



On Logical Structures

by

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Submitted

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To Thakur, Maa and Swamiji
To Dadubhai

Certificate

This is to certify that the thesis entitled *On Logical Structures* being submitted by **Mr. Sayantan Roy** to the **Indraprastha Institute of Information Technology Delhi**, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under our supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
July 2024



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Candidate's Declaration

The author hereby declares that the work presented in this thesis entitled *On Logical Structures*, submitted as partial fulfilment for the degree of **Doctor of Philosophy** to **Indraprastha Institute of Information Technology, Delhi**, has been carried out under the supervision of **Prof. Sankha S. Basu** and **Prof. Mihir K. Chakraborty**.

The work done in this thesis is original and has not been submitted earlier as a whole or in part for a degree or diploma at this or any other Institution or University.

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Sayantan Roy

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Abstract

Universal logic, as described by Béziau, is a structural approach to logic which aims to unify the large kaleidoscopic variety of logics, while preserving their diversity. In order to do this, Béziau’s original proposal was to consider logics irrespective of how they are generated – beyond, e.g., the dichotomy of syntax/semantics – via logical structures (which, according to his initial proposal, are pairs of the form $(\mathcal{L}, \vdash)$ where $\mathcal{L}$ is a set and $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$). In this thesis, we investigate the fruitfulness of this idea in detail. The results we obtain can be divided into three classes. The first two chapters (after the introductory one) are concerned with the properties of some particular types of logical structures. The fourth one hovers around Suszko’s Thesis and many-valued logics. In this chapter, we provide a definition of many-valued logical structures taking into account the language/metalanguage hierarchy; a generalised version of Suszko’s Thesis is outlined as well. The fifth chapter deals with the question: is it possible to describe paraconsistency independently of the language or connectives? We show that the answer to this question is affirmative, and propose several syntax independent definitions of paraconsistency. The sixth and the final one is a contribution to proof theory. In this chapter, we investigate CUT-elimination theorems in some detail. We provide two non-algorithmic proofs of the same for the propositional fragment of **LK**, followed by an addressal of the more general question of elimination of an arbitrary rule. This forces us to consider the notions of ‘rules’, ‘sequent systems’ and related notions in more detail. We generalise these concepts and manage to prove such ‘RULE-elimination theorems’ in these generalised settings. It is important to notice that in spite of the diversity of the areas to which these results are generally assumed to belong, the underlying unity rests entirely upon the very simple definition of logical structures. The establishment of this simple yet nontrivial fact, is perhaps one of the most important contributions of this thesis.

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Research Publications

Published Articles:

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Chapter 1

Introduction

1.1 Overview

The field of logic has grown phenomenally since the turn of the twentieth century. This might be, at least to some extent, attributed to the involvement of mathematics and computer science with logic, which is still continuing. There was also an unprecedented upsurge in the number of logics, which continues even to the present day, making it practically impossible to even have an idea of all the branches of the subject. Therefore, it is necessary to have a unifying notion of logic. This is necessary for establishing logic as a bonafide and independent branch of mathematics, described by Bourbaki in [30], as the study of mathematical structures and also to provide a bird's eye view of the subject for general patterns that were otherwise buried deep under the heap of particularities.

Perhaps the first progress towards this end was made by Paul Hertz via the notion of *Satzsysteme* (see [62, 63, 64] for details). Indeed, as Arnold Koslow pointed out in [69],

Hertz intended the inquiry to be very general, and as a result he deliberately treated the sentences* [expressions of the form $A_1, \dots, A_n \longrightarrow B$] as composed of elements that were not assumed to have any special logical form. The logical complexity of the elements was deliberately omitted, for that would have delimited the scope of generality.

Hertz's work on *Satzsysteme* was extended further, by Gerhard Gentzen (see, e.g., [59]), and later by Arnold Koslow (see, e.g., [69, 70]).

At around the same time, Alfred Tarski in [103], developed a similar abstract approach to the study of logic via the notion of *consequence operators*. In this paper, Tarski's aim was to provide an outline of the general framework for the *methodology of the deductive sciences*. In this theory, the nature of the objects of $\mathcal{L}$ is also not specified (even though Tarski himself called these objects 'meaningful sentences').

Tarski's idea was investigated in great detail by logicians such as Lindenbaum, Łoś, Suszko, Wójcicki, Pogorzelski, Wojtylak, Śłupeki etc., to name a few (see [89, 107] for a collection of the results developed during this period). In the hands of these logicians, the

theory of consequence operators took an algebraic turn. In [108], e.g., the notions of *structural consequence* and *sentential logic* were introduced, both of which depended on the language being an algebra (unlike in Tarski’s case). A generalisation of the notion of sentential logic, under the name of *abstract logics* were investigated by Suszko and his co-workers in [26, 32]. Today, this line of research falls under the scope of *abstract algebraic logic* (see [52, 54] for details).

The modern notion of *universal logic*, introduced in [14] with the aim of providing a general theory of logics in the same way as universal algebra is a general study of algebraic structures, can be seen as an extension of the same ideas and spirit.¹ Béziau’s original proposal in [14], was to focus on pairs of the form $(\mathcal{L}, \vdash)$ where $\mathcal{L}$ is a set, $\mathcal{P}(\mathcal{L})$ is its powerset and $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$. However, in later years, approaches based on other mathematical theories, e.g., category theory, have also been considered (see, e.g., [3, 49, 85]). All of these are part of the *universal logic project* (see [22] for details).

In this thesis, however, by the term *logical structure*, we will mean pairs of the form just described. Sometimes, we will also need to talk about pairs of the form $(\mathcal{L}, \vdash)$ where $\mathcal{L}$ is an algebra. In that case, $(\mathcal{L}, \vdash)$ will be referred to as a *logic*. For this thesis, the term *universal logic* will be assumed to mean a theory of logical structures.

1.2 Our Contribution

We begin with a discussion of *Tarski-* and *Lindenbaum-type* logical structures, in Chapter 2. A new characterisation theorem for the Tarski-type logical structures (Theorem 2.3.9) based on the notion of *strongly closed sets* is proved in Section 2.3. We then investigate the notion of α -excessive sets and show that this notion offers a generalisation to some better-known ones, viz., those of the maximal consistent sets in classical propositional logic, and the implication-saturated sets introduced by Batens in [12] (see Subsection 2.4.1 for details). Towards the end, we introduce the four Lindenbaum-type logical structures and show, via multiple examples, that these are, in general, not identical.

Continuing this study further, in Chapter 3, we show that the class of Tarski-type logical structures does not coincide with any of the Lindenbaum-type logical structures. The nonempty overlap with each of the classes of Lindenbaum-type logical structures prompts us to investigate logical structures which are simultaneously of Tarski- as well as a Lindenbaum-type, in other words, of *Tarski-Lindenbaum-type*. These have been named as the TL_i -type logical structures, where $i = 1, 2, 3, 4$ corresponding to each of the four Lindenbaum-type logical structures introduced earlier. The TL_4 -type logical structures have been studied in detail. We have proved characterisation, adequacy, minimality, and representation theorems for these (see Theorem 3.3.2, Theorem 3.3.6, Theorem 3.3.7 and Theorem 3.3.8, respectively). It has been pointed out that these are generalisations of some results proved by Béziau in [16], for finitary Tarski-type logical structures.

Next, we try to formulate a definition of *many-valued logical structure* in Chapter 4. For

¹This is the sense in which we use the term ‘universal logic’ in this thesis. There have been other uses of the term, such as the one in [31], but these are not connected to the present study.

this, we embark on a deeper study of Suszko’s Thesis² (**ST**) and show that the truth or falsity of **ST** depends, at least, on the precise notion of semantics. We propose two different notions of semantics (see [Definition 4.1.1](#) and [Definition 4.3.1](#)) and three different notions of entailment. The first one helps us formulate a precise definition of inferentially many-valued logical structures, and we show that every monotonic logical structure is functionally 4-valued (see [Section 4.2](#) for details). The second and the third help us to generalise Suszko Reduction and provide adequate bivalent semantics for monotonic and a couple of nonmonotonic logical structures. All these lead us to a closer examination of the language/metalanguage hierarchy. We conclude that many-valued logical structures can be obtained if the bivalence of all the higher-order metalogics of the logic under consideration is discarded, building formal bridges between the theory of graded consequence (see [\[38\]](#)) and the theory of many-valued logical structures, culminating in generalisations of Suszko’s Thesis.

The genesis of [Chapter 5](#) was the observation that paraconsistency – commonly defined and/or characterised as the failure of a principle of explosion – should not necessarily depend on the presence of a ‘negation’ operator in the language (of the logic under consideration). Two such notions of negation-free paraconsistency were introduced in [\[10\]](#). In [Chapter 5](#), we have extended this work. The earlier article introduced the *NF*-paraconsistency as the failure of the generalised explosion principle, denoted by **gECQ** (see [Section 5.2](#) for details). Several new examples of *NF*-paraconsistent logics have been provided here, the most notable of which are the ones involving variable inclusion logics, an area of research that is seeing a lot of interest lately.

We have defined a notion of *quasi-negation* and investigated its properties. We have then focused on principles of explosion and, in the process, further generalised **gECQ**. These principles gradually move away from the idea that an explosion requires a statement and its opposite. Thus, these principles can capture the explosion observed in logics where a statement and its negation explode only in the presence of additional information, such as in the *logics of formal inconsistency* (see [\[36, 37\]](#) for details). It is thus established that one can indeed talk about explosion in a broader sense without necessarily involving *any* logical connectives/operators. We show the interconnections, or the lack thereof, among these various principles of explosion via examples.

One of the major goals of universal logic, according to [\[21\]](#), is “to clarify the fundamental concepts of logic and to construct general proofs.” In the final chapter ([Chapter 6](#)), we aim to do this for the CUT-elimination theorems. Since Gentzen’s proof of the CUT-elimination theorem for the system **LK**, several other proofs have been proposed. Even though the techniques of these proofs can be modified to sequent systems other than **LK**, they are essentially of a very particular nature; each of them describes an algorithm to transform a given proof to a CUT-free proof. However, due to its reliance on heavy syntactic arguments and case distinctions, such an algorithm makes the fundamental structures of the arguments rather opaque. We, therefore, consider rules abstractly, and aim to clarify the essence of the so-called ‘elimination theorems’. To do this, we first give two non-algorithmic proofs of the CUT-elimination theorem for the propositional fragment of **LK**. From these proofs, we abstract the essential features of the argument and define something called *normal sequent*

²See [\[99\]](#) for a comprehensive survey.

structures. Next, a proof of the RULE-elimination theorem for these is given. Generalising even more, we define *abstract sequent structures* and show that for these structures, the corresponding version of the RULE-elimination theorem has a converse as well.

Chapter 2

Tarski-type and Lindenbaum-type Logical Structures¹

2.1 Introduction

In this chapter, we study two classes of logical structures, viz., logical structures of the Tarski- and of the Lindenbaum-types. The Tarski-type (see [Definition 2.2.1](#)) refers to a single type of logical structures and has been widely studied (referred to as normal logics in [\[17\]](#)). On the other hand, there are four possible definitions of a Lindenbaum-type logical structure (see [Definition 2.4.24](#)) based on the four versions of Lindenbaum's law discussed in [\[16\]](#). We build upon and extended the study of the Lindenbaum-type logical structures initiated by Béziau in [\[16\]](#). Questions regarding the possible relationships between these five classes (the Tarski- and the four Lindenbaum-types) have also been answered.

Central to our development of the theory of Lindenbaum-type logical structures, is the notion of α -excessive sets (see [Definition 2.4.1](#)). This is not a new concept and is defined as α -excessive theories in [\[14\]](#); a similar concept has also been described in [\[72\]](#). We study these here in full generality and show that in case of known logics, this notion corresponds nicely to familiar ones. For example, it has been proved in [Subsection 2.4.1](#) that the α -excessive sets of a certain kind are precisely the implication-saturated sets described by Batens in [\[12\]](#) (see [Theorem 2.4.13](#) and [Theorem 2.4.14](#)).

2.2 Fundamental notions

Definition 2.2.1 (LOGICAL STRUCTURE). A *logical structure* is a pair $(\mathcal{L}, \vdash)$, where $\mathcal{L}$ is a set and $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$. Here $\mathcal{P}(\mathcal{L})$ denotes the power set of $\mathcal{L}$.

In this chapter, we will always assume that $\mathcal{L}$ and $\vdash$ are nonempty.

¹A version of this chapter forms part of the article [\[96\]](#).

Remark 2.2.2. As mentioned earlier, logical structures were defined in [14]. However, although the terms ‘logic’ and ‘logical structure’ have been considered synonymous in [14], we distinguish between these here. The term ‘logic’ is reserved for referring to those pairs $(\mathcal{L}, \vdash)$, where $\mathcal{L}$ is an algebra. It is customary to assume some properties on $\vdash$ when considering logics (see for example, [32]); however, we deviate from this practice and do not assume any property of $\vdash$ even in such cases. We also do not use the term ‘formula’ to refer to the elements of $\mathcal{L}$. This is mainly to avoid introducing any potentially psychologically binding intuition about the elements of $\mathcal{L}$ (see [14, Section 3.2] for further discussion on this matter).

It is, however, easy to see that any logic $(\mathcal{L}, \vdash)$ has an underlying logical structure, the one obtained by ignoring the nature of, and any relationship between, the elements of $\mathcal{L}$. The underlying logical structure of a logic $(\mathcal{L}, \vdash)$ will be referred to as such, and denoted by $(\|\mathcal{L}\|, \vdash)$. To avoid any confusion, it will be clearly indicated at the outset of every discussion whether $(\mathcal{L}, \vdash)$ is a logic or a logical structure.

Let $(\mathcal{L}, \vdash)$ be a logical structure and $\Gamma \subseteq \mathcal{L}$. Then $W_{\mathcal{L}}^{\vdash}(\Gamma)$ denotes the set of all elements of $\mathcal{L}$ that are $\vdash$ -related to Γ , i.e.

$$W_{\mathcal{L}}^{\vdash}(\Gamma) = \{\alpha \in \mathcal{L} : \Gamma \vdash \alpha\}.$$

The subscript $\mathcal{L}$ on $W^{\vdash}$ is dropped when the logical structure under consideration is clear.

Remark 2.2.3. It is easy to see that given a $\vdash$, $W^{\vdash}$, as described above is an operator from $\mathcal{P}(\mathcal{L})$ to $\mathcal{P}(\mathcal{L})$. Conversely, given an operator $W : \mathcal{P}(\mathcal{L}) \rightarrow \mathcal{P}(\mathcal{L})$, we can define a relation $\vdash_W \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ such that $W = W^{\vdash_W}$ as follows. For all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, $\Gamma \vdash_W \alpha$ iff $\alpha \in W(\Gamma)$. Henceforth we will use this two descriptions interchangeably.

The following four definitions are obtained by unravelling [17, Definition 2]. We have chosen to divide the content for the sake of clarity.

Definition 2.2.4 (BIVALUATION). Let $\mathcal{L}$ be a set. Then any function $v \in \{0, 1\}^{\mathcal{L}}$ is said to be an $\mathcal{L}$ -*bivaluation*, or simply a *bivaluation*, when the set under consideration is understood.

Since $\mathcal{L}$ -bivaluations are functions from $\mathcal{L}$ into $\{0, 1\}$ (treated just as a set of two distinct elements), these can be understood as characteristic functions of subsets of $\mathcal{L}$ as well. Thus, given any $\Sigma \subseteq \mathcal{L}$, the characteristic function χ_{Σ} is a bivaluation and conversely, any bivaluation v is the characteristic function of $\{\alpha \in \mathcal{L} \mid v(\alpha) = 1\} \subseteq \mathcal{L}$. We will use this alternative description of bivaluations later in the chapter.

Definition 2.2.5 (SATISFACTION BY BIVALUATIONS). Suppose $\mathcal{L}$ is a set. An $\mathcal{L}$ -bivaluation v is said to *satisfy* $\Gamma \subseteq \mathcal{L}$ if $v(\alpha) = 1$ for every $\alpha \in \Gamma$. Given a set $\mathcal{V}$ of $\mathcal{L}$ -bivaluations, we say that $\mathcal{V}$ *satisfies* Γ if every $v \in \mathcal{V}$ satisfies Γ .

Given a set $\mathcal{L}$, a logical structure can also be obtained using a set of $\mathcal{L}$ -bivaluations $\mathcal{V} \subseteq \{0, 1\}^{\mathcal{L}}$. The *logical structure induced by* $\mathcal{V}$ is the pair $(\mathcal{L}, \vdash_{\mathcal{V}})$, where $\vdash_{\mathcal{V}} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ is defined as follows. For any $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$,

$$\Gamma \vdash_{\mathcal{V}} \alpha \iff \text{for all } v \in \mathcal{V}, \text{ if } v \text{ satisfies } \Gamma \text{ then } v(\alpha) = 1.$$

Definition 2.2.6 (BIVALUATION SOUNDNESS, COMPLETENESS AND ADEQUACY). Let $(\mathcal{L}, \vdash)$ is a logical structure and let $\mathcal{V}$ be a set of bivaluations. Then,

- (i) $(\mathcal{L}, \vdash)$ is said to be *sound* with respect to $(\mathcal{L}, \vdash_{\mathcal{V}})$ if for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, $\Gamma \vdash \alpha \implies \Gamma \vdash_{\mathcal{V}} \alpha$, i.e. $\vdash \subseteq \vdash_{\mathcal{V}}$.
- (ii) $(\mathcal{L}, \vdash)$ is said to be *complete* with respect to $(\mathcal{L}, \vdash_{\mathcal{V}})$ if for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, $\Gamma \vdash_{\mathcal{V}} \alpha \implies \Gamma \vdash \alpha$, i.e. $\vdash_{\mathcal{V}} \subseteq \vdash$.
- (iii) $(\mathcal{L}, \vdash)$ is said to be *adequate* with respect to $(\mathcal{L}, \vdash_{\mathcal{V}})$ if it is both sound and complete with respect to $(\mathcal{L}, \vdash_{\mathcal{V}})$, i.e. if $\vdash = \vdash_{\mathcal{V}}$.

2.3 Tarski-type Logical Structures

In this section, we describe the Tarski-type logical structures and prove a characterisation theorem for the same. As mentioned earlier, these were defined as *normal logics* in [17]. We begin with the definition, which is essentially the same as that of these normal logics.

Definition 2.3.1 (TARSKI-TYPE LOGICAL STRUCTURE). A logical structure $(\mathcal{L}, \vdash)$ is said to be of *Tarski-type*, or a *Tarski-type logical structure*, if $\vdash$ satisfies the following properties.

- (i) For all $\Gamma \subseteq \mathcal{L}$ and $\alpha \in \mathcal{L}$, if $\alpha \in \Gamma$ then $\Gamma \vdash \alpha$. (Reflexivity)
- (ii) For all $\Gamma, \Sigma \subseteq \mathcal{L}$ and $\alpha \in \mathcal{L}$, if $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Sigma$ then $\Sigma \vdash \alpha$. (Monotonicity)
- (iii) For all $\Gamma, \Sigma \subseteq \mathcal{L}$ and $\alpha \in \mathcal{L}$, if $\Gamma \vdash \beta$ for all $\beta \in \Sigma$ and $\Sigma \vdash \alpha$ then $\Gamma \vdash \alpha$. (Transitivity)

If $\vdash$ satisfies reflexivity/ monotonicity/ transitivity, the logical structure $(\mathcal{L}, \vdash)$ is said to be *reflexive/ monotonic/ transitive*, respectively.

Remark 2.3.2. The above definition of a Tarski-type logical structure can be rephrased in terms of $W^\vdash$ as follows. A logical structure $(\mathcal{L}, \vdash)$ is of Tarski-type if the following conditions are satisfied.

- (i) For all $\Gamma \subseteq \mathcal{L}$, $\Gamma \subseteq W^\vdash(\Gamma)$. (Reflexivity)
- (ii) For all $\Gamma, \Sigma \subseteq \mathcal{L}$, if $\Gamma \subseteq \Sigma$ then $W^\vdash(\Gamma) \subseteq W^\vdash(\Sigma)$. (Monotonicity)
- (iii) For all $\Gamma, \Sigma \subseteq \mathcal{L}$, if $\Sigma \subseteq W^\vdash(\Gamma)$ then $W^\vdash(\Sigma) \subseteq W^\vdash(\Gamma)$. (Transitivity)

It may be noted here that a similar definition of a Tarskian consequence operator has been given in [104]. We now adapt a couple of results connecting the Tarskian consequence operators with the concept of a *Suszko set* of bivaluations from [104].

Definition 2.3.3 (SUSZKO SET). [104, Subsection 3.1] Let $(\mathcal{L}, \vdash)$ be a logical structure. A nonempty set of bivaluations $\mathcal{V}$ is called a *Suszko set for* $(\mathcal{L}, \vdash)$, or simply a *Suszko set* (when the logical structure is clear from the context), if $(\mathcal{L}, \vdash)$ is adequate with respect to $(\mathcal{L}, \vdash_{\mathcal{V}})$.

Now, by [104, Corollary 1], if $(\mathcal{L}, \vdash)$ is a Tarski-type logical structure, then there exists a Suszko set for $(\mathcal{L}, \vdash)$.

Conversely, by [104, Corollary 2], if there exists a Suszko set for a logical structure $(\mathcal{L}, \vdash)$, then it is reflexive. Moreover, it is, in fact, straightforward to verify that, given any set $\mathcal{L}$ and a nonempty $\mathcal{V} \subseteq \{0, 1\}^{\mathcal{L}}$, $(\mathcal{L}, \vdash_{\mathcal{V}})$ is of Tarski-type. So in particular, if there exists a Suszko set $\mathcal{V}$ for a logical structure $(\mathcal{L}, \vdash)$, then $\vdash = \vdash_{\mathcal{V}}$, and so $(\mathcal{L}, \vdash)$ must be of Tarski-type. Thus we get the following representation theorem for Tarski-type logical structures.

Theorem 2.3.4 (REPRESENTATION THEOREM FOR TARSKI-TYPE). *Suppose $(\mathcal{L}, \vdash)$ is a logical structure. Then $(\mathcal{L}, \vdash)$ is of Tarski-type iff there exists a Suszko set for $(\mathcal{L}, \vdash)$.*

Remark 2.3.5. A proof of the same result, using an abstract notion of ‘semantics’, is given in [15]. A more general version of the above theorem, although in a many-valued context, is available in [38].

We now come to our main result in this section, the characterisation of Tarski-type logical structures. We begin with the definitions of closed sets and strongly closed sets in a given logical structure.

Definition 2.3.6 (CLOSED SETS). Let $(\mathcal{L}, \vdash)$ be a logical structure and $\Gamma \subseteq \mathcal{L}$. Γ is said to be *closed in* $\mathcal{L}$ if $W^{\vdash}(\Gamma) = \Gamma$, i.e. if for all $\alpha \in \mathcal{L}$, $\Gamma \vdash \alpha$ iff $\alpha \in \Gamma$.

Definition 2.3.7 (STRONGLY CLOSED SETS). Let $(\mathcal{L}, \vdash)$ be a logical structure and $\Gamma \subseteq \mathcal{L}$. Then Γ is called *strongly closed in* $\mathcal{L}$ if it satisfies the following conditions.

- (i) For all $\alpha \in \Gamma$, $\Gamma \vdash \alpha$, i.e. $\Gamma \subseteq W^{\vdash}(\Gamma)$.
- (ii) For all $\alpha \in \mathcal{L}$, if there exists a $\Gamma' \subseteq \Gamma$ with $\Gamma' \vdash \alpha$ then $\alpha \in \Gamma$.

Remark 2.3.8. It is easy to see that every strongly closed set is closed, but the converse is not true for *arbitrary* logical structures. However, if the logical structure under consideration is monotonic, then the two concepts are equivalent.

Theorem 2.3.9 (CHARACTERISATION OF TARSKI-TYPE). *Let $(\mathcal{L}, \vdash)$ be a logical structure. Then the following statements are equivalent.*

- (1) $(\mathcal{L}, \vdash)$ is of Tarski-type.
- (2) There exists a Suszko set for $(\mathcal{L}, \vdash)$.
- (3) For all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\Gamma \not\vdash \alpha$, there exists a strongly closed extension $\Sigma \supseteq \Gamma$ such that $\Sigma \not\vdash \alpha$.

(4) For all $\Gamma, \Sigma \subseteq \mathcal{L}$, $\Sigma \subseteq W^\perp(\Gamma)$ iff $W^\perp(\Sigma) \subseteq W^\perp(\Gamma)$.

Proof. The equivalence of (1) and (2) constituted [Theorem 2.3.4](#). Thus, to complete the proof, it is enough to show that statements (3) and (4) are equivalent to (1).

(1) $\implies$ (3): Suppose $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \not\vdash \alpha$. We will show that $W^\perp(\Gamma)$ is a strongly closed extension of Γ such that $W^\perp(\Gamma) \not\vdash \alpha$.

Since $\vdash$ is reflexive, $\Gamma \subseteq W^\perp(\Gamma)$. Then by monotonicity of $\vdash$, we have $W^\perp(\Gamma) \subseteq W^\perp(W^\perp(\Gamma))$. Moreover, by transitivity of $\vdash$, it follows that, $W^\perp(W^\perp(\Gamma)) \subseteq W^\perp(\Gamma)$. Thus $W^\perp(W^\perp(\Gamma)) = W^\perp(\Gamma)$, and hence $W^\perp(\Gamma)$ is closed. Since $\vdash$ satisfies monotonicity, by [Remark 2.3.8](#), we can conclude that $W^\perp(\Gamma)$ is also strongly closed.

Finally, to prove that $W^\perp(\Gamma) \not\vdash \alpha$, we note that since $\Gamma \not\vdash \alpha$, $\alpha \notin W^\perp(\Gamma)$. This implies that $W^\perp(\Gamma) \not\vdash \alpha$ as $W^\perp(\Gamma)$ is closed. Thus statement (3) holds.

(3) $\implies$ (1): Let $\Gamma \subseteq \mathcal{L}$ and $\beta \in \Gamma$. We claim that $\Gamma \vdash \beta$. Suppose the contrary, i.e. $\Gamma \not\vdash \beta$. Then there exists a strongly closed $\Sigma \supseteq \Gamma$ with $\Sigma \not\vdash \beta$. Now, since Σ is strongly closed, and hence closed, $\Sigma \not\vdash \beta$ implies that $\beta \notin \Sigma$. Thus $\beta \notin \Gamma$ as well, contrary to our hypothesis. Hence $\Gamma \vdash \beta$. Thus $\vdash$ is reflexive.

