for $x^{237/196-\epsilon} \leq y < x^{98/45-\epsilon}$,

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq y}\left(\sum_{0<\mathcal{N}(\mathcal{J})\leq x} C_{\mathcal{J}}(\mathcal{I})\right)^2 = \frac{\rho_{\mathbb{K}}^2}{2\zeta_{\mathbb{K}}(2)}yx^2 + \frac{\rho_{\mathbb{K}}^2\zeta_{\mathbb{K}}(0)}{4\zeta_{\mathbb{K}}(2)^2}x^4 + \mathrm{O}_{\mathfrak{q}}\left(x^2y^{45/49+\epsilon}\log^3 x\right) + \mathrm{O}_{\mathfrak{q}}\left(x^{1021/392+\epsilon}y^{1/2}\log^{20}x\right),$$

and for $y \geq x^{98/45-\epsilon}$,

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq y}\left(\sum_{0<\mathcal{N}(\mathcal{J})\leq x} C_{\mathcal{J}}(\mathcal{I})\right)^2 = \frac{\rho_{\mathbb{K}}^2}{2\zeta_{\mathbb{K}}(2)}yx^2 + \mathrm{O}_{\mathfrak{q}}\left(yx^{25/14+\epsilon}\log^{10}x\right) + \mathrm{O}_{\mathfrak{q}}\left(x^2y^{45/49+\epsilon}\log^3 x\right).$$

Furthermore, we provide an estimation for cyclotomic number fields.

Theorem 4.2.5. *Let $\mathbb{K} = \mathbb{Q}(\zeta_m)$ be a cyclotomic number field, then for $y < x^2$,*

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq y}\left(\sum_{0<\mathcal{N}(\mathcal{J})\leq x} C_{\mathcal{J}}(\mathcal{I})\right)^2 = \frac{\rho_{\mathbb{K}}^2\zeta_{\mathbb{K}}(0)}{4\zeta_{\mathbb{K}}(2)^2}x^4 + \mathrm{O}_{\mathfrak{q}}\left(x^{2-1/4\phi(m)}y\log^{4\phi(m)+1}x\right) + \mathrm{O}_{\mathfrak{q}}\left(x^2y^{5/6}\log^{4\phi(m)}x\right),$$

and for $y > x^2$,

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq y}\left(\sum_{0<\mathcal{N}(\mathcal{J})\leq x} C_{\mathcal{J}}(\mathcal{I})\right)^2 = \frac{\rho_{\mathbb{K}}^2}{2\zeta_{\mathbb{K}}(2)}yx^2 + \mathrm{O}_{\mathfrak{q}}\left(x^{5/2-1/2\phi(m)}y^{1/2}\log^{4\phi(m)}x\right) + \mathrm{O}_{\mathfrak{q}}\left((x^2y^{5/6} + x^{2-1/4\phi(m)}y\log x)\log^{4\phi(m)}x\right).$$

The constants in Theorems 4.2.1, 4.2.2, 4.2.3, 4.2.4, and 4.2.5 depend on the discriminant of the corresponding number fields. We derive asymptotic results for the second moment of Ramanujan sums over Prüfer domains in the following theorem.

Definition 4.2.1. An integral domain R is called a Prüfer domain if every finitely generated non-zero ideal of R is invertible.

For our computations of the second moment, we use the following ideal property of Prüfer domains: If $\mathcal{I}$ and $\mathcal{J}$ are two ideals of a Prüfer domain, then

$$(\mathcal{I}+\mathcal{J})(\mathcal{I}\cap\mathcal{J}) = \mathcal{I}\mathcal{J}. \tag{4.2.5}$$

Some examples of a Prüfer domain consist of the ring of algebraic integers, the ring of entire functions in $\mathbb{C}$. For more on multiplicative ideal theory and Prüfer domains, see [26, Chapter 4].

Theorem 4.2.6. *Let $\mathbb{K}$ be a number field such that its ring of integers $\mathcal{O}_{\mathbb{K}}$ is a Prüfer domain. If*

$$\sum_{1\leq\mathcal{N}(\mathcal{I})\leq y} 1 = \rho_{\mathbb{K}} y + \mathrm{O}\left(y^{\alpha}\right),$$

then for $x^{\lambda} < y$ for some $\lambda > 1$, we have

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq y}\left(\sum_{0<\mathcal{N}(\mathcal{J})\leq x} C_{\mathcal{J}}(\mathcal{I})\right)^2 = \frac{\rho_{\mathbb{K}}^2}{2\zeta_{\mathbb{K}}(2)}x^2 y + \mathrm{O}\left(xy\log x + x^{3-\alpha}y^{\alpha}\right).$$

The value of α was estimated by Landau [38] to be $(n-1)/(n+1)$, where n is the degree of $\mathbb{K}$ over $\mathbb{Q}$. This was later improved by Nowak [50] and Müller [46] for the case $n = 2$ and $n = 3$, respectively.

Remark 4.2.1. Theorem 4.2.6 holds for any number field whose corresponding ring of integers satisfies property (4.2.5). For the ring of integers $\mathbb{Z}$, (4.2.5) reduces to the fact that the gcd times lcm of any two integers is equal to the product of the integers. This property is not valid for more than two integers, complicating the computations for higher moments ($k \geq 3$).

Remark 4.2.2. Theorems 4.2.3 and 4.2.4 are special cases of Theorem 4.2.6 with additional main terms in certain ranges of y, as the ring of integers for both quadratic and cubic number field is a Prüfer domain.

4.3 Preliminaries Lemmas

In this section, we state and prove some results related to the Dirichlet series of functions appearing in the proofs of our main Theorems. We start by recalling the Dirichlet series of $C_{\mathcal{J}}(\mathcal{I})$.

Lemma 4.3.1. *For a number field* $\mathbb{K}$ *and for* $\mathrm{Re}(s) > 1$, *one has*

$$\sum_{\mathcal{J}\subseteq\mathcal{O}_{\mathbb{K}}} \frac{C_{\mathcal{J}}(\mathcal{I})}{\mathcal{N}(\mathcal{J})^s} = \frac{\sigma_{\mathbb{K},(1-s)}(\mathcal{I})}{\zeta_{\mathbb{K}}(s)}, \tag{4.3.1}$$

where $\sigma_{\mathbb{K},(1-s)}(\mathcal{I}) = \sum_{\mathcal{I}_1|\mathcal{I}} \mathcal{N}(\mathcal{I}_1)^{1-s}$.

Proof. From the definition of $C_{\mathcal{J}}(\mathcal{I})$ in (1.3.4), we have

$$\begin{aligned}\sum_{\mathcal{J}\subseteq\mathcal{O}_{\mathbb{K}}} \frac{C_{\mathcal{J}}(\mathcal{I})}{\mathcal{N}(\mathcal{J})^s} &= \sum_{\mathcal{J}\subseteq\mathcal{O}_{\mathbb{K}}} \frac{1}{\mathcal{N}(\mathcal{J})^s} \sum_{\substack{\mathcal{I}_1|\mathcal{J}\\ \mathcal{I}_1|\mathcal{I}}} \mathcal{N}(\mathcal{I}_1)\mu(\frac{\mathcal{J}}{\mathcal{I}_1}) = \sum_{\mathcal{I}_1|\mathcal{I}} \frac{1}{\mathcal{N}(\mathcal{I}_1)^{s-1}} \sum_{\mathcal{I}_2\subseteq\mathcal{O}_{\mathbb{K}}} \frac{\mu(\mathcal{I}_2)}{\mathcal{N}(\mathcal{I}_2)^s}\\ &= \frac{\sigma_{\mathbb{K},(1-s)}(\mathcal{I})}{\zeta_{\mathbb{K}}(s)}.\end{aligned}$$

The above series is absolutely convergent for $\mathrm{Re}(s) > 1$. □

Lemma 4.3.2. *For* $z \in \mathbb{C}$, *and a number field* $\mathbb{K}$,

$$\sum_{\mathcal{I}\subseteq\mathcal{O}_{\mathbb{K}}} \frac{\sigma_{\mathbb{K},z}(\mathcal{I})}{\mathcal{N}(\mathcal{I})^s} = \zeta_{\mathbb{K}}(s)\zeta_{\mathbb{K}}(s-z), \tag{4.3.2}$$

for $\mathrm{Re}(s) > \max(1 + \mathrm{Re}(z), 1)$.

Proof. Using the definition of $\sigma_{\mathbb{K},z}(\mathcal{I})$, we have

$$\begin{aligned}\sum_{\mathcal{I}\subseteq\mathcal{O}_{\mathbb{K}}} \frac{\sigma_{\mathbb{K},z}(\mathcal{I})}{\mathcal{N}(\mathcal{I})^s} &= \sum_{\mathcal{I}\subseteq\mathcal{O}_{\mathbb{K}}} \frac{1}{\mathcal{N}(\mathcal{I})^s} \sum_{\mathcal{I}_1|\mathcal{I}} \mathcal{N}(\mathcal{I}_1)^z\\ &= \zeta_{\mathbb{K}}(s)\zeta_{\mathbb{K}}(s-z),\end{aligned}$$

and it is absolutely convergent in $\mathrm{Re}(s) > \max(1 + \mathrm{Re}(z), 1)$. □

Lemma 4.3.3. *For* $\mathrm{Re}(s) > \max(1, 1+\mathrm{Re}(z_1), 1+\mathrm{Re}(z_2), 1+\mathrm{Re}(z_1+z_2))$, *we have*

$$\sum_{\mathcal{I}\subseteq\mathcal{O}_{\mathbb{K}}} \frac{\sigma_{\mathbb{K},z_1}(\mathcal{I})\sigma_{\mathbb{K},z_2}(\mathcal{I})}{\mathcal{N}(\mathcal{I})^s} = \frac{\zeta_{\mathbb{K}}(s)\zeta_{\mathbb{K}}(s-z_1)\zeta_{\mathbb{K}}(s-z_2)\zeta_{\mathbb{K}}(s-z_1-z_2)}{\zeta_{\mathbb{K}}(2s-z_1-z_2)}. \tag{4.3.3}$$

Proof. Fix a prime ideal $\mathcal{P}$, then for a positive integer k,

$$\sigma_{\mathbb{K},z}(\mathcal{P}^k) = \frac{\mathcal{N}(P)^{z(k+1)} - 1}{\mathcal{N}(P)^z - 1}.$$

Both functions $\sigma_{\mathbb{K},z}(\mathcal{I})$, and $\sigma_{\mathbb{K},z_1}(\mathcal{I})\sigma_{\mathbb{K},z_2}(\mathcal{I})$ are multiplicative, and hence the infinite series has an Euler product representation given by

$$\begin{aligned}\sum_{\mathcal{I}\subseteq\mathcal{O}_{\mathbb{K}}} \frac{\sigma_{\mathbb{K},z_1}(\mathcal{I})\sigma_{\mathbb{K},z_2}(\mathcal{I})}{\mathcal{N}(\mathcal{I})^s} &= \prod_{\mathcal{P}\subseteq\mathcal{O}_{\mathbb{K}}} \left(1 + \sum_{k=1}^{\infty} \frac{\sigma_{\mathbb{K},z_1}(\mathcal{P}^k)\sigma_{\mathbb{K},z_2}(\mathcal{P}^k)}{\mathcal{N}(\mathcal{P})^{ks}}\right) \\ &= \prod_{\mathcal{P}\subseteq\mathcal{O}_{\mathbb{K}}} \left(1 + \sum_{k=1}^{\infty} \frac{(\mathcal{N}(P)^{z_1(k+1)} - 1)(\mathcal{N}(P)^{z_2(k+1)} - 1)}{\mathcal{N}(\mathcal{P})^{ks}(\mathcal{N}(P)^{z_1} - 1)(\mathcal{N}(P)^{z_2} - 1)}\right).\end{aligned}$$

Let $\mathcal{N}(\mathcal{P})^{-s} = x$, $\mathcal{N}(\mathcal{P})^{z_1} = y$, and $\mathcal{N}(\mathcal{P})^{z_2} = z$, then

$$\begin{aligned}\sum_{\mathcal{I}\subseteq\mathcal{O}_{\mathbb{K}}} \frac{\sigma_{\mathbb{K},z_1}(\mathcal{I})\sigma_{\mathbb{K},z_2}(\mathcal{I})}{\mathcal{N}(\mathcal{I})^s} &= \prod_{\mathcal{P}\subseteq\mathcal{O}_{\mathbb{K}}} \left(\frac{1}{(y-1)(z-1)} \sum_{k=0}^{\infty} x^k(y^{k+1} - 1)(z^{k+1} - 1)\right) \\ = \prod_{\mathcal{P}\subseteq\mathcal{O}_{\mathbb{K}}} &\left(\frac{1}{(y-1)(z-1)} \left\{\frac{yz}{1-xyz} - \frac{z}{1-xz} - \frac{y}{1-xy} + \frac{1}{1-x}\right\}\right) \\ = \prod_{\mathcal{P}\subseteq\mathcal{O}_{\mathbb{K}}} &\frac{1 - x^2yz}{(1-x)(1-xy)(1-xz)(1-xyz)}.\end{aligned}$$

On substituting the values of x, y, and z in the above equation, we obtain Lemma 4.3.3. □

Next, we cite two lemmas that will be useful in the next sections. The first one is a Brun-Titchmarsh theorem proved by Shiu [62]. We will use it to estimate the partial sums

$$\sum_{\substack{\mathcal{I}\subseteq\mathcal{O}_{\mathbb{K}} \\ \mathcal{N}(\mathcal{I})=n}} \sigma_{\mathbb{K},z} \text{ and } \sum_{\substack{\mathcal{I}\subseteq\mathcal{O}_{\mathbb{K}} \\ \mathcal{N}(\mathcal{I})=n}} \sigma_{\mathbb{K},z_1}(\mathcal{I})\sigma_{\mathbb{K},z_2}(\mathcal{I}).$$

In [62], the author derives the theorem for a larger class M of arithmetic functions f which are non-negative and multiplicative and which satisfy the following conditions:

1. For a prime p, and integer $l \geq 1$, there exists a positive constant C_1 such that

$$f(p^l) \leq C_1^l.$$

2. For every $\epsilon > 0$, and for $n \geq 1$, there exists a positive constant $C_2 = C_2(\epsilon)$ such that
$$f(n) \leq C_2 n^{\epsilon}.$$

Lemma 4.3.4. *[62, Theorem 1] Let $f \in M$, $0 < \alpha, \beta < 1/2$, and a, k be integers. If $0 < a < k$, and $(a,k) = 1$, then as $x \to \infty$*

$$\sum_{\substack{x-y<n\leq x \\ n\equiv a \bmod q}} f(n) \ll \frac{y}{\phi(q)\log x} \exp\left(\sum_{\substack{p\leq x \\ p\nmid q}} \frac{f(p)}{p}\right),$$

uniformly in a, q, and y provided that $q \leq y^{1-\alpha}$, and $x^{\beta} < y \leq x$.

The second lemma is a Perron-type formula for a sequence of complex numbers.

Lemma 4.3.5. *[55, Lemma 2.8] Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ be any sequence of real numbers, and let $\{a_n\}$ be a sequence of complex numbers. Let the Dirichlet series $g(s) := \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ be absolutely convergent for σ_a. If $\sigma_0 > \max(0, \sigma_a)$ and $x > 0$, then*

$$\sum_{\lambda_n \leq x} a_n = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} g(s)\frac{x^s}{s}ds + R,$$

where

$$R \ll \sum_{\substack{x/2<\lambda_n<2x \\ n\neq x}} |a_n| \min\left(1, \frac{x}{T|x-\lambda_n|}\right) + \frac{4^{\sigma_0} + x^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{\lambda_n^{\sigma_0}}.$$

Next, recall the Phragmén-Lindelöf principle given by

Theorem 4.3.6. *[34, Theorem 5.53] Let $f(\sigma + it)$ be analytic in the strip $a \leq \sigma \leq b$ with $f(\sigma + it) \ll \exp(\epsilon|t|)$. If $|f(a+it)| \ll |t|^{c_1}$ and $|f(b+it)| \ll |t|^{c_2}$, then*

$$|f(\sigma + it)| \ll |t|^{c(\sigma)},$$

uniformly in $a \leq \sigma \leq b$, where $c(\sigma)$ is linear in σ with $c(a) = c_1$ and $c(b) = c_2$.

Finally, we give the bound of Dedekind zeta function. For a quadratic number field $\mathbb{K}$ with discriminant $\mathfrak{q}$,

$$\zeta_{\mathbb{K}}(s) = \zeta(s)L(s, \chi_{\mathfrak{q}}), \tag{4.3.4}$$

where $\zeta(s)$ is the Riemann zeta function, and $L(s,\chi_{\mathfrak{q}})$ is the ordinary Dirichlet L-series corresponding to $\chi_{\mathfrak{q}}$ and $\chi_{\mathfrak{q}}$ is the Kronecker symbol of $\mathfrak{q}$. We use bounds of $\zeta(s)$ [65, Chapter II.3] and derive the following bounds for $\zeta_{\mathbb{K}}(s)$:

$$\zeta_{\mathbb{K}}(\sigma+it) \ll_{\mathfrak{q}} \begin{cases} |t|^{1-2\sigma}\log^2|t| & -1\le\sigma\le 0, \\ |t|^{1-\frac{4\sigma}{3}}\log^4|t| & 0\le\sigma\le 1/2, \\ |t|^{\frac{2-2\sigma}{3}}\log^4|t| & 1/2\le\sigma\le 1, \\ \log^2|t| & 1\le\sigma\le 2, \\ 1 & \sigma\ge 2, \end{cases} \tag{4.3.5}$$

and

$$\frac{1}{\zeta_{\mathbb{K}}(\sigma+it)} \ll_{\mathfrak{q}} \log^2|t|, \quad 1\le\sigma\le 2. \tag{4.3.6}$$

The next lemma expresses the Dedekind zeta function for a cubic number field with discriminant $D=d\mathfrak{q}^2$ (d squarefree).

Lemma 4.3.7. *[46, Lemma 1] Let $\mathbb{K}$ be a cubic number field and $D=d\mathfrak{q}^2$ (d squarefree) its discriminant; then*

1. *$\mathbb{K}$ is normal extension if and only if $D=\mathfrak{q}^2$. In this case*

$$\zeta_{\mathbb{K}}(s)=\zeta(s)L(s,\chi_1)\overline{L(s,\chi_1)}, \tag{4.3.7}$$

where $\zeta(s)$ is the Riemann zeta function and $L(s,\chi_1)$ is the ordinary Dirichlet series corresponding to the primitive character χ_1 modulo $\mathfrak{q}$.

2. *If $\mathbb{K}$ is not a normal extension, then $d\neq 1$, and*

$$\zeta_{\mathbb{K}}(s)=\zeta(s)L(s,\chi_2), \tag{4.3.8}$$

where $L(s,\chi_2)$ is the Dirichlet L-series over the quadratic number field $\mathbb{Q}(\sqrt{d})$:

$$L(s,\chi_2)=\sum_{\mathcal{I}}\chi_2(\mathcal{I})\mathcal{N}(\mathcal{I})^{-s}.$$

Here summation is taken over all ideals $\mathcal{I}\neq 0$ in $\mathbb{Q}(\sqrt{d})$.

Using the above lemma, Theorem 4.3.6, and the bounds given in [33], we arrive at the following bounds in the cubic case.

$$\zeta_{\mathbb{K}}(\sigma+it) \ll_{\mathfrak{q}} \begin{cases} |t|^{3(1/2-\sigma)}\log^3|t| & -1 \le \sigma \le 0, \\ |t|^{\frac{63-85\sigma}{42}+\epsilon} & 0 \le \sigma \le 1/2, \\ |t|^{2(13/84+1/3)(1-\sigma)+\epsilon} = |t|^{\frac{41(1-\sigma)}{42}+\epsilon} & 1/2 \le \sigma \le 1, \\ \log^3|t| & 1 \le \sigma \le 2, \\ 1 & \sigma \ge 2, \end{cases} \tag{4.3.9}$$

and

$$\frac{1}{\zeta_{\mathbb{K}}(\sigma+it)} \ll_{\mathfrak{q}} \log^3|t|, \quad 1 \le \sigma \le 2. \tag{4.3.10}$$

Note that from [68, Theorem 4.3], for a cyclotomic number field $\mathbb{K} = \mathbb{Q}(\zeta_m)$ with discriminant $\mathfrak{q}$, we have

$$\zeta_{\mathbb{K}}(s) = \zeta(s)\prod_{\chi\neq\chi_0} L(s,\chi).$$

Here, χ is a non-principle Dirichlet character associated to $\mathbb{K}$. Next, we use the bounds of zeta function and L-function [33] and Theorem 4.3.6 to bound $\zeta_{\mathbb{K}}(\sigma+it)$:

$$\zeta_{\mathbb{K}}(\sigma+it) \ll_{\mathfrak{q}} \begin{cases} t^{\frac{(3-4\sigma)\phi(m)}{6}} & 0 \le \sigma \le 1/2, \\ t^{\frac{(1-\sigma)\phi(m)}{3}} & 1/2 \le \sigma \le 1, \\ (\log t)^{\phi(m)} & 1 \le \sigma \le 2, \\ 1 & \sigma \ge 2. \end{cases} \tag{4.3.11}$$

Using Generalized Lindelöf Hypothesis and the Phragmén-Lindelöf principle, we have the following upper bounds of a Dedekind zeta function associated with a number field $\mathbb{K}$ of degree m and discriminant $\mathfrak{q}$.

$$|\zeta_{\mathbb{K}}(\sigma+it)| \ll_{\mathfrak{q}} \begin{cases} |t|^{m(1/2-\sigma)+\epsilon} & \text{if } 0 \le \sigma \le 1/2, \\ |t|^{\epsilon} & \text{if } 1/2 \le \sigma \le 1, \\ \log|t| & \text{if } 1 \le \sigma \le 2. \end{cases} \tag{4.3.12}$$

4.4 A key estimate for the first moment

In this section, we estimate the average value of the generalized divisor function for number fields. This is also the key estimate for Theorem 4.2.1.

Lemma 4.4.1. *Let $\mathbb{K}$ be a number field, $-1/2 < \mathrm{Re}(z) \le 0$, and $|\mathrm{Im}(z)| \le x$, then* *under GLH and for arbitrary small $\epsilon > 0$, we have*

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq n} \sigma_{\mathbb{K},z}(\mathcal{I}) = \rho_{\mathbb{K}}\zeta_{\mathbb{K}}(1-z)x + \rho_{\mathbb{K}}\zeta_{\mathbb{K}}(1+z)\frac{x^{1+z}}{1+z} + \mathrm{O}_{\mathfrak{q}}\left(x^{1/2+\epsilon}\right).$$

Proof. We write

$$\sum_{\mathcal{I}\subseteq\mathcal{O}_{\mathbb{K}}} \frac{\sigma_{\mathbb{K},z}(\mathcal{I})}{\mathcal{N}(\mathcal{I})^s} = \sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{\mathcal{N}(\mathcal{I})=n}\sigma_{\mathbb{K},z}(\mathcal{I}) = \sum_{n=1}^{\infty}\frac{A(n,z)}{n^s},$$

where $A(n,z) = \sum_{\mathcal{N}(\mathcal{I})=n}\sigma_{\mathbb{K},z}(\mathcal{I})$. Consider $c = 1 + 1/\log x$ and $z = a + ib$ with $-1/2 < a < 0$. Therefore, from Lemma 4.3.5, we have

$$\sum_{n\leq x} A(n,z) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta_{\mathbb{K}}(s)\zeta_{\mathbb{K}}(s-z)\frac{x^s}{s}ds + R(x,z), \tag{4.4.1}$$

where

$$R(x,z) \ll \sum_{x/2<n<2x} |A(n,z)| \min\left(1, \frac{x}{T|x-n|}\right) + \frac{x^c}{T}\sum_{n=1}^{\infty}\frac{|A(n,z)|}{n^c}. \tag{4.4.2}$$

Next, from (4.3.2), we conclude that

$$\begin{aligned} |A(n,z)| &\leq \sum_{d|n}\left(\sum_{d_1\cdots d_m|d} 1 \sum_{l_1\cdots l_m|n/d} l_1^a\cdots l_m^a\right) \\ &\leq \sum_{d|n}\left(\sum_{d_1\cdots d_m|d} 1 \sum_{l_1\cdots l_m|n/d} 1\right). \end{aligned} \tag{4.4.3}$$

For α is a positive real number, we choose $T = x^{\alpha}$. Now, if $0 < |x-n| < x^{1-\alpha}$, then (4.4.3) gives

$$\sum_{0<|x-n|<x^{1-\alpha}} |A(n,z)| \min\left(1, \frac{x}{T|x-n|}\right) \ll \sum_{0<|x-n|<x^{1-\alpha}}\sum_{d|n}\left(\sum_{d_1\cdots d_m|d} 1 \sum_{l_1\cdots l_m|n/d} 1\right). \tag{4.4.4}$$

Moreover, using Lemma 4.3.4 and (4.4.4), we have

$$\sum_{0<|x-n|<x^{1-\alpha}} |A(n,z)| \min\left(1, \frac{x}{T|x-n|}\right) \ll x^{1-\alpha} \log^{2m-1} x.$$

Next, if $x + x^{1-\alpha} < n < 2x$, then

$$\begin{aligned}
&\sum_{x+x^{1-\alpha}<n<2x} |A(n,z)| \min\left(1, \frac{x}{T|x-n|}\right) \\
&\ll \frac{x}{T} \sum_{x+x^{1-\alpha}<n<2x} \frac{\sum_{d|n} \left(\sum_{d_1\cdots d_m|d} 1 \sum_{l_1\cdots l_m|n/d} 1\right)}{n-x} \\
&\ll \frac{x}{T} \sum_{l \ll \log x} \frac{1}{U} \sum_{\substack{U<n-x<2U \\ U=2^l x^{1-\alpha}}} \frac{\sum_{d|n} \left(\sum_{d_1\cdots d_m|d} 1 \sum_{l_1\cdots l_m|n/d} 1\right)}{n-x} \\
&\ll \frac{x}{T} \log^{2m} x. \qquad (4.4.5)
\end{aligned}$$

For $x/2 < n < x - x^{1-\alpha}$, we get the same bound. From Lemma 4.3.2, we have

$$\frac{x^c}{T} \sum_{n=1}^{\infty} \frac{|A(n,z)|}{n^c} \ll \frac{x}{T} \log^{2m} x. \qquad (4.4.6)$$

Thus, from (4.4.4), (4.4.5), and (4.4.6), we obtain

$$R(x,z) \ll x^{1-\alpha} \log^{2m} x. \qquad (4.4.7)$$

Next, we evaluate the integral in (4.4.1) by shifting the line integration $c - iT$ to $c+iT$ into the rectangular contour consists the line segments $I_1 : c-iT$ to $1/2-iT$, $I_2 : 1/2 - iT$ to $1/2 + iT$, $I_3 : 1/2 + iT$ to $c + iT$, and $I_4 : c + iT$ to $c - iT$. By Cauchy residue theorem

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta_{\mathbb{K}}(s)\zeta_{\mathbb{K}}(s-z) \frac{x^s}{s} ds = \rho_{\mathbb{K}}\zeta_{\mathbb{K}}(1-z)x + \rho_{\mathbb{K}}\zeta_{\mathbb{K}}(1+z)\frac{x^{1+z}}{1+z} + \sum_{i=1}^{3} J_i, \qquad (4.4.8)$$

where $\rho_{\mathbb{K}}$ is the residue of $\zeta_{\mathbb{K}}(s)$ at the pole $s = 1$. In the right side of (4.4.8), the first and the second terms are the residues at the poles 1 and $1+z$, respectively, and the last term is the sum of integration along the line segments I_i. From (4.3.12), if $|b| \leq T$ then we have

$$|J_1|, |J_3| \ll \int_{1/2}^{c} \frac{|\zeta_{\mathbb{K}}(s)||\zeta_{\mathbb{K}}(s-z)|x^{\sigma}}{T} d\sigma \ll_{\mathfrak{q}} \frac{x^{1/2}\log x}{T^{1-\epsilon}}. \tag{4.4.9}$$

The line integral J_2 along the line segment I_2 is given by

$$\begin{aligned} |J_2| &\ll \int_{-T}^{T} \frac{|\zeta_{\mathbb{K}}(1/2+it)||\zeta_{\mathbb{K}}(1/2-a+it-ib)||x^{1/2+it}|}{|1/2+it|} dt \\ &\ll_{\mathfrak{q}} x^{1/2} \int_{-T}^{T} t^{-1+\epsilon} dt \ll_{\mathfrak{q}} x^{1/2} T^{\epsilon}. \end{aligned} \tag{4.4.10}$$

Choose $T = x^2$ and collecting all the estimates from (4.4.7), (4.4.9), and (4.4.10) and inserting in (4.4.1), provides us the required result. □

4.5 The first moment

In this section we prove Theorem 4.2.1 using Lemma 4.4.1 and a Perron's formula.

Proof of Theorem 4.2.1. Let $\beta = 1 + \frac{1}{\log y}$, then from Lemma 4.3.1 and Lemma 4.3.5, we have

$$\sum_{0<\mathcal{N}(\mathcal{J})\leq x} C_{\mathcal{J}}(\mathcal{I}) = \frac{1}{2\pi i} \int_{\beta-iT}^{\beta+iT} \frac{\sigma_{\mathbb{K},(1-s)}(\mathcal{I})}{\zeta_{\mathbb{K}}(s)} \frac{x^s}{s} ds + R_1(x,\mathcal{I}), \tag{4.5.1}$$

where

$$R_1(x,\mathcal{I}) \ll \frac{x}{T} \sigma_{\mathbb{K},(-1/\log y)}(\mathcal{I}) + \frac{x\log y}{T} \sigma_{\mathbb{K},0}(\mathcal{I}). \tag{4.5.2}$$

Next, summing both sides of (4.5.1) over the ideals with $\mathcal{N}(\mathcal{I})$ and applying Lemma 4.4.1, we have

$$\begin{aligned} \sum_{0<\mathcal{N}(\mathcal{I})\leq y} \sum_{0<\mathcal{N}(\mathcal{J})\leq x} C_{\mathcal{J}}(\mathcal{I}) &= \frac{\rho_{\mathbb{K}} y}{2\pi i} \int_{\beta-iT}^{\beta+iT} \frac{x^s}{s} ds + \rho_{\mathbb{K}} y^2 \int_{\beta-iT}^{\beta+iT} \frac{\zeta_{\mathbb{K}}(2-s)}{\zeta_{\mathbb{K}}(s)} \frac{y^{-s}x^s}{(2-s)s} ds \\ &\quad + \mathrm{O}\left(y^{1/2+\epsilon} \int_{\beta-iT}^{\beta+iT} \frac{1}{\zeta_{\mathbb{K}}(s)} \frac{x^s}{s} ds + \frac{xy}{T} \right) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.5.3}$$

We see that

$$I_1 = \rho_{\mathbb{K}} y + \mathrm{O}\left(\frac{yx}{T}\right),$$

and

$$I_3 \ll_{\mathfrak{q}} xy^{1/2+\epsilon} \log^m T.$$

Also, from (4.3.12), we have

$$I_2 \ll_{\mathfrak{q}} xy \int_{-T}^{T} t^{-2} \log t dt \ll_{\mathfrak{q}} \frac{xy \log T}{T}.$$

Collecting all the above estimates and choosing $T = x^2$ gives the required result. □

4.6 A key estimate for the second moment

In this section, we shall compute the average of the product of divisor functions in a number field, analogous to [55, Theorem 1.5] for divisor functions over rationals.