Next, let $\Gamma \subseteq \Sigma \subseteq \mathcal{L}$ and $\beta \in \mathcal{L}$ such that $\Gamma \vdash \beta$. We claim that $\Sigma \vdash \beta$. Suppose the contrary, i.e. $\Sigma \not\vdash \beta$. Then there exists a strongly closed $\Delta \supseteq \Sigma$ with $\Delta \not\vdash \beta$. Now, $\Gamma \subseteq \Delta$ with $\Gamma \vdash \beta$. So, $\Delta \vdash \beta$ since Δ is strongly closed. This contradiction proves that $\Sigma \vdash \beta$. Thus $\vdash$ satisfies monotonicity.

Finally, let $\Gamma, \Sigma \subseteq \mathcal{L}$ and $\beta \in \mathcal{L}$ such that $\Gamma \vdash \gamma$ for all $\gamma \in \Sigma$, and $\Sigma \vdash \beta$. We claim that $\Gamma \vdash \beta$. Suppose the contrary, i.e. $\Gamma \not\vdash \beta$. Then there exists a strongly closed set $\Delta \supseteq \Gamma$ with $\Delta \not\vdash \beta$. Now, since $\Gamma \vdash \gamma$ for all $\gamma \in \Sigma$, $\Gamma \subseteq \Delta$ and Δ is strongly closed, it follows that $\gamma \in \Delta$ for all $\gamma \in \Sigma$. Thus $\Sigma \subseteq \Delta$. Now, since $\Sigma \vdash \beta$, we must have $\Delta \vdash \beta$, since Δ is strongly closed. This is a contradiction. Hence $\Gamma \vdash \beta$. So $\vdash$ is transitive.

Thus $(\mathcal{L}, \vdash)$ is a Tarski-type logical structure.

We now show that statements (1) and (4) are equivalent using the definition of a Tarski-type logical structure in terms of $W^\perp$.

(1) $\implies$ (4): Let $\Gamma, \Sigma \subseteq \mathcal{L}$. Now, if $\Sigma \subseteq W^\perp(\Gamma)$, then $W^\perp(\Sigma) \subseteq W^\perp(\Gamma)$ by transitivity. Conversely, suppose $W^\perp(\Sigma) \subseteq W^\perp(\Gamma)$. Then by reflexivity, we get $\Sigma \subseteq W^\perp(\Sigma) \subseteq W^\perp(\Gamma)$. Thus $\Sigma \subseteq W^\perp(\Gamma)$ iff $W^\perp(\Sigma) \subseteq W^\perp(\Gamma)$. Hence statement (4) holds.

(4) $\implies$ (1): We first note that reflexivity and transitivity follow immediately from the assumed condition. To prove monotonicity, let $\Sigma \subseteq \Gamma \subseteq \mathcal{L}$. By reflexivity, we have $\Gamma \subseteq W^\perp(\Gamma)$. Thus $\Sigma \subseteq W^\perp(\Gamma)$. Then by the assumed condition, it follows that $W^\perp(\Sigma) \subseteq W^\perp(\Gamma)$. Thus $(\mathcal{L}, \vdash)$ is a Tarski-type logical structure. $\square$

Remark 2.3.10. The statement (3) in the above theorem can be tightened to show that $(\mathcal{L}, \vdash)$ is of Tarski-type iff for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\Gamma \not\vdash \alpha$, there exists a $\subseteq$ -minimum strongly closed extension $\Delta \supseteq \Gamma$ such that $\Delta \not\vdash \alpha$. The forward direction follows by choosing $\Delta = W^\perp(\Gamma)$ as in the proof of (1) $\implies$ (3) above. The converse is immediate.

2.4 Lindenbaum-type Logical Structures

2.4.1 Excessive and Maximal Nontrivial Sets

We devote this section to the *excessive* and *maximal nontrivial* sets which play crucial roles in the characterisations of Lindenbaum-type logical structures introduced in [Subsection 2.4.2](#). The very notion of excessive sets already occurred in [\[72\]](#) as *α -saturated sets*, but they were working in the context of Tarski-type logical structures. The notion of excessive sets for arbitrary logical structures was considered for the first time in [\[14\]](#) as *a -excessive theories*, from which we have taken the terminology, albeit with slight modification.

2.4.1.1 Excessive Sets

Definition 2.4.1 (EXCESSIVE SET). Let $(\mathcal{L}, \vdash)$ be a logical structure and $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$. Γ is called *α -excessive* in $(\mathcal{L}, \vdash)$ if $\Gamma \not\vdash \alpha$ but for any $\beta \in \mathcal{L} \setminus \Gamma$, $\Gamma \cup \{\beta\} \vdash \alpha$. Γ is called *excessive* in $(\mathcal{L}, \vdash)$ if it is α -excessive for some $\alpha \in \mathcal{L}$.

Definition 2.4.2 (RELATIVELY MAXIMAL SET). [\[17, Definition 7\]](#) Let $(\mathcal{L}, \vdash)$ be a logical structure and $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$. Γ is said to be *relatively maximal in α* in $(\mathcal{L}, \vdash)$ if $\Gamma \not\vdash \alpha$ but for any $\Sigma \supsetneq \Gamma$, $\Sigma \vdash \alpha$. Γ is called *relatively maximal* in $(\mathcal{L}, \vdash)$ if it is relatively maximal in α for some $\alpha \in \mathcal{L}$.

Theorem 2.4.3. *Suppose $(\mathcal{L}, \vdash)$ is a logical structure and $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$. Then the following statements hold.*

- (1) *If Γ is relatively maximal, then it is excessive.*
- (2) *If Γ is relatively maximal in α , then it is α -excessive.*
- (3) *If $(\mathcal{L}, \vdash)$ is monotonic, then every α -excessive set is relatively maximal in α .*

Proof. (1) Suppose that Γ is relatively maximal. Then there exists some $\beta \in \mathcal{L}$ such that it is relatively maximal in β . Since it is relatively maximal in β , it follows $\Gamma \not\vdash \beta$ and for every $\Sigma \supsetneq \Gamma$, $\Sigma \vdash \beta$. Now, for any $\gamma \notin \Gamma$ since $\Gamma \cup \{\gamma\} \supsetneq \Gamma$, it follows that for any $\gamma \notin \Gamma$, $\Gamma \cup \{\gamma\} \vdash \beta$. Since γ was chosen arbitrarily, we have shown that Γ is β -excessive, and hence excessive.

(2) Similar argument.

(3) Let Γ be an α -excessive set, i.e. $\Gamma \not\vdash \alpha$ and for all $\gamma \notin \Gamma$, $\Gamma \cup \{\gamma\} \vdash \alpha$. Choose any $\Sigma \supsetneq \Gamma$. Since $\Sigma \supsetneq \Gamma$, it follows that there exists $\delta \in \Sigma \setminus \Gamma$. Choose any such δ and consider $\Gamma \cup \{\delta\}$. Note that, since $\delta \notin \Gamma$, it follows that $\Gamma \cup \{\delta\} \vdash \alpha$. Since $\Gamma \cup \{\delta\} \subseteq \Sigma$, by monotonicity it follows that $\Sigma \vdash \alpha$. Since Σ was chosen arbitrarily, we have thus shown that Γ is relatively maximal in α . Furthermore, since Γ was chosen arbitrarily, we have thus shown (3). □

Example 2.4.4. Let $(\mathcal{L}_{\mathbf{CL}}, \vdash_{\mathbf{CL}})$ denote the classical propositional logic. Then it is a straightforward observation that every maximal consistent set Σ of $(\mathcal{L}_{\mathbf{CL}}, \vdash_{\mathbf{CL}})$ is α -excessive, and also relatively maximal in α , in the underlying logical structure $(\|\mathcal{L}_{\mathbf{CL}}\|, \vdash_{\mathbf{CL}})$ for each $\alpha \notin \Sigma$.

Definition 2.4.5 (CUT/MIXED-CUT SATISFACTION). Let $(\mathcal{L}, \vdash)$ be a logical structure.

- (i) It is said to satisfy *cut* if for all $\Gamma \cup \{\alpha, \beta\} \subseteq \mathcal{L}$, $\Gamma \vdash \alpha$ and $\Gamma \cup \{\alpha\} \vdash \beta$ implies that $\Gamma \vdash \beta$.
- (ii) It is said to satisfy *mixed-cut* if for all $\Gamma \cup \Sigma \cup \{\alpha, \beta\} \subseteq \mathcal{L}$, $\Gamma \vdash \alpha$ and $\Sigma \cup \{\alpha\} \vdash \beta$ implies that $\Gamma \cup \Sigma \vdash \beta$.

Remark 2.4.6. It is straightforward to see that satisfying cut is a special case of satisfying mixed-cut, i.e. if a logical structure satisfies mixed-cut then it also satisfies cut. The converse holds if the logical structure under consideration is monotonic, as shown in the next theorem.

The above notions of cut and mixed-cut have been adopted from those in [65], but we have not assumed any underlying proof system.

Theorem 2.4.7. *Let $(\mathcal{L}, \vdash)$ be a monotonic logical structure satisfying cut. Then it satisfies mixed-cut as well.*

Proof. Let $\Gamma \cup \Sigma \cup \{\alpha, \beta\} \subseteq \mathcal{L}$ be such that $\Gamma \vdash \alpha$ and $\Sigma \cup \{\alpha\} \vdash \beta$. Now, as $(\mathcal{L}, \vdash)$ is monotonic, $\Gamma \vdash \alpha$ implies that $\Gamma \cup \Sigma \vdash \alpha$ and $\Sigma \cup \{\alpha\} \vdash \beta$ implies that $\Gamma \cup \Sigma \cup \{\alpha\} \vdash \beta$. Since $(\mathcal{L}, \vdash)$ satisfies cut, we have $\Gamma \cup \Sigma \vdash \beta$. This completes the proof. $\square$

Theorem 2.4.8. *Let $(\mathcal{L}, \vdash)$ be a logical structure satisfying cut. Then every excessive set is closed.*

Proof. Let $\Sigma \subseteq \mathcal{L}$ be an excessive set. So, Σ is β -excessive for some $\beta \in \mathcal{L}$. If possible, suppose Σ is not closed. Then there exists $\alpha \in \mathcal{L}$ such that $\Sigma \vdash \alpha$ but $\alpha \notin \Sigma$. Now, since Σ is β -excessive and $\alpha \notin \Sigma$, we have $\Sigma \cup \{\alpha\} \vdash \beta$. Thus $\Sigma \vdash \alpha$ and $\Sigma \cup \{\alpha\} \vdash \beta$. Then by cut, it follows that $\Sigma \vdash \beta$. This is a contradiction as Σ is β -excessive. Hence Σ must be closed. $\square$

Corollary 2.4.9. *Let $(\mathcal{L}, \vdash)$ be a logical structure satisfying cut. Then $W^\vdash(\mathcal{L}) = \mathcal{L}$.*

Proof. Suppose $W^\vdash(\mathcal{L}) \neq \mathcal{L}$. Then there exists an $\alpha \in \mathcal{L}$ such that $\mathcal{L} \not\vdash \alpha$. This implies that $\mathcal{L}$ is excessive. Then as $(\mathcal{L}, \vdash)$ satisfies cut, it follows by the above theorem that $\mathcal{L}$ is closed. However, this implies that $\mathcal{L} \vdash \alpha$, since $\alpha \in \mathcal{L}$. This is a contradiction. Hence $W^\vdash(\mathcal{L}) = \mathcal{L}$. $\square$

Theorem 2.4.10. *Let $(\mathcal{L}, \vdash)$ be a logical structure satisfying mixed-cut. Then every excessive set is strongly closed.*

Proof. Let $\Sigma \subseteq \mathcal{L}$ be an excessive set. Now, since $(\mathcal{L}, \vdash)$ satisfies mixed-cut, it also satisfies cut. Then by [Theorem 2.4.8](#), Σ is closed. Thus, for any $\alpha \in \mathcal{L}$, $\Sigma \vdash \alpha$ iff $\alpha \in \Sigma$. Hence to show that Σ is strongly closed, it suffices to show that, for any $\alpha \in \mathcal{L}$, if $\Gamma \subsetneq \Sigma$ with $\Gamma \vdash \alpha$, then $\alpha \in \Sigma$.

Suppose the contrary, i.e. there exists $\alpha \in \mathcal{L}$ such that $\Gamma \subsetneq \Sigma$ and $\Gamma \vdash \alpha$ but $\alpha \notin \Sigma$. Now, Σ is β -excessive for some $\beta \in \mathcal{L}$. Thus $\Sigma \not\vdash \beta$ but since $\alpha \notin \Sigma$, $\Sigma \cup \{\alpha\} \vdash \beta$. So, we have $\Gamma \vdash \alpha$ and $\Sigma \cup \{\alpha\} \vdash \beta$, and hence by mixed-cut it follows that $\Gamma \cup \Sigma \vdash \beta$. This implies that $\Sigma \vdash \beta$ since $\Gamma \subsetneq \Sigma$. This is a contradiction. Hence Σ must be strongly closed. $\square$

2.4.1.2 $\rightarrow$ -saturated Sets

We now digress a bit and show that a certain generalisation of *implication-saturated* sets, introduced by Batens in [\[12\]](#), constitute an example of maximal nontrivial sets, which in turn are excessive sets as well, by [Corollary 2.4.21](#). In [\[12\]](#), the implication-saturated sets have been used to develop an alternative method for proving the completeness results for a certain class of logics. It has been argued there that since this method avoids the use of traditional maximal consistent sets, it can be applied to a wide class of non-classical logics, especially the paraconsistent logics described in [\[4, 46, 47\]](#).

Definition 2.4.11 ($\rightarrow$ -SATURATED Set). Let $(\mathcal{L}, \vdash)$ be a logic with a binary connective, $\rightarrow$, and $\Gamma \subseteq \|\mathcal{L}\|$. Γ is called $\rightarrow$ -saturated if $\xi_\alpha = \{\alpha \rightarrow \beta : \beta \in \mathcal{L}\} \subseteq \Gamma$ whenever $\alpha \notin \Gamma$.

Remark 2.4.12. The above definition, although motivated by the notion of implication-saturated sets, is a slight generalisation of the latter since $\rightarrow$ here could be any binary connective.

It may also be noted here that the discussion in [\[12\]](#) is based on logics and not logical structures. However, as mentioned in [Remark 2.2.2](#), every logic has an underlying logical structure. Hence a connection can be drawn regardless of this difference.

The concept of *deductively closed* sets used in [\[12\]](#) is the same as that of our closed sets described in [Definition 2.3.6](#).

We differ in the definition of trivial sets as well. While in [\[12\]](#), a set Γ in a logic $(\mathcal{L}, \vdash)$ is called trivial if $\Gamma = \mathcal{L}$ ($\Gamma = \|\mathcal{L}\|$ in our notation), we call a set Γ in a logical structure $(\mathcal{L}, \vdash)$ trivial if $W^\vdash(\Gamma) = \mathcal{L}$. However, this difference is inconsequential if we are dealing with closed sets.

Suppose $(\mathcal{L}, \vdash)$ is a logic with $\rightarrow$ such that, for every $\Gamma \cup \{\alpha, \beta\} \subseteq \|\mathcal{L}\|$, $\Gamma \vdash \alpha$ and $\Gamma \vdash \alpha \rightarrow \beta$ implies that $\Gamma \vdash \beta$. Then we say that $\rightarrow$ -elimination holds in $(\mathcal{L}, \vdash)$.

Theorem 2.4.13. Suppose $(\mathcal{L}, \vdash)$ is a reflexive logic with $\rightarrow$ such that $\rightarrow$ -elimination holds in $(\mathcal{L}, \vdash)$. Then every nontrivial $\rightarrow$ -saturated $\Gamma \subseteq \|\mathcal{L}\|$ is maximal nontrivial² and closed.

²A set is said to be *maximal nontrivial* if it is nontrivial, and every proper subset of it is trivial. The formal definition is given in [Definition 2.4.19](#).

Proof. Let $\Gamma \subseteq \|\mathcal{L}\|$ be nontrivial and $\rightarrow$ -saturated. Suppose Γ is not maximal nontrivial. Then there exists $\Sigma \supsetneq \Gamma$ such that Σ is nontrivial. Since $\Gamma \subsetneq \Sigma$, there exists $\gamma \in \Sigma \setminus \Gamma$. Then $\xi_\gamma \subseteq \Gamma$, as Γ is $\rightarrow$ -saturated, and hence $\xi_\gamma \subseteq \Sigma$. Thus $\gamma \in \Sigma$ and $\gamma \rightarrow \alpha \in \Sigma$ for every $\alpha \in \|\mathcal{L}\|$. Therefore by reflexivity of $(\mathcal{L}, \vdash)$, we have $\Sigma \vdash \gamma$ and $\Sigma \vdash \gamma \rightarrow \alpha$ for every $\alpha \in \|\mathcal{L}\|$. Then by $\rightarrow$ -elimination, we get $\Sigma \vdash \alpha$ for every $\alpha \in \|\mathcal{L}\|$, i.e. Σ is trivial, a contradiction. Since Γ is nontrivial, this implies that Γ is maximal nontrivial.

Next, to show that Γ is closed, suppose the contrary. Then there exists a $\beta \in \|\mathcal{L}\|$ such that $\Gamma \vdash \beta$ but $\beta \notin \Gamma$. Since Γ is implication-saturated, this implies that $\xi_\beta \subseteq \Gamma$, i.e. $\beta \rightarrow \delta \in \Gamma$ for all $\delta \in \|\mathcal{L}\|$. So, by reflexivity, $\Gamma \vdash \beta \rightarrow \delta$ for all $\delta \in \|\mathcal{L}\|$. Hence by $\rightarrow$ -elimination, $\Gamma \vdash \delta$ for all $\delta \in \|\mathcal{L}\|$. In other words, Γ is trivial, a contradiction. Thus Γ must be closed. $\square$

Theorem 2.4.14. *Let $(\mathcal{L}, \vdash)$ be a reflexive logic with $\rightarrow$. Suppose that for all $\Sigma \cup \{\beta\} \subseteq \|\mathcal{L}\|$ with $\beta \notin \Sigma$, $W^\vdash(\Sigma \cup \xi_\beta) \subseteq W^\vdash(\Sigma)$. Then any closed $\Gamma \subseteq \|\mathcal{L}\|$ is $\rightarrow$ -saturated.*

Proof. Suppose $\Gamma \cup \{\alpha\} \subseteq \|\mathcal{L}\|$ such that Γ is closed and $\alpha \notin \Gamma$. Now,

$$\begin{aligned} \xi_\alpha &\subseteq \Gamma \cup \xi_\alpha \\ &\subseteq W^\vdash(\Gamma \cup \xi_\alpha) \quad (\text{by reflexivity}) \\ &\subseteq W^\vdash(\Gamma) \quad (\text{by the given condition}) \\ &= \Gamma \quad (\text{since } \Gamma \text{ is closed}) \end{aligned}$$

Thus Γ is $\rightarrow$ -saturated. $\square$

Remark 2.4.15. The proviso that for all $\Sigma \cup \{\beta\} \subseteq \|\mathcal{L}\|$ with $\beta \notin \Sigma$, $W^\vdash(\Sigma \cup \xi_\beta) \subseteq W^\vdash(\Sigma)$, in [Theorem 2.4.14](#) is what [\[12, Lemma 4\]](#) states.

We can now conclude that in a logic that satisfies the conditions of [Theorem 2.4.13](#) and [Theorem 2.4.14](#), a nontrivial set is $\rightarrow$ -saturated iff it is closed and maximal nontrivial in the corresponding underlying logical structure.

Remark 2.4.16. It is noteworthy that while the notions of excessive and maximal nontrivial sets do not depend on the existence of any connective like $\rightarrow$, and hence make sense for any logical structure, implication-saturated sets can only exist in logics with such a logical operator. The above results thus provide a more general alternative understanding of the discussions in [\[12\]](#).

2.4.1.3 Maximal Nontrivial Sets

Definition 2.4.17. Let $(\mathcal{L}, \vdash)$ be a logical structure and $\Gamma \subseteq \mathcal{L}$. Γ is called *trivial* if $W^\vdash(\Gamma) = \mathcal{L}$, and *nontrivial* otherwise. Thus $\Gamma \subseteq \mathcal{L}$ is nontrivial if there exists an $\alpha \in \mathcal{L}$ such that $\Gamma \not\vdash \alpha$.

Remark 2.4.18. A trivial set is also called an *absolutely inconsistent* set (see, e.g. [\[83\]](#)).

Definition 2.4.19. Let $(\mathcal{L}, \vdash)$ be a logical structure and $\Gamma \subseteq \mathcal{L}$. Γ is said to be *maximal nontrivial* in $(\mathcal{L}, \vdash)$ if Γ is nontrivial but every $\Sigma \subseteq \mathcal{L}$ with $\Gamma \subsetneq \Sigma$ is trivial.

Theorem 2.4.20. *Let $(\mathcal{L}, \vdash)$ be a logical structure and $\Gamma \subseteq \mathcal{L}$. If Γ is maximal nontrivial then it is relatively maximal in α for every $\alpha \notin W^\vdash(\Gamma)$.*

Proof. Suppose $\Gamma \subseteq \mathcal{L}$ is maximal nontrivial. Let $\alpha \notin W^\vdash(\Gamma)$ (such an α exists since Γ is nontrivial, and hence $W^\vdash(\Gamma) \neq \mathcal{L}$). Then $\Gamma \not\vdash \alpha$. Let $\Sigma \subseteq \mathcal{L}$ such that $\Gamma \subsetneq \Sigma$. Then as Γ is maximal nontrivial, Σ must be trivial. So, in particular, $\Sigma \vdash \alpha$. Hence Γ is relatively maximal in α for every $\alpha \notin W^\vdash(\Gamma)$. $\square$

Corollary 2.4.21. *Let $(\mathcal{L}, \vdash)$ be a logical structure and $\Gamma \subseteq \mathcal{L}$. If Γ is maximal nontrivial then it is α -excessive for every $\alpha \notin W^\vdash(\Gamma)$, and hence excessive.*

Proof. Straightforward from the above theorem and [Theorem 2.4.3](#). $\square$

Remark 2.4.22. Suppose $(\mathcal{L}, \vdash)$ is a logical structure and $\Gamma \subseteq \mathcal{L}$ is excessive. Then there exists an $\alpha \in \mathcal{L}$ such that Γ is α -excessive. This implies that $\Gamma \not\vdash \alpha$. Hence Γ is nontrivial. Thus every excessive (α -excessive) set is nontrivial.

The converse of [Theorem 2.4.20](#) also holds for monotonic logical structures as shown below.

Theorem 2.4.23. *Let $(\mathcal{L}, \vdash)$ be a monotonic logical structure and $\Gamma \subseteq \mathcal{L}$. If Γ relatively maximal in α for every $\alpha \notin W^\vdash(\Gamma)$ then Γ is maximal nontrivial.*

Proof. Suppose $\Gamma \subseteq \mathcal{L}$ is relatively maximal in α for every $\alpha \notin W^\vdash(\Gamma)$ and $\Sigma \supsetneq \Gamma$. If possible, let Σ be nontrivial, i.e. there exists $\beta \in \mathcal{L}$ such that $\Sigma \not\vdash \beta$. This implies that $\Gamma \not\vdash \beta$ as well, since $(\mathcal{L}, \vdash)$ is monotonic and $\Gamma \subsetneq \Sigma$. So, $\beta \notin W^\vdash(\Gamma)$, and hence by our assumption, Γ is relatively maximal in β . Then it must be that $\Sigma \vdash \beta$. This is a contradiction. So, Σ cannot be trivial and thus Γ is maximal nontrivial. $\square$

2.4.2 Lindenbaum-type Logical Structures

In [\[16, Section 4\]](#), four versions of Lindenbaum's law were listed. We now list the definitions of the logical structures induced by them, albeit with some terminological modifications.

Definition 2.4.24. A logical structure $(\mathcal{L}, \vdash)$ is said to be of:

- (i) *Lindenbaum-I-type* if for all nontrivial $\Gamma \subseteq \mathcal{L}$, there exists an excessive $\Sigma \supseteq \Gamma$.
- (ii) *Lindenbaum-II-type* if for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, with $\Gamma \not\vdash \alpha$ there exists an α -excessive $\Sigma \supseteq \Gamma$.
- (iii) *Lindenbaum-III-type* if for all nontrivial $\Gamma \subseteq \mathcal{L}$, there exists a maximal nontrivial $\Sigma \supseteq \Gamma$.
- (iv) *Lindenbaum-IV-type* if for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, with $\Gamma \not\vdash \alpha$ there exists a $\Sigma \supseteq \Gamma$ which is relatively maximal in α .

Remark 2.4.25. There are important terminological differences between the above definitions and those of [16]. These are summarised below.

In [16]	In this chapter
$(\mathcal{L}, \vdash)$ is a <i>logic</i>	$(\mathcal{L}, \vdash)$ is a <i>logical structure</i>
$\Gamma \subseteq \mathcal{L}$ is a <i>limited theory</i>	$\Gamma \subseteq \mathcal{L}$ is <i>nontrivial</i>
Γ is α - <i>limited</i> , where $\alpha \in \mathcal{L}$	$\Gamma \not\vdash \alpha$
Γ is <i>maximal</i>	Γ is <i>maximal nontrivial</i>
Γ is α - <i>maximal</i>	Γ is <i>relatively maximal in α</i>
$(\mathcal{L}, \vdash)$ is a <i>weak Lindenbaum-Asser logic</i>	$(\mathcal{L}, \vdash)$ is a <i>Lindenbaum-I-type logical structure</i>
$(\mathcal{L}, \vdash)$ is a <i>Lindenbaum-Asser logic</i>	$(\mathcal{L}, \vdash)$ is a <i>Lindenbaum-II-type logical structure</i>
$(\mathcal{L}, \vdash)$ is a <i>Lindenbaum logic</i>	$(\mathcal{L}, \vdash)$ is a <i>Lindenbaum-III-type logical structure</i>
$(\mathcal{L}, \vdash)$ is a <i>strong Lindenbaum logic</i>	$(\mathcal{L}, \vdash)$ is a <i>Lindenbaum-IV-type logical structure</i>

We have chosen to go with these new names since the previous terminology might suggest that every ‘strong Lindenbaum logic’ is a ‘Lindenbaum logic’, but that is not the case, as we have shown in [Example 2.4.50](#).

2.4.2.1 Lindenbaum-II-type Logical Structures

Theorem 2.4.26. *Let $(\mathcal{L}, \vdash)$ be a Lindenbaum-II-type logical structure and $\alpha \in \mathcal{L}$. Then every maximal α -excessive set is relatively maximal in α .*

Proof. Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that Γ is maximal α -excessive. If possible, suppose Γ is not relatively maximal in α . Then there exists $\Sigma \supsetneq \Gamma$ such that $\Sigma \not\vdash \alpha$. Since $(\mathcal{L}, \vdash)$ is of Lindenbaum-II-type, this implies that there must exist an α -excessive $\Delta \supseteq \Sigma$ by [Definition 2.4.24\(ii\)](#). Thus Δ is an α -excessive set containing Γ . This contradicts the maximality of Γ . Hence Γ must be relatively maximal in α . $\square$

Remark 2.4.27. The following result (rephrased using our terminology) is quoted from [16, Section 5].