Lemma 4.6.1. *Let $\mathbb{K}$ be a number field. Then,*

$$\sum_{\substack{0<\mathcal{N}(\mathcal{I})\leq y\\ \mathcal{I}\subseteq\mathcal{O}_{\mathbb{K}}}} \sigma_{\mathbb{K},z_1}(\mathcal{I})\sigma_{\mathbb{K},z_2}(\mathcal{I}) = R_{\mathbb{K}} + E_{\mathbb{K}}.$$

For a ***quadratic number field*** $\mathbb{K}$*: for $-1/13 < a_1 < 0$, $-1/9 < a_2 < 0$, $-1/13 < a_1 + a_2 < 0$, and $|b_1|, |b_2| \ll y^{1/3}$,*

$$\begin{aligned} R_{\mathbb{K}} = &\rho_{\mathbb{K}} \frac{\zeta_{\mathbb{K}}(1-z_1)\zeta_{\mathbb{K}}(1-z_2)\zeta_{\mathbb{K}}(1-z_1-z_2)}{\zeta_{\mathbb{K}}(2-z_1-z_2)} y \\ &+ \rho_{\mathbb{K}} \frac{\zeta_{\mathbb{K}}(1+z_1)\zeta_{\mathbb{K}}(1+z_1-z_2)\zeta_{\mathbb{K}}(1-z_2)}{\zeta_{\mathbb{K}}(2+z_1-z_2)} \frac{y^{1+z_1}}{1+z_1} \\ &+ \rho_{\mathbb{K}} \frac{\zeta_{\mathbb{K}}(1+z_2)\zeta_{\mathbb{K}}(1+z_2-z_1)\zeta_{\mathbb{K}}(1-z_1)}{\zeta_{\mathbb{K}}(2-z_1+z_2)} \frac{y^{1+z_2}}{1+z_2} \\ &+ \rho_{\mathbb{K}} \frac{\zeta_{\mathbb{K}}(1+z_1+z_2)\zeta_{\mathbb{K}}(1+z_2)\zeta_{\mathbb{K}}(1+z_1)}{\zeta_{\mathbb{K}}(2+z_1+z_2)} \frac{y^{1+z_1+z_2}}{1+z_1+z_2}, \end{aligned}$$

and

$$E_{\mathbb{K}} = \mathrm{O}_{\mathfrak{q}}\left(y^{\frac{17+5a_1+9a_2}{18}} \log^{18} y \right).$$

For ***a cubic number field*** $\mathbb{K}$*: for $-16/183 < a_1 < 0$, $-8/49 < a_2 < 0$, $-16/183 < a_1 + a_2 < 0$, and $|b_1|, |b_2| \ll y^{3/14}$, $R_{\mathbb{K}}$ is same as above, and*

$$E_{\mathbb{K}} = \mathrm{O}_{\mathfrak{q}}\left(y^{\frac{180+13a_1+98a_2}{196}+\epsilon} \log^3 y \right).$$

*For **a cyclotomic number field** $\mathbb{K} = \mathbb{Q}(\zeta_m)$: for $-1/6 < \mathrm{Re}(z_1) = a_1 < 0$, $-1/6 < \mathrm{Re}(z_2) = a_2 < 0$, and $-1/6\phi(m) < \mathrm{Re}(z_1 + z_2) = a_1 + a_2 < 0$, then $R_{\mathbb{K}}$ is same as above, and*

$$E_{\mathbb{K}} = \mathrm{O}_{\mathfrak{q}}\left(y^{5/6+a_1/2-a_2/6}\log^{5\phi(m)} y + y^{1-1/\phi(m)}\log^{5\phi(m)}\right).$$

*For **a general number field** $\mathbb{K}$: for $-1/2 \le \mathrm{Re}(z_1) = a_1 < 0$, $-1/2m^2 \le \mathrm{Re}(z_2) = a_2 < 0$, and $-1/2 < \mathrm{Re}(z_1 + z_2) = a_1 + a_2 < 0$, then under GLH $R_{\mathbb{K}}$ is same as above, and for $\epsilon > 0$*

$$E_{\mathbb{K}} = \mathrm{O}_{\mathfrak{q}}\left(y^{\frac{1+a_1+a_2-4a_2m}{2}+\epsilon}\right),$$

and m is the degree of $\mathbb{K}$.

Proof. For any number field $\mathbb{K}$, we have

$$\sum_{\mathcal{I}\subseteq\mathcal{O}_{\mathbb{K}}} \frac{\sigma_{\mathbb{K},z_1}(\mathcal{I})\sigma_{\mathbb{K},z_2}(\mathcal{I})}{\mathcal{N}(\mathcal{I})^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\substack{\mathcal{I}\subseteq\mathcal{O}_{\mathbb{K}}\\ \mathcal{N}(\mathcal{I})=n}} \sigma_{\mathbb{K},z_1}(\mathcal{I})\sigma_{\mathbb{K},z_2}(\mathcal{I}).$$

Define $A(n, z_1, z_2) := \sum_{\substack{\mathcal{I}\subseteq\mathcal{O}_{\mathbb{K}}\\ \mathcal{N}(\mathcal{I})=n}} \sigma_{\mathbb{K},z_1}(\mathcal{I})\sigma_{\mathbb{K},z_2}(\mathcal{I})$ and

$$f(z_1, z_2, s) := \frac{\zeta_{\mathbb{K}}(s)\zeta_{\mathbb{K}}(s - z_1)\zeta_{\mathbb{K}}(s - z_2)\zeta_{\mathbb{K}}(s - z_1 - z_2)}{\zeta_{\mathbb{K}}(2s - z_1 - z_2)}.$$

Let $\mathrm{Re}(z_1) = a_1$ and $\mathrm{Re}(z_2) = a_2$ be such that $a_1, a_2 < 0$, and $a_1 + a_2 > -1$. Consider $\alpha = 1 + \frac{1}{\log y}$. Using Lemma 4.3.5, we have

$$\sum_{\substack{0<\mathcal{N}(\mathcal{I})\le y\\ \mathcal{I}\subseteq\mathcal{O}_{\mathbb{K}}}} \sigma_{\mathbb{K},z_1}(\mathcal{I})\sigma_{\mathbb{K},z_2}(\mathcal{I}) = \sum_{n\le y} A(n, z_1, z_2) = \frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} f(z_1, z_2, s)\frac{y^s}{s}ds + R(y; z_1, z_2), \tag{4.6.1}$$

where

$$R(y; z_1, z_2) \ll \sum_{y/2<n<2y} |A(n, z_1, z_2)| \min\left(1, \frac{y}{T|y-n|}\right) + \frac{y^\alpha}{T}\sum_{n=1}^{\infty}\frac{|A(n, z_1, z_2)|}{n^\alpha}. \tag{4.6.2}$$

For $\lambda = (1+a_1+a_2)/2$, we solve integral in (4.6.1) by modifying the line integral into a rectangular path C with vertices $\alpha \pm iT$, and $\lambda \pm iT$. The poles of the integrand inside the contour C are $s_1 = 1$, $s_2 = z_1+1$, $s_3 = z_2+1$, and $s_4 = z_1+z_2+1$. Hence by Cauchy's residue theorem, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_C f(z_1,z_2,s)\frac{y^s}{s}ds = {}& \rho_{\mathbb{K}}\frac{\zeta_{\mathbb{K}}(1-z_1)\zeta_{\mathbb{K}}(1-z_2)\zeta_{\mathbb{K}}(1-z_1-z_2)}{\zeta_{\mathbb{K}}(2-z_1-z_2)}y \\ &+ \rho_{\mathbb{K}}\frac{\zeta_{\mathbb{K}}(1+z_1)\zeta_{\mathbb{K}}(1+z_1-z_2)\zeta_{\mathbb{K}}(1-z_2)}{\zeta_{\mathbb{K}}(2+z_1-z_2)}\frac{y^{1+z_1}}{1+z_1} \\ &+ \rho_{\mathbb{K}}\frac{\zeta_{\mathbb{K}}(1+z_2)\zeta_{\mathbb{K}}(1+z_2-z_1)\zeta_{\mathbb{K}}(1-z_1)}{\zeta_{\mathbb{K}}(2-z_1+z_2)}\frac{y^{1+z_2}}{1+z_2} \\ &+ \rho_{\mathbb{K}}\frac{\zeta_{\mathbb{K}}(1+z_1+z_2)\zeta_{\mathbb{K}}(1+z_2)\zeta_{\mathbb{K}}(1+z_1)}{\zeta_{\mathbb{K}}(2+z_1+z_2)}\frac{y^{1+z_1+z_2}}{1+z_1+z_2}.\end{aligned} \tag{4.6.3}$$

This implies

$$\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} f(z_1,z_2,s)\frac{y^s}{s}ds = R_0 + \sum_{i=1}^{3} J_i, \tag{4.6.4}$$

where R_0 is equal to the right side of (4.6.3), and J_i's are the line integrals along the lines $[\lambda+iT, \alpha+iT]$, $[\lambda-iT, \lambda+iT]$ and $[\alpha-iT, \lambda-iT]$ respectively. By Holder's inequality

$$\begin{aligned}&\left(\int_\lambda^\alpha\int_{T_0/2}^{T_0} f(z_1,z_2,\sigma+it)\frac{y^{\sigma+it}}{\sigma+it}d\sigma dt\right)^4 \\ &\ll \int_\lambda^\alpha\int_{T_0/2}^{T_0}\frac{|\zeta_{\mathbb{K}}(\sigma+it)|^4 y^\sigma}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|}d\sigma dt \\ &\times \int_\lambda^\alpha\int_{T_0/2}^{T_0}\frac{|\zeta_{\mathbb{K}}(\sigma+it-z_1)|^4 y^\sigma}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|}d\sigma dt \\ &\times \int_\lambda^\alpha\int_{T_0/2}^{T_0}\frac{|\zeta_{\mathbb{K}}(\sigma+it-z_2)|^4 y^\sigma}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|}d\sigma dt \\ &\times \int_\lambda^\alpha\int_{T_0/2}^{T_0}\frac{|\zeta_{\mathbb{K}}(\sigma+it-z_1-z_2)|^4 y^\sigma}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|}d\sigma dt.\end{aligned} \tag{4.6.5}$$

We estimate the above integrals and the remainder $R(y; z_1, z_2)$ in (4.6.2) separately for the quadratic and cubic number fields.

4.6.1 Quadratic number fields

For $\mathbb{K}$ a quadratic number field, the Dirichlet series for $A(n, z_1, z_2)$ can be expressed in terms of $\zeta(s)$ and $L(s, \chi)$ using (4.3.3) and (4.3.4). Consequently, $A(n, z_1, z_2)$ is written as a Dirichlet convolution of coefficients of its Dirichlet series. This exercise yields the following elementary but essential bound.

$$\begin{aligned} |A(n, z_1, z_2)| &\leq \left| n^{a_1+a_2} \sum_{\mathcal{N}(\mathcal{I})=n} \left(\sum_{\mathcal{I}_1|\mathcal{I}} 1 \right)^2 \right| \\ &\leq \sum_{d|n} \left\{ \sum_{d'|d} \left(\sigma_0(d') \sum_{d_1|d/d'} \sigma_0(d_1)\sigma_0(\frac{d}{d'd_1}) \right) \right\} \sigma_0(n/d). \end{aligned} \tag{4.6.6}$$

This leads to $|A(p, z_1, z_2)| \leq 9$. Taking $T = y^c$ where c is a fixed real number, and dividing the interval $y/2 < n < 2y$ according to $\min\left(1, \frac{y}{T|y-n|}\right)$, we arrive at

$$\sum_{|y-n|<y^{1-c}} |A(n, z_1, z_2)| \min\left(1, \frac{y}{T|y-n|}\right) = \sum_{|y-n|<y^{1-c}} |A(n, z_1, z_2)| \ll \frac{y}{T} \log^8 T. \tag{4.6.7}$$

The last estimate follows from an application of Lemma 4.3.4 on the function $A(n, z_1, z_2)$. Note that (4.6.6) ensures that the hypothesis in the lemma is satisfied. For the interval $y + y^{1-c} < n < 2y$, we have

$$\begin{aligned} &\sum_{y+y^{1-c}<n<2y} |A(n, z_1, z_2)| \min\left(1, \frac{y}{T|y-n|}\right) \\ &\ll \frac{y}{T} \sum_{y+y^{1-c}<n<2y} \frac{\sum_{d|n} \left\{ \sum_{d'|d} \left(\sigma_0(d') \sum_{d_1|d/d'} \sigma_0(d_1)\sigma_0(\frac{d}{d'd_1}) \right) \right\} \sigma_0(n/d)}{n-y} \\ &\ll \frac{y}{T} \sum_{l \ll \log y} \frac{1}{U} \sum_{\substack{U<n-y<2U \\ U=2^l y^{1-c}}} \frac{\sum_{d|n} \left\{ \sum_{d'|d} \left(\sigma_0(d') \sum_{d_1|d/d'} \sigma_0(d_1)\sigma_0(\frac{d}{d'd_1}) \right) \right\} \sigma_0(n/d)}{n-y} \\ &\ll \frac{y}{T} \log^8 y. \end{aligned} \tag{4.6.8}$$

We obtain similar bounds for $y/2 < n < y - y^{1-c}$. Moreover

$$\frac{y^\alpha}{T}\sum_{n=1}^{\infty}\frac{|A(n,z_1,z_2)|}{n^\alpha} \ll \frac{y}{T}\log^8 T. \tag{4.6.9}$$

Finally, from (4.6.7), (4.6.8), and (4.6.9), we deduce that

$$R(y;z_1,z_2) \ll \frac{y}{T}\log^8 T. \tag{4.6.10}$$

Next, solving the line integrals using bounds in (4.3.5) and (4.3.6), we have

$$\begin{aligned}
&\int_\lambda^\alpha \int_{T_0/2}^{T_0} \frac{|\zeta_{\mathbb{K}}(\sigma+it)|^4 y^\sigma}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|} d\sigma dt \\
&\ll_{\mathfrak{q}} \int_{T_0/2}^{T_0}\int_\lambda^{1/2} t^{4-16\sigma/3}\log^{18} t \frac{y^\sigma}{t} d\sigma dt + \int_{T_0/2}^{T_0}\int_{1/2}^{\alpha} t^{8(1-\sigma)/3}\log^{18} t \frac{y^\sigma}{t} d\sigma dt \\
&\ll_{\mathfrak{q}} \int_{T_0/2}^{T_0} t^3 \log^{18} t \int_\lambda^{1/2}\left(\frac{y}{t^{16/3}}\right)^\sigma d\sigma dt + \int_{T_0/2}^{T_0} t^{5/3}\log^{18} t \int_{1/2}^{\alpha}\left(\frac{y}{t^{8/3}}\right)^\sigma d\sigma dt,
\end{aligned}$$

for $y^{3/16} < T_0 < y^{3/8}$, we have

$$\int_\lambda^\alpha \int_{T_0/2}^{T_0} \frac{|\zeta_{\mathbb{K}}(\sigma+it)|^4 y^\sigma}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|} d\sigma dt \ll_{\mathfrak{q}} (T_0^{4-16\lambda/3} y^\lambda + y)\log^{18} T_0.$$

If $a_1 - a_2 > 0$, then we have

$$\begin{aligned}
&\int_\lambda^\alpha \int_{T_0/2}^{T_0} \frac{|\zeta_{\mathbb{K}}(\sigma+it-z_1)|^4 y^\sigma}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|} d\sigma dt \\
&\ll_{\mathfrak{q}} \int_{T_0/2}^{T_0}\int_\lambda^{1/2+a_1} t^{4-16(\sigma-a_1)/3}\log^{18} t \frac{y^\sigma}{t} d\sigma dt \\
&+ \int_{T_0/2}^{T_0}\int_{1/2+a_1}^{1+a_1} t^{(8-8\sigma+8a_1)/3}\log^{18} t \frac{y^\sigma}{t} d\sigma dt + \int_{T_0/2}^{T_0}\int_{1+a_1}^{\alpha} \log^{10} t \frac{y^\sigma}{t} d\sigma dt \\
&\ll_{\mathfrak{q}} \int_{T_0/2}^{T_0} t^{3+16a_1/3}\log^{18} t \int_\lambda^{\frac{1}{2}+a_1}\left(\frac{y}{t^{16/3}}\right)^\sigma d\sigma dt \\
&+ \int_{T_0/2}^{T_0} t^{(5+8a_1)/3}\log^{18} t \int_{\frac{1}{2}+a_1}^{1+a_1}\left(\frac{y}{t^{8/3}}\right)^\sigma d\sigma dt + \int_{T_0/2}^{T_0}\int_{1+a_1}^{\alpha}\log^{10} t \frac{y^\sigma}{t} d\sigma dt \\
&\ll_{\mathfrak{q}} (T_0^{4+\frac{16a_1-16\lambda}{3}} y^\lambda + y^{1+a_1})\log^{18} T_0 + y\log^{10} T_0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{\lambda}^{\alpha}\int_{T_0/2}^{T_0}\frac{|\zeta_{\mathbb{K}}(\sigma+it-z_2)|^4y^{\sigma}}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|}d\sigma dt\\
&\ll_{\mathfrak{q}}\int_{T_0/2}^{T_0}\int_{\lambda}^{1+a_2}t^{(8-8\sigma+8a_2)/3}\log^{18}t\frac{y^{\sigma}}{t}d\sigma dt+\int_{T_0/2}^{T_0}\int_{1+a_2}^{\alpha}\log^{10}t\frac{y^{\sigma}}{t}d\sigma dt\\
&\ll_{\mathfrak{q}}\int_{T_0/2}^{T_0}t^{(5+8a_2)/3}\log^{18}t\int_{\lambda}^{1+a_2}\left(\frac{y}{t^{8/3}}\right)^{\sigma}d\sigma dt+\int_{T_0/2}^{T_0}\int_{1+a_2}^{\alpha}\log^{10}t\frac{y^{\sigma}}{t}d\sigma dt\\
&\ll_{\mathfrak{q}}y^{1+a_2}\log^{18}T_0+y\log^{10}T_0,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\lambda}^{\alpha}\int_{T_0/2}^{T_0}\frac{|\zeta_{\mathbb{K}}(\sigma+it-z_1-z_2)|^4y^{\sigma}}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|}d\sigma dt\\
&\ll_{\mathfrak{q}}\int_{T_0/2}^{T_0}\int_{\lambda}^{1+a_1+a_2}t^{(8-8\sigma+8a_1+8a_2)/3}\log^{18}t\frac{y^{\sigma}}{t}d\sigma dt+\int_{T_0/2}^{T_0}\int_{1+a_1+a_2}^{\alpha}\log^{10}t\frac{y^{\sigma}}{t}d\sigma dt\\
&\ll_{\mathfrak{q}}y^{1+a_1+a_2}\log^{18}T_0+y\log^{10}T_0.
\end{aligned}$$

Collecting all the above results and substituting in (4.6.5), we obtain

$$\int_{\lambda}^{\alpha}\int_{T_0/2}^{T_0}f(z_1,z_2,\sigma+it)\frac{y^{\sigma+it}}{\sigma+it}d\sigma dt\ll_{\mathfrak{q}}y\log^{12}T_0.$$

Next, we choose T such that $T_0/2<T<T_0$, which gives

$$\int_{\lambda}^{\alpha}f(z_1,z_2,\sigma+iT)\frac{y^{\sigma+iT}}{\sigma+iT}d\sigma\ll_{\mathfrak{q}}\frac{y}{T}\log^{12}T.$$

Integral along the vertical line $[\lambda-iT,\lambda+iT]$ is given by

$$\begin{aligned}
\int_{\lambda-iT}^{\lambda+iT}f(z_1,z_2,s)\frac{y^s}{s}ds&\ll_{\mathfrak{q}}\int_{-T}^{T}t^{\frac{4-2a_1}{3}}\log^{18}t\frac{y^{\lambda}}{t}dt\\
&\ll_{\mathfrak{q}}T^{\frac{4-2a_1}{3}}y^{\lambda}\log^{18}T.
\end{aligned}$$

We get the required result by putting $T=y^{1/3}$ in the above estimates.