“In a Lindenbaum-II-type logical structure every set coincides with the intersection of its excessive extensions.”

We point out here that the above result is not true in general. This is witnessed in the following example. Let $(\mathcal{L}_{\mathbf{CL}}, \vdash_{\mathbf{CL}})$ be as in [Example 2.4.4](#). We define a new logical structure $(\mathcal{L}, \vdash)$, where $\mathcal{L} = \|\mathcal{L}_{\mathbf{CL}}\|$, and for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, $\Gamma \vdash \alpha$ iff $\Gamma \not\vdash_{\mathbf{CL}} \alpha$. It then immediately follows that $\mathcal{L} \not\vdash \alpha$ for all $\alpha \in \mathcal{L}$, and hence $\mathcal{L}$ is trivially α -excessive in $(\mathcal{L}, \vdash)$ for every $\alpha \in \mathcal{L}$. So, for every $\Gamma \cup \{\alpha\}$ with $\Gamma \not\vdash \alpha$, there exists an α -excessive $\Sigma \supseteq \Gamma$. Thus $(\mathcal{L}, \vdash)$ is a Lindenbaum-II-type logical structure.

Now, suppose $\Gamma \subseteq \mathcal{L}$ is α -excessive for some $\alpha \in \mathcal{L}$. Then $\Gamma \not\vdash \alpha$ which implies that $\Gamma \vdash_{\mathbf{CL}} \alpha$. This in turn implies that $\Gamma \cup \{\delta\} \vdash_{\mathbf{CL}} \alpha$, and hence $\Gamma \cup \{\delta\} \not\vdash \alpha$, for any $\delta \in \mathcal{L} \setminus \Gamma$.

This is impossible unless $\Gamma = \mathcal{L}$. Thus $\mathcal{L}$ is the only excessive set in $(\mathcal{L}, \vdash)$. Hence in this case, for any $\Gamma \subseteq \mathcal{L}$, the intersection of its excessive extensions is $\mathcal{L}$, and so if $\Gamma \subsetneq \mathcal{L}$, it cannot be equal to the intersection of its excessive extensions.

However, the following weaker versions of the above result hold.

Theorem 2.4.28. *Let $(\mathcal{L}, \vdash)$ be a Lindenbaum-II-type logical structure satisfying cut. If $\Gamma \subsetneq \mathcal{L}$ is closed, then it coincides with the intersection of its excessive extensions.*

Proof. Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \subsetneq \mathcal{L}$ is closed and $\alpha \notin \Gamma$. Since Γ is closed, we have $\Gamma \not\vdash \alpha$. Then, as $(\mathcal{L}, \vdash)$ is of Lindenbaum-II-type, by Definition 2.4.24(ii), there exists an α -excessive $\Delta \supseteq \Gamma$. This implies that $\Delta \not\vdash \alpha$, i.e. $\alpha \notin W^\vdash(\Delta)$. Now, as Δ is excessive and $(\mathcal{L}, \vdash)$ satisfies cut, by Theorem 2.4.8, Δ is closed. Hence $W^\vdash(\Delta) = \Delta$ and so $\alpha \notin \Delta$. Let $\mathbb{E}\mathbb{X}\mathbb{C}$ denote the collection of all excessive sets in $(\mathcal{L}, \vdash)$. Then this implies that $\alpha \notin \bigcap_{\substack{\Gamma \subseteq \Sigma \\ \Sigma \in \mathbb{E}\mathbb{X}\mathbb{C}}} \Sigma$.

Since α was chosen arbitrarily, we have $\Gamma \supseteq \bigcap_{\substack{\Gamma \subseteq \Sigma \\ \Sigma \in \mathbb{E}\mathbb{X}\mathbb{C}}} \Sigma$. The reverse inclusion is immediate.

Hence $\Gamma = \bigcap_{\substack{\Gamma \subseteq \Sigma \\ \Sigma \in \mathbb{E}\mathbb{X}\mathbb{C}}} \Sigma$. □

Theorem 2.4.29. *Let $(\mathcal{L}, \vdash)$ be a reflexive Lindenbaum-II-type logical structure. If $\Gamma \subsetneq \mathcal{L}$ is closed, then $\Gamma = \bigcap_{\substack{\Gamma \subseteq \Sigma \\ \Sigma \in \mathbb{E}\mathbb{X}\mathbb{C}}} W^\vdash(\Sigma)$, where $\mathbb{E}\mathbb{X}\mathbb{C}$ denotes, as before, the set of excessive subsets of $\mathcal{L}$.*

Proof. Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \subsetneq \mathcal{L}$ is closed and $\alpha \notin \Gamma$. Since Γ is closed, we have $\Gamma \not\vdash \alpha$. Then, as $(\mathcal{L}, \vdash)$ is of Lindenbaum-II-type, by Definition 2.4.24(ii), there exists an α -excessive $\Delta \supseteq \Gamma$. This implies that $\Delta \not\vdash \alpha$, i.e. $\alpha \notin W^\vdash(\Delta)$. So, $\alpha \notin \bigcap_{\substack{\Gamma \subseteq \Sigma \\ \Sigma \in \mathbb{E}\mathbb{X}\mathbb{C}}} W^\vdash(\Sigma)$.

Since α was chosen arbitrarily, we have $\Gamma \supseteq \bigcap_{\substack{\Gamma \subseteq \Sigma \\ \Sigma \in \mathbb{E}\mathbb{X}\mathbb{C}}} W^\vdash(\Sigma)$. Now, as $(\mathcal{L}, \vdash)$ is reflexive,

$\Sigma \subseteq W^\vdash(\Sigma)$ for all $\Sigma \subseteq \mathcal{L}$. Thus

$$\Gamma \subseteq \bigcap_{\Gamma \subseteq \Sigma} \Sigma \subseteq \bigcap_{\Gamma \subseteq \Sigma} W^\vdash(\Sigma) \subseteq \bigcap_{\substack{\Gamma \subseteq \Sigma \\ \Sigma \in \mathbb{E}\mathbb{X}\mathbb{C}}} W^\vdash(\Sigma).$$

Hence $\Gamma = \bigcap_{\substack{\Gamma \subseteq \Sigma \\ \Sigma \in \mathbb{E}\mathbb{X}\mathbb{C}}} W^\vdash(\Sigma)$. □

Theorem 2.4.30. *Let $(\mathcal{L}, \vdash)$ be a Lindenbaum-II-type logical structure satisfying cut. Then it is reflexive.*

Proof. Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\alpha \in \Gamma$. If possible, suppose $\Gamma \not\vdash \alpha$. Then, as $(\mathcal{L}, \vdash)$ is of Lindenbaum-II-type, by Definition 2.4.24(ii), there exists an α -excessive $\Sigma \supseteq \Gamma$. So,

$\Sigma \not\vdash \alpha$. Now, since Σ is excessive and $(\mathcal{L}, \vdash)$ satisfies cut, by [Theorem 2.4.8](#), Σ is closed. Thus $\alpha \notin \Sigma$. This implies that $\alpha \notin \Gamma$. This is a contradiction. Hence $\Gamma \vdash \alpha$. Thus $(\mathcal{L}, \vdash)$ is reflexive. $\square$

Theorem 2.4.31. *Let $(\mathcal{L}, \vdash)$ be a Lindenbaum-II-type logical structure satisfying mixed-cut. Then it is of Tarski-type.*

Proof. Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \not\vdash \alpha$. Then, since $(\mathcal{L}, \vdash)$ is of Lindenbaum-II-type, by [Definition 2.4.24\(ii\)](#), there exists an α -excessive $\Sigma \supseteq \Gamma$. So, $\Sigma \not\vdash \alpha$. Now, as Σ is excessive and $(\mathcal{L}, \vdash)$ satisfies mixed-cut, by [Theorem 2.4.10](#), Σ is strongly closed. Thus Σ is a strongly closed extension of Γ such that $\Sigma \not\vdash \alpha$. Thus by [Theorem 2.3.9](#), $(\mathcal{L}, \vdash)$ is of Tarski-type. $\square$

2.4.2.2 Lindenbaum-IV-type Logical Structures

We now prove the following characterisation theorem for Lindenbaum-IV-type logical structures.

Theorem 2.4.32 (CHARACTERISATION OF LINDENBAUM-IV). *Let $(\mathcal{L}, \vdash)$ be a logical structure. Then the following statements are equivalent.*

- (1) $(\mathcal{L}, \vdash)$ is of Lindenbaum-IV-type.
- (2) For all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, if $\Gamma \not\vdash \alpha$ then there exists a maximal α -excessive set $\Sigma \supseteq \Gamma$.
- (3) For all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, the set $\text{Lim}_{\mathcal{L}}(\Gamma, \alpha) = \{\Sigma : \Gamma \subseteq \Sigma \text{ and } \Sigma \not\vdash \alpha\}$ (i.e. the set of all extensions $\Sigma \supseteq \Gamma$ such that $\Sigma \not\vdash \alpha$) has a maximal element whenever it is nonempty.

Proof. We will follow the following scheme to show that the above three statements are equivalent: (1) $\implies$ (2); (2) $\implies$ (3); (3) $\implies$ (1).

(1) $\implies$ (2): Suppose $(\mathcal{L}, \vdash)$ is a Lindenbaum-IV-type logical structure and $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \not\vdash \alpha$. Then, by [Definition 2.4.24\(iv\)](#), there exists $\Sigma \supseteq \Gamma$ which is relatively maximal in α . So, Σ is α -excessive by [Theorem 2.4.3](#). Since Σ is relatively maximal in α , $\Delta \vdash \alpha$, and hence not α -excessive, for every $\Delta \supsetneq \Sigma$. Thus Σ is maximal α -excessive.

(2) $\implies$ (3): Suppose $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\text{Lim}_{\mathcal{L}}(\Gamma, \alpha) \neq \emptyset$. Let $\Sigma \in \text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$. Then, $\Sigma \not\vdash \alpha$. So by the assumed condition (2), there exists a maximal α -excessive $\Delta \supseteq \Sigma$. We claim that Δ is a maximal element of $\text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$.

Suppose the contrary. Then there exists $\Delta' \in \text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$ such that $\Delta \subsetneq \Delta'$. Now, since $\Delta' \in \text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$, $\Delta' \not\vdash \alpha$. Again by the assumed condition (2), this implies that there exists a maximal α -excessive $\Delta'' \supseteq \Delta'$. Thus $\Delta \subsetneq \Delta''$ and Δ'' is α -excessive. This contradicts the fact that Δ is maximal α -excessive. Hence Δ must be a maximal element of $\text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$.

(3) $\implies$ (1): Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \not\vdash \alpha$. Then $\Gamma \in \text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$, and hence $\text{Lim}_{\mathcal{L}}(\Gamma, \alpha) \neq \emptyset$. So, by the assumed condition (3), $\text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$ has a maximal element, say Σ . Clearly, $\Gamma \subseteq \Sigma$. We claim that Σ is also relatively maximal in α .

We note that $\Sigma \not\vdash \alpha$ since $\Sigma \in \text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$. Now, suppose there exists $\Delta \supsetneq \Sigma$ such that $\Delta \not\vdash \alpha$. Then, $\Delta \in \text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$. However, this contradicts the maximality of Σ in

$\text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$. Hence $\Delta \vdash \alpha$ for every $\Delta \supseteq \Sigma$. Thus $\Sigma \supseteq \Gamma$ is relatively maximal in α . Therefore, $(\mathcal{L}, \vdash)$ is of Lindenbaum-IV-type. $\square$

Remark 2.4.33. It follows from the equivalence of the statements (1) and (2) of the above theorem that, if a logical structure $(\mathcal{L}, \vdash)$ is of Lindenbaum-IV-type, then for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\Gamma \not\vdash \alpha$, there exists an α -excessive $\Sigma \supseteq \Gamma$. Now, the condition $\Gamma \not\vdash \alpha$ implies that Γ is nontrivial. As pointed out in [Example 2.4.4](#), in case of classical propositional logic, every maximal consistent set Σ is α -excessive for all $\alpha \notin \Sigma$. Moreover, a set of classical propositional formulas is nontrivial iff it is consistent. So, in this special case, the statement (2) assumes the usual form of the Lindenbaum lemma for classical propositional logic, i.e. every consistent set of formulas can be extended to a maximal consistent set.

Theorem 2.4.34. *Let $(\mathcal{L}, \vdash)$ be a reflexive logical structure satisfying cut. Suppose further that for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\alpha \notin \Gamma$, if $W^\vdash(\Gamma \cup \{\alpha\}) \subseteq W^\vdash(\Gamma)$ then Γ is closed. Then $(\mathcal{L}, \vdash)$ is of Lindenbaum-IV-type.*

Proof. Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ be such that $\text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$ is nonempty. Let $\mathcal{C}$ be any chain in $\text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$ and $\Delta = \bigcup_{\Sigma \in \mathcal{C}} \Sigma$. Clearly, $\Gamma \subseteq \Delta$. We claim that $\Delta \not\vdash \alpha$.

Suppose the contrary, i.e. $\Delta \vdash \alpha$. We first note that for each $\Sigma \in \mathcal{C} \subseteq \text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$, $\Sigma \not\vdash \alpha$. Since $(\mathcal{L}, \vdash)$ is reflexive, this implies that $\alpha \notin \Sigma$ for all $\Sigma \in \mathcal{C}$. Thus $\alpha \notin \Delta$. Now, let $\beta \in W^\vdash(\Delta \cup \{\alpha\})$, i.e. $\Delta \cup \{\alpha\} \vdash \beta$. Then by cut, we have $\Delta \vdash \beta$. So, $\beta \in W^\vdash(\Delta)$. Thus $W^\vdash(\Delta \cup \{\alpha\}) \subseteq W^\vdash(\Delta)$. Hence by our hypothesis, Δ must be closed. Now, since $\Delta \vdash \alpha$, this implies that $\alpha \in \Delta$, which contradicts our earlier conclusion that $\alpha \notin \Delta$. So, $\Delta \not\vdash \alpha$.

This implies that $\Delta \in \text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$. Hence Δ is an upper bound for $\mathcal{C}$ in $\text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$. Since $\mathcal{C}$ was an arbitrary chain in $\text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$, we can conclude that every chain in $\text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$ has an upper bound. So, by Zorn's Lemma, it follows that $\text{Lim}_{\mathcal{L}}(\Gamma, \alpha)$ has a maximal element. Hence by [Theorem 2.4.32](#), it follows that $(\mathcal{L}, \vdash)$ is of Lindenbaum-IV-type. $\square$

We conclude the following connection between the Lindenbaum-IV and the Lindenbaum-II-type logical structures.

Theorem 2.4.35. *Let $(\mathcal{L}, \vdash)$ be a monotonic logical structure. Then $(\mathcal{L}, \vdash)$ is of Lindenbaum-IV-type iff it is of Lindenbaum-II-type.*

Proof. Suppose $(\mathcal{L}, \vdash)$ is a monotonic logical structure. Then by [Theorem 2.4.3](#), any $\Sigma \subseteq \mathcal{L}$ is α -excessive iff it is relatively maximal in α . The theorem now follows from the definitions of Lindenbaum-IV and Lindenbaum-II-type structures. $\square$

2.4.2.3 Lindenbaum-I and Lindenbaum-III-type Logical Structures

Some theorems of the previous subsections hold for Lindenbaum-I and/or Lindenbaum-III-type logical structures as well, of course with suitable modifications. We mention these below. One may compare the [Theorem 2.4.36](#) and [Theorem 2.4.37](#) below with the [Theorem 2.4.26](#) and [Theorem 2.4.32](#), respectively, from the previous subsections.

Theorem 2.4.36. *Let $(\mathcal{L}, \vdash)$ be a Lindenbaum-I-type logical structure. Then every maximal excessive set is maximal nontrivial.*

Proof. Let $\Gamma \subseteq \mathcal{L}$ be a maximal excessive set. Then by [Remark 2.4.22](#), Γ is nontrivial. We claim that Γ is, in fact, maximal nontrivial. Suppose the contrary. Then there exists a nontrivial $\Sigma \supsetneq \Gamma$. Now, as $(\mathcal{L}, \vdash)$ is of Lindenbaum-I-type, by [Definition 2.4.24\(i\)](#), there exists an excessive $\Delta \supseteq \Sigma$. This implies that $\Delta \supsetneq \Gamma$, and hence contradicts the assumption that Γ is a maximal excessive set. Thus Γ must be maximal nontrivial. $\square$

Theorem 2.4.37 (CHARACTERISATION OF LINDENBAUM-III). *Let $(\mathcal{L}, \vdash)$ be a logical structure. Then the following statements are equivalent.*

- (1) $(\mathcal{L}, \vdash)$ is of Lindenbaum-III-type.
- (2) For all nontrivial $\Gamma \subseteq \mathcal{L}$, there exists a maximal excessive set $\Sigma \supseteq \Gamma$.
- (3) For all $\Gamma \subseteq \mathcal{L}$ the set $\text{Lim}_{\mathcal{L}}(\Gamma) = \{\Sigma : \Gamma \subseteq \Sigma \text{ and } \Sigma \text{ is nontrivial}\}$ (i.e. the set of all nontrivial extensions $\Sigma \supseteq \Gamma$) has a maximal element whenever it is nonempty.

Proof. We will follow the following scheme to show that the above three statements are equivalent: (1) $\implies$ (2); (2) $\implies$ (3); (3) $\implies$ (1).

(1) $\implies$ (2): Suppose $(\mathcal{L}, \vdash)$ is a Lindenbaum-III-type logical structure and $\Gamma \subseteq \mathcal{L}$ is nontrivial. Then by [Definition 2.4.24\(iii\)](#), there exists a maximal nontrivial $\Sigma \supseteq \Gamma$. So, by [Corollary 2.4.21](#), Σ is excessive. We claim that Σ is, in fact, maximal excessive.

Suppose the contrary. Then there exists an excessive $\Delta \supsetneq \Sigma$. So, by [Remark 2.4.22](#), Δ is nontrivial. Thus, again by [Definition 2.4.24\(iii\)](#), there exists a maximal nontrivial $\Delta' \supseteq \Delta$. However, this implies that $\Delta' \supsetneq \Sigma$, and hence contradicts the fact that Σ is maximal nontrivial. Thus Σ is maximal excessive.

(2) $\implies$ (3): Suppose $\Gamma \subseteq \mathcal{L}$ such that $\text{Lim}_{\mathcal{L}}(\Gamma) \neq \emptyset$. Let $\Sigma \in \text{Lim}_{\mathcal{L}}(\Gamma)$. Then $\Gamma \subseteq \Sigma$ and Σ is nontrivial. Hence by the assumed condition (2), there exists a maximal excessive $\Delta \supseteq \Sigma$. By [Remark 2.4.22](#), Δ is nontrivial and since $\Gamma \subseteq \Sigma \subseteq \Delta$, $\Delta \in \text{Lim}_{\mathcal{L}}(\Gamma)$. We claim that Δ is a maximal element of $\text{Lim}_{\mathcal{L}}(\Gamma)$.

Suppose the contrary. Then there exists a $\Delta' \supsetneq \Delta$ in $\text{Lim}_{\mathcal{L}}(\Gamma)$. So, Δ' is nontrivial, and hence again by the assumed condition (2), there exists a maximal excessive $\Delta'' \supseteq \Delta'$. This implies that $\Sigma \subsetneq \Delta''$, and hence contradicts the maximality of Σ . Thus Δ must be a maximal element of $\text{Lim}_{\mathcal{L}}(\Gamma)$.

(3) $\implies$ (1): Suppose $\Gamma \subseteq \mathcal{L}$ is nontrivial. Then $\Gamma \in \text{Lim}_{\mathcal{L}}(\Gamma)$ and hence $\text{Lim}_{\mathcal{L}}(\Gamma) \neq \emptyset$. Thus by the assumed condition (3), $\text{Lim}_{\mathcal{L}}(\Gamma)$ must have a maximal element. Let $\Sigma \subseteq \mathcal{L}$ be such a maximal element of $\text{Lim}_{\mathcal{L}}(\Gamma)$. Then $\Gamma \subseteq \Sigma$ and Σ is nontrivial. So, Σ is a maximal nontrivial set containing Γ . Hence $\mathcal{L}$ is of Lindenbaum-III-type. $\square$

Theorem 2.4.38. *Let $(\mathcal{L}, \vdash)$ be a Lindenbaum-I-type logical structure satisfying mixed-cut. Then the set of trivial sets is upward closed.*

Proof. Let $\Gamma \subseteq \Sigma \subseteq \mathcal{L}$ such that Γ is trivial. If possible, suppose Σ is nontrivial. Then, since $(\mathcal{L}, \vdash)$ is of Lindenbaum-I-type, by [Definition 2.4.24\(i\)](#), there exists an excessive $\Delta \supseteq \Sigma$. So, by [Remark 2.4.22](#), Δ is nontrivial. Now, as $(\mathcal{L}, \vdash)$ satisfies mixed-cut, by [Theorem 2.4.10](#), Δ must be strongly closed. This, however, implies that Δ is trivial since Γ is so and $\Gamma \subseteq \Delta$. This is a contradiction. Hence Σ must be trivial. This proves that the set of trivial sets is upward closed. $\square$

2.4.2.4 Relationships between the Lindenbaum-types

We end our investigation of Lindenbaum-type logical structures by discussing the relationships between the different Lindenbaum-types introduced so far. We begin with the straightforward observations listed in the following theorem.

Theorem 2.4.39. *Let $(\mathcal{L}, \vdash)$ be a logical structure.*

- (1) *If $(\mathcal{L}, \vdash)$ is of Lindenbaum-II-type then it is of Lindenbaum-I-type.*
- (2) *If $(\mathcal{L}, \vdash)$ is of Lindenbaum-IV-type then it is of Lindenbaum-II-type, and hence also of Lindenbaum-I-type.*
- (3) *If $(\mathcal{L}, \vdash)$ is of Lindenbaum-III-type then it is of Lindenbaum-I-type.*

Proof. Statement (1) follows from the facts that for every nontrivial set $\Gamma \subseteq \mathcal{L}$, there exists an $\alpha \in \mathcal{L}$ such that $\Gamma \not\vdash \alpha$ and that every α -excessive set is excessive.

Statement (2) follows from [Theorem 2.4.3](#) and statement (1).

Statement (3) follows from [Corollary 2.4.21](#). $\square$

We devote the rest of this chapter to showing that none of the other possible inclusions between the classes of Lindenbaum-type logical structures holds.

Example 2.4.40 (LINDENBAUM-I/II $\not\Rightarrow$ LINDENBAUM-III/IV). Let $(\mathcal{L}, \vdash)$ be a logical structure, where $\mathcal{L}$ is an infinite set and $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ be such that for any $\Gamma \subseteq \mathcal{L}$,

$$W^\vdash(\Gamma) = \begin{cases} \emptyset, & \text{if } \Gamma \text{ is finite and } \text{Cardinality}(\Gamma) \text{ is odd} \\ \mathcal{L}, & \text{otherwise.} \end{cases}$$

We first show that $(\mathcal{L}, \vdash)$ is of Lindenbaum-I and Lindenbaum-II-type. Let $\Gamma \subseteq \mathcal{L}$. If Γ is nontrivial, then Γ must be finite with $|\Gamma|$ odd, and hence there must exist an $\alpha \in \mathcal{L}$ such that $\Gamma \not\vdash \alpha$. On the other hand, if there exists an $\alpha \in \mathcal{L}$ such that $\Gamma \not\vdash \alpha$, then Γ must be nontrivial, and so Γ must be finite with $|\Gamma|$ odd.

Now, suppose Γ is nontrivial and $\alpha \in \mathcal{L}$ such that $\Gamma \not\vdash \alpha$. Since $|\Gamma|$ is odd, for any $\beta \in \mathcal{L} \setminus \Gamma$ (such a β exists since Γ is finite while $\mathcal{L}$ is infinite), $|\Gamma \cup \{\beta\}| = |\Gamma| + 1$ is even, and hence $\Gamma \cup \{\beta\}$ is trivial. So, in particular, $\Gamma \cup \{\beta\} \vdash \alpha$ for any $\beta \in \mathcal{L} \setminus \Gamma$. Hence Γ is α -excessive and hence excessive. Thus every nontrivial $\Gamma \subseteq \mathcal{L}$ is contained in an excessive set, viz., Γ itself. So, $(\mathcal{L}, \vdash)$ is of Lindenbaum-I-type.

We note that, instead of starting with the assumption that Γ is nontrivial, if we assume that $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \not\vdash \alpha$, the same argument leads us to the conclusion that Γ is α -excessive. So, Γ is an α -excessive set containing itself. Hence $(\mathcal{L}, \vdash)$ is also of Lindenbaum-II-type.

Next, to show that $(\mathcal{L}, \vdash)$ is not of Lindenbaum-III-type, let $\Gamma \subseteq \mathcal{L}$ be nontrivial. If possible, let there be a maximal nontrivial $\Sigma \supseteq \Gamma$. Since Σ is nontrivial, it must be finite with $|\Sigma|$ odd. Suppose $\beta, \gamma \in \mathcal{L} \setminus \Sigma$ with $\beta \neq \gamma$ (such β, γ exist since Σ is finite and $\mathcal{L}$ is infinite). Then $|\Sigma \cup \{\beta, \gamma\}| = |\Sigma| + 2$ is also odd, and hence $\Sigma \cup \{\beta, \gamma\}$ is nontrivial. Thus Σ is not maximal nontrivial. This is a contradiction. Hence no nontrivial set $\Gamma \subseteq \mathcal{L}$ is contained in a maximal nontrivial set. This implies that $(\mathcal{L}, \vdash)$ cannot be of Lindenbaum-III-type.

Finally, to show that $(\mathcal{L}, \vdash)$ is not of Lindenbaum-IV-type, let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \not\vdash \alpha$. If possible, suppose there exists a $\Sigma \supseteq \Gamma$ that is relatively maximal in α . Then $\Sigma \not\vdash \alpha$ which implies that Σ is nontrivial, and hence finite with $|\Sigma|$ odd. Then using the same argument as in the above proof of the fact that $(\mathcal{L}, \vdash)$ is not of Lindenbaum-III-type, we have that $\Sigma \cup \{\beta, \gamma\}$ is nontrivial for $\beta, \gamma \in \mathcal{L} \setminus \Sigma$ with $\beta \neq \gamma$. Now, by definition of the logical structure $(\mathcal{L}, \vdash)$, this implies that $W^\vdash(\Sigma \cup \{\beta, \gamma\}) = \emptyset$, and hence in particular, $\Sigma \cup \{\beta, \gamma\} \not\vdash \alpha$. This, however, contradicts the assumption that Σ is relatively maximal in α . Hence, there cannot be any $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that there exists a $\Sigma \supseteq \Gamma$ that is relatively maximal in α . This implies that $(\mathcal{L}, \vdash)$ cannot be of Lindenbaum-IV-type.

We can thus conclude that not every Lindenbaum-I/II-type logical structure is of Lindenbaum-III/IV-type.

Remark 2.4.41. It may be noted here that instead of having the finite subsets of odd cardinality as the only nontrivial subsets of $\mathcal{L}$, we could define $\vdash$ such that the only nontrivial sets are the ones in the collection $\mathcal{A} \subsetneq \{\Gamma \subseteq \mathcal{L} \mid \Gamma \text{ is finite}\}$, where $\mathcal{A}$ satisfies the following two conditions.

- For every $\Gamma \in \mathcal{A}$ and $\alpha \notin \Gamma$, $\Gamma \cup \{\alpha\} \notin \mathcal{A}$.
- For every finite $\Gamma \notin \mathcal{A}$ and $\alpha \notin \Gamma$, $\Gamma \cup \{\alpha\} \in \mathcal{A}$.

Similar arguments as in the above example then lead to the same conclusions.

Example 2.4.42 (LINDENBAUM-III $\not\Rightarrow$ LINDENBAUM-II). Let $(\mathcal{L}, \vdash)$ be a logical structure, where $\mathcal{L}$ is an infinite set and $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ be such that for any $\Gamma \subseteq \mathcal{L}$,

$$W^\vdash(\Gamma) = \begin{cases} \Lambda_0 & \text{if } \Gamma \text{ is finite,} \\ \mathcal{L} & \text{if } \Gamma \text{ is infinite and } \Gamma \subsetneq \mathcal{L}, \\ \mathcal{L} \setminus \Lambda_0 & \text{otherwise,} \end{cases}$$

where $\emptyset \neq \Lambda_0 \subseteq \mathcal{L}$ is fixed.

We first show that $(\mathcal{L}, \vdash)$ is not of Lindenbaum-II-type. Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that Γ is finite and $\alpha \notin \Lambda_0$ (such an α exists since Λ_0 is finite while $\mathcal{L}$ is infinite). Then, by definition of $\vdash$, $\Gamma \not\vdash \alpha$. If possible, suppose there exists an α -excessive $\Sigma \supseteq \Gamma$. This implies that $\Sigma \not\vdash \alpha$, and so, Σ must be finite. Let $\beta \in \mathcal{L} \setminus \Sigma$ (again, such a β exists as $\mathcal{L}$ is infinite). However, this implies that $\Sigma \cup \{\beta\}$ is finite, and hence $\Sigma \cup \{\beta\} \not\vdash \alpha$. This is a contradiction to the assumption that Σ is α -excessive. Thus Γ is not contained in any α -excessive set. Hence $(\mathcal{L}, \vdash)$ is not of Lindenbaum-II-type.

Now, to show that $(\mathcal{L}, \vdash)$ is of Lindenbaum-III-type, we first note that $\mathcal{L} \not\vdash \beta$ for all $\beta \in \Lambda_0$, and hence is nontrivial. Clearly, $\mathcal{L}$ is also maximal nontrivial and contains every nontrivial subset of itself.

We can thus conclude that not every Lindenbaum-III-type logical structure is of Lindenbaum-II-type.

Remark 2.4.43. It may be noted here that instead of using a fixed nonempty finite set $\Lambda_0 \subseteq \mathcal{L}$, as in the above example, we could define $W^\vdash(\Gamma) = \emptyset$ for each finite $\Gamma \subseteq \mathcal{L}$. This would lead to a different logical structure $(\mathcal{L}, \vdash)$ that is also of Lindenbaum-III-type but not of Lindenbaum-II-type.

Corollary 2.4.44. *We can now use the results obtained thus far to conclude the following.*

- (1) *Not every Lindenbaum-I-type logical structure is of Lindenbaum-II-type.*
- (2) *Not every Lindenbaum-III-type logical structure is of Lindenbaum-IV-type.*

Proof. (1) Suppose the contrary. By [Theorem 2.4.39](#), every Lindenbaum-III-type logical structure is also of Lindenbaum-I-type. Thus, every Lindenbaum-III-type logical structure must also be of Lindenbaum-II-type. This, however, is not the case as shown by [Example 2.4.42](#). Hence, not every Lindenbaum-I-type logical structure is of Lindenbaum-II-type.

- (2) Suppose the contrary. Again, by [Theorem 2.4.39](#), every Lindenbaum-IV-type logical structure is also of Lindenbaum-II-type. Thus, every Lindenbaum-III-type logical structure must also be of Lindenbaum-II-type. This is not the case as shown by [Example 2.4.42](#). Hence, not every Lindenbaum-III-type logical structure is of Lindenbaum-IV-type.

□

We now introduce a couple of notions necessary for our next example.

Definition 2.4.45 (FINITARY LOGICAL STRUCTURE). A logical structure $(\mathcal{L}, \vdash)$ is said to be *finitary* if for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, whenever $\Gamma \vdash \alpha$, there exists a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \alpha$.

We state the following well-known theorem without proof.

Theorem 2.4.46 (LINDENBAUM-ASSER THEOREM). *Let $(\mathcal{L}, \vdash)$ be a finitary monotonic logical structure. Then for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\Gamma \not\vdash \alpha$, there exists a $\Sigma \supseteq \Gamma$ that is relatively maximal in α .*

Remark 2.4.47. The proof of the above theorem requires the axiom of choice, as pointed out in [17]. In fact, the Lindenbaum-Asser Theorem (Theorem 2.4.46) is equivalent to the axiom of choice. For a proof of this result, see [51, 84].

The following observation is now a straightforward corollary of Theorem 2.4.46.

Corollary 2.4.48. *Any finitary monotonic logical structure is of Lindenbaum-IV-type, and hence, also of Lindenbaum-II/I-type, by Theorem 2.4.39. Hence, any finitary Tarski-type logical structure is of Lindenbaum-IV-type, and hence also of Lindenbaum-II/I-type.*

Remark 2.4.49. Thus, it follows that the underlying logical structure of any finitary Tarski-type logic is of Lindenbaum-IV-type. The prototypical example of such a logical structure is the one underlying classical propositional logic. The underlying logical structures of the logics mentioned in [2] are also of Lindenbaum-IV-type and hence, also of Lindenbaum-II/I-type.

Example 2.4.50 (LINDENBAUM-IV $\not\Rightarrow$ LINDENBAUM-III). We consider the logical structure $(\mathbb{N}, \vdash)$, where $\mathbb{N}$ denotes the set of natural numbers and $\vdash \subseteq \mathcal{P}(\mathbb{N}) \times \mathbb{N}$ is such that for any $\Gamma \subseteq \mathbb{N}$,

$$W^\vdash(\Gamma) = \begin{cases} \mathbb{N} & \text{if } \Gamma \text{ is infinite} \\ \mathbb{N} \setminus \{n\} & \text{if } \Gamma \text{ is finite with } |\Gamma| = n \end{cases}$$

We first show that $(\mathbb{N}, \vdash)$ is of Lindenbaum-IV-type. Let $\Gamma \cup \{n\} \subseteq \mathbb{N}$ such that $\Gamma \not\vdash n$. Then Γ must be finite with $|\Gamma| = n$. Now, suppose $\Sigma \supsetneq \Gamma$. If Σ is infinite, then by definition of $\vdash$, Σ is trivial, and so, in particular, $\Sigma \vdash n$. On the other hand, if Σ is finite, then since $\Sigma \supsetneq \Gamma$, $|\Gamma| = n < |\Sigma|$. Let $|\Sigma| = m$. Then, again by definition of $\vdash$, $W^\vdash(\Sigma) = \mathbb{N} \setminus \{m\}$. This implies that $n \in W^\vdash(\Sigma)$, and hence $\Sigma \vdash n$. Thus $\Sigma \vdash n$ for any $\Sigma \supsetneq \Gamma$. Hence Γ is relatively maximal in n . In other words, Γ is an extension of itself that is relatively maximal in n . So, we can conclude that $(\mathbb{N}, \vdash)$ is a Lindenbaum-IV-type logical structure.

Now, to show that $(\mathbb{N}, \vdash)$ is not of Lindenbaum-III-type, suppose the contrary. Let $\Gamma \subseteq \mathbb{N}$ be nontrivial. Then, there must exist a maximal nontrivial $\Sigma \supseteq \Gamma$. This implies that Σ is nontrivial and hence finite. Let $r \in \mathbb{N} \setminus \Sigma$ (such an r exists, since Σ is finite). Clearly, $\Sigma \cup \{r\} \supsetneq \Sigma$ is finite, and hence nontrivial. However, this contradicts the assumption that Σ is maximal nontrivial. Hence $(\mathbb{N}, \vdash)$ is not of Lindenbaum-III-type.

The results of this whole discussion have been summarized in Figure 2.1.

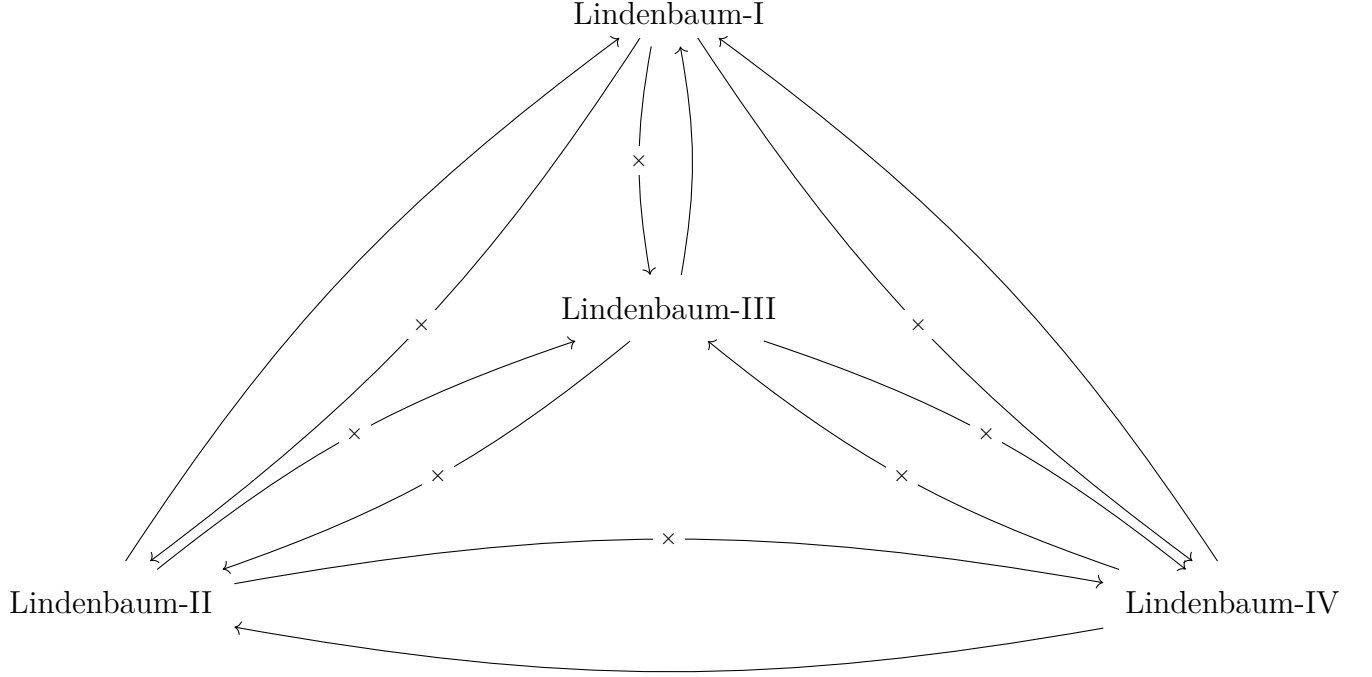


Figure 2.1: Comparison between Lindenbaum-type logical structures

2.5 Future Works

- The characterisation theorems for Lindenbaum-I- and Lindenbaum-II type logical structures remain as topics for future projects.
- Let $(\mathcal{L}, \vdash)$ be a logical structure and $\alpha \in \mathcal{L}$. Let $\{\mathcal{X}\}_{\alpha \in \mathcal{L}}$ and $\{\mathcal{Y}\}_{\alpha \in \mathcal{L}}$ be two families of subsets of $\mathcal{P}(\mathcal{L})$. We say that *the elements of $\{\mathcal{X}\}_{\alpha \in \mathcal{L}}$ is extendable to the elements of $\{\mathcal{Y}\}_{\alpha \in \mathcal{L}}$* if for all $\Gamma \cup \{\alpha\}$, $\Gamma \in \mathcal{X}_\alpha$, implies that there exists $\Sigma \cup \beta \in \mathcal{L}$ such that $\Gamma \subseteq \Sigma$ and $\Sigma \in \mathcal{Y}_\beta$. A logical structure $(\mathcal{L}, \vdash)$ to be *extendable* if there exist two families $\{\mathcal{X}\}_{\alpha \in \mathcal{L}}$ and $\{\mathcal{Y}\}_{\alpha \in \mathcal{L}}$ of subsets of $\mathcal{P}(\mathcal{L})$ such that the elements of $\{\mathcal{X}\}_{\alpha \in \mathcal{L}}$ is extendable to $\{\mathcal{Y}\}_{\alpha \in \mathcal{L}}$.

It is easy to see that depending upon suitable choices of $\{\mathcal{X}\}_{\alpha \in \mathcal{L}}$ and $\{\mathcal{Y}\}_{\alpha \in \mathcal{L}}$, each logical structure introduced in this chapter is an example of extendable logical structure. We would like to work on a general theory of extendable logical structures in future.

Chapter 3

Tarski-Lindenbaum-type Logical Structures¹

3.1 Introduction

In the previous chapter we have discussed the Tarski-type logical structures [Section 2.3](#) and various Lindenbaum-type logical structures in [Section 2.4](#). In this chapter, we investigate logical structures which are both of Tarski- as well as Lindenbaum-type. Notice that by [Corollary 2.4.48](#) it follows that every finitary Tarski-type logical structure is Lindenbaum-IV-type. So, there are plenty of logical structures which are both of Tarski- as well as Lindenbaum-type. There are several natural questions that arise at this point. We list some of them below.

- Is every Tarski-type logical structure also of Lindenbaum-type?
- Is every Lindenbaum-type logical structure also of Tarski-type?
- What are the relationships between the classes of various Tarski-Lindenbaum-type logical structures?

We show via examples that the answers to both the first questions are negative.²

Example 3.1.1 (TARSKI $\not\Rightarrow$ LINDENBAUM-I). Let $(\mathcal{L}_{\mathbf{CL}}, \vdash_{\mathbf{CL}})$ denote the classical propositional logic, as before. We define a new logical structure $(\|\mathcal{L}_{\mathbf{CL}}\|, \vdash_f)$, where $\vdash_f \subseteq \mathcal{P}(\|\mathcal{L}_{\mathbf{CL}}\|) \times \|\mathcal{L}_{\mathbf{CL}}\|$ is such that for all $\Gamma \subseteq \|\mathcal{L}_{\mathbf{CL}}\|$,

$$W_{\vdash_f}(\Gamma) = \begin{cases} \Gamma & \text{if } \Gamma \text{ is finite,} \\ \|\mathcal{L}_{\mathbf{CL}}\| & \text{otherwise.} \end{cases}$$

¹A version of this chapter forms part of the article [\[96\]](#).

²The fact that not every Tarski-type logical structure is of Lindenbaum-I-type is also proved in [\[16\]](#). Our example, given below, is different from the one given in that paper.

Clearly, $\|\mathcal{L}_{\mathbf{CL}}\|$ is infinite. Now, to show that $(\|\mathcal{L}_{\mathbf{CL}}\|, \vdash_f)$ is of Tarski-type, let $\Gamma, \Sigma \subseteq \|\mathcal{L}_{\mathbf{CL}}\|$.

Suppose $\Sigma \subseteq W_{\vdash_f}(\Gamma)$. Now, if Γ is finite, then $W_{\vdash_f}(\Gamma) = \Gamma$ is finite. So, Σ must be finite. Thus in this case, $W_{\vdash_f}(\Sigma) = \Sigma \subseteq W_{\vdash_f}(\Gamma)$. On the other hand, if Γ is infinite, then $W_{\vdash_f}(\Gamma) = \|\mathcal{L}_{\mathbf{CL}}\|$, and hence $W_{\vdash_f}(\Sigma) \subseteq W_{\vdash_f}(\Gamma)$.

Conversely, suppose $W_{\vdash_f}(\Sigma) \subseteq W_{\vdash_f}(\Gamma)$. Now, if Σ is finite, then $W_{\vdash_f}(\Sigma) = \Sigma$. So, $\Sigma \subseteq W_{\vdash_f}(\Gamma)$ in this case. On the other hand, if Σ is infinite, then $W_{\vdash_f}(\Sigma) = \|\mathcal{L}_{\mathbf{CL}}\|$, which implies that $W_{\vdash_f}(\Gamma) = \|\mathcal{L}_{\mathbf{CL}}\|$. Hence $\Sigma \subseteq W_{\vdash_f}(\Gamma)$.

Thus, for all $\Gamma, \Sigma \subseteq \|\mathcal{L}_{\mathbf{CL}}\|$, $\Sigma \subseteq W_{\vdash_f}(\Gamma)$ iff $W_{\vdash_f}(\Sigma) \subseteq W_{\vdash_f}(\Gamma)$. Hence by using statement (4) of [Theorem 2.3.9](#), we can conclude that $(\|\mathcal{L}_{\mathbf{CL}}\|, \vdash_f)$ is of Tarski-type.

We claim that $(\|\mathcal{L}_{\mathbf{CL}}\|, \vdash_f)$ is, however, not of Lindenbaum-I-type. Suppose the contrary.

Let $\Gamma \subseteq \|\mathcal{L}_{\mathbf{CL}}\|$ be nontrivial (any finite subset of $\|\mathcal{L}_{\mathbf{CL}}\|$ is nontrivial). Then, by definition of Lindenbaum-I-type logical structures, there must exist an excessive $\Sigma \supseteq \Gamma$. Suppose Σ is α -excessive. So, $\Sigma \not\vdash_f \alpha$, and hence $W_{\vdash_f}(\Sigma) \neq \|\mathcal{L}_{\mathbf{CL}}\|$. Then, $W_{\vdash_f}(\Sigma) = \Sigma$. Thus Σ must be finite with $\alpha \notin \Sigma$. Now, let $\beta \in \|\mathcal{L}_{\mathbf{CL}}\| \setminus (\Sigma \cup \{\alpha\})$. Then, since $\Sigma \cup \{\beta\}$ is finite and $\alpha \notin \Sigma \cup \{\beta\}$, $\Sigma \cup \{\beta\} \not\vdash_f \alpha$. This implies that Σ is not α -excessive, a contradiction. So, we can conclude that Σ cannot be α -excessive for any $\alpha \in \|\mathcal{L}_{\mathbf{CL}}\|$, and hence is not excessive. Thus $(\|\mathcal{L}_{\mathbf{CL}}\|, \vdash_f)$ is not of Lindenbaum-I-type.

Remark 3.1.2. We know, by [Theorem 2.4.39](#), that every logical structure which is of Lindenbaum-II/III/IV-type is of Lindenbaum-I-type as well. Thus, we can conclude, from the above example, that not every Tarski-type logical structure is of a Lindenbaum type. However, as pointed out in [Corollary 2.4.48](#), every finitary Tarski-type logical structure is of Lindenbaum-IV/II/I-type.

Example 3.1.3 (LINDENBAUM-III/IV $\not\Rightarrow$ TARSKI). Let $(\mathcal{L}, \vdash)$ be a logical structure, where $\mathcal{L} = \{0, 1, 2\}$ and $\vdash$ is such that

$$W^+(\{0\}) = W^+(\{0, 1\}) = \{0, 1\},$$

and $W^+(\Gamma) = \mathcal{L}$ for any other $\Gamma \subseteq \mathcal{L}$.

We note that a set $\Gamma \subseteq \mathcal{L}$ is nontrivial iff $\Gamma \not\vdash 2$ and $\{0, 1\}$ is a maximal 2-excessive set that contains every nontrivial $\Gamma \subseteq \mathcal{L}$. Thus by using statement (2) of [Theorem 2.4.32](#), we can conclude that $(\mathcal{L}, \vdash)$ is of Lindenbaum-IV-type.

Moreover, $\{0, 1\}$ is a maximal nontrivial set that extends every nontrivial $\Gamma \subseteq \mathcal{L}$. Hence $(\mathcal{L}, \vdash)$ is also of Lindenbaum-III-type.

However, $\emptyset$ is trivial but is contained in the nontrivial sets $\{0\}$ and $\{0, 1\}$. Thus $(\mathcal{L}, \vdash)$ is not monotonic, and hence not of Tarski-type.

Remark 3.1.4. We can now conclude the following from the above counterexample.

- (1) Not every Lindenbaum-I-type logical structure is of Tarski-type.

To prove this, suppose the contrary. By [Theorem 2.4.39](#), we know that every Lindenbaum-III-type logical structure is of Lindenbaum-I-type, and hence of Tarski-type. This, however, is not the case as established by the above example. Hence not every Lindenbaum-I-type logical structure is of Tarski-type.

- (2) Not every Lindenbaum-II-type logical structure is of Tarski-type.

To prove this, suppose the contrary. By [Theorem 2.4.39](#), we know that every Lindenbaum-IV-type logical structure is of Lindenbaum-II-type, and hence of Tarski-type. This, however, is not the case as established by the above example. Hence not every Lindenbaum-II-type logical structure is of Tarski-type.

In other words, all five types of logical structures, viz., the four Lindenbaum-types and the Tarski-type are distinct.

We will, therefore, focus on the last question for the rest of this chapter. This means, in particular, that we focus on logical structures which are both Tarski- as well as a Lindenbaum-type. We will also investigate in detail a particular type of Tarski-Lindenbaum-type logical structure. More specifically, we will prove the characterisation, adequacy, minimality, and representation theorems for these logical structures.

3.2 Tarski-Lindenbaum-type Logical Structures

We devote this section to the discussion of logical structures that are of Tarski-type as well as of a Lindenbaum-type. Rather unsurprisingly, these are referred to as *Tarski-Lindenbaum-type* logical structures.

Definition 3.2.1 (TARSKI-LINDENBAUM-TYPE LOGICAL STRUCTURE). A logical structure $(\mathcal{L}, \vdash)$ is said to be of:

- (i) *Tarski-Lindenbaum-I-type* (henceforth abbreviated as TL_1 -type) if it is of Tarski- as well as of Lindenbaum-I-type.
- (ii) *Tarski-Lindenbaum-II-type* (henceforth abbreviated as TL_2 -type) if it is of Tarski- as well as of Lindenbaum-II-type.
- (iii) *Tarski-Lindenbaum-III-type* (henceforth abbreviated as TL_3 -type) if it is of Tarski- as well as of Lindenbaum-III-type.
- (iv) *Tarski-Lindenbaum-IV-type* (henceforth abbreviated as TL_4 -type) if it is of Tarski- as well as of Lindenbaum-IV-type.

We first discuss the relationships between these classes of logical structures. To begin with, we list some straightforward observations in the following theorem.

Theorem 3.2.2. *Let $(\mathcal{L}, \vdash)$ be a logical structure.*

- (1) If $(\mathcal{L}, \vdash)$ is of TL_i -type, where $i = 2, 3, 4$, then it is of TL_1 -type.
- (2) $(\mathcal{L}, \vdash)$ is of TL_2 -type iff it is of TL_4 -type.

Proof. (1) follows from [Theorem 2.4.39](#) and (2) follows from [Theorem 2.4.35](#). $\square$

The following examples now show that none of the other possible inclusions between the classes of Tarski-Lindenbaum-type logical structures holds.

For the first two examples, we use the concept of ordinal numbers, more specifically, the *von Neumann ordinal numbers*. The definition and some important properties of these are provided below for the readers' sake. These are based on the discussion on this in [\[71, Chapter I, Section 7\]](#).

Definition 3.2.3 (VON NEUMANN ORDINAL NUMBER). A set α is called a (*von Neumann*) *ordinal number* or, simply an *ordinal*, if it satisfies the following conditions³.

- (i) For all $x \in \alpha$, $x \subseteq \alpha$.
- (ii) For all $x \in \alpha$, $x \notin x$.
- (iii) For all $x, y, z \in \alpha$, if $x \in y$ and $y \in z$ then $x \in z$.
- (iv) For all $x, y \in \alpha$ either $x \in y$, or $x = y$, or $y \in x$.
- (v) For every nonempty $x \subseteq \alpha$, there exists $z \in x$ such that $x \cap z = \emptyset$.

Theorem 3.2.4. [\[71, Lemmas I.7.5, I.7.8, I.7.10, I.7.11, I.7.14\]](#) Suppose α, β, γ are ordinals. Then the following statements hold.

- (1) $\alpha \notin \alpha$.
- (2) If $\delta \in \alpha$, then δ is also an ordinal.
- (3) $\beta \subseteq \alpha$ iff $\beta \in \alpha$ or $\beta = \alpha$.
- (4) $\alpha \in \beta$ and $\beta \in \gamma$ implies $\alpha \in \gamma$.
- (5) Either $\alpha \in \beta$ or $\alpha = \beta$ or $\beta \in \alpha$.
- (6) $\alpha \cup \beta$ and $\alpha \cap \beta$ are ordinals. Moreover, if X is a nonempty set of ordinals then $\bigcup X$ and $\bigcap X$ are ordinals.

Lastly, the following are the von Neumann ordinals that we use in the following examples (see [\[71\]](#) for more details).

- $0 = \emptyset, 1 = \{\emptyset\}, \dots$, and in general, for any natural number n , $n + 1 = n \cup \{n\}$. These are the finite ordinals.

³These conditions are usually stated in a compact form as follows. A transitive set well-ordered by $\in$ is a von Neumann ordinal. We avoid the terminology here as this is beside the main aim of the present chapter.

- $\omega = \{n \mid n \text{ is a natural number}\} = \mathbb{N}$ is the first infinite ordinal.
- $\omega + 1 = \omega \cup \{\omega\}$, and for any natural number n , $\omega + (n + 1) = (\omega + n) \cup \{\omega + n\}$.
- $\omega + \omega = \bigcup_{n \in \omega} (\omega + n)$.