4.6.2 Cubic number fields

If $\mathbb{K}$ is a cubic number field, then like the case for quadratic, we write $A(n, z_1, z_2)$ using (4.3.3) and (4.3.7) to obtain

$$|A(n, z_1, z_2)| \leq \left| n^{a_1+a_2} \sum_{\mathcal{N}(\mathcal{I})=n} \left(\sum_{\mathcal{I}_1|\mathcal{I}} 1 \right)^2 \right|$$
$$\leq \sum_{d|n} \left[\sum_{d'|d} \left\{ \sum_{d_1|d'} \sigma_0(d_1) \sum_{d_2|d/d'} \left(\sum_{d_{21}|d_2} \sigma_0(d_{21}) \sum_{d_{22}|d/d_2} \sigma_0(d_{22}) \right) \right\} \right] \sum_{d''|n/d} \sigma_0(d'').$$

For a prime p, a direct computation of the right-hand side of the above inequality gives the bound

$$|A(p, z_1, z_2)| \leq 13.$$

Choosing $T = y^c$ where c is fixed real number, then using Lemma 4.3.4, we have

$$\sum_{|y-n|<y^{1-c}} |A(n, z_1, z_2)| \min \left(1, \frac{y}{T|y-n|}\right) = \sum_{|y-n|<y^{1-c}} |A(n, z_1, z_2)| \ll \frac{y}{T} \log^{12} T. \tag{4.6.11}$$

The above bounds also hold true for the intervals: $y/2 < T < y - y^{1-c}$ and $y + y^{1-c} < n < 2y$. Moreover,

$$\frac{y^\alpha}{T} \sum_{n=1}^{\infty} \frac{|A(n, z_1, z_2)|}{n^\alpha} \ll \frac{y}{T} \log^{12} T. \tag{4.6.12}$$

These estimates yield,

$$R(y; z_1, z_2) \ll \frac{y}{T} \log^{12} T. \tag{4.6.13}$$

In the following computations, we employ Dedekind zeta bounds (4.3.9) and (4.3.10) to obtain estimates of the line integrals in (4.6.5). For $y^{7/170} \leq T \leq y^{21/82}$

$$\int_\lambda^\alpha \int_{T_0/2}^{T_0} \frac{|\zeta_{\mathbb{K}}(\sigma+it)|^4 y^\sigma}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|} d\sigma dt$$
$$\ll_{\mathfrak{q}} \int_{T_0/2}^{T_0} \int_\lambda^{1/2} t^{2(63-85\sigma)/21+\epsilon} \log^3 t \frac{y^\sigma}{t} d\sigma dt + \int_{T_0/2}^{T_0} \int_{1/2}^\alpha t^{82(1-\sigma)/21+\epsilon} \log^3 t \frac{y^\sigma}{t} d\sigma dt$$
$$\ll_{\mathfrak{q}} (T_0^{6-170\lambda/21+\epsilon} y^\lambda + T_0^\epsilon y) \log^3 T_0.$$

If $a_1 - a_2 > 0$, then we have

$$\begin{aligned}
&\int_{\lambda}^{\alpha}\int_{T_0/2}^{T_0} \frac{|\zeta_{\mathbb{K}}(\sigma+it-z_1)|^4 y^{\sigma}}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|} d\sigma dt \\
&\ll_{\mathfrak{q}} \int_{T_0/2}^{T_0}\int_{\lambda}^{1/2+a_1} t^{2(63-85\sigma+85a_1)/21+\epsilon} \log^3 t \frac{y^{\sigma}}{T_0} d\sigma dt \\
&+ \int_{T_0/2}^{T_0}\int_{1/2+a_1}^{1+a_1} t^{\frac{82(1-\sigma+a_1)}{21}+\epsilon} \log^3 t \frac{y^{\sigma}}{t} d\sigma dt + \int_{T_0/2}^{T_0}\int_{1+a_1}^{\alpha} \log^{15} t \frac{y^{\sigma}}{t} d\sigma dt \\
&\ll_{\mathfrak{q}} (T_0^{6-(170\lambda-170a_1)/21+\epsilon} y^{\lambda} + T_0^{\epsilon} y^{1+a_1}) \log^3 T_0 + y\log^{15} T_0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{\lambda}^{\alpha}\int_{T_0/2}^{T_0} \frac{|\zeta_{\mathbb{K}}(\sigma+it-z_2)|^4 y^{\sigma}}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|} d\sigma dt \\
&\ll_{q} \mathfrak{q} \int_{T_0/2}^{T_0}\int_{\lambda}^{1+a_2} t^{\frac{81(1-\sigma+a_2)}{21}+\epsilon} \log^3 t \frac{y^{\sigma}}{t} d\sigma dt + \int_{T_0/2}^{T_0}\int_{1+a_2}^{\alpha} \log^{15} t \frac{y^{\sigma}}{t} d\sigma dt \\
&\ll_{\mathfrak{q}} T_0^{\epsilon} y^{1+a_2} \log^3 T_0 + y\log^{15} T_0,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\lambda}^{\alpha}\int_{T_0/2}^{T_0} \frac{|\zeta_{\mathbb{K}}(\sigma+it-z_1-z_2)|^4 y^{\sigma}}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|} d\sigma dt \\
&\ll_{\mathfrak{q}} \int_{T_0/2}^{T_0}\int_{\lambda}^{1+a_1+a_2} t^{\frac{82(1-\sigma+a_1+a_2)}{21}+\epsilon} \log^3 t \frac{y^{\sigma}}{t} d\sigma dt + \int_{T_0/2}^{T_0}\int_{1+a_1+a_2}^{\alpha} \log^{15} t \frac{y^{\sigma}}{t} d\sigma dt \\
&\ll_{\mathfrak{q}} T_0^{\epsilon} y^{1+a_1+a_2} \log^3 T_0 + y\log^{15} T_0.
\end{aligned}$$

The above estimates show that the double integral in (4.6.5) is bounded by

$$\int_{\lambda}^{\alpha}\int_{T_0/2}^{T_0} f(z_1,z_2,\sigma+it)\frac{y^{\sigma+it}}{\sigma+it} d\sigma dt \ll_{\mathfrak{q}} yT_0^{\epsilon}\log^{12} T_0.$$

Next, we choose a T such that $T_0/2 < T < T_0$, which gives

$$\int_{\lambda}^{\alpha} f(z_1,z_2,\sigma+it)\frac{y^{\sigma+iT}}{\sigma+iT} d\sigma \ll_{\mathfrak{q}} \frac{y}{T^{1-\epsilon}} \log^{12} T.$$

Finally, the integral along the vertical line $[\lambda - iT, \lambda + iT]$ is estimated as

$$\int_{\lambda-iT}^{\lambda+iT} f(z_1, z_2, s)\frac{y^s}{s}ds \ll_q \int_{-T}^{T} t^{\frac{82-85a_1}{42}+\epsilon}\log^3 t\frac{y^\lambda}{t}dt \ll_{\mathfrak{q}} T^{\frac{82-85a_1}{42}+\epsilon}y^\lambda \log^3 T.$$

We get the required result by putting $T = y^{3/14}$ in the above bounds.

4.6.3 Cyclotomic number fields

Let $\mathbb{K}$ be a cyclotomic number field, then from (4.3.3), we write

$$|A(n, z_1, z_2)|$$
$$\leq \sum_{d|n}\left(\sum_{d_1|d}\left(\sum_{d_{11}\cdots d_{1\phi(m)}|d_1} 1 \sum_{l_{11}\cdots l_{1\phi(m)}|n/d} 1\right)\sum_{d_2|n/d_1}\left(\sum_{d_{21}\cdots d_{2\phi(m)}|d} 1 \sum_{l_{21}\cdots l_{2\phi(m)}|n/d_2} 1\right)\right).$$

This implies

$$|A(p, z_1, z_2)| \leq 4\phi(m).$$

Taking $T = y^c$ where c is a fixed real number. From Lemma 4.3.4, we have

$$\sum_{|y-n|<y^{1-c}} |A(n, z_1, z_2)| \min\left(1, \frac{y}{T|y-n|}\right) = \sum_{|y-n|<y^{1-c}} |A(n, z_1, z_2)| \ll \frac{y}{T}\log^{4\phi(m)} T. \tag{4.6.14}$$

For the interval $y + y^{1-c} < n < 2y$, we have

$$\sum_{y+y^{1-c}<n<2y} |A(n, z_1, z_2)| \min\left(1, \frac{x}{T|y-n|}\right) \ll \frac{y}{T}\log^{4\phi(m)} y. \tag{4.6.15}$$

We obtain similar bounds for $y/2 < T < y - y^{1-c}$. We have

$$\frac{y^\alpha}{T}\sum_{n=1}^{\infty}\frac{|A(n, z_1, z_2)|}{n^\alpha} \ll \frac{y}{T}\log^{4\phi(m)} T. \tag{4.6.16}$$

Finally, from (4.6.14), (4.6.15), and (4.6.16), we deduce that

$$R(y; z_1, z_2) \ll \frac{y}{T}\log^{4\phi(m)} T. \tag{4.6.17}$$

Next, we estimate line integrals in (4.6.5) using (4.3.11).

$$\begin{aligned}&\int_{\lambda}^{\alpha}\int_{T_0/2}^{T_0}\frac{|\zeta_{\mathbb{K}}(\sigma+it)|^4y^{\sigma}}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|}d\sigma dt\\&\ll_{\mathfrak{q}}\int_{T_0/2}^{T_0}\int_{\lambda}^{1/2}t^{(3-4\sigma)\phi(m)/6}\log^{\phi(m)}t\frac{y^{\sigma}}{t}d\sigma dt\\&+\int_{T_0/2}^{T_0}\int_{1/2}^{\alpha}t^{(1-\sigma)\phi(m)/3}\log^{\phi(m)}t\frac{y^{\sigma}}{t}d\sigma dt\\&\ll_{\mathfrak{q}}(T_0^{(3-4\lambda)\phi(m)/6}y^{\lambda}+y)\log^{\phi(m)}T_0.\end{aligned}$$

If $a_1-a_2>0$, then we have

$$\begin{aligned}&\int_{\lambda}^{\alpha}\int_{T_0/2}^{T_0}\frac{|\zeta_{\mathbb{K}}(\sigma+it-z_1)|^4y^{\sigma}}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|}d\sigma dt\\&\ll_{\mathfrak{q}}\int_{T_0/2}^{T_0}\int_{\lambda}^{1/2+a_1}t^{(3-4\sigma+4a_1)\phi(m)/6}\log^{5\phi(m)}t\frac{y^{\sigma}}{t}d\sigma dt\\&+\int_{T_0/2}^{T_0}\int_{1/2+a_1}^{1+a_1}t^{(1-\sigma+a_1)\phi(m)/3}\log^{5\phi(m)}t\frac{y^{\sigma}}{t}d\sigma dt+\int_{T_0/2}^{T_0}\int_{1+a_1}^{\alpha}\log^{5\phi(m)}t\frac{y^{\sigma}}{t}d\sigma dt\\&\ll_{\mathfrak{q}}(T_0^{(3-4\lambda+4a_1)\phi(m)/6}y^{\lambda}+y^{1+a_1}+y)\log^{5\phi(m)}T_0.\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_{\lambda}^{\alpha}\int_{T_0/2}^{T_0}\frac{|\zeta_{\mathbb{K}}(\sigma+it-z_2)|^4y^{\sigma}}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|}d\sigma dt\\&\ll_{\mathfrak{q}}\int_{T_0/2}^{T_0}\int_{\lambda}^{1+a_2}t^{(1-\sigma+a_2)\phi(m)/3}\log^{5\phi(m)}t\frac{y^{\sigma}}{t}d\sigma dt+\int_{T_0/2}^{T_0}\int_{1+a_2}^{\alpha}\log^{5\phi(m)}t\frac{y^{\sigma}}{t}d\sigma dt\\&\ll_{\mathfrak{q}}(y^{1+a_2}+y)\log^{5\phi(m)}T_0,\end{aligned}$$

and

$$\begin{aligned}&\int_{\lambda}^{\alpha}\int_{T_0/2}^{T_0}\frac{|\zeta_{\mathbb{K}}(\sigma+it-z_1-z_2)|^4y^{\sigma}}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|}d\sigma dt\\&\ll_{\mathfrak{q}}\int_{T_0/2}^{T_0}\int_{\lambda}^{1+a_1+a_2}t^{(1-\sigma+a_1+a_2)\phi(m)/3}\log^{5\phi(m)}t\frac{y^{\sigma}}{t}d\sigma dt\\&+\int_{T_0/2}^{T_0}\int_{1+a_1+a_2}^{\alpha}\log^{5\phi(m)}t\frac{y^{\sigma}}{t}d\sigma dt\end{aligned}$$

$$\ll_{\mathfrak{q}} (y^{1+a_1+a_2} + y) \log^{5\phi(m)} T_0.$$

Collecting all the above results and substituting in (4.6.5), we obtain

$$\int_{\lambda}^{\alpha} \int_{T_0/2}^{T_0} f(z_1, z_2, \sigma + it) \frac{y^{\sigma+it}}{\sigma + it} d\sigma dt \ll_q y \log^{5\phi(m)} T_0.$$

Next, we choose T such that $T_0/2 < T < T_0$, which gives T

$$\int_{\lambda}^{\alpha} f(z_1, z_2, \sigma + iT) \frac{y^{\sigma+iT}}{\sigma + iT} d\sigma \ll_q \frac{y}{T} \log^{5\phi(m)} T.$$

The integral along the vertical line $[\lambda - iT, \lambda + iT]$ is given by

$$\begin{aligned}\int_{\lambda-iT}^{\lambda+iT} f(z_1, z_2, s) \frac{y^s}{s} ds &\ll_q \int_{-T}^{T} t^{(2+3a_1+a_2)\phi(m)/3} \log^{5\phi(m)} t \frac{y^\lambda}{t} dt \\ &\ll_{\mathfrak{q}} T^{(2+3a_1+a_2)\phi(m)/3} y^\lambda \log^{5\phi(m)} T.\end{aligned}$$

Putting $T = y^{1/\phi(m)}$ in the above estimates, we get the required result.

4.6.4 General number fields under GLH

For a general number field, from (4.3.3), we have

$$|A(n, z_1, z_2)| \le \sum_{d|n} \left(\sum_{d_1|d} \left(\sum_{d_{11}\cdots d_{1m}|d_1} 1 \sum_{l_{11}\cdots l_{1m}|n/d} 1 \right) \sum_{d_2|n/d_1} \left(\sum_{d_{21}\cdots d_{2m}|d} 1 \sum_{l_{21}\cdots l_{2m}|n/d_2} 1 \right) \right). \tag{4.6.18}$$

This implies $|A(p, z_1, z_2)| \le 4m$. Taking $T = y^c$ where c is a fixed real number yields

$$\sum_{|y-n|<y^{1-c}} |A(n, z_1, z_2)| \min\left(1, \frac{y}{T|y-n|}\right) = \sum_{|y-n|<y^{1-c}} |A(n, z_1, z_2)|.$$

We intend to use Lemma 4.3.4 to estimate the sum on the right side of the above estimate. From (4.6.18), we ensure that $A(n, z_1, z_2)$ satisfies the hypothesis in Lemma

4.3.4 and therefore

$$\sum_{|y-n|<y^{1-c}} |A(n,z_1,z_2)| \min\left(1, \frac{y}{T|y-n|}\right) \ll \frac{y}{T}\log^{4m} T.$$

For the interval $y + y^{1-c} < n < 2y$, we have

$$\begin{aligned}&\sum_{y+y^{1-c}<n<2y} |A(n,z_1,z_2)| \min\left(1, \frac{y}{T|y-n|}\right)\\ &\ll \frac{y}{T}\sum_{y+y^{1-\alpha}<n<2y} \frac{|A(n,z_1,z_2)|}{n-y} \ll \frac{y}{T}\log^{4m} y.\end{aligned}$$

Moreover, we have the same bounds for $y/2 < T < y - y^{1-c}$. Therefore, from the above estimates, we deduce that

$$R(y;z_1,z_2) \ll \frac{y}{T}\log^{4m} T. \tag{4.6.19}$$

In the following computations, we employ Dedekind zeta bounds (4.3.12) to obtain estimates of the line integrals in (4.6.5).

$$\begin{aligned}&\int_\lambda^\alpha \int_{T_0/2}^{T_0} \frac{|\zeta_{\mathbb{K}}(\sigma+it)|^4 y^\sigma}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|} d\sigma dt\\ &\ll_{\mathfrak{q}} \int_{T_0/2}^{T_0}\int_\lambda^{1/2} t^{m(2-4\sigma)+\epsilon}\frac{y^\sigma}{t} d\sigma dt + \int_{T_0/2}^{T_0}\int_{1/2}^{\alpha} t^{4\epsilon}\frac{y^\sigma}{t} d\sigma dt\\ &\ll_{\mathfrak{q}} \int_{T_0/2}^{T_0} t^{2m-1+\epsilon}\int_\lambda^{1/2}\left(\frac{y}{t^{4m}}\right)^\sigma d\sigma dt + \int_{T_0/2}^{T_0} t^{4\epsilon-1}\int_{1/2}^{\alpha}(y)^\sigma d\sigma dt\\ &\ll_{\mathfrak{q}} T_0^{2m-4m\lambda+\epsilon} y^\lambda + yT_0^\epsilon.\end{aligned}$$

If $a_1 - a_2 > 0$, then we have

$$\begin{aligned}&\int_\lambda^\alpha \int_{T_0/2}^{T_0} \frac{|\zeta_{\mathbb{K}}(\sigma+it-z_1)|^4 y^\sigma}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|} d\sigma dt\\ &\ll_{\mathfrak{q}} \int_{T_0/2}^{T_0}\int_\lambda^{1/2+a_1} t^{m(2-4\sigma+4a_1)+\epsilon}\frac{y^\sigma}{t} d\sigma dt + \int_{T_0/2}^{T_0}\int_{1/2+a_1}^{\alpha} t^{4\epsilon}\frac{y^\sigma}{t} d\sigma dt\\ &\ll_{\mathfrak{q}} \int_{T_0/2}^{T_0} t^{2m-1+4a_1m+\epsilon}\int_\lambda^{\frac{1}{2}+a_1}\left(\frac{y}{t^{4m}}\right)^\sigma d\sigma dt + \int_{T_0/2}^{T_0} t^{4\epsilon-1}\int_{\frac{1}{2}+a_1}^{\alpha}(y)^\sigma d\sigma dt\end{aligned}$$

$$\ll_{\mathfrak{q}} T_0^{(2a_1-2a_2)m+\epsilon} y^{\lambda} + yT_0^{\epsilon}.$$

Similarly,

$$\begin{aligned}
&\int_{\lambda}^{\alpha}\int_{T_0/2}^{T_0} \frac{|\zeta_{\mathbb{K}}(\sigma+it-z_2)|^4 y^{\sigma}}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|} d\sigma dt \\
&\ll_{\mathfrak{q}} \int_{T_0/2}^{T_0}\int_{\lambda}^{1+a_2} t^{m(2-4\sigma+4a_2)+\epsilon}\frac{y^{\sigma}}{t} d\sigma dt + \int_{T_0/2}^{T_0}\int_{1+a_2}^{\alpha} t^{4\epsilon}\frac{y^{\sigma}}{t} d\sigma dt \\
&\ll_{\mathfrak{q}} T_0^{-2m+\epsilon} y^{1+a_2} + yT_0^{\epsilon},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\lambda}^{\alpha}\int_{T_0/2}^{T_0} \frac{|\zeta_{\mathbb{K}}(\sigma+it-z_1-z_2)|^4 y^{\sigma}}{|\zeta_{\mathbb{K}}(2(\sigma+it)-z_1-z_2)(\sigma+it)|} d\sigma dt \\
&\ll_{\mathfrak{q}} \int_{T_0/2}^{T_0}\int_{\lambda}^{1+a_1+a_2} t^{m(2-4\sigma+4a_1+4a_2)+\epsilon}\frac{y^{\sigma}}{t} d\sigma dt + \int_{T_0/2}^{T_0}\int_{1+a_1+a_2}^{\alpha} t^{4\epsilon}\frac{y^{\sigma}}{t} d\sigma dt \\
&\ll_{\mathfrak{q}} y^{1+a_1+a_2} T_0^{-2m+\epsilon} + yT_0^{\epsilon}.
\end{aligned}$$

Collecting all the above results and substituting in (4.6.5), we obtain

$$\int_{\lambda}^{\alpha}\int_{T_0/2}^{T_0} f(z_1,z_2,\sigma+it)\frac{y^{\sigma+it}}{\sigma+it} d\sigma dt \ll_{\mathfrak{q}} yT_0^{\epsilon}.$$

Next, we choose T such that $T_0/2 < T < T_0$, which gives

$$\int_{\lambda}^{\alpha} f(z_1,z_2,\sigma+iT)\frac{y^{\sigma+iT}}{\sigma+iT} d\sigma \ll_{\mathfrak{q}} \frac{y}{T^{1-\epsilon}}.$$

Integral along the vertical line $[\lambda - iT, \lambda + iT]$ is given by

$$\int_{\lambda-iT}^{\lambda+iT} f(z_1,z_2,s)\frac{y^s}{s} ds \ll \int_{-T}^{T} t^{-a_2m+\epsilon}\frac{y^{\lambda}}{t} dt \ll_{\mathfrak{q}} T^{-a_2m+\epsilon} y^{\lambda}.$$

Putting $T = y^2$ in the above estimates, we get the required result. □

4.7 The second moment

Our arguments for the asymptotics of second moments for quadratic and cubic fields use ideas from [12] and [55] with several adaptations required to extend the proof for the number field. Let

$$\beta_i = 1 + \frac{i}{\log y},$$

where $i \in \{1, 2\}$. Using Lemma 4.3.5, we have

$$\sum_{0<\mathcal{N}(\mathcal{J})\le x} C_{\mathcal{J}}(\mathcal{I}) = \frac{1}{2\pi i}\int_{\beta_i - iT}^{\beta_i + iT} \frac{\sigma_{\mathbb{K},(1-s)}(\mathcal{I})}{\zeta_{\mathbb{K}}(s)}\frac{x^s}{s} ds + \mathrm{O}\left(\frac{x\log y}{T}\sigma_{\mathbb{K},0}(\mathcal{I})\right).$$

Squaring the two sides yields

$$\left(\sum_{0<\mathcal{N}(\mathcal{J})\le x} C_{\mathcal{J}}(\mathcal{I})\right)^2 = \frac{1}{(2\pi i)^2}\int_{\beta_1 - iT}^{\beta_1 + iT}\int_{\beta_2 - iT}^{\beta_2 + iT} \frac{\sigma_{\mathbb{K},(1-s_1)}(\mathcal{I})\sigma_{\mathbb{K},(1-s_2)}(\mathcal{I})}{\zeta_{\mathbb{K}}(s_1)\zeta_{\mathbb{K}}(s_2)}\frac{x^{s_1+s_2}}{s_1 s_2} ds_1 ds_2 + R(x,\mathcal{I}), \tag{4.7.1}$$

Inserting (4.7.4) in (4.7.1), and summing both sides over ideals $\mathcal{I}$ with $\mathcal{N}(\mathcal{I}) \le y$, we have

$$\begin{aligned}\sum_{0<\mathcal{N}(\mathcal{I})\le y}\left(\sum_{0<\mathcal{N}(\mathcal{J})\le x} C_{\mathcal{J}}(\mathcal{I})\right)^2 &= \frac{1}{(2\pi i)^2}\int_{\beta_1 - iT}^{\beta_1 + iT}\int_{\beta_2 - iT}^{\beta_2 + iT} \frac{G(s_1,s_2,y)}{\zeta_{\mathbb{K}}(s_1)\zeta_{\mathbb{K}}(s_2)}\frac{x^{s_1+s_2}}{s_1 s_2} ds_1 ds_2 + \\ &\quad \mathrm{O}\left(\sum_{0<\mathcal{N}(\mathcal{I})\le y} R(x,\mathcal{I})\right) \\ &= I + \mathrm{O}\left(\sum_{0<\mathcal{N}(\mathcal{I})\le y} R(x,\mathcal{I})\right),\end{aligned} \tag{4.7.2}$$

where $G(s_1, s_2, y) := \sum_{0<\mathcal{N}(\mathcal{I})\le y} \sigma_{\mathbb{K},(1-s_1)}(\mathcal{I})\sigma_{\mathbb{K},(1-s_2)}(\mathcal{I})$. Using Lemma 4.6.1, for $|T| \ll x^{\alpha}$ where we obtain different value of α for different number fields, the integral I in (4.7.2) can be written as

$$I = I_1 + I_2 + I_3 + I_4 + E(x,y), \tag{4.7.3}$$

where I_1, I_2, I_3, and I_4 are the integrals corresponding to the four terms appearing in R_0 in Lemma 4.6.1 and $E(x,y)$ is corresponding error term. We compute each integral separately below.

Evaluation of I_1:

The integral I_1 is given by

$$I_1 = \frac{\rho_{\mathbb{K}} y}{(2\pi i)^2} \int_{\beta_1 - iT}^{\beta_1 + iT} \int_{\beta_2 - iT}^{\beta_2 + iT} \frac{\zeta_{\mathbb{K}}(s_1 + s_2 - 1)}{\zeta_{\mathbb{K}}(s_1 + s_2)} \frac{x^{s_1+s_2}}{s_1 s_2} ds_1 ds_2.$$

We first shift the line integral in s_2-plane to a rectangular contour containing the lines $[\beta_2 + iT, 3/2 - \beta_1 + iT]$, $[3/2 - \beta_1 + iT, 3/2 - \beta_1 - iT]$, $[3/2 - \beta_1 - iT, \beta_2 - iT]$, and $[\beta_2 - iT, \beta_2 + iT]$. We see that $s_2 = 2 - s_1$ is the pole of the integrand inside the contour and the residue at pole is

$$\frac{\rho_{\mathbb{K}} x^2}{\zeta_{\mathbb{K}}(2) s_1 (2 - s_1)}.$$

If $L_{1,1}$, $L_{1,2}$, and $L_{1,3}$ are the integrals along the lines $[3/2 - \beta_1 + iT, \beta_2 + iT]$, $[3/2 - \beta_1 - iT, 3/2 - \beta_1 + iT]$, and $[3/2 - \beta_1 - iT, \beta_2 - iT]$ respectively. Therefore, we have

$$\begin{aligned} I_1 &= \frac{\rho_{\mathbb{K}}^2 y x^2}{\zeta_{\mathbb{K}}(2)} \frac{1}{2\pi i} \int_{\beta_1 - iT}^{\beta_1 + iT} \frac{1}{s_1(2 - s_1)} ds_1 + \sum_{i=1}^{3} L_{1i} \\ &= \frac{\rho_{\mathbb{K}}^2 y x^2}{2\zeta_{\mathbb{K}}(2)} + \sum_{i=1}^{3} L_{1,i}. \end{aligned}$$

Evaluation of I_2:

The integral I_2 is given by

$$I_2 = \frac{\rho_{\mathbb{K}} y^2}{(2\pi i)^2} \int_{\beta_1 - iT}^{\beta_1 + iT} \int_{\beta_2 - iT}^{\beta_2 + iT} \frac{\zeta_{\mathbb{K}}(2 - s_1)\zeta_{\mathbb{K}}(1 - s_1 + s_2)}{(2 - s_1)\zeta_{\mathbb{K}}(2 - s_1 + s_2)\zeta_{\mathbb{K}}(s_1)} \frac{x^{s_1+s_2}}{y^{s_1} s_1 s_2} ds_1 ds_2.$$

We move the line integral over s_1-plane to a contour with vertices $\beta_1 + iT$, $3/2 + iT$, $3/2 - iT$, and $\beta_1 + iT$. The integrand has a simple at $s_1 = s_2$ with residue

$$-\rho_{\mathbb{K}}^2 \frac{\zeta_{\mathbb{K}}(2 - s_2) x^{2s_2} / y^{s_2}}{s_2^2 (2 - s_2)\zeta_{\mathbb{K}}(2)\zeta_{\mathbb{K}}(s_2)}$$

Let $L_{2,1}$, $L_{2,3}$ be the integrals along the horizontal lines of the contour, and $L_{2,2}$ be the integral along the vertical line. Therefore, we have

$$\begin{aligned} I_2 &= \frac{\rho_{\mathbb{K}}^2 y^2}{\zeta_{\mathbb{K}}(2)(2\pi i)} \int_{\beta_2 - iT}^{\beta_2 + iT} \frac{\zeta_{\mathbb{K}}(2-s_2) x^{2s_2}/y^{s_2}}{s_2^2(2-s_2)\zeta_{\mathbb{K}}(s_2)} ds_2 + \sum_{i=1}^{3} L_{2i} \\ &= \frac{\rho_{\mathbb{K}}^2 x^4 \zeta_{\mathbb{K}}(0)}{4\zeta_{\mathbb{K}}(2)^2} + \sum_{i=1}^{3} L_{2,i}. \end{aligned}$$

Evaluation of I_3:

The integral I_3 is given as

$$I_3 = \frac{\rho_{\mathbb{K}} y^2}{(2\pi i)^2} \int_{\beta_1 - iT}^{\beta_1 + iT} \int_{\beta_2 - iT}^{\beta_2 + iT} \frac{\zeta_{\mathbb{K}}(2-s_2)\zeta_{\mathbb{K}}(1-s_2+s_1)}{(2-s_2)\zeta_{\mathbb{K}}(2-s_2+s_1)\zeta_{\mathbb{K}}(s_2)} \frac{x^{s_1+s_2}}{y^{s_2} s_1 s_2} ds_1 ds_2.$$

To estimate the integral I_3, we modify the line integration over s_2 to the contour containing the vertices $\beta_2 + iT$, $\beta_2 - iT$, $3/2 + iT$, and $3/2 - iT$, and denote the integration along the lines $[\beta_2 + iT, 3/2 + iT]$, $[3/2 + iT, 3/2 - iT]$, and $[3/2 - iT, \beta_2 - iT]$ by $L_{3,1}$, $L_{3,2}$, and $L_{3,3}$, respectively. There is no pole of the integrand inside the contour, so

$$I_3 = \sum_{i=1}^{3} L_{3,i.}$$

Evaluation of I_4:

Finally, the integral I_4 is given by

$$\begin{aligned} I_4 &= \frac{\rho_{\mathbb{K}}}{(2\pi i)^2} \\ &\times \int_{\beta_1 - iT}^{\beta_1 + iT} \int_{\beta_2 - iT}^{\beta_2 + iT} \frac{\zeta_{\mathbb{K}}(2-s_1)\zeta_{\mathbb{K}}(2-s_2)\zeta_{\mathbb{K}}(3-s_1-s_2)}{(3-s_1-s_2)\zeta_{\mathbb{K}}(4-s_1-s_2)\zeta_{\mathbb{K}}(s_1)\zeta_{\mathbb{K}}(s_2)} \frac{y^{3-s_1-s_2} x^{s_1+s_2}}{s_1 s_2} ds_1 ds_2. \end{aligned}$$

We estimate I_4 by shifting the integration over s_2 to the contour with vertices $\beta_2 + iT$, $\beta_2 - iT$, $5/2 - \beta_1 + iT$, and $5/2 - \beta_1 - iT$. We denote the integration along the lines $[\beta_2 + iT, 5/2 - \beta_1 + iT]$, $[5/2 - \beta_1 + iT, 5/2 - \beta_1 - iT]$, and $[5/2 - \beta_1 - iT, \beta_2 - iT]$ are $L_{4,1}$, $L_{4,2}$, and $L_{4,3}$, respectively. Since there is no pole of integrand inside the contour, therefore we have

$$I_4 = \sum_{i=1}^{3} L_{4,i}.$$

Since the bound of integrals $L_{j,i}$ for $1 \leq j \leq 4$ and $1 \leq i \leq 3$ depends on the bound of the Dedekind zeta function, we estimate the bound of $L_{j,i}$ for quadratic, cubic, cyclotomic number fields, and arbitrary number field under GLH separately using (4.3.5), (4.3.9), (4.3.11), and (4.3.12), respectively to obtain the second moment of respective number fields.