Example 3.2.5 ($\text{TL}_1 \not\Rightarrow \text{TL}_2/\text{TL}_3$). We consider the logical structure $(\omega + \omega, \vdash)$, where $\vdash \subseteq \mathcal{P}(\omega + \omega) \times (\omega + \omega)$ is such that for any $\Gamma \subseteq \omega + \omega$,

$$W^\vdash(\Gamma) = \begin{cases} \bigcap \{\beta \in \omega + \omega \mid \Gamma \subseteq \beta\} & \text{if } \bigcap \{\beta \in \omega + \omega \mid \Gamma \subseteq \beta\} \neq \omega \\ \omega + 1 & \text{otherwise.} \end{cases}$$

We define, for our convenience, $\text{OrCl}(\Gamma) = \{\beta \in \omega + \omega \mid \Gamma \subseteq \beta\}$ for each $\Gamma \subseteq \omega + \omega$. So, the above definition of $W^\vdash$ can be rephrased as follows.

$$W^\vdash(\Gamma) = \begin{cases} \bigcap \text{OrCl}(\Gamma) & \text{if } \bigcap \text{OrCl}(\Gamma) \neq \omega \\ \omega + 1 & \text{otherwise.} \end{cases}$$

Suppose $\Gamma \subseteq \omega + \omega$. Then the following are easy observations.

(a) $\text{OrCl}(\Gamma) \neq \emptyset$.

Suppose $\Gamma = \emptyset$. Then $\text{OrCl}(\Gamma) = \{\beta \in \omega + \omega \mid \emptyset = \Gamma \subseteq \beta\} = \omega + \omega \neq \emptyset$.

Now, suppose $\Gamma \neq \emptyset$. So, there exists $\alpha \in \Gamma$. Moreover, since $\alpha \in \Gamma \subseteq \omega + \omega$, by [Theorem 3.2.4\(2\)](#), α is an ordinal, i.e. Γ is a set of ordinals. Then, by [Theorem 3.2.4\(6\)](#), $\lambda = \bigcup_{\alpha \in \Gamma} \alpha$ is an ordinal. Now, for any $\beta \in \Gamma$, $\beta \subseteq \lambda$. This implies, by [Theorem 3.2.4\(3\)](#), that either $\beta \in \lambda$ or $\beta = \lambda$. Thus, $\Gamma \subseteq \lambda \cup \{\lambda\}$, which is also an ordinal. So, $\lambda \cup \{\lambda\} \in \text{OrCl}(\Gamma)$.

Hence, in all cases, $\text{OrCl}(\Gamma)$ is nonempty.

(b) $W^\vdash(\Gamma)$ is an ordinal in $\omega + \omega$.

Since $\text{OrCl}(\Gamma) \neq \emptyset$, by [Theorem 3.2.4\(6\)](#), $\bigcap \text{OrCl}(\Gamma)$ is an ordinal. Clearly, $\bigcap \text{OrCl}(\Gamma)$ is in $\omega + \omega$, since each member of $\text{OrCl}(\Gamma)$ is in $\omega + \omega$. So, if $\bigcap \text{OrCl}(\Gamma) \neq \omega$, then $W^\vdash(\Gamma)$ is an ordinal in $\omega + \omega$. On the other hand, if $\bigcap \text{OrCl}(\Gamma) = \omega$, then $W^\vdash(\Gamma) = \omega + 1$, also an ordinal in $\omega + \omega$.

(c) $W^\vdash(\Gamma) \neq \omega$.

If $\bigcap \text{OrCl}(\Gamma) = \omega$, then $W^\vdash(\Gamma) = \omega + 1 \neq \omega$. On the other hand, if $\bigcap \text{OrCl}(\Gamma) \neq \omega$, then $W^\vdash(\Gamma) = \bigcap \text{OrCl}(\Gamma) \neq \omega$.

(d) $\Gamma \subseteq W^\vdash(\Gamma)$.

We first note that $\Gamma \subseteq \beta$ for all $\beta \in \text{OrCl}(\Gamma)$ and so, $\Gamma \subseteq \bigcap \text{OrCl}(\Gamma)$.

Now, if $\bigcap \text{OrCl}(\Gamma) \neq \omega$, then $W^\vdash(\Gamma) = \bigcap \text{OrCl}(\Gamma)$, and hence, $\Gamma \subseteq W^\vdash(\Gamma)$. On the other hand, if $\bigcap \text{OrCl}(\Gamma) = \omega$, then $\Gamma \subseteq \bigcap \text{OrCl}(\Gamma) = \omega \subseteq \omega + 1 = W^\vdash(\Gamma)$.

Now, let $\Gamma, \Sigma \subseteq \omega + \omega$. If $W^\perp(\Gamma) \subseteq W^\perp(\Sigma)$ then by (d) above, $\Gamma \subseteq W^\perp(\Gamma) \subseteq W^\perp(\Sigma)$. Conversely, suppose $\Gamma \subseteq W^\perp(\Sigma)$. We first note that, by (b) above, $W^\perp(\Sigma)$ is an ordinal in $\omega + \omega$. Thus $\Gamma \subseteq W^\perp(\Sigma)$ implies that $W^\perp(\Sigma) \in \text{OrCl}(\Gamma)$, and hence, $\bigcap \text{OrCl}(\Gamma) \subseteq W^\perp(\Sigma)$.

Now, if $\bigcap \text{OrCl}(\Gamma) \neq \omega$, then clearly, $W^\perp(\Gamma) = \bigcap \text{OrCl}(\Gamma) \subseteq W^\perp(\Sigma)$.

On the other hand, if $\bigcap \text{OrCl}(\Gamma) = \omega$, then $\omega = \bigcap \text{OrCl}(\Gamma) \subseteq W^\perp(\Sigma)$. As noted in (b) and (c) above, $W^\perp(\Sigma)$ is an ordinal such that $W^\perp(\Sigma) \neq \omega$. So, by [Theorem 3.2.4\(3\)](#), $\omega \in W^\perp(\Sigma)$, which implies that $\{\omega\} \subseteq W^\perp(\Sigma)$. Hence,

$$W^\perp(\Gamma) = \omega + 1 = \omega \cup \{\omega\} \subseteq W^\perp(\Sigma).$$

Thus, for all $\Gamma, \Sigma \subseteq \omega + \omega$, $\Gamma \subseteq W^\perp(\Sigma)$ iff $W^\perp(\Gamma) \subseteq W^\perp(\Sigma)$. So, by [Theorem 2.3.9](#), $(\omega + \omega, \perp)$ is of Tarski-type.

Next, before proceeding to prove that $(\omega + \omega, \perp)$ is of Lindenbaum-I-type, we establish the following.

CLAIM: For any $\alpha \in \omega + \omega$ such that $\alpha \neq \omega$, α is α -excessive. (*)

We first note that, since α is an ordinal in $\omega + \omega$, $\alpha \in \text{OrCl}(\alpha)$. Thus, $\alpha \subseteq \bigcap \text{OrCl}(\alpha) \subseteq \alpha$, i.e. $\bigcap \text{OrCl}(\alpha) = \alpha$. Now, since $\alpha \neq \omega$, $W^\perp(\alpha) = \bigcap \text{OrCl}(\alpha) = \alpha$. Then, as $\alpha \notin \alpha$, $\alpha \notin W^\perp(\alpha)$, i.e. $\alpha \not\perp \alpha$.

Now, if possible, suppose there exists $\delta \in (\omega + \omega) \setminus \alpha$ such that $\alpha \cup \{\delta\} \not\perp \alpha$. If $\bigcap \text{OrCl}(\alpha \cup \{\delta\}) = \omega$, then we have $W^\perp(\alpha \cup \{\delta\}) = \omega + 1$. This implies that $\alpha \subsetneq \alpha \cup \{\delta\} \subseteq W^\perp(\alpha \cup \{\delta\}) = \omega + 1$. So, by [Theorem 3.2.4\(3\)](#), $\alpha \in \omega + 1$, which means that $\alpha \in W^\perp(\alpha \cup \{\delta\})$. This is a contradiction. Hence, $\bigcap \text{OrCl}(\alpha \cup \{\delta\}) \neq \omega$, and so, $W^\perp(\alpha \cup \{\delta\}) = \bigcap \text{OrCl}(\alpha \cup \{\delta\})$. Then $\alpha \cup \{\delta\} \not\perp \alpha$ implies that $\alpha \notin \bigcap \text{OrCl}(\alpha \cup \{\delta\})$. So, there exists $\beta \in \omega + \omega$ such that $\alpha \cup \{\delta\} \subseteq \beta$ but $\alpha \notin \beta$. Thus, by [Theorem 3.2.4\(5\)](#), either $\beta \in \alpha$ or $\beta = \alpha$, i.e. $\beta \subseteq \alpha$ by [Theorem 3.2.4\(3\)](#). This implies that $\alpha \cup \{\delta\} \subseteq \alpha$, which is impossible since $\delta \notin \alpha$. Thus we can conclude that for every $\delta \in (\omega + \omega) \setminus \alpha$, $\alpha \cup \{\delta\} \perp \alpha$. This completes the proof of the claim that for every $\alpha \in \omega + \omega$, if $\alpha \neq \omega$ then α is α -excessive.

Now, to show that $(\omega + \omega, \perp)$ is of Lindenbaum-I-type, let Γ be a nontrivial subset of $\omega + \omega$. Then there exists $\alpha \in \omega + \omega$ such that $\Gamma \not\perp \alpha$. We have the following cases.

Case 3.2.5.1. $\bigcap \text{OrCl}(\Gamma) = \omega$.

Then $\Gamma \subseteq W^\perp(\Gamma) = \omega + 1$, and by Claim (*), $\omega + 1$ is $(\omega + 1)$ -excessive, and hence excessive.

Case 3.2.5.2. $\bigcap \text{OrCl}(\Gamma) \neq \omega$.

In this case, $W^\perp(\Gamma) = \bigcap \text{OrCl}(\Gamma)$. Then since $\Gamma \not\perp \alpha$, i.e. $\alpha \notin W^\perp(\Gamma)$, there exists $\beta \in \omega + \omega$ such that $\Gamma \subseteq \beta$ but $\alpha \notin \beta$. Then, by [Theorem 3.2.4\(5\)](#), either $\beta \in \alpha$ or $\beta = \alpha$, i.e. $\beta \subseteq \alpha$ by [Theorem 3.2.4\(3\)](#). This implies that $\Gamma \subseteq \alpha$.

Now, if $\alpha \neq \omega$, then α is α -excessive, by Claim (*), and hence excessive. On the other hand, if $\alpha = \omega$, then $\Gamma \subseteq \omega \subseteq \omega + 1$, and again by Claim (*), $\omega + 1$ is $(\omega + 1)$ -excessive, and hence excessive.

So, in all cases, Γ is contained in an excessive set. This implies that $(\omega + \omega, \vdash)$ is of Lindenbaum-I-type and hence, of TL_1 -type.

Moreover, we note that in all the cases above the nontrivial set Γ has a proper excessive extension. This is clear in [Case 3.2.5.1](#), where $\Gamma \subseteq \bigcap \text{OrCl}(\Gamma) = \omega$, and in [Case 3.2.5.2](#), where $\bigcap \text{OrCl}(\Gamma) \neq \omega$ and $\Gamma \not\vdash \omega$ – in both scenarios, Γ is properly contained in the excessive set, $\omega + 1$. In the remaining case, where $\Gamma \not\vdash \alpha$ and $\alpha \neq \omega$, α is an α -excessive extension of Γ . If $\Gamma \subsetneq \alpha$, then we are done. Suppose not, i.e. $\Gamma = \alpha$. Now, since $\alpha \in \omega + \omega = \bigcup_{n \in \omega} (\omega + n)$, there exists $n \in \omega$ such that $\alpha \in \omega + n$. This implies that $\{\alpha\} \subseteq \omega + n$, and by definition of an ordinal, $\alpha \subseteq \omega + n$. So,

$$\alpha \cup \{\alpha\} \subseteq \omega + n \subsetneq (\omega + n) \cup \{\omega + n\} = \omega + (n + 1) \subseteq \omega + \omega.$$

Thus $\alpha \cup \{\alpha\} \subsetneq \omega + \omega$, and by [Theorem 3.2.4\(6\)](#), it is an ordinal, often denoted by $\alpha + 1$. Hence, by [Theorem 3.2.4\(3\)](#) $\alpha \cup \{\alpha\} = \alpha + 1 \in \omega + \omega$. Now, if possible, suppose $\alpha + 1 = \omega$. Then $\alpha \in \omega$. This implies that $\alpha = n$ for some $n \in \omega$. Then $\alpha \in n + 1$. This implies that $\{\alpha\} \subsetneq n + 1$. Hence, $\alpha \cup \{\alpha\} = \alpha + 1 \subseteq n + 1 \subsetneq n + 2 \subseteq \omega$. However, this means that $\omega = \alpha + 1 \subsetneq \omega$, which is impossible. Hence, $\alpha + 1 \neq \omega$. So, by Claim (*), $\alpha + 1$ is $(\alpha + 1)$ -excessive and hence, nontrivial. Now, $\Gamma = \alpha \subsetneq \alpha + 1$. So, $\alpha + 1$ is a proper excessive extension of Γ .

Hence, every nontrivial $\Gamma \subseteq \omega + \omega$ has a proper excessive, and hence nontrivial, extension. This implies that there does not exist any maximal nontrivial subset of $\omega + \omega$. Therefore, we can conclude that $(\omega + \omega, \vdash)$ is not of Lindenbaum-III-type, and thus, not of TL_3 -type.

Finally, to show that $(\omega + \omega, \vdash)$ is not of Lindenbaum-II-type, and hence, not of TL_2 -type, we consider the finite ordinal 1, and note that $W^\vdash(1) = \bigcap \text{OrCl}(1) = 1$. Thus, $1 \not\vdash \omega$. Now, if possible, suppose $\Sigma \subseteq \omega + \omega$ be an ω -excessive set containing 1. So, $\Sigma \not\vdash \omega$, i.e. $\omega \notin W^\vdash(\Sigma)$. Since $W^\vdash(\Sigma)$ is an ordinal in $\omega + \omega$, as noted in Observation (b) above, this implies, by [Theorem 3.2.4\(5\)](#), that either $W^\vdash(\Sigma) \in \omega$ or $W^\vdash(\Sigma) = \omega$. Since $W^\vdash(\Sigma) \neq \omega$, by Observation (c) above, $W^\vdash(\Sigma) \in \omega$. This implies that $W^\vdash(\Sigma) = n$ for some $n \in \omega$. Then, by reflexivity, $\Sigma \subseteq W^\vdash(\Sigma) = n$. So, $\Sigma \cup \{n\} \subseteq n \cup \{n\} = n + 1$. Hence, by monotonicity, $W^\vdash(\Sigma \cup \{n\}) \subseteq W^\vdash(n + 1) = n + 1$. Clearly, $\omega \notin W^\vdash(\Sigma \cup \{n\})$, i.e. $\Sigma \cup \{n\} \not\vdash \omega$. This contradicts the assumption that Σ is ω -excessive. Thus, there is no ω -excessive set containing 1, even though $1 \not\vdash \omega$. Thus, $(\omega + \omega, \vdash)$ is not of Lindenbaum-II-type, and hence, not of TL_2 -type.

The next example uses a similar logical structure as in the above one, the only difference is in the definition of $\vdash$. We remove the separate cases now, and define $W^\vdash(\Gamma)$ in a uniform way for all $\Gamma \subseteq \omega + \omega$.

Example 3.2.6 ($\text{TL}_2 \not\Rightarrow \text{TL}_3$). As in the above example, we consider a logical structure $(\omega + \omega, \vdash)$, where $\vdash \subseteq \mathcal{P}(\omega + \omega) \times (\omega + \omega)$ is such that for any $\Gamma \subseteq \omega + \omega$,

$$W^\vdash(\Gamma) = \bigcap \{\beta \in \omega + \omega \mid \Gamma \subseteq \beta\} = \bigcap \text{OrCl}(\Gamma).$$

It is fairly straightforward to see that $(\omega + \omega, \vdash)$ is of Tarski-type, but not of Lindenbaum-III-type, and hence not of TL_3 -type, due to the same reasons as described in the previous example. We claim that $(\omega + \omega, \vdash)$ is, however, of Lindenbaum-II-type.

To prove this let $\Gamma \cup \{\alpha\} \subseteq \omega + \omega$ be such that $\Gamma \not\vdash \alpha$. Now, since $\alpha \in \omega + \omega$, it is an ordinal, and hence, $\alpha \in \text{OrCl}(\alpha)$. Thus, as in the previous example, $\alpha \subseteq W^\vdash(\alpha) = \bigcap \text{OrCl}(\alpha) \subseteq \alpha$. Thus, $W^\vdash(\alpha) = \alpha$, and since $\alpha \notin \alpha$, $\alpha \notin W^\vdash(\alpha)$, i.e. $\alpha \not\vdash \alpha$. We claim that α is, in fact, α -excessive.

Suppose the contrary. Then there exists $\delta \in (\omega + \omega) \setminus \alpha$ such that $\alpha \cup \{\delta\} \not\vdash \alpha$, i.e. $\alpha \notin W^\vdash(\alpha \cup \{\delta\}) = \bigcap \text{OrCl}(\alpha \cup \{\delta\})$. So, there exists $\beta \in \omega + \omega$ such that $\alpha \cup \{\delta\} \subseteq \beta$ but $\alpha \notin \beta$. Now, $\alpha \notin \beta$ implies, by [Theorem 3.2.4\(5\)](#), that either $\beta \in \alpha$ or $\beta = \alpha$, i.e. $\beta \subseteq \alpha$, by [Theorem 3.2.4\(3\)](#). Thus, we have $\alpha \cup \{\delta\} \subseteq \alpha$, which implies that $\delta \in \alpha$. This is a contradiction. Hence, α is α -excessive.

Thus, for any $\Gamma \cup \{\alpha\} \subseteq \omega + \omega$ with $\Gamma \not\vdash \alpha$, there exists an α -excessive extension of Γ , viz., α . Hence, $(\omega + \omega, \vdash)$ is of Lindenbaum-II-type, and consequently of TL_2 -type.

Example 3.2.7 ($\text{TL}_3 \not\Rightarrow \text{TL}_2$). We consider $(\mathbb{Z}^+, \vdash)$ as a logical structure, where $\mathbb{Z}^+$ is the set of positive integers and $\vdash \subseteq \mathcal{P}(\mathbb{Z}^+) \times \mathbb{Z}^+$ is such that for any $\Gamma \subseteq \mathbb{Z}^+$,

$$W^\vdash(\Gamma) = \begin{cases} \Gamma & \text{if } \Gamma \subseteq p\mathbb{Z}^+, \text{ for some prime } p, \text{ and is finite} \\ \bigcap \{q\mathbb{Z}^+ \mid \Gamma \subseteq q\mathbb{Z}^+, q \text{ prime}\} & \text{if } \Gamma \subseteq p\mathbb{Z}^+, \text{ for some prime } p, \text{ and is infinite} \\ \mathbb{Z}^+ & \text{otherwise} \end{cases}$$

We first note that for any $\Gamma \subseteq \mathbb{Z}^+$, $\Gamma \subseteq W^\vdash(\Gamma)$. Now, to show that $(\mathbb{Z}^+, \vdash)$ is of Tarski-type, let $\Sigma, \Delta \subseteq \mathbb{Z}^+$ such that $W^\vdash(\Sigma) \subseteq W^\vdash(\Delta)$. Then, $\Sigma \subseteq W^\vdash(\Sigma) \subseteq W^\vdash(\Delta)$.

Conversely, suppose $\Sigma, \Delta \subseteq \mathbb{Z}^+$ such that $\Sigma \subseteq W^\vdash(\Delta)$. Then the following cases arise.

Case 3.2.7.1. $W^\vdash(\Delta) = \Delta$

This implies that Δ must be finite and $\Delta \subseteq p\mathbb{Z}^+$, for some prime p . Now, since $\Sigma \subseteq W^\vdash(\Delta) = \Delta$, it follows that Σ is finite and $\Sigma \subseteq p\mathbb{Z}^+$ as well. Thus $W^\vdash(\Sigma) = \Sigma \subseteq W^\vdash(\Delta)$.

Case 3.2.7.2. $W^\vdash(\Delta) = \bigcap \{q\mathbb{Z}^+ \mid \Delta \subseteq q\mathbb{Z}^+, q \text{ prime}\}$

Then it follows that $\Delta \subseteq p\mathbb{Z}^+$, for some prime p , and is infinite. So, $W^\vdash(\Delta) \subseteq p\mathbb{Z}^+$, and hence $\Sigma \subseteq W^\vdash(\Delta) \subseteq p\mathbb{Z}^+$.

Now, if Σ is finite, then $W^\vdash(\Sigma) = \Sigma \subseteq W^\vdash(\Delta)$.

On the other hand, if Σ is infinite, then $W^\vdash(\Sigma) = \bigcap \{q\mathbb{Z}^+ \mid \Sigma \subseteq q\mathbb{Z}^+, q \text{ prime}\}$. Then, since $\Sigma \subseteq W^\vdash(\Delta)$, $\Sigma \subseteq q\mathbb{Z}^+$ for every prime q such that $\Delta \subseteq q\mathbb{Z}^+$. This implies that

$$W^\vdash(\Sigma) = \bigcap \{q\mathbb{Z}^+ \mid \Sigma \subseteq q\mathbb{Z}^+, q \text{ prime}\} \subseteq \bigcap \{q\mathbb{Z}^+ \mid \Delta \subseteq q\mathbb{Z}^+, q \text{ prime}\} = W^\vdash(\Delta).$$

Case 3.2.7.3. $W^\vdash(\Delta) = \mathbb{Z}^+$

In this case, it immediately follows that $W^\vdash(\Sigma) \subseteq W^\vdash(\Delta)$.

Thus in all cases, $\Sigma \subseteq W^\vdash(\Delta)$ implies that $W^\vdash(\Sigma) \subseteq W^\vdash(\Delta)$. Hence for any $\Sigma, \Delta \subseteq \mathbb{Z}^+$, $\Sigma \subseteq W^\vdash(\Delta)$ iff $W^\vdash(\Sigma) \subseteq W^\vdash(\Delta)$, and so, by [Theorem 2.3.9](#), $(\mathbb{Z}^+, \vdash)$ is of Tarski-type.

We next show that $(\mathbb{Z}^+, \vdash)$ is of Lindenbaum-III-type. Suppose $\Sigma \subseteq \mathbb{Z}^+$ is nontrivial. Then there must exist a prime p such that $\Sigma \subseteq p\mathbb{Z}^+$. We claim that $p\mathbb{Z}^+$ is maximal nontrivial.

Suppose the contrary. Let $\Delta \supsetneq p\mathbb{Z}^+$ be a nontrivial subset of $\mathbb{Z}^+$. This implies that there exists a prime p' such that $\Delta \subseteq p'\mathbb{Z}^+$. So, we have $p\mathbb{Z}^+ \subsetneq \Delta \subseteq p'\mathbb{Z}^+$. Then $p' \mid p$. Since p, p' are both prime, this implies that $p = p'$, and hence $p\mathbb{Z}^+ = p'\mathbb{Z}^+$. This is, however, a contradiction. Hence $p\mathbb{Z}^+$ is maximal nontrivial. Thus every nontrivial $\Sigma \subseteq \mathbb{Z}^+$ has a maximal nontrivial extension. So, $(\mathbb{Z}^+, \vdash)$ is of Lindenbaum-III-type, and hence of TL_3 -type.

Lastly, to show that $(\mathbb{Z}^+, \vdash)$ is not of Lindenbaum-II-type, and hence not of TL_2 -type, consider the set $\{2\} \subseteq 2\mathbb{Z}^+ \subseteq \mathbb{Z}^+$. Then as $\{2\}$ is finite, $W^\vdash(\{2\}) = \{2\}$, and so, in particular, $\{2\} \not\vdash 4$. If possible, suppose there exists a 4-excessive $\Delta \supseteq \{2\}$.

Then $\Delta \not\vdash 4$ and hence is nontrivial. So, there must exist a prime q such that $\Delta \subseteq q\mathbb{Z}^+$. Since $2 \in \Delta$, q must be 2, i.e. $\Delta \subseteq 2\mathbb{Z}^+$. Moreover, we claim that Δ is infinite. Suppose the contrary. Then $W^\vdash(\Delta) = \Delta$, and so, $4 \notin \Delta$. Now, let $k \in \mathbb{Z}^+$ such that $k > 2$ and $2k \in \mathbb{Z}^+ \setminus \Delta$ (clearly, such a k exists as Δ is finite). Then $\Delta \cup \{2k\} \subseteq 2\mathbb{Z}^+$ is finite, and so $W^\vdash(\Delta \cup \{2k\}) = \Delta \cup \{2k\}$. Since $2k \neq 4$ and $4 \notin \Delta$, this implies that $\Delta \cup \{2k\} \not\vdash 4$. This, however, contradicts the assumption that Δ is 4-excessive. So, Δ must be infinite. Then $W^\vdash(\Delta) = \bigcap \{q\mathbb{Z}^+ \mid \Delta \subseteq q\mathbb{Z}^+, q \text{ prime}\}$. Now, since $2 \in \Delta$, we must have $2 \in q\mathbb{Z}^+$ for every prime q such that $\Delta \subseteq q\mathbb{Z}^+$, which implies that $q = 2$. Thus $W^\vdash(\Delta) = \bigcap \{q\mathbb{Z}^+ \mid \Delta \subseteq q\mathbb{Z}^+, q \text{ prime}\} = 2\mathbb{Z}^+$. However, this implies that $\Delta \vdash 4$, a contradiction. Hence the set $\{2\}$ is such that $\{2\} \not\vdash 4$ but not contained in a 4-excessive set. Thus, $(\mathbb{Z}^+, \vdash)$ is not of Lindenbaum-II-type, and hence not of TL_2 -type.

Corollary 3.2.8. *We can now use the results obtained thus far to conclude the following.*

- (1) *Not every TL_1 -type structure is of TL_4 -type.*
- (2) *Not every TL_3 -type structure is of TL_4 -type.*
- (3) *Not every TL_4 -type structure is of TL_3 -type.*

Proof. (1) Suppose the contrary. By [Theorem 3.2.2](#), every TL_4 -type logical structure is of TL_2 -type. Thus, every TL_1 -type logical structure is of TL_2 -type. This, however, is not the case as shown by [Example 3.2.5](#). Hence, not every TL_1 -type structure is of TL_4 -type.

(2) Suppose the contrary. Again, by [Theorem 3.2.2](#), every TL_4 -type logical structure is of TL_2 -type. Thus, every TL_3 -type logical structure is of TL_2 -type. This, however, is not the case as shown by [Example 3.2.7](#). Hence, not every TL_3 -type structure is of TL_4 -type.