4.7.1 Quadratic number fields

Proof of theorem 4.2.3. For a quadratic number field, the error term in (4.7.1) is estimated as

$$\begin{aligned} R(x,\mathcal{I}) &\ll \frac{x\log y}{T}\sigma_{\mathbb{K},0}(\mathcal{I})\int_{\beta_i-iT}^{\beta_i+iT}\frac{\sigma_{\mathbb{K},(1-s)}(\mathcal{I})}{\zeta_{\mathbb{K}}(s)}\frac{x^s}{s}ds+\frac{x^2\log^2 y}{T^2}(\sigma_{\mathbb{K},0}(\mathcal{I}))^2 \\ &\ll \frac{x^2\log y\log^3 T}{T}(\sigma_{\mathbb{K},0}(\mathcal{I}))^2. \end{aligned} \tag{4.7.4}$$

Therefore, we have

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq y} R(x,\mathcal{I}) \ll \frac{x^2\log y\log^3 T}{T}\sum_{0<\mathcal{N}(\mathcal{I})\leq y}(\sigma_{\mathbb{K},0}(\mathcal{I}))^2.$$

We take $a_1 = a_2 = 0$ in (4.6.6), and then use Lemma 4.3.4 to get

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq y}(\sigma_{\mathbb{K},0}(\mathcal{I}))^2 \ll y\log^9 y,$$

and

$$E(x,y) \ll_{\mathfrak{q}} x^2 y^{17/18}\log^{24} T.$$

Next, from lemma 4.6.1, for a quadratic number field $\alpha = 1/3$. To estimate the bounds of the integrals $L_{i,j}$ for $1 \leq i \leq 4$ and $1 \leq j \leq 3$, in the estimation of integrals I_i for $1 \leq i \leq 4$, we use (4.3.5). In the estimation of I_1, the integral along the horizontal lines as

$$\begin{aligned} |L_{1,1}|,|L_{1,3}| &\ll_{\mathfrak{q}} yx\int_{-T}^{T}\left(\int_{3/2-\beta_1}^{\beta_2}T^{2(2-\beta_1-\sigma)/3}\log^8 T\frac{x^\sigma}{T}d\sigma\right)\frac{1}{1+|t|}dt \\ &\ll_{\mathfrak{q}} \frac{xy\log^8 T}{T^{1/3}}\int_{-T}^{T}\left(\int_{3/2-\beta_1}^{\beta_2}\left(\frac{x}{T^{2/3}}\right)^\sigma d\sigma\right)\frac{1}{1+|t|}dt \end{aligned}$$

$$\ll_{\mathfrak{q}} \frac{x^2 y \log^9 T}{T} + \frac{x^{3/2} y \log^9 T}{T^{2/3}}.$$

Furthermore, the integral along the vertical line is given by

$$\begin{aligned}
|L_{1,2}| &\ll yx^{3/2} \int_{-T}^{T} \int_{-T}^{T} \frac{|\zeta_{\mathbb{K}}(\beta_1 - 1/2 + i(t_1+t_2))|}{|\zeta_{\mathbb{K}}(\beta_1 + 1/2 + i(t_1+t_2))|} \frac{1}{(1+|t_1|)(1+|t_2|)} dt_1 dt_2 \\
&\ll yx^{3/2} \int_{-2T}^{2T} \frac{|\zeta_{\mathbb{K}}(\beta_1 - 1/2 + it)|}{|\zeta_{\mathbb{K}}(\beta_1 + 1/2 + it)|} \int_{-T}^{T} \frac{1}{(1+|t_1|)(1+|t-t_1|)} dt_1 dt \\
&\ll_{\mathfrak{q}} yx^{3/2} \log T \int_{-2T}^{2T} \frac{t^{1/3} \log^6 t}{(1+|t|)} dt \\
&\ll_{\mathfrak{q}} yx^{3/2} T^{1/3} \log^7 T.
\end{aligned}$$

Therefore, we have

$$I_1 = \frac{\rho_{\mathbb{K}}^2 y x^2}{2\zeta_{\mathbb{K}}(2)} + \mathrm{O}_{\mathfrak{q}} \left(\frac{x^2 y \log^9 T}{T} + yx^{3/2} T^{1/3} \log^7 T \right). \tag{4.7.5}$$

For the integral I_2, we have

$$\begin{aligned}
&|L_{2,1}|, |L_{2,3}| \\
&\ll y^2 \int_{-T}^{T} \left(\int_{\beta_1}^{3/2} \frac{\zeta_{\mathbb{K}}(2-\sigma-iT)\zeta_{\mathbb{K}}(1-\sigma+\beta_2-iT+it)}{\zeta_{\mathbb{K}}(2-\sigma+\beta_2)\zeta_{\mathbb{K}}(\sigma)} \frac{x^{\sigma+\beta_2}}{y^\sigma T^2} d\sigma \right) \frac{1}{1+|t|} dt \\
&\ll_{\mathfrak{q}} \frac{xy^2}{T^2} \int_{-T}^{T} \left(\int_{\beta_1}^{3/2} T^{\frac{2-2(2-\sigma)}{3}} T^{\frac{2-2(1-\sigma+\beta_2)}{3}} \log^{10} T \frac{x^\sigma}{y^\sigma} d\sigma \right) \frac{1}{1+|t|} dt \\
&\ll_{\mathfrak{q}} \frac{xy^2 \log^{10} T}{T^{2+4/3}} \int_{-T}^{T} \left(\int_{\beta_1}^{3/2} \left(\frac{xT^{4/3}}{y} \right)^{\sigma} d\sigma \right) \frac{1}{1+|t|} dt \\
&\ll_{\mathfrak{q}} \frac{x^{5/2} y^{1/2} \log^{11} T}{T^{4/3}} + \frac{x^2 y \log^{11} T}{T^2},
\end{aligned}$$

and

$$\begin{aligned}
&|L_{2,2}| \ll x^{5/2} y^{1/2} \\
&\times \int_{-T}^{T} \int_{-T}^{T} \frac{|\zeta_{\mathbb{K}}(1/2 - it_1)||\zeta_{\mathbb{K}}(-1/2+\beta_2+i(-t_1+t_2))|}{|\zeta_{\mathbb{K}}(3/2+it_1)||\zeta_{\mathbb{K}}(1/2+\beta_2+i(-t_1+t_2))|(1+|t_1|)^2(1+|t_2|)} dt_1 dt_2 \\
&\ll_{\mathfrak{q}} x^{5/2} y^{1/2} \int_{-2T}^{2T} \frac{|\zeta_{\mathbb{K}}(-1/2+\beta_2+it)|}{|\zeta_{\mathbb{K}}(1/2+\beta_2+it)|} \int_{-T}^{T} \frac{t_1^{1/3} \log^6 t_1}{(1+|t_1|)^2(1+|t_1+t|)} dt_1 dt
\end{aligned}$$

$$\begin{aligned}
&\ll_{\mathfrak{q}} x^{5/2}y^{1/2}\log^6 T\int_{-2T}^{2T}\frac{t^{1/3}\log^6 t}{(1+|t|)}dt\\
&\ll_{\mathfrak{q}} x^{5/2}y^{1/2}T^{1/3}\log^{12} T. \qquad (4.7.6)
\end{aligned}$$

Combining the above estimates, we have

$$I_2=\frac{\rho_{\mathbb{K}}^2 x^4\zeta_{\mathbb{K}}(0)}{4\zeta_{\mathbb{K}}(2)^2}+\mathrm{O}_{\mathfrak{q}}\left(\frac{x^2y\log^{11}T}{T^2}+x^{5/2}y^{1/2}T^{1/3}\log^{12}T\right). \qquad (4.7.7)$$

In the estimation of integral I_3, the integral along the horizontal lines are

$$\begin{aligned}
&|L_{3,1}|,|L_{3,3}|\\
&\ll y^2\int_{-T}^{T}\left(\int_{\beta_2}^{3/2}\frac{\zeta_{\mathbb{K}}(2-\sigma-iT)\zeta_{\mathbb{K}}(1-\sigma+\beta_1-iT+it)}{\zeta_{\mathbb{K}}(2-\sigma+\beta_1)\zeta_{\mathbb{K}}(\sigma)}\frac{x^{\sigma+\beta_1}}{y^{\sigma}T^2}d\sigma\right)\frac{1}{1+|t|}dt\\
&\ll_{\mathfrak{q}} \frac{xy^2}{T^2}\int_{-T}^{T}\left(\int_{\beta_2}^{3/2}T^{\frac{2-2(2-\sigma)}{3}}T^{\frac{2-2(1-\sigma+\beta_1)}{3}}\log^{10}T\frac{x^{\sigma}}{y^{\sigma}}d\sigma\right)\frac{1}{1+|t|}dt\\
&\ll_{\mathfrak{q}} \frac{xy^2\log^{10}T}{T^{2+4/3}}\int_{-T}^{T}\left(\int_{\beta_2}^{3/2}\left(\frac{xT^{4/3}}{y}\right)^{\sigma}d\sigma\right)\frac{1}{1+|t|}dt\\
&\ll_{\mathfrak{q}} \frac{x^{5/2}y^{1/2}\log^{11}T}{T^{4/3}}+\frac{x^2y\log^{11}T}{T^2}.
\end{aligned}$$

We evaluate the integral along the vertical line same as (4.7.6). Therefore

$$|L_{3,2}|\ll_{\mathfrak{q}} x^{5/2}y^{1/2}T^{1/3}\log^{12}T,$$

and

$$I_3=\mathrm{O}_{\mathfrak{q}}\left(\frac{x^2y\log^{11}T}{T^2}+x^{5/2}y^{1/2}T^{1/3}\log^{12}T\right). \qquad (4.7.8)$$

In the estimation of I_4, we have

$$\begin{aligned}
&|L_{4,1}|,|L_{4,3}|\\
&\ll y^2x\int_{-T}^{T}\int_{\beta_2}^{5/2-\beta_1}\frac{\zeta_{\mathbb{K}}(2-\beta_1-it)\zeta_{\mathbb{K}}(2-\sigma-iT)\zeta_{\mathbb{K}}(3-\beta_1-\sigma-iT-it)x^{\sigma}}{\zeta_{\mathbb{K}}(4-\sigma-\beta_1)\zeta_{\mathbb{K}}(\beta_1)\zeta_{\mathbb{K}}(\sigma)y^{\sigma}T^2(1+|t|)}d\sigma dt\\
&\ll_{\mathfrak{q}} \frac{xy^2}{T^2}\int_{-T}^{T}\left(\int_{\beta_1}^{5/2-\beta_1}T^{\frac{2-2(2-\sigma)}{3}}T^{\frac{2-2(1-\sigma-\beta_1)}{3}}\log^{14}T\frac{x^{\sigma}}{y^{\sigma}}d\sigma\right)\frac{1}{1+|t|}dt
\end{aligned}$$

$$\ll_{\mathfrak{q}} \frac{x^{5/2}y^{1/2}\log^{15} T}{T^{4/3}} + \frac{x^2 y \log^{15} T}{T^2},$$

and

$$\begin{aligned}
|L_{4,2}| &\ll x^{5/2}y^{1/2} \\
&\times \int_{-T}^{T}\int_{-T}^{T} \frac{|\zeta_{\mathbb{K}}(2-\beta_1-it_1)||\zeta_{\mathbb{K}}(-1/2+\beta_1-it_1)||\zeta_{\mathbb{K}}(1/2+i(-t_1-t_2))|dt_1dt_2}{|\zeta_{\mathbb{K}}(3/2+i(-t_1-t_2))||\zeta_{\mathbb{K}}(\beta_1+it_1)||\zeta_{\mathbb{K}}(5/2-\beta_1+it_2)|(1+|t_1|)^2(1+|t_2|)} \\
&\ll_{\mathfrak{q}} x^{5/2}y^{1/2}\int_{-2T}^{2T}\frac{|\zeta_{\mathbb{K}}(1/2+it)|}{|\zeta_{\mathbb{K}}(3/2+\beta_2+it)|}\int_{-T}^{T}\frac{t_1^{1/3}\log^{12} t_1}{(1+|t_1|)^2(1+|t-t_1|)}dt_1dt \\
&\ll_{\mathfrak{q}} x^{5/2}y^{1/2}\log^{12} T\int_{-2T}^{2T}\frac{t^{1/3}\log^6 t}{(1+|t|)}dt \\
&\ll_{\mathfrak{q}} x^{5/2}y^{1/2}T^{1/3}\log^{12} T.
\end{aligned}$$

Thus,

$$I_4 = \mathrm{O}_{\mathfrak{q}}\left(\frac{x^2 y\log^{15} T}{T^2} + x^{5/2}y^{1/2}T^{1/3}\log^{12} T\right). \tag{4.7.9}$$

Collecting the results from (4.7.5), (4.7.7), (4.7.8), (4.7.9) and inserting in (4.7.3), we deduce

$$\begin{aligned}
I = &\frac{\rho_{\mathbb{K}}^2 y x^2}{\zeta_{\mathbb{K}}(2)} + \frac{\rho_{\mathbb{K}}^2 x^4\zeta_{\mathbb{K}}(0)}{4\zeta_{\mathbb{K}}(2)^2} + \mathrm{O}_{\mathfrak{q}}\left(\frac{x^2y\log^9 T}{T} + yx^{3/2}T^{1/3}\log^7 T\right) \\
&+ \mathrm{O}_{\mathfrak{q}}\left(x^{5/2}y^{1/2}T^{1/3}\log^{12} T + x^2y^{17/18}\log^{24} T\right).
\end{aligned}$$

Taking $T = x^{1/3-\epsilon}$, we obtain the required result. □

4.7.2 Cubic number fields

Proof of Theorem 4.2.4. The proof uses the same steps as in the degree two case, except for several technical changes that arise due to the difference between the bounds of the Dedekind zeta function for cubic (4.3.9) and quadratic number fields (4.3.5). The bound of $R(x,\mathcal{I})$ for cubic is given as

$$\begin{aligned}
R(x,\mathcal{I}) &\ll \frac{x\log y}{T}\sigma_{\mathbb{K},0}(\mathcal{I})\int_{\beta_i-iT}^{\beta_i+iT}\frac{\sigma_{\mathbb{K},(1-s)}(\mathcal{I})}{\zeta_{\mathbb{K}}(s)}\frac{x^s}{s}ds + \frac{x^2\log^2 y}{T^2}(\sigma_{\mathbb{K},0}(\mathcal{I}))^2 \\
&\ll \frac{x^2\log y\log^3 T}{T}(\sigma_{\mathbb{K},0}(\mathcal{I}))^2.
\end{aligned} \tag{4.7.10}$$

Put $a_1 = a_2 = 0$ in Lemma 4.6.1, this implies

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq y} (\sigma_{\mathbb{K},0}(\mathcal{I}))^2, \ll y\log^9 y.$$

Therefore,

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq y} R(x,\mathcal{I}) \ll \frac{x^2y\log y\log^3 T\log^9 y}{T}.$$

Also, from Lemma 4.6.1, we have

$$E(x,y) \ll_{\mathfrak{q}} x^2y^{\frac{45}{49}}\log^3 y.$$

From lemma 4.6.1, for a cubic number field $\alpha = 3/14$. Next, we estimate the bounds of the integrals $L_{i,j}$ for $1 \leq i \leq 4$ and $1 \leq j \leq 3$ using (4.3.9) for cubic number fields. In the estimation of I_1, the integral along the horizontal lines as

$$\begin{aligned}|L_{1,1}|, |L_{1,3}| &\ll_{\mathfrak{q}} yx\int_{-T}^{T}\left(\int_{1/2}^{\beta_2} T^{41(2-\beta_1-\sigma)/42}\log^{10} T\frac{x^\sigma}{T}d\sigma\right)\frac{1}{1+|t|}dt\\ &\ll_{\mathfrak{q}} \frac{x^2y\log^{11}T}{T} + \frac{x^{3/2}y\log^{11}T}{T^{43/84}}.\end{aligned}$$

Furthermore, the integral along the vertical line is given by

$$\begin{aligned}|L_{1,2}| &\ll_{\mathfrak{q}} yx^{3/2}\log T\int_{-2T}^{2T}\frac{t^{41/84}\log^6 t}{(1+|t|)}dt\\ &\ll_{\mathfrak{q}} yx^{3/2}T^{41/84}\log^7 T.\end{aligned}$$

Therefore, the integral I_1 becomes

$$I_1 = \frac{\rho_{\mathbb{K}}^2}{2\zeta_{\mathbb{K}}(2)}yx^2 + \mathrm{O}_{\mathfrak{q}}\left(\frac{x^2y\log^{11}T}{T} + yx^{3/2}T^{41/84}\log^7 T\right). \tag{4.7.11}$$

The line integral along horizontal lines for I_2 are given as:

$$\begin{aligned}|L_{2,1}|, |L_{2,3}| &\ll_{\mathfrak{q}} \frac{xy^2}{T^2}\int_{-T}^{T}\left(\int_{\beta_1}^{3/2} T^{\frac{41(2-\sigma)}{42}}T^{\frac{41(1-\sigma+\beta_2)}{42}}\log^3 T\frac{x^\sigma}{y^\sigma}d\sigma\right)\frac{1}{1+|t|}dt\\ &\ll_{\mathfrak{q}} \frac{x^{5/2}y^{1/2}\log^4 T}{T^{43/42}} + \frac{x^2y\log^4 T}{T^{1/21}}.\end{aligned}$$

The integral along the vertical line is given by

$$
\begin{aligned}
|L_{2,2}| &\ll_{\mathfrak{q}} x^{5/2}y^{1/2}\int_{-2T}^{2T}\frac{|\zeta_{\mathbb{K}}(-1/2+\beta_2+it)|}{|\zeta_{\mathbb{K}}(1/2+\beta_2+it)|}\int_{-T}^{T}\frac{t_1^{41/84}\log^3 t_1}{(1+|t_1|)^2(1+|t_1+t|)}dt_1dt \\
&\ll_{\mathfrak{q}} x^{5/2}y^{1/2}T^{41/84}\log^4 T. \qquad (4.7.12)
\end{aligned}
$$

Therefore, the integral I_2 becomes

$$
I_2 = \frac{\rho_{\mathbb{K}}^2\zeta_{\mathbb{K}}(0)}{4\zeta_{\mathbb{K}}(2)^2}x^4 + \mathrm{O}_{\mathfrak{q}}\left(\frac{x^2y\log^4 T}{T^{1/21}} + x^{5/2}y^{1/2}T^{41/84}\log^4 T\right). \qquad (4.7.13)
$$

For I_3, the integral along the horizontal lines are

$$
|L_{3,1}|, |L_{3,3}| \ll_{\mathfrak{q}} \frac{x^{5/2}y^{1/2}\log^5 T}{T^{41/84}} + \frac{x^2y\log^4 T}{T^{1/21}}.
$$

We can evaluate the integral along the vertical line the same as (4.7.12). Therefore

$$
|L_{3,2}| \ll_{\mathfrak{q}} x^{5/2}y^{1/2}T^{41/84}\log^4 T,
$$

and therefore

$$
I_3 = \mathrm{O}_{\mathfrak{q}}\left(\frac{x^2y\log^4 T}{T^{1/21}} + x^{5/2}y^{1/2}T^{41/84}\right). \qquad (4.7.14)
$$

For I_4, the integral along the horizontal lines are

$$
|L_{4,1}|, |L_{4,3}| \ll_{\mathfrak{q}} \frac{x^{5/2}y^{1/2}\log^5 T}{T^{41/84}} + \frac{x^2y\log^4 T}{T^{1/21}},
$$

and the integral along the vertical line the same as (4.7.12)

$$
|L_{4,2}| \ll_{\mathfrak{q}} x^{5/2}y^{1/2}T^{41/84}.
$$

Thus,

$$
I_4 = \mathrm{O}_{\mathfrak{q}}\left(\frac{x^2y\log^4 T}{T^{1/21}} + x^{5/2}y^{1/2}T^{41/84}\right). \qquad (4.7.15)
$$

Collecting the results from (4.7.11), (4.7.13), (4.7.14), (4.7.15), and insert in (4.7.3).

We get

$$I = \frac{\rho_{\mathbb{K}}^2}{\zeta_{\mathbb{K}}(2)} yx^2 + \frac{\rho_{\mathbb{K}}^2 \zeta_{\mathbb{K}}(0)}{4\zeta_{\mathbb{K}}(2)^2} x^4 + \mathrm{O}_{\mathfrak{q}}\left(\frac{x^2 y \log^3 T}{T} + x^2 y^{\frac{45}{49}} \log^3 y\right)$$
$$+ \mathrm{O}_{\mathfrak{q}}\left(\frac{x^2 y \log^4 T}{T^{1/21}} + x^{5/2} y^{1/2} T^{41/84}\right).$$

Substitute $T = x^{3/14-\epsilon}$ for a fixed $\epsilon > 0$ gives the required result. □

4.7.3 Cyclotomic number fields

Proof of Theorem 4.2.5. For a cyclotomic number field, we have

$$R(x, \mathcal{I}) \ll \frac{x}{T} \sigma_{\mathbb{K},(-1/\log y)}(\mathcal{I}) + \frac{x \log y}{T} \sigma_{\mathbb{K},0}(\mathcal{I}). \tag{4.7.16}$$

Therefore, we have

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq y} R(x, \mathcal{I}) \ll \frac{x^2 \log y \log^m T}{T} \sum_{0<\mathcal{N}(\mathcal{I})\leq y} (\sigma_{\mathbb{K},0}(\mathcal{I}))^2.$$

We take $a_1 = a_2 = 0$ in Lemma 4.3.4 to get

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq y} (\sigma_{\mathbb{K},0}(\mathcal{I}))^2 \ll y,$$

and from Lemma 4.3.4, we have

$$E(x, y) \ll_{\mathfrak{q}} x^2 (y^{5/6} + y^{1-1/\phi(m)}) \log^{5\phi(m)}.$$

To estimate the bounds of the integrals $L_{i,j}$ for $1 \leq i \leq 4$ and $1 \leq j \leq 3$ for cyclotomic number fields, we use (4.3.11). The integrals along horizontal lines for I_1 are given by:

$$|L_{1,1}|, |L_{1,3}| \ll_{\mathfrak{q}} y \int_{-T}^{T} \left(\int_{3/2-\beta_1}^{\beta_2} T^{(1-\sigma)\phi(m)/3} \log^{\phi(m)} T \frac{x^\sigma}{T} d\sigma\right) \frac{1}{1+|t|} dt$$
$$\ll_{\mathfrak{q}} \frac{x^2 y \log^{\phi(m)} T}{T} + x^{3/2} y T^{\phi(m)/6-1} \log^{\phi(m)+1} T.$$

Furthermore, the integral along the vertical line is given by

$$|L_{1,2}| \ll_{\mathfrak{q}} yx^{3/2} \int_{-T}^{T}\int_{-T}^{T} \frac{|\zeta_{\mathbb{K}}(\beta_1 - 1/2 + i(t_1+t_2))|}{|\zeta_{\mathbb{K}}(\beta_1 + 1/2 + i(t_1+t_2))|} \frac{1}{(1+|t_1|)(1+|t_2|)} dt_1 dt_2$$

$$\ll_{\mathfrak{q}} yx^{3/2} T^{\phi(m)/6} \log^{2\phi(m)+1} T.$$

Therefore, the integral I_1 becomes

$$I_1 = \frac{\rho^2 y x^2}{2\zeta_{\mathbb{K}}(2)} + \mathrm{O}_{\mathfrak{q}}\left(\frac{x^2 y \log^{\phi(m)} T}{T} + yx^{3/2} T^{\phi(m)/6} \log^{\phi(m)+1} T \right). \tag{4.7.17}$$

For the integral I_2, we have

$$|L_{2,1}|, |L_{2,3}| \ll_{\mathfrak{q}} y^2 \int_{-T}^{T} \left(\int_{\beta_1}^{3/2} \frac{\zeta_{\mathbb{K}}(2-\sigma)\zeta_{\mathbb{K}}(1-\sigma+\beta_2)}{\zeta_{\mathbb{K}}(2-\sigma+\beta_2)\zeta_{\mathbb{K}}(\sigma)} \frac{x^{\sigma+\beta_2}}{y^{\sigma}T^2} d\sigma \right) \frac{1}{1+|t|} dt$$

$$\ll_{\mathfrak{q}} x^{5/2} y^{1/2} T^{\phi(m)/3-2} \log^{4\phi(m)+1} T + \frac{x^2 y \log^{4\phi(m)+1} T}{T^2}.$$

The integral along the vertical line is given by

$$|L_{2,2}| \ll_{\mathfrak{q}} x^{5/2} y^{1/2}$$

$$\times \int_{-T}^{T}\int_{-T}^{T} \frac{|\zeta_{\mathbb{K}}(1/2 - it_1)||\zeta_{\mathbb{K}}(-1/2+\beta_2+i(-t_1+t_2))|}{|\zeta_{\mathbb{K}}(3/2+it_1)||\zeta_{\mathbb{K}}(1/2+\beta_2+i(-t_1+t_2))|(1+|t_1|)^2(1+|t_2|)} dt_1 dt_2$$

$$\ll_{\mathfrak{q}} x^{5/2} y^{1/2} T^{\phi(m)/3-1} \log^{4\phi(m)} T. \tag{4.7.18}$$

Therefore, the integral I_2 becomes

$$I_2 = \frac{\rho^2 x^4 \zeta_{\mathbb{K}}(0)}{4\zeta_{\mathbb{K}}(2)^2} + \mathrm{O}_{\mathfrak{q}}\left(\frac{x^2 y \log^{4\phi(m)+1} T}{T^2} + x^{5/2} y^{1/2} T^{\phi(m)/3-1} \log^{4\phi(m)} T \right). \tag{4.7.19}$$

The integral along the horizontal lines of I_3 are bounded as

$$|L_{3,1}|, |L_{3,3}| \ll_{\mathfrak{q}} y^2 \int_{-T}^{T} \left(\int_{\beta_2}^{3/2} \frac{\zeta_{\mathbb{K}}(2-\sigma)\zeta_{\mathbb{K}}(1-\sigma+\beta_1)}{\zeta_{\mathbb{K}}(2-\sigma+\beta_1)\zeta_{\mathbb{K}}(\sigma)} \frac{x^{\sigma+\beta_1}}{y^{\sigma}T^2} d\sigma \right) \frac{1}{1+|t|} dt$$

$$\ll_{\mathfrak{q}} x^{5/2} y^{1/2} T^{\phi(m)/3-2} \log^{4\phi(m)+1} T + \frac{x^2 y \log^{4\phi(m)+1} T}{T^2}.$$

We can evaluate the integral along the vertical line the same as (4.7.18). Therefore

$$|L_{3,2}| \ll_{\mathfrak{q}} x^{5/2}y^{1/2}T^{\phi(m)/3-1}\log^{4\phi(m)} T,$$

and

$$I_3 = \mathrm{O}_{\mathfrak{q}}\left(\frac{x^2y\log^{4\phi(m)+1} T}{T^2} + x^{5/2}y^{1/2}T^{\phi(m)/3-1}\log^{4\phi(m)} T\right). \tag{4.7.20}$$

The integral along the horizontal lines for the integral I_4 are bounded as

$$|L_{4,1}|, |L_{4,3}| \ll_{\mathfrak{q}} x^{5/2}y^{1/2}T^{\phi(m)/3-2}\log^{4\phi(m)+1} T + \frac{x^2y\log^{4\phi(m)+1} T}{T^2},$$

and the integral along the vertical line for the integral I_3 is bounded as

$$|L_{4,2}| \ll_{\mathfrak{q}} x^{5/2}y^{1/2}T^{\phi(m)/3-1}\log^{4\phi(m)} T.$$

Thus,

$$I_4 = \mathrm{O}_{\mathfrak{q}}\left(\frac{x^2y\log^{4\phi(m)+1} T}{T^2} + x^{5/2}y^{1/2}T^{\phi(m)/3-1}\log^{4\phi(m)} T\right). \tag{4.7.21}$$

Collecting the results from (4.7.17), (4.7.19), (4.7.20), (4.7.21), and insert in (4.7.3). We get

$$\begin{aligned} I = \frac{\rho^2yx^2}{\zeta_{\mathbb{K}}(2)} + \frac{\rho^2x^4\zeta_{\mathbb{K}}(0)}{4\zeta_{\mathbb{K}}(2)^2} + \mathrm{O}_{\mathfrak{q}}\left(\frac{x^2y\log^{2\phi(m)} T}{T} + x^{3/2}yT^{\phi(m)/6}\log^{2\phi(m)+1} T\right) \\ + \mathrm{O}_{\mathfrak{q}}\left(+x^{5/2}y^{1/2}T^{\phi(m)/3-1}\log^{4\phi(m)} T + x^2y^{1-k(\phi(m)+1/2)}\log^{5\phi(m)} x\right). \end{aligned}$$

Putting $T = x^{1/4\phi(m)}$ gives the required result. □

4.7.4 General number fields under GLH

Proof of Theorem 4.2.2. For a general number field, under GLH, we have

$$R(x, \mathcal{I}) \ll \frac{x}{T}\sigma_{\mathbb{K},(-1/\log y)}(\mathcal{I}) + \frac{x\log y}{T}\sigma_{\mathbb{K},0}(\mathcal{I}). \tag{4.7.22}$$

Therefore, we have

$$\sum_{0<\mathcal{N}(\mathcal{I})\le y} R(x,\mathcal{I}) \ll \frac{x^2\log y\log^m T}{T}\sum_{0<\mathcal{N}(\mathcal{I})\le y}(\sigma_{\mathbb{K},0}(\mathcal{I}))^2.$$

We take $a_1=a_2=0$ in Lemma 4.3.4, we have

$$\sum_{0<\mathcal{N}(\mathcal{I})\le y}(\sigma_{\mathbb{K},0}(\mathcal{I}))^2 \ll y.$$

Also, from Lemma 4.3.4, we have

$$E(x,y)\ll_{\mathfrak{q}} x^2y^{1/2+\epsilon}.$$

Next, from lemma 4.6.1, under GLH, we choose $\alpha=2$. To estimate the bounds of the integrals $L_{i,j}$ for $1\le i\le 4$ and $1\le j\le 3$, in the estimation of integrals I_i for $1\le i\le 4$, we use (4.3.12). In the estimation of I_1, the integral along the horizontal lines as

$$|L_{1,1}|,|L_{1,3}|\ll_{\mathfrak{q}} xy\int_{-T}^{T}\left(\int_{1/2}^{\beta_2}T^{\epsilon}\frac{x^{\sigma}}{T}d\sigma\right)\frac{1}{1+|t|}dt\ll_{\mathfrak{q}}\frac{x^2y}{T^{1-\epsilon}},$$

and the integral along $L_{1,2}$ is given by

$$\begin{aligned}|L_{1,2}| &\ll_{\mathfrak{q}} yx^{3/2}\int_{-T}^{T}\int_{-T}^{T}\frac{|\zeta_{\mathbb{K}}(\beta_1-1/2+i(t_1+t_2))|}{|\zeta_{\mathbb{K}}(\beta_1+1/2+i(t_1+t_2))|}\frac{1}{(1+|t_1|)(1+|t_2|)}dt_1dt_2\\ &\ll_{\mathfrak{q}} yx^{3/2}\log T\int_{-2T}^{2T}\frac{t^{\epsilon}}{(1+|t|)}dt\ll yx^{3/2}T^{\epsilon}. \end{aligned}\tag{4.7.23}$$

Therefore, the integral I_1 is

$$I_1=\frac{\rho_{\mathbb{K}}^2}{2\zeta_{\mathbb{K}}(2)}yx^2+\mathrm{O}_{\mathfrak{q}}\left(\frac{x^2y}{T^{1-\epsilon}}+yx^{3/2}T^{\epsilon}\right).\tag{4.7.24}$$