(3) Suppose the contrary. Again, by [Theorem 3.2.2](#), every TL_2 -type logical structure is of TL_4 -type. Thus, every TL_2 -type logical structure is of TL_3 -type. This, however, is not the case as shown by [Example 3.2.6](#). Hence, not every TL_4 -type structure is of TL_3 -type.

□

The results in this section have been summarised in [Figure 3.1](#).

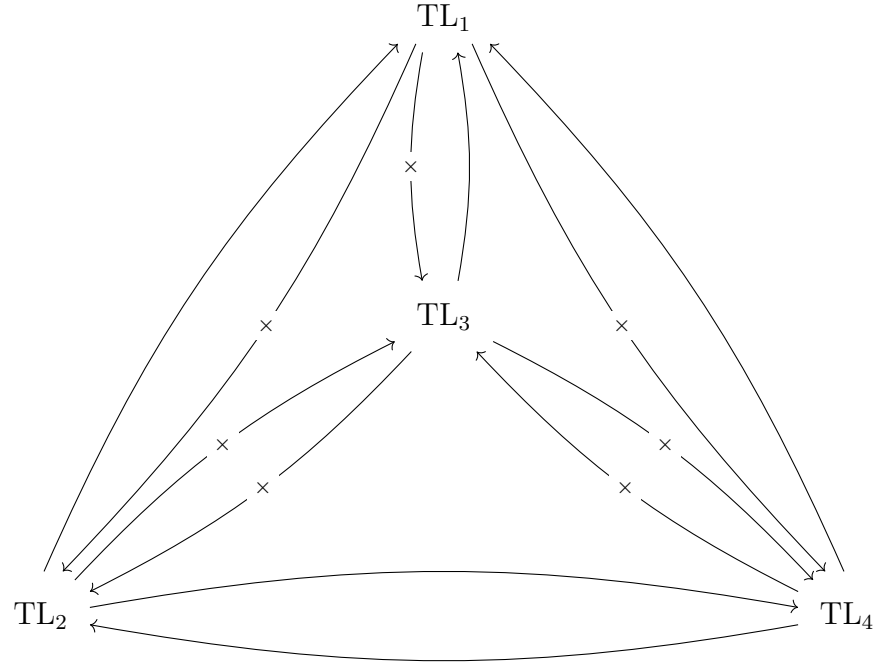


Figure 3.1: Comparison between Tarski-Lindenbaum-type logical structures

3.3 TL_4 -Logical Structures

We devote this section to the study of TL_4 -type logical structures. Since a logical structure is of TL_4 -type iff it is of TL_2 -type, by [Theorem 2.4.35](#), the results of this section apply to TL_2 -type logical structures as well.

Theorem 3.3.1. *Let $(\mathcal{L}, \vdash)$ be a Tarski-type logical structure. Then it satisfies mixed-cut.*

Proof. Immediate. □

Theorem 3.3.2 (CHARACTERISATION OF TL_4). *Let $(\mathcal{L}, \vdash)$ be a logical structure. Then the following statements are equivalent.*

- (1) $(\mathcal{L}, \vdash)$ is TL_4 .
- (2) For all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\Gamma \not\vdash \alpha$, there exists a strongly closed $\Sigma \supseteq \Gamma$ which is relatively maximal in α .
- (3) For all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\Gamma \not\vdash \alpha$, there exists a strongly closed α -excessive $\Sigma \supseteq \Gamma$.

- (4) For all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\Gamma \not\vdash \alpha$, there exists $\beta \in \mathcal{L}$ and a strongly closed β -excessive $\Sigma \supseteq \Gamma$ such that $\{\alpha\} \vdash \beta$ and $\{\beta\} \vdash \alpha$.
- (5) For all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\Gamma \not\vdash \alpha$, there exists $\beta \in \mathcal{L}$ and a strongly closed β -excessive $\Sigma \supseteq \Gamma$ such that $\Sigma \not\vdash \alpha$ and $\{\beta\} \vdash \alpha$.
- (6) For each $\alpha \in \mathcal{L}$, there exists a $\beta \in \mathcal{L}$ such that for any $\Gamma \subseteq \mathcal{L}$, if $\Gamma \not\vdash \alpha$ then there exists a strongly closed β -excessive $\Sigma \supseteq \Gamma$ with $\Sigma \not\vdash \alpha$.

Proof. We will follow the following scheme to show that the above statements are equivalent:

$$(1) \implies (2) \implies (3) \implies (4) \implies (5) \implies (3) \implies (6) \implies (1).$$

(1) $\implies$ (2): Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ be such that $\Gamma \not\vdash \alpha$. Since $(\mathcal{L}, \vdash)$ is of Lindenbaum-IV-type, there exists $\Sigma \supseteq \Gamma$ which is relatively maximal in α . Now, by [Theorem 2.4.3](#), Σ is also α -excessive, and hence, excessive. Since $(\mathcal{L}, \vdash)$ is of Tarski-type as well, it satisfies mixed-cut, by [Theorem 3.3.1](#). Then, by [Theorem 2.4.10](#), Σ is strongly closed. Thus Σ is a strongly closed extension of Γ that is relatively maximal in α .

(2) $\implies$ (3): This follows immediately, by [Theorem 2.4.3](#).

(3) $\implies$ (4): Suppose $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \not\vdash \alpha$. Then, by statement (3), there exists a strongly closed α -excessive $\Sigma \supseteq \Gamma$. Since Σ is α -excessive, $\Sigma \not\vdash \alpha$. Thus, by [Theorem 2.3.9](#), $(\mathcal{L}, \vdash)$ is of Tarski-type, and hence reflexive. So, $\{\alpha\} \vdash \alpha$. The statement (4) now follows with $\beta = \alpha$.

(4) $\implies$ (5): Suppose $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \not\vdash \alpha$. Then, by statement (4), there exists $\beta \in \mathcal{L}$ and a strongly closed β -excessive $\Sigma \supseteq \Gamma$ such that $\{\alpha\} \vdash \beta$ and $\{\beta\} \vdash \alpha$. We now claim that $\Sigma \not\vdash \alpha$.

Suppose the contrary, then $\Sigma \vdash \alpha$. But then, since by hypothesis $\{\alpha\} \vdash \beta$. Then by transitivity, it follows that $\Sigma \vdash \beta$, a contradiction. This contradiction ensures that $\Sigma \not\vdash \alpha$. Thus $\Sigma \supseteq \Gamma$ is a strongly closed β -excessive set such that $\Sigma \not\vdash \alpha$ and $\{\beta\} \vdash \alpha$.

(5) $\implies$ (3): Suppose $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \not\vdash \alpha$. Then, by statement (5), there exists $\beta \in \mathcal{L}$ and a strongly closed β -excessive $\Sigma \supseteq \Gamma$ such that $\Sigma \not\vdash \alpha$ and $\{\beta\} \vdash \alpha$. Thus, in particular, Σ is a strongly closed extension of Γ such that $\Sigma \not\vdash \alpha$. So, by [Theorem 2.3.9](#), $(\mathcal{L}, \vdash)$ is of Tarski-type. Now, let $\gamma \in \mathcal{L} \setminus \Sigma$. Since Σ is β -excessive, $\Sigma \cup \{\gamma\} \vdash \beta$. Then, as $\{\beta\} \vdash \alpha$, we have, by transitivity, $\Sigma \cup \{\gamma\} \vdash \alpha$. Thus Σ is α -excessive as well.

(3) $\implies$ (6): The statement (6) follows from (3) by taking $\beta = \alpha$.

(6) $\implies$ (1): Suppose $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \not\vdash \alpha$. Then, by statement (5), there exists a $\beta \in \mathcal{L}$ (dependent only on α), and a strongly closed β -excessive $\Sigma \supseteq \Gamma$ with $\Sigma \not\vdash \alpha$. Thus, in particular, Σ is a strongly closed extension of Γ such that $\Sigma \not\vdash \alpha$. So, by [Theorem 2.3.9](#), $(\mathcal{L}, \vdash)$ is of Tarski-type.

Next, to show that $(\mathcal{L}, \vdash)$ is of Lindenbaum-IV-type as well, let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \not\vdash \alpha$. So, by the assumed condition, there exists a $\beta \in \mathcal{L}$ (dependent only on α) and a strongly closed β -excessive $\Sigma \supseteq \Gamma$ with $\Sigma \not\vdash \alpha$. If possible, suppose Σ is not relatively maximal in α . Then there exists a $\Delta \supsetneq \Sigma$ such that $\Delta \not\vdash \alpha$. So, again by the assumed

condition, there exists a strongly closed β -excessive $\Lambda \supseteq \Delta$ with $\Lambda \not\models \alpha$. We note that $\Sigma \subsetneq \Delta \subseteq \Lambda$. So, there exists a $\gamma \in \Lambda \setminus \Sigma$. Now, since Σ is β -excessive, $\Sigma \cup \{\gamma\} \vdash \beta$. Then, as Λ is strongly closed and $\Sigma \cup \{\gamma\} \subseteq \Lambda$, $\Lambda \vdash \beta$. This, however, contradicts that Λ is β -excessive. Thus Σ must be relatively maximal in α . Hence $(\mathcal{L}, \vdash)$ is of Lindenbaum-IV-type, and consequently, of TL_4 -type. $\square$

Theorem 3.3.3. *Let $(\mathcal{L}, \vdash)$ be a logical structure and SCS (strongly closed and excessive) be a set of bivaluations defined as follows.*

$$\text{SCS} = \bigcup_{\beta \in \mathcal{L}} \{\chi_\Sigma \in \{0, 1\}^\mathcal{L} : \Sigma \text{ is strongly closed and } \beta\text{-excessive}\},$$

where χ_Σ is the characteristic function of Σ . Then for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, $\Gamma \vdash \alpha$ implies that $\Gamma \vdash_{\text{SCS}} \alpha$.

Proof. Suppose $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \vdash \alpha$. If possible, let $\Gamma \not\vdash_{\text{SCS}} \alpha$. This implies that there exists a $v \in \text{SCS}$ such that $v(\Gamma) = \{1\}$ but $v(\alpha) = 0$. Then, by the definition of SCS , there exists $\Sigma \cup \{\beta\} \subseteq \mathcal{L}$ such that Σ is a strongly closed β -excessive set and $v = \chi_\Sigma$. Thus $\chi_\Sigma(\Gamma) = \{1\}$ implies that $\Gamma \subseteq \Sigma$. Then, as Σ is strongly closed and $\Gamma \vdash \alpha$, it follows that $\alpha \in \Sigma$. Hence $\chi_\Sigma(\alpha) = v(\alpha) = 1$. This is a contradiction. Hence $\Gamma \vdash_{\text{SCS}} \alpha$. $\square$

Corollary 3.3.4 (SOUNDNESS THEOREM FOR TL_4). *Let $(\mathcal{L}, \vdash)$ be a TL_4 -type logical structure and SCS be as in the above theorem. Then $\text{SCS} \neq \emptyset$ (assuming that there is at least one nontrivial $\Gamma \subseteq \mathcal{L}$) and $(\mathcal{L}, \vdash)$ is sound with respect to $(\mathcal{L}, \vdash_{\text{SCS}})$.*

Theorem 3.3.5 (COMPLETENESS THEOREM FOR TL_4). *Let $(\mathcal{L}, \vdash)$ be a TL_4 -type logical structure and SCS be as defined in Theorem 3.3.3. Then for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, $\Gamma \vdash_{\text{SCS}} \alpha$ implies that $\Gamma \vdash \alpha$, i.e. $(\mathcal{L}, \vdash)$ is complete with respect to $(\mathcal{L}, \vdash_{\text{SCS}})$.*

Proof. Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \vdash_{\text{SCS}} \alpha$ but $\Gamma \not\vdash \alpha$. Since $(\mathcal{L}, \vdash)$ is of TL_4 -type, this implies, by Theorem 3.3.2, that there exists a $\beta \in \mathcal{L}$ (dependent only on α) and a strongly closed β -excessive $\Sigma \supseteq \Gamma$ with $\Sigma \not\vdash \alpha$. Now, as Σ is strongly closed and β -excessive, $\chi_\Sigma \in \text{SCS}$, and since $\Gamma \subseteq \Sigma$, $\chi_\Sigma(\Gamma) = \{1\}$. Then, using the assumption $\Gamma \vdash_{\text{SCS}} \alpha$, we have $\chi_\Sigma(\alpha) = 1$, which implies that $\alpha \in \Sigma$. Now, since Σ is strongly closed, it follows that $\Sigma \vdash \alpha$. This is a contradiction. Hence, if $\Gamma \vdash_{\text{SCS}} \alpha$ then $\Gamma \vdash \alpha$. $\square$

Theorem 3.3.6 (ADEQUACY THEOREM FOR TL_4). *Let $(\mathcal{L}, \vdash)$ be a logical structure. If $(\mathcal{L}, \vdash)$ is of TL_4 -type then $\vdash = \vdash_{\text{SCS}}$.*

Proof. Suppose $(\mathcal{L}, \vdash)$ is a TL_4 -type logical structure. Then it follows that $\vdash = \vdash_{\text{SCS}}$ from Corollary 3.3.4 and Theorem 3.3.5. $\square$

Theorem 3.3.7 (MINIMALITY THEOREM FOR TL_4). *Let $(\mathcal{L}, \vdash)$ be a TL_4 -type logical structure. Then for any $\mathcal{B} \subseteq \text{SCS}$ we have $\vdash \subseteq \vdash_{\mathcal{B}}$. Moreover, if $\mathcal{B} \subsetneq \text{SCS}$, then $\vdash \subsetneq \vdash_{\mathcal{B}}$.*

Proof. Suppose $\mathcal{B} \subseteq \text{SCS}$. We claim that this implies $\vdash_{\text{SCS}} \subseteq \vdash_{\mathcal{B}}$. Suppose the contrary. Then there exists $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \vdash_{\text{SCS}} \alpha$ while $\Gamma \not\vdash_{\mathcal{B}} \alpha$. This implies that

there exists $v \in \mathcal{B}$ such that $v(\Gamma) = \{1\}$ but $v(\alpha) \neq 1$. Now, since $\mathcal{B} \subseteq \text{SCS}$, $v \in \text{SCS}$. Then $v(\Gamma) = \{1\}$ while $v(\alpha) \neq 1$ implies that $\Gamma \not\vdash_{\text{SCS}} \alpha$. This is a contradiction. Hence, $\vdash_{\text{SCS}} \subseteq \vdash_{\mathcal{B}}$.

Now, suppose $\mathcal{B} \subsetneq \text{SCS}$. Then there exists $\beta \in \mathcal{L}$ and a strongly closed β -excessive $\Sigma \subseteq \mathcal{L}$ such that $\chi_\Sigma \in \text{SCS}$ but $\chi_\Sigma \notin \mathcal{B}$. Since Σ is β -excessive, $\Sigma \not\vdash \beta$. This implies, by [Theorem 3.3.5](#), that $\Sigma \not\vdash_{\text{SCS}} \beta$.

We claim that $\Sigma \vdash_{\mathcal{B}} \beta$. Suppose the contrary, i.e. $\Sigma \not\vdash_{\mathcal{B}} \beta$. This implies that there exists $\delta \in \mathcal{L}$ and a strongly closed δ -excessive $\Delta \subseteq \mathcal{L}$ such that $\chi_\Delta \in \mathcal{B}$ with $\chi_\Delta(\Sigma) = \{1\}$ but $\chi_\Delta(\beta) \neq 1$. This implies that $\Sigma \subseteq \Delta$ but $\beta \notin \Delta$. Now, since $\chi_\Delta \in \mathcal{B}$, we can conclude that $\Delta \neq \Sigma$, as $\chi_\Sigma \notin \mathcal{B}$. Thus $\Sigma \subsetneq \Delta$, which implies that there exists $\sigma \in \Delta \setminus \Sigma$. Then, as Σ is β -excessive, $\Sigma \cup \{\sigma\} \vdash \beta$. Now, since Δ is strongly closed and $\Sigma \cup \{\sigma\} \subseteq \Delta$, $\Delta \vdash \beta$, and hence $\beta \in \Delta$. This is a contradiction. So, $\Sigma \vdash_{\mathcal{B}} \beta$ while $\Sigma \not\vdash_{\text{SCS}} \beta$, which implies that $\vdash_{\text{SCS}} \subsetneq \vdash_{\mathcal{B}}$. Hence, by [Theorem 3.3.6](#), $\vdash \subsetneq \vdash_{\mathcal{B}}$. $\square$

Theorem 3.3.8 (REPRESENTATION THEOREM FOR TL_4). *Let $(\mathcal{L}, \vdash)$ be a logical structure. We define*

$$\text{SCS}[\beta, \alpha] = \{\chi_\Sigma \in \text{SCS} \mid \Sigma \text{ is } \beta\text{-excessive, } \Sigma \not\vdash \alpha \text{ and } \{\beta\} \vdash \alpha\} \quad \text{and} \quad \text{SCS}^* = \bigcup_{\alpha, \beta \in \mathcal{L}} \text{SCS}[\beta, \alpha].$$

Then $(\mathcal{L}, \vdash)$ is of TL_4 -type iff $\vdash = \vdash_{\text{SCS}^*}$.

Proof. Suppose $(\mathcal{L}, \vdash)$ is of TL_4 -type. We note that $\text{SCS}^* \subseteq \text{SCS}$. Then, by [Theorem 3.3.7](#), $\vdash \subseteq \vdash_{\text{SCS}^*}$. To prove the reverse inclusion, let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \not\vdash \alpha$. Then, since $(\mathcal{L}, \vdash)$ is of TL_4 -type, by statement (5) of [Theorem 3.3.2](#), there exists $\beta \in \mathcal{L}$ and a strongly closed β -excessive $\Sigma \supseteq \Gamma$ such that $\Sigma \not\vdash \alpha$ and $\{\beta\} \vdash \alpha$. This implies that $\chi_\Sigma \in \text{SCS}[\beta, \alpha] \subseteq \text{SCS}^*$. Now, as $\Gamma \subseteq \Sigma$, $\chi_\Sigma(\Gamma) = \{1\}$. However, since Σ is strongly closed and $\Sigma \not\vdash \alpha$, $\alpha \notin \Sigma$, which implies that $\chi_\Sigma(\alpha) = 0$. This means that $\Gamma \not\vdash_{\text{SCS}^*} \alpha$. Thus, $\Gamma \not\vdash \alpha$ implies $\Gamma \not\vdash_{\text{SCS}^*} \alpha$. So, we can conclude that $\vdash_{\text{SCS}^*} \subseteq \vdash$. Hence, $\vdash = \vdash_{\text{SCS}^*}$.

Conversely, suppose $\vdash = \vdash_{\text{SCS}^*}$. Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ be such that $\Gamma \not\vdash \alpha$. Then, as $\vdash = \vdash_{\text{SCS}^*}$, $\Gamma \not\vdash_{\text{SCS}^*} \alpha$. So, there exists $\beta \in \mathcal{L}$ and a strongly closed, β -excessive, $\Sigma \supseteq \Gamma$ such that $\Sigma \not\vdash \alpha$ and $\{\beta\} \vdash \alpha$. Hence, by statement (5) of [Theorem 3.3.2](#), $(\mathcal{L}, \vdash)$ is of TL_4 -type. $\square$

The following two results have been proved in [\[16\]⁴](#),

- (1) Let $(\mathcal{L}, \vdash)$ be a logical structure and

$$\text{RELMAX} = \bigcup_{\alpha \in \mathcal{L}} \{\chi_\Sigma : \Sigma \text{ is relatively maximal in } \alpha\}$$

If $(\mathcal{L}, \vdash)$ is of finitary Tarski-type then $\vdash_{\text{RELMAX}} \subseteq \vdash$.

- (2) For a given finitary Tarski-type logical structure $(\mathcal{L}, \vdash)$ and for any $\mathcal{B} \subsetneq \text{RELMAX}$ we have $\vdash_{\mathcal{B}} \not\subseteq \vdash$.

⁴The presentations of these results here differ from those in [\[16\]](#) mainly due to differing terminology. One can, however, easily prove that our formulations are equivalent to the original ones.

The above results can now be seen as special cases of Theorem [Theorem 3.3.6](#) and [Theorem 3.3.7](#) by the following observations.

- Every finitary Tarski-type logical structure is of TL_4 -type by [Theorem 2.4.46](#).
- Let $(\mathcal{L}, \vdash)$ be a Tarski-type logical structure and $\Gamma \cup \{\alpha\}$ be such that Γ is relatively maximal in α . Then Γ is strongly closed.

To prove this, we note that since Γ is relatively maximal in α , $\Gamma \not\vdash \alpha$. Then, since $(\mathcal{L}, \vdash)$ is of Tarski-type, by [Theorem 2.3.9](#), there exists a strongly closed $\Sigma \supseteq \Gamma$ such that $\Sigma \not\vdash \alpha$. Since Γ is relatively maximal in α , this implies $\Sigma = \Gamma$. Hence, Σ is strongly closed.

3.4 Future Works

- Although the study of specific classes of logical structures is important, the spirit of universal logic lies in developing a general theory. This is embodied by results such as our [Theorem 3.3.3](#). We expect to have more such results in future.
- In this chapter, we have provided a number of examples to separate the various classes of logical structures. It will be interesting to see if these examples can be characterised in some way. In particular, one might ask if the use of von Neumann ordinals is essential to separate the TL_1 - from TL_2 - and TL_3 -type logical structures.
- It is expected that results similar to the ones proved in [Section 3.3](#) can be proved for TL_i -structures, where $i = 1, 2, 3$. These are also left for the future.
- Lastly, one could work with more general sets instead of $\{0, 1\}$. Universal logic has deep connections with Suszko's Thesis and many-valued logics (see [\[23\]](#)). Although, we have pointed out some connections in [Chapter 4](#), more investigations along these lines remain a job for the future.

Chapter 4

Suszko's Thesis and Many-valued Logical Structures

Suszko's Thesis (**ST**), named after the Polish logician Roman Suszko, is the (philosophical) claim that “there are but two logical values, true and false” (see [35]). This thesis has been given formal contents by the so-called *Suszko Reduction* (the particular case of Łukasiewicz's 3-valued logic $\mathbf{L}_3$ is outlined in [102] by Suszko himself). However, as has been pointed out in [34], “there seems to be no paper where Suszko explicitly formulates (SR)[the Suszko Reduction] in full generality!” Furthermore, there seems to be no *explicit definition* of ‘logical value’. This makes **ST** rather ambiguous, difficult to justify and open to interpretation. The situation is explained succinctly in [21] as follows (emphasis ours).

The terminology “Suszko's thesis” has been put forward (see e.g. [76]). This is a quite ambiguous terminology. First all there is [is] a result according to which every logic is bivalent. *Such a result is not a thesis. What can be considered as a thesis is rather the interpretation of this result.*

Interpretations differ. For example, the ‘real’ reason behind the existence of adequate bivalent semantics for every Tarski-type logical structure – in other words, the ‘logical 2-valuedness’ of Tarski-type logical structures – has been explained by Malinowski in [77] as follows.

[L]ogical two-valuedness . . . is obviously related to the division of the universe of interpretation into two subsets of elements: distinguished and others.

Thus, according to him, ‘logical value’ corresponds to the number of divisions of the universe of interpretation (cf. [104] and [106]). This particular conceptualisation of ‘logical value’ is termed by Malinowski as *inferential value*. This notion of many-valuedness is different from the more traditional conception of many-valuedness – *algebraic many-valuedness*¹. The difference between these is explained in [102] as follows (emphasis ours).

¹Sometimes ‘algebraic many-valuedness’ is also referred to as ‘referential many-valuedness’ (see, e.g., [78, 79]). However, we will not use this terminology in this chapter.

[T]he logical valuations are morphisms (of formulas to the zero-one model) in some exceptional cases, only. Thus, the logical valuations and the algebraic valuations are functions of quite different conceptual nature. *The former relate to the truth and falsity, and the latter represent the reference assignments.*

Introducing the notion of q -consequence operators in [74], Malinowski showed that the logical structures induced by them are inferentially 3-valued, i.e., **ST** is false in general. Going further, in [80, 81], he investigates the problem of constructing inferentially n -valued logical structures (where n is a natural number) and proposes a partial solution.

Even though much research has been done on inferential many-valuedness, surprisingly there is no *explicit* or *formal* definition of the same!² The underlying intuition has been provided several times (see [75, Section 4], [80, Section 2] and [81, Section 3, Section 5] e.g.), of course. The lack of a formal definition, however, is problematic if, e.g., one wants to characterise the class of inferentially n -valued structures for a natural number n .

One aim of this chapter is to remedy this situation. Based on a rather general notion of *semantics* (Definition 4.1.1), we define the notion of inferentially many-valued logical structures (Definition 4.1.13) and *prove* that the class of monotonic logical structures can be split into four sub-classes (not necessarily disjoint), each of which has a certain type of adequate 3-valued semantics (see Theorem 4.2.8, Theorem 4.2.14 and Theorem 4.2.29 for details). Moreover, we identify several classes of logical structures, for which these 3-valued semantics are respectively minimal (see Theorem 4.2.10, Theorem 4.2.16 and Theorem 4.2.31 for details). It turns out that Malinowski's notion of many-valuedness is just a particular one among a galaxy of such possible notions (see Remark 4.1.18). These are the contents of Section 4.1 and Section 4.2.

Section 4.3 is motivated by the observation that most of the research concerning **ST** is focused on trying to show that **ST** does *not* hold in general. This has been done either by changing the notion of semantics (e.g. in [74]) or by changing the notion of logical structure (e.g. in [106]). In Section 4.3, we introduce a different notion of semantics than the one already mentioned (Definition 4.3.1) and show that several logical structures can have adequate bivalent semantics. We illustrate this first by providing adequate bivalent semantics for monotonic logical structures and then doing the same for certain particular types of non-monotonic logical structures (Theorem 4.3.22, Theorem 4.3.23). We also discuss some problems regarding the existence of adequate bivalent semantics for *arbitrary* non-monotonic logical structures. Thus, the contents of this section lead to new questions regarding the true nature of logical value, many-valuedness, and most importantly, semantics.

Even though we do not explicitly define *logical value*, in Section 4.4, we define what we

²In [100], the author defines and examines *inferential many-valued inference systems*. However, this is not the same as Malinowski's notion of inferential many-valuedness, as is evident from the following quote.

We define and investigate the notion of an inferentially many-valued inference system. In this definition we try to capture the idea of the proof-theoretical representation of truth-functional many-valuedness.

On the other hand, the notion of *referentially k -valued logic*, as investigated in the article, can be seen as a special case of our notion algebraically k -valued logic (see Definition 4.1.9).

mean by a κ -valued logical structure of order λ for any cardinal κ and ordinal λ (Definition 4.4.2). For this, we delve into a deeper analysis of the *principle of bivalence* and explicate how **ST** is related to it. The ensuing analysis culminates in the conclusion that many-valued logical structures can be obtained if the bivalence of the higher-order metalogics of the logic under consideration up to a certain level is discarded. This helps to build formal bridges between the theory of graded consequence (see [38]) and the theory of many-valued logical structures, leading to a generalisation of **ST**.

In the final section, we point out some directions for future research.

4.1 Fundamental Notions

In the discussions on **ST**, one important thing that comes out regularly but is not often emphasised may be stated as follows: *the justification (or refutation) of ST depends on the notion of semantics*. Usually, the term ‘semantics’ is accompanied by an adjective, e.g., *truth-functional semantics*, *matrix semantics*, *referential semantics*, etc. However, these terms are rather ambiguous because they make sense only if one has a definition of semantics. In [53], e.g., Font says the following (emphasis ours).

Suszko avoids speaking of *truth values*. Instead, one of his main points is to distinguish between *algebraic values* (the members of the algebra of truth values, introduced by Łukasiewicz and others and usually interpreted as such in the literature) and *logical values* (of which his intuition admits only two, **true** and **false**). *The mathematical result ... can be paraphrased ... only after the adoption of a peculiar conception of what is a “two-valued semantics”, and in particular of what is, or can be, a “semantics”...*

So, we propose the following definition of semantics.

Definition 4.1.1 (SEMANTICS). Given a set $\mathcal{L}$, a *semantics for $\mathcal{L}$* is a tuple $(\mathbf{M}, \{\models_i\}_{i \in I}, S, \mathcal{P}(\mathcal{L}))$, where

- (i) $\mathbf{M}$ is a set,
- (ii) $I \neq \emptyset$ and $\models_i \subseteq \mathbf{M} \times \mathcal{P}(\mathcal{L})$ for each $i \in I$,
- (iii) $\emptyset \subsetneq S \subseteq \{(\models_i, \models_j) : (i, j) \in I \times I\}$.

If, in particular, $I = \{1, \dots, n\}$ with $n \in \mathbb{N} \setminus \{0\}$, instead of writing $(\mathbf{M}, \{\models_i\}_{i \in I}, S, \mathcal{P}(\mathcal{L}))$ we write $(\mathbf{M}, \models_1, \dots, \models_n, S, \mathcal{P}(\mathcal{L}))$. If $(m, \Gamma) \in \models_i$, we write $m \models_i \Gamma$. Given a semantics $\mathfrak{S} = (\mathbf{M}, \{\models_i\}_{i \in I}, S, \mathcal{P}(\mathcal{L}))$ the *entailment relation induced by $\mathfrak{S}$* , $\vdash_{\mathfrak{S}} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ is defined as follows.

$$\Gamma \vdash_{\mathfrak{S}} \alpha \iff \text{for all } (\models_i, \models_j) \in S \text{ and } m \in \mathbf{M}, m \models_i \Gamma \text{ implies that } m \models_j \alpha$$

A logical structure $(\mathcal{L}, \vdash)$ (respectively, $(\mathcal{L}, W)$) is said to be *adequate with respect to a semantics $\mathfrak{S}$* if $\vdash = \vdash_{\mathfrak{S}}$ (respectively, $W = W_{\mathfrak{S}}$; where $W_{\mathfrak{S}}$ is the consequence operator

induced by $\vdash_{\mathfrak{S}}$). A logical structure is said to have an *adequate semantics* if there exists a semantics $\mathfrak{S}$ for $\mathcal{L}$ such that $\vdash = \vdash_{\mathfrak{S}}$ (respectively, $W = W_{\mathfrak{S}}$).

Remark 4.1.2. Let $(\mathcal{L}, W_{\mathfrak{S}})$ be a logical structure induced by the entailment relation corresponding to the semantics $\mathfrak{S} = (\mathbf{M}, \{\models_i\}_{i \in I}, S, \mathcal{P}(\mathcal{L}))$ for $\mathcal{L}$ such that $\mathbf{M} = \emptyset$. If there exists $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\alpha \notin W_{\mathfrak{S}}(\Gamma)$, then there exists $m \in \mathbf{M}$ such that $m \models_i \Gamma$ but $m \not\models_j \{\alpha\}$. However, since $\mathbf{M} = \emptyset$, $W_{\mathfrak{S}}(\Gamma) = \mathcal{L}$ for all $\Gamma \subseteq \mathcal{L}$.

Example 4.1.3. Consider the following semantics for a nonempty set $\mathcal{L}$, $\mathfrak{B}_{\mathcal{V}} = (\mathcal{V}, \models, S, \mathcal{P}(\mathcal{L}))$, where

- (i) $\emptyset \subsetneq \mathcal{V} \subseteq \{0, 1\}^{\mathcal{L}}$,
- (ii) $\models \subseteq \mathcal{V} \times \mathcal{P}(\mathcal{L})$,
- (iii) for all $v \in \mathcal{V}$, $\Gamma \subseteq \mathcal{L}$, $v \models \Gamma$ iff $v(\Gamma) \subseteq \{1\}$,
- (iv) $S = \{(\models, \models)\}$.

Then, one can show that $(\mathcal{L}, \vdash_{\mathfrak{B}_{\mathcal{V}}})$ is of Tarski-type. In fact, given a Tarski-type logical structure $(\mathcal{L}, \vdash)$, one can prove that $\vdash = \vdash_{\mathfrak{B}_{\mathcal{S}}}$, where $\mathcal{S} = \{\chi_{W \vdash (\Sigma)} \in \{0, 1\}^{\mathcal{L}} : \Sigma \subseteq \mathcal{L}\}$ (see [Theorem 2.3.4](#) for details).

Example 4.1.4. Let $\mathcal{L}$ be a set, $\emptyset \subsetneq \mathcal{V} \subseteq \{0, 1\}^{\mathcal{L}}$, and $(P, \preceq)$ is a *complete lattice* (i.e., a poset where every subset has a supremum as well as an infimum). Consider the following semantics for $\mathcal{L}$, $\mathfrak{P}_{\mathcal{V}} = (\mathcal{L}, \{\models_v\}_{v \in \mathcal{V}}, S, \mathcal{P}(\mathcal{L}))$, where

- (i) for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, $v \in \mathcal{V}$, $\alpha \models_v \Gamma$ iff $\inf(v(\Gamma)) \leq v(\alpha)$ (here $\inf(v(\Gamma))$ denotes the infimum of $v(\Gamma)$),
- (ii) $S = \{(\models_v, \models_v) : v \in \mathcal{V}\}$.

Then, one can easily show that $(\mathcal{L}, \vdash_{\mathfrak{P}_{\mathcal{V}}})$ is of Tarski-type. In fact, given a Tarski-type logical structure $(\mathcal{L}, \vdash)$, one can prove that $\vdash = \vdash_{\mathfrak{P}_{\mathcal{S}}}$ where $P = \{0, 1\}$, $\preceq = \{(0, 0), (0, 1), (1, 1)\}$, and $\mathcal{S}$ is as defined in the previous example.

Remark 4.1.5. Given a set $\mathcal{L}$, and a semantics $\mathfrak{S}$ for $\mathcal{L}$, the entailment relation $\vdash_{\mathfrak{S}}$ as we have defined above, is not the only possible one. Below we mention two other possibilities.

$$\Gamma \vdash_{\mathfrak{S}} \alpha \iff \text{for all } (\models_i, \models_j) \in S \text{ and } (m, n) \in \mathbf{M} \times \mathbf{M}, m \models_i \Gamma \text{ implies that } n \models_j \{\alpha\}$$

or,

$$\Gamma \vdash_{\mathfrak{S}} \alpha \iff \text{there exists } (\models_i, \models_j) \in S \text{ such that for some } m \in \mathbf{M}, m \models_i \Gamma \text{ implies that } m \models_j \{\alpha\}$$

However, every logical structure is adequate with respect to some semantics, as the following theorem shows.

Theorem 4.1.6 (CANONICAL SEMANTICS OF A LOGICAL STRUCTURE). *Let $(\mathcal{L}, W)$ be a logical structure. Consider the following semantics for $\mathcal{L}$, $\mathfrak{S} = (\mathcal{P}(\mathcal{L}), \models_1, \models_2, S, \mathcal{P}(\mathcal{L}))$ where $S = \{(\models_1, \models_2)\}$, and for all $\Gamma \cup \Sigma \subseteq \mathcal{L}$ $\models_1$ and $\models_2$ are defined respectively as follows:*

$$\Sigma \models_1 \Gamma \text{ iff } \Sigma = \Gamma$$

and

$$\Sigma \models_2 \Gamma \text{ iff } \Gamma \subseteq W(\Sigma).$$

Then, $W = W_{\mathfrak{S}}$.

Proof. Suppose the contrary, i.e., suppose that $W \neq W_{\mathfrak{S}}$. Then, for some $\Gamma \subseteq \mathcal{L}$, $W(\Gamma) \neq W_{\mathfrak{S}}(\Gamma)$. If possible, let $W(\Gamma) \not\subseteq W_{\mathfrak{S}}(\Gamma)$. Then there exists $\alpha \in W(\Gamma)$ such that $\alpha \notin W_{\mathfrak{S}}(\Gamma)$. So, there exists $\Sigma \subseteq \mathcal{L}$ such that $\Sigma \models_1 \Gamma$ but $\Sigma \not\models_2 \{\alpha\}$. Now, $\Sigma \models_1 \Gamma$ implies that $\Sigma = \Gamma$. But then, since $\Sigma \not\models_2 \{\alpha\}$, we must have $\{\alpha\} \not\subseteq W(\Gamma)$, i.e., $\alpha \notin W(\Gamma)$ – a contradiction. Thus, $W_{\mathfrak{S}}(\Gamma) \not\subseteq W(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$.

So, there exists $\alpha \in W_{\mathfrak{S}}(\Gamma)$ such that $\alpha \notin W(\Gamma)$. Choose this Γ , and note that $\Gamma \models_1 \Gamma$. Therefore, since $\alpha \in W_{\mathfrak{S}}(\Gamma)$, we must have $\Gamma \models_2 \{\alpha\}$, i.e., $\alpha \in W(\Gamma)$ – a contradiction. Thus, $W_{\mathfrak{S}}(\Gamma) = W(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$, i.e. $W = W_{\mathfrak{S}}$. $\square$

Remark 4.1.7. Generally, the notion of semantics is not defined explicitly. A notable exception³ is [15] where a semantics is defined as a triple $\langle \mathfrak{F}; \mathbf{U}; \text{mod} \rangle$, where

- $\mathfrak{F}$ is a set, called the *domain* of the semantics,
- $\mathbf{U}$ is a set, called the *universe* of the semantics,
- mod is a map from $\mathfrak{F}$ to $\mathcal{P}(\mathbf{U})$:

$$\begin{aligned} \text{mod} : \mathfrak{F} &\longrightarrow \mathcal{P}(\mathbf{U}), \\ F &\mapsto \text{mod}(F) \end{aligned}$$

which can be extended to a function $\mathbf{mod} : \mathcal{P}(\mathfrak{F}) \longrightarrow \mathcal{P}(\mathbf{U})$ as follows:

$$\begin{aligned} \mathbf{mod} : \mathcal{P}(\mathfrak{F}) &\longrightarrow \mathcal{P}(\mathbf{U}), \\ T &\mapsto \mathbf{mod}(T) = \bigcap_{F \in T} \text{mod}(F) \end{aligned}$$

Any semantics $\langle \mathfrak{F}; \mathbf{U}; \text{mod} \rangle$ induces a semantics (in the sense of Definition 4.1.1) $(\mathbf{M}, \models, S, \mathcal{P}(\mathfrak{F}))$ where $\mathbf{M} = \mathcal{P}(\mathbf{U})$, $S = \{(\models, \models)\}$ and for all $\Sigma \subseteq \mathfrak{F}$, $\Gamma \subseteq \mathbf{U}$, $\Gamma \models \Sigma$ iff $\mathbf{mod}(\Sigma) = \Gamma$.

By Theorem 4.1.6, every logical structure has an adequate semantics. Even though this result might be of some theoretical interest, the construction of the semantics may seem ad hoc. We, therefore, introduce several less abstract notions of semantics and investigate their inter-relations in some detail. We will return to a deeper analysis of results like this in Section 4.4 when we discuss the relationships between different formulations of the principle of bivalence and **ST**.

³Equivalent notions under the names of *model-based structure* and *abstract model logic* have been defined in [105] and [58] respectively.

Definition 4.1.8 (ALGEBRAICALLY κ -VALUED SEMANTICS). Given an algebra $\mathcal{L}$, a semantics $\mathfrak{S} = (\mathbf{M}, \{\models_i\}_{i \in I}, S, \mathcal{P}(\mathcal{L}))$ for $\mathcal{L}$, and a cardinal κ , $\mathfrak{S}$ is said to be *algebraically κ -valued* if there exists an algebra $\mathcal{A}$ of the same similarity type as $\mathcal{L}$, with universe A such that the following properties hold.

- (i) $\mathbf{M}$ is the set of all homomorphisms from $\mathcal{L}$ to $\mathcal{A}$.
- (ii) $|A| = \kappa$.
- (iii) $\bigcup_{m \in \mathbf{M}} m(\mathcal{L}) = A$.
- (iv) For all $m \in \mathbf{M}, i \in I$ and $\Gamma \subseteq \mathcal{L}$,

$$m \models_i \Gamma \text{ iff } m(\Gamma) \subseteq D_i$$

where $\emptyset \subsetneq D_i$ for each $i \in I$ and $\bigcup_{i \in I} D_i \subsetneq A$.

Given an algebra $\mathcal{L}$, the class of all algebraically κ -valued semantics for $\mathcal{L}$ will be denoted by $\mathcal{A}_\kappa(\mathcal{L})$.

Definition 4.1.9 (ALGEBRAICALLY κ -VALUED LOGIC). Given a cardinal κ , a logic $(\mathcal{L}, \vdash)$ (i.e., where $\mathcal{L}$ is an algebra) is said to be *algebraically κ -valued* if there exists an $\mathfrak{S} = (\mathbf{M}, \{\models_i\}_{i \in I}, S, \mathcal{P}(\mathcal{L})) \in \mathcal{A}_\kappa(\mathcal{L})$ such that the following statements hold.

- (i) $\vdash = \vdash_{\mathfrak{S}}$.
- (ii) $\kappa > 1$
- (iii) For all algebraically μ -valued semantics $\mathfrak{T} = (\mathbf{M}', \{\models_i\}_{i \in I}, S, \mathcal{P}(\mathcal{L}))$ with $\mu < \kappa$, $\vdash \neq \vdash_{\mathfrak{T}}$.

Remark 4.1.10. In other words, a logic $(\mathcal{L}, \vdash)$ is said to be algebraically κ -valued if it is adequate with respect to an algebraically κ -valued semantics, where

$$\kappa = \min \{ \lambda : \lambda \text{ is a cardinal, and there exists } \mathfrak{S} \in \mathcal{A}_\lambda(\mathcal{L}) \text{ such that } \vdash = \vdash_{\mathfrak{S}} \} \quad (4.1)$$

This can be seen as a generalisation of the usual understanding of κ -valued logics (see, e.g., [18] and [82]).

Definition 4.1.11 (FUNCTIONALLY κ -VALUED SEMANTICS). Given a set $\mathcal{L}$, a semantics $\mathfrak{S} = (\mathbf{M}, \{\models_i\}_{i \in I}, S, \mathcal{P}(\mathcal{L}))$ for $\mathcal{L}$, and a cardinal κ , $\mathfrak{S}$ is said to be *functionally κ -valued* if there exists a nonempty set A such that the following properties hold.

- (i) $\emptyset \subsetneq \mathbf{M} \subseteq A^{\mathcal{L}}$ for some nonempty set A .
- (ii) $|A| = \kappa$.
- (iii) $\bigcup_{m \in \mathbf{M}} m(\mathcal{L}) = A$.

(iv) For all $m \in \mathbf{M}, i \in I$ and $\Gamma \subseteq \mathcal{L}$,

$$m \models_i \Gamma \text{ iff } m(\Gamma) \subseteq D_i$$

where $\emptyset \subsetneq D_i$ for each $i \in I$ and $\bigcup_{i \in I} D_i \subsetneq A$.

Given a set $\mathcal{L}$, the class of all functionally κ -valued semantics for $\mathcal{L}$ will be denoted by $\mathcal{F}_\kappa(\mathcal{L})$.

Remark 4.1.12. The entailment relation induced by a functionally κ -valued semantics (for a given cardinal κ) is more general than the ones induced by t -, f -, p - and q -entailment relations (see [25] for details).

Definition 4.1.13 (INFERENTIALLY κ -VALUED LOGICAL STRUCTURE). Given a cardinal κ , a logical structure $(\mathcal{L}, \vdash)$ said to be *inferentially κ -valued* if there exists $\mathfrak{S} = (\mathbf{M}, \{\models_i\}_{i \in I}, S, \mathcal{P}(\mathcal{L})) \in \mathcal{F}_\kappa(\mathcal{L})$ such that the following statements hold.

- (i) $\vdash = \vdash_{\mathfrak{S}}$.
- (ii) $\kappa > 1$
- (iii) For all functionally μ -valued semantics $\mathfrak{T} = (\mathbf{M}', \{\models_i\}_{i \in I}, S, \mathcal{P}(\mathcal{L}))$ with $\mu < \kappa$, $\vdash \neq \vdash_{\mathfrak{T}}$.

Definition 4.1.14 (GRANULAR SEMANTICS). A semantics $\mathfrak{S} = (\mathbf{M}, \{\models_i\}_{i \in I}, S, \mathcal{P}(\mathcal{L}))$ for $\mathcal{L}$ is said to be *granular at the i -th component* if for all $\Gamma \subseteq \mathcal{L}, m \in \mathbf{M}$, the following statement holds.

$$m \models_i \Gamma \text{ iff } m \models_i \{\alpha\} \text{ for all } \alpha \in \Gamma$$

If $\mathfrak{S}$ is granular at the i -th component for all $i \in I$, then $\mathfrak{S}$ is said to be *strongly granular for $\mathcal{L}$* .

Remark 4.1.15. For all cardinal κ , every functionally κ -valued semantics is strongly granular. The converse of this statement forms the basis of a general form of Suszko Reduction, which we discuss in [Section 4.3](#).

Remark 4.1.16. We would like to point out that the definition of inferentially κ -valued logical structures, as we have given in [Definition 4.1.13](#) is *not* the same as Malinowski's treatment of the same. Nevertheless, we have chosen to adopt the same terminology because we think that this definition captures the essential idea: a logical structure $(\mathcal{L}, \vdash)$ is inferentially κ -valued if it is adequate with respect to a functionally κ -valued semantics, where

$$\kappa = \min \{ \lambda : \lambda \text{ is a cardinal, and there exists } \mathfrak{S} \in \mathcal{F}_\lambda(\mathcal{L}) \text{ such that } \vdash = \vdash_{\mathfrak{S}} \} \quad (4.2)$$

Remark 4.1.17. Let $(\mathcal{L}, \vdash)$ be a logic that is algebraically κ -valued. Suppose that the corresponding logical structure $(\mathcal{L}, \vdash)$ is inferentially λ -valued, and is adequate with respect to a functionally κ -valued semantics. Consequently, by the previous remark, it follows that $\lambda \leq \kappa$. In other words, in general, the logical structure corresponding to algebraically κ -valued logic is inferentially λ -valued for $\lambda \leq \kappa$.

Remark 4.1.18. It is possible to introduce more general notions of ‘many-valuedness’ by considering objects more general than functions in (i) of [Definition 4.1.11](#). For example, one may consider the set of all *relations*, instead of the set of all *functions* and obtain the corresponding notion of many-valuedness. The upshot of this discussion is that both the standard as well as Malinowski’s notion of inferential many-valuedness are just two particular conceptions of many possible ones. However, since in this chapter, we will be focusing only on inferential many-valuedness, we don’t pursue this issue in more detail.

4.2 Monotonic Logical Structures: A Case Study

In this section, we consider monotonic logical structures. We show that the class of such logical structures can be split into four subclasses, each functionally 3-valued at least. Later, in [Subsection 4.3.1](#), we show how each can be given a bivalent characterisation.

Interestingly enough, the entailment relations induced by strongly granular semantics give rise to monotonic logical structures. This is the content of the following theorem.

Theorem 4.2.1 (REPRESENTATION THEOREM FOR MONOTONIC LOGICAL STRUCTURES (PART I)). *Let $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S, \mathcal{P}(\mathcal{L}))$, where $S = \{(\models_1, \models_2)\}$, be a strongly granular semantics for $\mathcal{L}$. Then $W_{\mathfrak{S}}$ is monotonic.*

Proof. If possible, suppose $\Gamma \cup \Sigma \subseteq \mathcal{L}$ be such that $\Gamma \subseteq \Sigma$ but $W_{\mathfrak{S}}(\Gamma) \not\subseteq W_{\mathfrak{S}}(\Sigma)$. Hence, there exists $\alpha \in W_{\mathfrak{S}}(\Gamma)$ such that $\alpha \notin W_{\mathfrak{S}}(\Sigma)$. Since $\alpha \notin W_{\mathfrak{S}}(\Sigma)$, there exists $m \in \mathbf{M}$ such that $m \models_1 \Sigma$ but $m \not\models_2 \{\alpha\}$. However, since $\Gamma \subseteq \Sigma$, $m \models_1 \Gamma$ and $\mathfrak{S}$ is strongly granular – and hence granular at the 1st component – it follows that $m \models_1 \Gamma$ as well. But then, since $\alpha \in W_{\mathfrak{S}}(\Gamma)$, we must have $m \models_2 \{\alpha\}$ – a contradiction. Thus, $W_{\mathfrak{S}}(\Gamma) \subseteq W_{\mathfrak{S}}(\Sigma)$. Hence, $W_{\mathfrak{S}}$ is monotonic. $\square$

Every logical structure induced by a monotonic consequence operator has a functionally 4-valued characterisation, as we show below.⁴

Theorem 4.2.2 (REPRESENTATION THEOREM FOR MONOTONIC LOGICAL STRUCTURES (PART II)). *Let $(\mathcal{L}, W)$ be a monotonic logical structure and $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S, \mathcal{P}(\mathcal{L}))$ is a semantics for $\mathcal{L}$ satisfying the following properties.*

- $\mathbf{M} = \{\mu_{\Sigma} \in \{0, 1, 2, 3\}^{\mathcal{L}} : \Sigma \subseteq \mathcal{L}\}$ where

$$\mu_{\Sigma}(\beta) = \begin{cases} 0 & \text{if } \beta \notin W(\Sigma) \cup \Sigma \\ 1 & \text{if } \beta \in \Sigma \cap W(\Sigma) \\ 2 & \text{if } \beta \in \Sigma \setminus W(\Sigma) \\ 3 & \text{if } \beta \in W(\Sigma) \setminus \Sigma \end{cases}$$

⁴An equivalent version of the same result, in a different setting, can be obtained by combining [\[25, Theorem 2 and 11\]](#).

- For all $m \in \mathbf{M}, \Gamma \subseteq \mathcal{L}$,

$$m \models_1 \Gamma \text{ iff } m(\Gamma) \subseteq \{1, 2\}$$

$$m \models_2 \Gamma \text{ iff } m(\Gamma) \subseteq \{1, 3\}$$

- $S = \{(\models_1, \models_2)\}$.

Then, $W = W_{\mathfrak{S}}$.

Proof. Suppose the contrary, i.e., suppose that $W \neq W_{\mathfrak{S}}$. Then there exists $\Gamma \subseteq \mathcal{L}$ such that $W(\Gamma) \neq W_{\mathfrak{S}}(\Gamma)$. Therefore, $W(\Gamma) \not\subseteq W_{\mathfrak{S}}(\Gamma)$ or $W_{\mathfrak{S}}(\Gamma) \not\subseteq W(\Gamma)$.

If $W(\Gamma) \not\subseteq W_{\mathfrak{S}}(\Gamma)$, then there exists $\alpha \in W(\Gamma)$ such that $\alpha \notin W_{\mathfrak{S}}(\Gamma)$. So, there exists $\mu_{\Sigma} \in \mathbf{M}$ such that $\mu_{\Sigma} \models_1 \Gamma$ but $\mu_{\Sigma} \not\models_2 \{\alpha\}$. Now, $\mu_{\Sigma} \models_1 \Gamma$ implies that $\mu_{\Sigma}(\Gamma) \subseteq \{1, 2\}$, or, equivalently $\Gamma \subseteq \Sigma$. Then, since W is monotonic, we have $W(\Gamma) \subseteq W(\Sigma)$. Consequently, $\alpha \in W(\Gamma)$ implies that $\alpha \in W(\Sigma)$. But, $\alpha \in W(\Sigma)$ is equivalent to requiring that $\mu_{\Sigma}(\alpha) \in \{1, 3\}$, i.e., $\mu_{\Sigma} \models_2 \{\alpha\}$ – a contradiction. Consequently, $W(\Gamma) \subseteq W_{\mathfrak{S}}(\Gamma)$.

If, on the other hand, $W_{\mathfrak{S}}(\Gamma) \not\subseteq W(\Gamma)$, then there exists $\alpha \in W_{\mathfrak{S}}(\Gamma)$ such that $\alpha \notin W(\Gamma)$. Now, $\alpha \notin W(\Gamma)$ is equivalent to saying that $\mu_{\Gamma} \not\models_2 \{\alpha\}$. However, since $\mu_{\Gamma} \models_1 \Gamma$ and $\alpha \in W_{\mathfrak{S}}(\Gamma)$, we must have $\mu_{\Gamma} \models_2 \{\alpha\}$ – a contradiction. Consequently, $W_{\mathfrak{S}}(\Gamma) \subseteq W(\Gamma)$ as well. This, however, is contrary to our assumption that $W \neq W_{\mathfrak{S}}$. Thus, $W = W_{\mathfrak{S}}$. $\square$

Remark 4.2.3. The above result can be rephrased as saying that every monotonic logical structure has an adequate functionally 4-valued semantics. This can be seen by considering $A = \{0, 1, 2, 3\}$, $\kappa = 4$, $D_1 = \{1, 2\}$ and $D_2 = \{1, 3\}$ in [Definition 4.1.11](#).

We now show that every monotonic logical structure is induced by a q -, p -, s_{κ} - or r_{κ} -consequence operator (the last two consequence operators will be defined respectively in [Subsection 4.2.3](#) and [Subsection 4.2.3](#)).