For the integral I_2, we have

$$|L_{2,1}|,|L_{2,3}|\ll_{\mathfrak{q}}\frac{xy^2}{T^{2-\epsilon}}\int_{-T}^{T}\left(\int_{\beta_1}^{3/2}\frac{x^{\sigma}}{y^{\sigma}}d\sigma\right)\frac{1}{1+|t|}dt$$

$$\ll_{\mathfrak{q}} \frac{x^{5/2}y^{1/2}}{T^{2-\epsilon}} + \frac{x^2 y}{T^{2-\epsilon}},$$

and

$$\begin{aligned}|L_{2,2}| &\ll_{\mathfrak{q}} x^{5/2}y^{1/2} \int_{-2T}^{2T} \frac{|\zeta_{\mathbb{K}}(-1/2+\beta_2+it)|}{|\zeta_{\mathbb{K}}(1/2+\beta_2+it)|} \int_{-T}^{T} \frac{t_1^{\epsilon}}{(1+|t_1|)^2(1+|t_1+t|)} dt_1 dt \\ &\ll_{\mathfrak{q}} \frac{x^{5/2}y^{1/2}}{T^{1-\epsilon}}.\end{aligned}$$

Therefore, the integral I_2 becomes

$$I_2 = \frac{\rho_{\mathbb{K}}^2 \zeta_{\mathbb{K}}(0)}{4\zeta_{\mathbb{K}}(2)^2} x^4 + \mathrm{O}_{\mathfrak{q}}\left(\frac{x^2 y}{T^{2-\epsilon}} + \frac{x^{5/2}y^{1/2}}{T^{1-\epsilon}} \right). \tag{4.7.25}$$

For the integral I_3, we have

$$\begin{aligned}|L_{3,1}|, |L_{3,3}| &\ll_{\mathfrak{q}} \frac{xy^2}{T^2} \int_{-T}^{T} \left(\int_{\beta_2}^{3/2} T^{\epsilon} \frac{x^{\sigma}}{y^{\sigma}} d\sigma \right) \frac{1}{1+|t|} dt \\ &\ll_{\mathfrak{q}} \frac{x^{5/2}y^{1/2}}{T^{2-\epsilon}} + \frac{x^2 y}{T^{2-\epsilon}},\end{aligned}$$

and

$$|L_{3,2}| \ll_{\mathfrak{q}} \frac{x^{5/2}y^{1/2}}{T^{1-\epsilon}}.$$

Therefore , we have

$$I_3 = \mathrm{O}_{\mathfrak{q}}\left(\frac{x^{5/2}y^{1/2}}{T^{1-\epsilon}} + \frac{x^2 y}{T^{2-\epsilon}} \right). \tag{4.7.26}$$

Finally, for the integral I_4, we have

$$|L_{4,1}|, |L_{4,3}| \ll_{\mathfrak{q}} \frac{x^{5/2}y^{1/2}}{T^{2-\epsilon}} + \frac{x^2 y}{T^{2-\epsilon}},$$

and

$$|L_{4,2}| \ll_{\mathfrak{q}} \frac{x^{5/2}y^{1/2}}{T^{1-\epsilon}}.$$

Thus, The integral I_4 becomes

$$I_4 = \mathrm{O}_{\mathfrak{q}}\left(\frac{x^{5/2}y^{1/2}}{T^{1-\epsilon}} + \frac{x^2 y}{T^{2-\epsilon}}\right). \tag{4.7.27}$$

Collecting the results from (4.7.24), (4.7.25), (4.7.26), (4.7.27), and insert in (4.7.3). We have

$$I = \frac{\rho_{\mathbb{K}}^2}{\zeta_{\mathbb{K}}(2)} y x^2 + \frac{\rho_{\mathbb{K}}^2 \zeta_{\mathbb{K}}(0)}{4\zeta_{\mathbb{K}}(2)^2} x^4 + \mathrm{O}_{\mathfrak{q}}\left(\frac{x^2 y}{T^{1-\epsilon}} + y x^{3/2} T^{\epsilon} + \frac{x^{5/2}y^{1/2}}{T^{1-\epsilon}}\right). \tag{4.7.28}$$

Choose $T = x^{2-\epsilon}$ and substitute in the above expression; we obtain the required result. □

4.8 Proof of Theorem 4.2.6

In this section, using elementary techniques, we prove the second moment for number fields of any degree satisfying condition (4.2.5). As the degree of a number field increases, due to large bounds for the associated Dedekind zeta function in the required regions, the error terms originating from the line integrals in Perron's formula dominate over the main terms. Consequently, we avoid an analytic approach for higher degree number fields at the cost of losing a second main term.

Proof of Theorem 4.2.6. From (1.3.4), we have

$$\begin{aligned}\sum_{0<\mathcal{N}(\mathcal{I})\le y}\left(\sum_{0<\mathcal{N}(\mathcal{J})\le x} C_{\mathcal{J}}(\mathcal{I})\right)^2 &= \sum_{0<\mathcal{N}(\mathcal{I})\le y}\left(\sum_{0<\mathcal{N}(\mathcal{J})\le x}\ \sum_{\substack{\mathcal{I}_1|\mathcal{J}\\ \mathcal{I}_1|\mathcal{I}}} \mathcal{N}(\mathcal{I}_1)\mu\Big(\frac{\mathcal{J}}{\mathcal{I}_1}\Big)\right)^2\\ &= \sum_{0<\mathcal{N}(\mathcal{I}_1\mathcal{J}_1)\le x}\ \sum_{0<\mathcal{N}(\mathcal{I}_2\mathcal{J}_2)\le x} \mathcal{N}(\mathcal{I}_1)\mathcal{N}(\mathcal{I}_2)\mu(\mathcal{J}_1)\mu(\mathcal{J}_2) \sum_{\substack{0<\mathcal{N}\mathcal{I})\le y\\ \mathcal{I}_1|\mathcal{I},\mathcal{I}_2|\mathcal{I}}} 1 \end{aligned} \tag{4.8.1}$$

From the hypothesis in Theorem 4.2.6, the innermost sum is given by

$$\#\{\mathcal{I} : \mathcal{I} \subseteq O_{\mathbb{K}} : 0 < \mathcal{N}(\mathcal{I}) \le y,\ \mathcal{I}_1|\mathcal{I}, \mathcal{I}_2|\mathcal{I}\} = \frac{\rho_{\mathbb{K}} y}{\mathcal{N}(\mathcal{I}_1 \cap \mathcal{I}_2)} + \mathrm{O}\left(\left(\frac{y}{\mathcal{N}(\mathcal{I}_1 \cap \mathcal{I}_2)}\right)^{\alpha}\right).$$

Using this, the left-hand side of (4.8.1) equals

$$\begin{aligned}
&\rho_{\mathbb{K}} y \sum_{0<\mathcal{N}(\mathcal{I}_1\mathcal{J}_1)\le x}\ \sum_{0<\mathcal{N}(\mathcal{I}_2\mathcal{J}_2)\le x} \mathcal{N}(\mathcal{I}_1+\mathcal{I}_2)\mu(\mathcal{J}_1)\mu(\mathcal{J}_2) \\
&+ \mathrm{O}\left(y^{\alpha} \sum_{0<\mathcal{N}(\mathcal{I}_1\mathcal{J}_1)\le x}\ \sum_{0<\mathcal{N}(\mathcal{I}_2\mathcal{J}_2)\le x} \mathcal{N}(\mathcal{I}_1+\mathcal{I}_2)^{\alpha}\mathcal{N}(\mathcal{I}_1)^{1-\alpha}\mathcal{N}(\mathcal{I}_2)^{1-\alpha}\right) \\
&=: I_1 + I_2 \qquad (4.8.2)
\end{aligned}$$

Let $\mathcal{I}_1+\mathcal{I}_2=\mathcal{A}$, then $\mathcal{I}_1=\mathcal{A}\mathcal{E}_1$, and $\mathcal{I}_2=\mathcal{A}\mathcal{E}_2$ such that $\mathcal{E}_1+\mathcal{E}_2=\mathcal{O}_{\mathbb{K}}$. This yields

$$\begin{aligned}
I_1 &= \rho_{\mathbb{K}} y \sum_{0<\mathcal{N}(\mathcal{A}\mathcal{E}_1\mathcal{J}_1)\le x}\ \sum_{\substack{0<\mathcal{N}(\mathcal{A}\mathcal{E}_2\mathcal{J}_2)\le x \\ \mathcal{E}_1+\mathcal{E}_2=\mathcal{O}_{\mathbb{K}}}} \mathcal{N}(\mathcal{A})\mu(\mathcal{J}_1)\mu(\mathcal{J}_2) \\
&= \rho_{\mathbb{K}} y \sum_{0<\mathcal{N}(\mathcal{A}\mathcal{E}_1\mathcal{J}_1)\le x}\ \sum_{0<\mathcal{N}(\mathcal{A}\mathcal{E}_2\mathcal{J}_2)\le x} \mathcal{N}(\mathcal{A})\mu(\mathcal{J}_1)\mu(\mathcal{J}_2) \sum_{\mathcal{M}\mid\mathcal{E}_1+\mathcal{E}_2} \mu(\mathcal{M}) \\
&= \rho_{\mathbb{K}} y \sum_{0<\mathcal{N}(\mathcal{A}\mathcal{M})\le x} \mathcal{N}(\mathcal{A})\mu(\mathcal{M}) \left(\sum_{0<\mathcal{N}(\mathcal{E}\mathcal{J})\le x/\mathcal{N}(\mathcal{A}\mathcal{M})} \mu(\mathcal{J}) \right)^2 \\
&= \rho_{\mathbb{K}} y \sum_{0<\mathcal{N}(\mathcal{A}\mathcal{M})\le x} \mathcal{N}(\mathcal{A})\mu(\mathcal{M} = \frac{\rho_{\mathbb{K}}^2 x^2 y}{2\zeta_{\mathbb{K}}(2)} + \mathrm{O}\left(xy\log x\right), \qquad (4.8.3)
\end{aligned}$$

and

$$\begin{aligned}
I_2 &\ll y^{\alpha} \sum_{0<\mathcal{N}(\mathcal{A}\mathcal{E}_1\mathcal{J}_1)\le x}\ \sum_{\substack{0<\mathcal{N}(\mathcal{A}\mathcal{E}_2\mathcal{J}_2)\le x \\ \mathcal{E}_1+\mathcal{E}_2=\mathcal{O}_{\mathbb{K}}}} \mathcal{N}(\mathcal{A})^{\alpha}\mathcal{N}(\mathcal{E}_1\mathcal{A})^{1-\alpha}\mathcal{N}(\mathcal{E}_2\mathcal{A})^{1-\alpha} \\
&\ll y^{\alpha} \sum_{0<\mathcal{N}(\mathcal{A}\mathcal{E}_1\mathcal{J}_1)\le x}\ \sum_{0<\mathcal{N}(\mathcal{A}\mathcal{E}_2\mathcal{J}_2)\le x} \mathcal{N}(\mathcal{A})^{2-\alpha}\mathcal{N}(\mathcal{E}_1)^{1-\alpha}\mathcal{N}(\mathcal{E}_2)^{1-\alpha} \sum_{\mathcal{M}\mid\mathcal{E}_1+\mathcal{E}_2} \mu(\mathcal{M}) \\
&\ll y^{\alpha} \sum_{0<\mathcal{N}(\mathcal{A}\mathcal{M})\le x} \mathcal{N}(\mathcal{A})^{2-\alpha} \left(\sum_{0<\mathcal{N}(\mathcal{E}\mathcal{J})\le x/\mathcal{N}(\mathcal{A}\mathcal{M})} \mathcal{N}(\mathcal{E})^{1-\alpha} \right)^2 \\
&\ll y^{\alpha} \sum_{0<\mathcal{N}(\mathcal{A}\mathcal{M})\le x} \mathcal{N}(\mathcal{A})^{2-\alpha} \frac{x^{(2-\alpha)^2}}{\mathcal{N}(\mathcal{A}\mathcal{M})^{(2-\alpha)^2}} \ll y^{\alpha}x^{3-\alpha}. \qquad (4.8.4)
\end{aligned}$$

On substitution of (4.8.3) and (4.8.4) in (4.8.2), we obtain the required result. □

5
Conclusions and Future Directions

The aim of this thesis is to study the distribution of Ramanujan sums by finding their moments of averages and convolution sums. We also used the convolutions of Ramanujan sums to derive a heuristic proof of the Hardy-Littlewood prime tuple conjecture. Ramanujan sums are defined over number fields in (1.3.4). For $y > x^{\delta}$, moments of Ramanujan sums over number fields are as follows:

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq y} \sum_{0<\mathcal{N}(\mathcal{J})\leq x} C_{\mathcal{J}}(\mathcal{I}) = \rho_{\mathbb{K}} y + E(x, y).$$

The second moment is given as:

$$\sum_{0<\mathcal{N}(\mathcal{I})\leq y} \left(\sum_{0<\mathcal{N}(\mathcal{J})\leq x} C_{\mathcal{J}}(\mathcal{I}) \right)^2 = \frac{\rho_{\mathbb{K}}^2}{\zeta_{\mathbb{K}}(2)} y x^2 + E(x, y).$$

Here, $E(x, y)$ is an error term. We summarize the results of moments in the following two tables:

Field	Moments		
($\mathbb{Q}$)	$k = 1$	$k = 2$	$k \geq 3$
$c_q(n)$	Chan and Kunchev (2012) $S_1(x, y) = y - \frac{x^2}{4\zeta(2)} + E(x, y)$	Chan and Kunchev (2012) $S_2(x, y) = \frac{yx^2}{2\zeta(2)} + E(x, y)$	Goel and Murty (2024) $S_k(x, y) = yx^k Q(\log x) + E(x, y)$,
$c_{q,\beta}(n)$	Robles and Roy (2017) $S_{1,\beta}(x, y) = y + E(x, y)$	Robles and Roy (2017) $S_{2,\beta}(x, y) = \frac{yx^{1+\beta}}{(1+b)\zeta(1+b)} + E(x, y)$	Goel and Murty (2024) $S_{k,\beta}(x, y) = yx^{k(\beta+1)/2} Q(\log x) + E(x, y)$ where $Q \in \mathbb{R}[X]$

Table 5.1: Moments of Ramanujan sums over $\mathbb{Q}$

Field	Moments	
	$k = 1$	$k = 2$
Quadratic number fields	Nowak (2012) improved by Zhai (2021)	Chaubey and Goel (2023)
Cubic number fields	Ma, Sun, Zhai (2021)	Chaubey and Goel (2023)
Arbitrary number fields	Fujisawa (2015) improved by Chaubey and Goel (2023) (Under Lindelof Hypothesis)	Chaubey and Goel (2023) (Under Lindelof Hypothesis)

Table 5.2: Moments of Ramanujan sums over number fields

In [9], Carlitz generalized Ramanujan sums over function fields as follows: Let $\mathbb{F}_q$ be a finite field with $q = p^l$ elements and $\mathbb{F}_q[x]$ be the polynomial ring. If G, H, and D are the polynomial in $A = \mathbb{F}_q[x]$ and $\beta \in \mathbb{N}$, then generalized Ramanujan sum is defined as:

$$\eta_\beta(G, H) := \sum_{\substack{D|G \\ D^\beta|H}} |D|^\beta \mu\left(\frac{G}{D}\right), \tag{5.0.1}$$

where $|D| = \mathfrak{q}^{\deg(D)}$, and $\mu(G)$ is the generalized Mobius function defined as $\mu(0) = 0$ and

$$\mu(G) = \begin{cases} 1, & \text{if } G \in \mathbb{F}_q^*, \\ (-1)^t, & \text{if } G = \alpha P_1 P_2 \cdots P_t \text{ such that } \alpha \in \mathbb{F}_q^* \text{ and } P_i's \text{ are distinct monic} \\ & \text{irreducible polynomials,} \\ 0, & \text{if there exists P such that } P^2|G. \end{cases}$$

Recently, Zheng [71] studied some interesting arithmetical and analytic properties of these sums. Our ongoing work studies the moments of Ramanujan sums over function fields.

During the estimation of the triple convolution of Ramanujan sums, we encountered two variable variant of Ramanujan sums defined as:

$$\mathscr{K}_r(h,j) := \sum_{\substack{(b,r)=(c,r)=1\\(b+c,r)=1}} e^{2\pi i(hb+jc)/r}.$$

We proved it is a multiplicative function of r and satisfies the orthogonality property. using the orthogonality property of these sums, we can express an arithmetical function as a series expansion involving the two-variable variant of Ramanujan sums. As a result, this function is worth an independent study.

Therefore, we are interested in finding a closed formula like Ramanujan sums in terms of arithmetic functions for this function and obtaining the series expansion of functions in terms of the two-variable variant of Ramanujan sums. Once one obtains the series expansion of a function, it would be interesting to study the convolution of the same function using the properties of generalized Ramanujan sums as we did in the case of Ramanujan sums. These problems parallel Ramanujan sums theory for this generalized variant and will appear in our forthcoming paper.

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