4.2.1 q -consequence Operators

As mentioned earlier, q -consequence operators were introduced by Malinowski to show that **ST** does not apply, in general, to the logical structures induced by these operators, provided ‘logical n -valuedness’ is understood as ‘inferential n -valuedness’. In this section, we delve into a detailed study of q -consequence operators and show that every logical structure induced by q -consequence operator has an adequate functionally 3-valued semantics.

Definition 4.2.4 (q -CONSEQUENCE OPERATOR). A q -consequence operator on $\mathcal{L}$ is a function $W : \mathcal{P}(\mathcal{L}) \rightarrow \mathcal{P}(\mathcal{L})$ satisfying the following properties.

- (i) For all $\Gamma \cup \Sigma \subseteq \mathcal{L}$, if $\Gamma \subseteq \Sigma$ then $W(\Gamma) \subseteq W(\Sigma)$. (Monotonicity)
- (ii) For all $\Gamma \subseteq \mathcal{L}$ $W(W(\Gamma) \cup \Gamma) = W(\Gamma)$. (Quasi-closure)

A logical structure $(\mathcal{L}, W)$ is said to be of q -type if W is a q -consequence operator on $\mathcal{L}$.

We now prove a characterisation theorem for q -type logical structures using downward q -closed sets, defined below. One might compare this result with [Theorem 2.3.9](#), and downward q -closed sets with strongly closed sets (see [Definition 2.3.7](#)).

Definition 4.2.5 (DOWNWARD q -CLOSED SET). Let $(\mathcal{L}, W)$ be a logical structure and $\Delta \subseteq \mathcal{L}$. The set Δ is said to be *downward q -closed* if for all $\Sigma \subseteq W^\infty(\Delta)$, $W(\Sigma) \subseteq W(\Delta)$. Here $W^0(\Delta) = \Delta$ and $W^\infty(\Delta) = \bigcup_{i=0}^{\infty} W^i(\Delta)$.

Theorem 4.2.6 (CHARACTERISATION OF q -TYPE). *Let $(\mathcal{L}, W)$ be a logical structure. Then, the following statements are equivalent.*

- (1) W is a q -consequence operator.
- (2) W is monotonic, and for all $\Gamma \subseteq \mathcal{L}$, $W(W^\infty(\Gamma)) = W(\Gamma)$. (For all $\Gamma \subseteq \mathcal{L}$, $W^0(\Gamma) = \Gamma$.)
- (3) For every $\Delta \subseteq \mathcal{L}$, Δ is downward q -closed.
- (4) For all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, if $\alpha \notin W(\Gamma)$ then there exists a downward q -closed $\Sigma \supseteq \Gamma$ such that $\alpha \notin W(\Sigma)$.
- (5) For all $\Gamma \cup \Sigma \subseteq \mathcal{L}$, $\Gamma \subseteq W(\Sigma) \cup \Sigma$ implies that $W(\Gamma) \subseteq W(\Sigma)$.

Proof. (1) $\implies$ (2): Since W is a q -consequence operator, it is monotonic. So, it suffices to show that $W(W^\infty(\Gamma)) = W(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$. Let $\Gamma \subseteq \mathcal{L}$. Then,

$$\begin{aligned} W^2(\Gamma) &= W(W(\Gamma)) \\ &\subseteq W(W(\Gamma) \cup \Gamma) && \text{(since } W \text{ is monotonic)} \\ &= W(\Gamma) && \text{(since } W \text{ satisfies quasi-closure)} \end{aligned}$$

Moreover, if for some $i \geq 1$ $W^i(\Gamma) \subseteq W(\Gamma)$ then $W^{i+1}(\Gamma) = W(W^i(\Gamma)) \subseteq W(W(\Gamma)) \subseteq W(\Gamma)$. Thus, for all $i \geq 1$, $W^i(\Gamma) \subseteq W(\Gamma)$. This implies $\bigcup_{i=1}^{\infty} W^i(\Gamma) \subseteq W(\Gamma)$, and hence

$$W^\infty(\Gamma) \subseteq W(\Gamma) \cup \Gamma \implies \overbrace{W(W^\infty(\Gamma))}^{\text{since } W \text{ is monotonic}} \subseteq \underbrace{W(W(\Gamma) \cup \Gamma)}_{\text{since } W \text{ satisfies quasi-closure}} = W(\Gamma)$$

(2) $\implies$ (1): Since W is monotonic, it suffices to show that W satisfies quasi-closure. For this, let $\Gamma \subseteq \mathcal{L}$ and consider $W(W(\Gamma) \cup \Gamma)$. Notice that $\Gamma \subseteq W(\Gamma) \cup \Gamma$ and $W(\Gamma) \cup \Gamma \subseteq W^\infty(\Gamma)$. Consequently,

$$\Gamma \subseteq W(\Gamma) \cup \Gamma \subseteq W^\infty(\Gamma) \implies \overbrace{W(\Gamma) \subseteq W(W(\Gamma) \cup \Gamma)}^{\text{since } W \text{ is monotonic}} \subseteq \underbrace{W(W^\infty(\Gamma))}_{\text{by (2)}} = W(\Gamma)$$

So, W satisfies quasi-closure, and hence is a q -consequence operator. Therefore, (1) holds.

(2) $\implies$ (3): Let $\Delta \subseteq \mathcal{L}$ and $\Sigma \subseteq W^\infty(\Delta)$. Then, by monotonicity of W and by (2), we have $W(\Sigma) \subseteq W(W^\infty(\Delta)) = W(\Delta)$. Thus, (3) holds.

(3) $\implies$ (2): Suppose that W is not monotonic. So, there exists $\Gamma \cup \Sigma \subseteq \mathcal{L}$ with $\Gamma \subseteq \Sigma$ such that $W(\Gamma) \not\subseteq W(\Sigma)$. This implies that there exists $\alpha \in W(\Gamma)$ such that $\alpha \notin W(\Sigma)$. Since $\Gamma \subseteq \Sigma \subseteq W^\infty(\Sigma)$ and Σ itself is downward q -closed by (3), it follows that $W(\Gamma) \subseteq W(\Sigma)$. Hence, $\alpha \in W(\Gamma)$ implies that $\alpha \in W(\Sigma)$ as well – a contradiction. Therefore, W is monotonic.

Now, let $\Gamma \subseteq \mathcal{L}$. Since $\Gamma \subseteq W^\infty(\Gamma)$, $W(\Gamma) \subseteq W(W^\infty(\Gamma))$ by monotonicity of W . Since Γ is downward q -closed by (3) and $W^\infty(\Gamma) \subseteq W^\infty(\Gamma)$, trivially, $W(W^\infty(\Gamma)) \subseteq W(\Gamma)$. Hence (2) holds.

(2) $\implies$ (4): Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ be such that $\alpha \notin W(\Gamma)$. Clearly $\Gamma \subseteq W^\infty(\Gamma)$. By (2), $W(W^\infty(\Gamma)) = W(\Gamma)$. So, $\alpha \notin W(W^\infty(\Gamma))$. Thus, it suffices to show that $W^\infty(\Gamma)$ is downward q -closed.

Now, it follows that for all $i \geq 1$ we have, $W^i(W^\infty(\Gamma)) = W^i(\Gamma)$. Hence, $W^\infty(W^\infty(\Gamma)) = W^\infty(\Gamma)$.

Let $\Lambda \subseteq W^\infty(W^\infty(\Gamma))$. Then $\Lambda \subseteq W^\infty(\Gamma)$. Now, since W is monotonic, $W(\Lambda) \subseteq W(W^\infty(\Gamma))$ for all $\Lambda \subseteq W^\infty(\Gamma)$. This implies that $W^\infty(\Gamma)$ is downward q -closed. This proves (4).

(4) $\implies$ (2): We first show that W is monotonic. For this, suppose that there exists $\Gamma \cup \Sigma \subseteq \mathcal{L}$ with $\Gamma \subseteq \Sigma$ such that $W(\Gamma) \not\subseteq W(\Sigma)$. This implies that there exists $\alpha \in W(\Gamma)$ such that $\alpha \notin W(\Sigma)$. Consequently, by (4), there exists $\Delta \supseteq \Sigma$ such that Δ is downward q -closed and $\alpha \notin W(\Delta)$. Now, since $\Gamma \subseteq \Sigma \subseteq \Delta \subseteq W^\infty(\Delta)$, and Δ is downward q -closed, $W(\Gamma) \subseteq W(\Delta)$. So, since $\alpha \in W(\Gamma)$, $\alpha \in W(\Delta)$ as well – a contradiction. Hence, W is monotonic.

We now show that W satisfies quasi-closure as well. Suppose the contrary, i.e., $W(W^\infty(\Gamma)) \neq W(\Gamma)$. Then we must have $W(W^\infty(\Gamma)) \not\subseteq W(\Gamma)$ for some $\Gamma \subseteq \mathcal{L}$. So, there exists $\alpha \in W(W^\infty(\Gamma))$ such that $\alpha \notin W(\Gamma)$. Then, by (4), there exists a downward q -closed $\Sigma \supseteq \Gamma$ such that $\alpha \notin W(\Sigma)$. Since Σ is downward q -closed and $W^\infty(\Sigma) \subseteq W^\infty(\Sigma)$, it follows that $\alpha \notin W(W^\infty(\Sigma))$. Now, as W is monotonic, $\Gamma \subseteq \Sigma$ implies that $W^\infty(\Gamma) \subseteq W^\infty(\Sigma)$, and hence $W(W^\infty(\Gamma)) \subseteq W(W^\infty(\Sigma))$. Thus, $\alpha \notin W(W^\infty(\Gamma))$ as well – a contradiction. Hence, $W(W^\infty(\Gamma)) \subseteq W(\Gamma)$. This proves (2).

(1) $\implies$ (5): Let $\Gamma \cup \Sigma \subseteq \mathcal{L}$ be such that $\Gamma \subseteq W(\Sigma) \cup \Sigma$. Then,

$$\begin{aligned} \Gamma \subseteq W(\Sigma) \cup \Sigma &\implies W(\Gamma) \subseteq W(W(\Sigma) \cup \Sigma) && \text{(by monotonicity)} \\ &\implies W(\Gamma) \subseteq W(\Sigma) && \text{(by quasi-closure)} \end{aligned}$$

Thus (5) holds.

(5) $\implies$ (1): Let $\Gamma \cup \Sigma \subseteq \mathcal{L}$ be such that $\Gamma \subseteq \Sigma$. Then, $\Gamma \subseteq W(\Sigma) \cup \Sigma$ which implies that $W(\Gamma) \subseteq W(\Sigma)$.

Hence, W is monotonic. Now, for any $\Gamma \subseteq \mathcal{L}$, $\Gamma \subseteq W(\Gamma) \cup \Gamma$, and hence by monotonicity of W , $W(\Gamma) \subseteq W(W(\Gamma) \cup \Gamma)$. The reverse inclusion is immediate since, by (5), $W(\Gamma) \cup \Gamma \subseteq$

$W(\Gamma) \cup \Gamma$ implies that $W(W(\Gamma) \cup \Gamma) \subseteq W(\Gamma)$. Hence, W is a q -consequence operator. This proves (1). $\square$

Theorem 4.2.7 (REPRESENTATION THEOREM FOR q -TYPE (PART I)). *Let $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S_q, \mathcal{P}(\mathcal{L}))$ be a strongly granular semantics for $\mathcal{L}$ such that $\models_2 \subseteq \models_1$ and $S_q = \{(\models_1, \models_2)\}$. Then $W_{\mathfrak{S}}$ is a q -consequence operator.*

Proof. Monotonicity of $W_{\mathfrak{S}}$ follows from [Theorem 4.2.1](#). Hence, it suffices to show that $W_{\mathfrak{S}}$ satisfies the quasi-closure. Suppose the contrary, i.e., suppose that for some $\Gamma \subseteq \mathcal{L}$, $W_{\mathfrak{S}}(W_{\mathfrak{S}}(\Gamma) \cup \Gamma) \neq W_{\mathfrak{S}}(\Gamma)$. Now, $W_{\mathfrak{S}}(\Gamma) \subseteq W_{\mathfrak{S}}(W_{\mathfrak{S}}(\Gamma) \cup \Gamma)$ since $W_{\mathfrak{S}}$ is monotonic. Thus, $W_{\mathfrak{S}}(W_{\mathfrak{S}}(\Gamma) \cup \Gamma) \not\subseteq W_{\mathfrak{S}}(\Gamma)$. So, there exists $\alpha \in W_{\mathfrak{S}}(W_{\mathfrak{S}}(\Gamma) \cup \Gamma)$ such that $\alpha \notin W_{\mathfrak{S}}(\Gamma)$. So, there exists $m \in \mathbf{M}$ such that $m \models_1 \Gamma$ but $m \not\models_2 \{\alpha\}$.

Let $\beta \in W_{\mathfrak{S}}(\Gamma)$. Then, since $m \models_1 \Gamma$, we have $m \models_2 \{\beta\}$. Furthermore, since $\models_2 \subseteq \models_1$ and $m \models_2 \{\beta\}$, it follows that $m \models_1 \{\beta\}$ as well. Thus, $m \models_1 \{\beta\}$. Now, since $m \models_1 \Gamma$ and $\mathfrak{S}$ is strongly granular, $m \models_1 \{\gamma\}$ for all $\gamma \in \Gamma$. Hence $m \models_1 \{\sigma\}$ for all $\sigma \in W_{\mathfrak{S}}(\Gamma) \cup \Gamma$. Again, since $\mathfrak{S}$ is strongly granular, $m \models_1 W_{\mathfrak{S}}(\Gamma) \cup \Gamma$. But then, since $\alpha \in W_{\mathfrak{S}}(W_{\mathfrak{S}}(\Gamma) \cup \Gamma)$, we must have $m \models_2 \{\alpha\}$ – a contradiction. Thus, $W_{\mathfrak{S}}(W_{\mathfrak{S}}(\Gamma) \cup \Gamma) \subseteq W_{\mathfrak{S}}(\Gamma)$. Hence, $W_{\mathfrak{S}}$ satisfies the quasi-closure property as well, and therefore is a q -consequence operator. $\square$

Let $(\mathcal{L}, W)$ be a logic where $\mathcal{L}$ is an algebra with universe $\mathcal{L}$. It is said to satisfy *structurality* if for all homomorphisms $\sigma : \mathcal{L} \rightarrow \mathcal{L}$ and all $\Gamma \subseteq \mathcal{L}$, $\sigma(W(\Gamma)) \subseteq W(\sigma(\Gamma))$. If a logic satisfies structurality, we call it *structural* (see [\[52, 54\]](#) for more details and context).

The proof that every structural q -type logic has a 3-valued description has already been given, e.g., in [\[80\]](#). However, a similar argument works for arbitrary q -type logical structures as well. Below we give a proof of the same, to keep the presentation self-contained.

Theorem 4.2.8 (REPRESENTATION THEOREM FOR q -TYPE (PART II)). *Let $(\mathcal{L}, W)$ be a q -type logical structure. Let $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S_q, \mathcal{P}(\mathcal{L}))$ be a semantics for $\mathcal{L}$ satisfying the following properties.*

- $\mathbf{M} = \{\mu_{\Sigma} \in \{0, 1, 2\}^{\mathcal{L}} : \Sigma \subseteq \mathcal{L}\}$, where

$$\mu_{\Sigma}(\beta) = \begin{cases} 0 & \text{if } \beta \notin W(\Sigma) \cup \Sigma \\ 1 & \text{if } \beta \in W(\Sigma) \\ 2 & \text{if } \beta \in (W(\Sigma) \cup \Sigma) \setminus W(\Sigma) \end{cases}$$

- For all $m \in \mathbf{M}, \Gamma \subseteq \mathcal{L}$,

$$m \models_1 \Gamma \text{ iff } m(\Gamma) \subseteq \{1, 2\}$$

$$m \models_2 \Gamma \text{ iff } m(\Gamma) \subseteq \{1\}$$

- S_q is as defined in the previous theorem.

Then, $W = W_{\mathfrak{S}}$.

Proof. Suppose, $\Gamma \subseteq \mathcal{L}$ is such that $W(\Gamma) \not\subseteq W_{\mathfrak{S}}(\Gamma)$. Then there exists $\alpha \in W(\Gamma)$ such that $\alpha \notin W_{\mathfrak{S}}(\Gamma)$. Hence, there exists $m \in \mathbf{M}$ such that $m \models_1 \Gamma$ but $m \not\models_2 \{\alpha\}$. Now, there exists $\Sigma \subseteq \mathcal{L}$ such that $m = \mu_{\Sigma}$. Then, $\mu_{\Sigma} \models_1 \Gamma$ implies that $\mu_{\Sigma}(\Gamma) \subseteq \{1, 2\}$, or equivalently $\Gamma \subseteq W(\Sigma) \cup \Sigma$. Since W is a q -consequence operator, it is monotonic. So, $W(\Gamma) \subseteq W(W(\Sigma) \cup \Sigma)$. Moreover, since W satisfies quasi-closure, it follows that $W(W(\Sigma) \cup \Sigma) = W(\Sigma)$. Consequently, $W(\Gamma) \subseteq W(\Sigma)$. Then, as $\alpha \in W(\Gamma)$, $\alpha \in W(\Sigma)$. But then $\mu_{\Sigma}(\alpha) = 1$, i.e., $\mu_{\Sigma} \models_2 \{\alpha\}$, i.e., $m \models_2 \{\alpha\}$ – a contradiction. Hence, $W(\Gamma) \subseteq W_{\mathfrak{S}}(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$.

Next, let $\Gamma \subseteq \mathcal{L}$ and $\alpha \in W_{\mathfrak{S}}(\Gamma)$. Now, $\mu_{\Gamma}(\Gamma) \subseteq \{1, 2\}$, i.e., $\mu_{\Gamma} \models_1 \Gamma$. Therefore, as $\alpha \in W_{\mathfrak{S}}(\Gamma)$, it follows that $\mu_{\Gamma} \models_2 \{\alpha\}$, which implies that $\alpha \in W(\Gamma)$. Thus, $W_{\mathfrak{S}}(\Gamma) \subseteq W(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$. Hence $W = W_{\mathfrak{S}}$. $\square$

Theorem 4.2.9 (REPRESENTATION THEOREM FOR q -TYPE (PART III)). *Let $(\mathcal{L}, W)$ be a logical structure where W is a q -consequence operator. Let $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S, \mathcal{P}(\mathcal{L}))$ where $S = \{(\models_1, \models_2)\}$ is a strongly granular semantics for $\mathcal{L}$. If $W = W_{\mathfrak{S}}$, then $\models_1$ and $\models_2$ satisfy the following property:*

For all $\Gamma \cup \Sigma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\Gamma \subseteq W(\Sigma) \cup \Sigma$, $\pi_1(\models_1 \cap (\mathbf{M} \times \{\Sigma\})) \not\subseteq \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\alpha\}\}))$ implies that $\pi_1(\models_1 \cap (\mathbf{M} \times \{\Gamma\})) \not\subseteq \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\alpha\}\}))$.

Here $\pi_1 : \mathbf{M} \times \mathcal{P}(\mathcal{L}) \rightarrow \mathbf{M}$ is defined as follows: $\pi_1(m, \Delta) = m$ for all $(m, \Delta) \in \mathbf{M} \times \mathcal{P}(\mathcal{L})$.

Proof. Let $\Gamma \cup \Sigma \subseteq \mathcal{L}$ be such that $\Gamma \subseteq W(\Sigma) \cup \Sigma$. Therefore, since W is a q -consequence operator, by Theorem 4.2.6(5) we have $W(\Gamma) \subseteq W(\Sigma)$. Since $W = W_{\mathfrak{S}}$, this implies $W_{\mathfrak{S}}(\Gamma) \subseteq W_{\mathfrak{S}}(\Sigma)$.

Let $\alpha \in \mathcal{L}$ be such that $\pi_1(\models_1 \cap (\mathbf{M} \times \{\Sigma\})) \not\subseteq \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\alpha\}\}))$. Then, for some $m \in \mathbf{M}$, $m \in \pi_1(\models_1 \cap (\mathbf{M} \times \{\Sigma\}))$ but $m \notin \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\alpha\}\}))$. Now, $m \in \pi_1(\models_1 \cap (\mathbf{M} \times \{\Sigma\}))$ implies that $m \models_1 \Sigma$. Since $\mathfrak{S}$ is strongly granular (and hence granular at the 1st component), $m \models_1 \Sigma$ implies that $m \models_1 \Gamma$. Consequently, $m \in \pi_1(\models_1 \cap (\mathbf{M} \times \{\Gamma\}))$ but $m \notin \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\alpha\}\}))$, i.e., $\pi_1(\models_1 \cap (\mathbf{M} \times \{\Gamma\})) \not\subseteq \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\alpha\}\}))$. $\square$

By Theorem 4.2.8, every q -consequence operator has an adequate functionally 3-valued semantics. However, not every q -consequence operator is inferentially 3-valued. For example, every logical structure induced by a reflexive q -consequence operator is of Tarski-type, and it is inferentially 2-valued. However, as shown below, every *non-reflexive* q -consequence operator is inferentially 3-valued.

Theorem 4.2.10 (INFERENTIAL IRREDUCIBILITY THEOREM FOR q -TYPE). *Every logical structure induced by a non-reflexive q -consequence operator is inferentially 3-valued.*

Proof. Let $(\mathcal{L}, W)$ be a logical structure induced by a non-reflexive q -consequence operator. Note that, by Theorem 4.2.8, it is functionally 3-valued. Suppose that it is not inferentially 3-valued. So, there exists some functionally 2-valued semantics for $\mathcal{L}$, say $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S_q, \subseteq, \mathcal{P}(\mathcal{L}))$ such that $W = W_{\mathfrak{S}}$. Since $\mathfrak{S}$ is functionally 2-valued, $\mathbf{M} \subseteq \{0, 1\}^{\mathcal{L}}$.

So, for each $i \in \{1, 2\}$, there exists $\emptyset \subsetneq D_i \subsetneq \{0, 1\}$ such that $D_1 \cup D_2 \subsetneq \{0, 1\}$. Since $\emptyset \subsetneq D_1 \cup D_2 \subsetneq \{0, 1\}$, either $D_1 \cup D_2 = \{1\}$ or $D_1 \cup D_2 = \{0\}$. Without loss of generality, let us assume that $D_1 \cup D_2 = \{1\}$. Consequently, since $\emptyset \subsetneq D_i$ for all $i \in \{1, 2\}$, it follows that $D_1 = \{1\} = D_2$.

Thus, for all $m \in \mathbf{M}, \Gamma \subseteq \mathcal{L}$, $m \models_1 \Gamma$ implies that $m(\Gamma) \subseteq \{1\}$. Similarly, for all $m \in \mathbf{M}, \Gamma \subseteq \mathcal{L}$, $m \models_2 \Gamma$ implies that $m(\Gamma) \subseteq \{1\}$. Therefore, $\models_1 = \models_2$.

We now claim that $W_{\mathfrak{E}}$ is reflexive. To prove this, we first show that $\alpha \in W_{\mathfrak{E}}(\{\alpha\})$ for all $\alpha \in \mathcal{L}$. Suppose the contrary. Then there exists $\alpha \in \mathcal{L}$ such that $\alpha \notin W_{\mathfrak{E}}(\{\alpha\})$. But this would mean that there exists $m \in \mathbf{M}$ such that $m \models_1 \{\alpha\}$ such that $m \not\models_2 \{\alpha\}$ – a contradiction since $\models_1 = \models_2$. Thus $\alpha \in W_{\mathfrak{E}}(\{\alpha\})$ for all $\alpha \in \mathcal{L}$. Now, let $\Gamma \subseteq \mathcal{L}$ and $\beta \in \Gamma$, i.e., $\{\beta\} \subseteq \Gamma$. Since $W = W_{\mathfrak{E}}$ and W is monotonic, $W_{\mathfrak{E}}$ is monotonic as well. This implies that $W_{\mathfrak{E}}(\{\beta\}) \subseteq W_{\mathfrak{E}}(\Gamma)$. Since $\beta \in W_{\mathfrak{E}}(\{\beta\})$, we must have $\beta \in W_{\mathfrak{E}}(\Gamma)$. So $\beta \in W_{\mathfrak{E}}(\Gamma)$ for all $\beta \in \Gamma$. Consequently, $W_{\mathfrak{E}}$, which implies that W is reflexive as well – a contradiction. This establishes the theorem. $\square$

4.2.2 p -consequence Operators

p -consequence operators were introduced by Frankowski in order to formalise the notion of ‘plausible inferences’ in [55] as follows.

The key intuition behind plausibility in the sense of Ajdukiewicz [1] is that it admits reasonings wherein the degree of strength of the conclusion (i.e. the conviction it is true) is smaller than that of the premises. In consequence in the formal description of plausible inference, the standard notions of consequence and rule of inference cannot be countenanced as entirely suitable.

Following is the formal definition of a p -consequence operator.

Definition 4.2.11 (p -CONSEQUENCE OPERATOR). A p -consequence operator on $\mathcal{L}$ is a function $W : \mathcal{P}(\mathcal{L}) \rightarrow \mathcal{P}(\mathcal{L})$ satisfying the following properties.

- (i) For all $\Gamma \subseteq \mathcal{L}$, $\Gamma \subseteq W(\Gamma)$. (Reflexivity)
- (ii) For all $\Gamma \cup \Sigma \subseteq \mathcal{L}$, if $\Gamma \subseteq \Sigma$ then $W(\Gamma) \subseteq W(\Sigma)$. (Monotonicity)

A logical structure $(\mathcal{L}, W)$ is said to be of p -type if W is a p -consequence operator on $\mathcal{L}$.

The following theorem provides another criteria for characterising p -consequence operators, which we will need in a later theorem.

Theorem 4.2.12 (CHARACTERISATION OF p -TYPE). Let $(\mathcal{L}, W)$ be a logical structure. Then, W is a p -consequence operator iff for all $\Gamma \cup \Sigma \subseteq \mathcal{L}$, $\Gamma \subseteq \Sigma$ implies that $\Gamma \cup W(\Gamma) \subseteq W(\Sigma)$.

Proof. Suppose W is a p -consequence operator. Let $\Gamma \cup \Sigma \subseteq \mathcal{L}$ such that $\Gamma \subseteq \Sigma$. Since W is reflexive, $\Sigma \subseteq W(\Sigma)$, which implies that $\Gamma \subseteq W(\Sigma)$. Moreover as W is monotonic, $\Gamma \subseteq \Sigma$ implies that $W(\Gamma) \subseteq W(\Sigma)$ as well. Consequently, $\Gamma \cup W(\Gamma) \subseteq W(\Sigma)$ for all $\Gamma \cup \Sigma \subseteq \mathcal{L}$.

Conversely, suppose that for all $\Gamma \cup \Sigma \subseteq \mathcal{L}$, $\Gamma \subseteq \Sigma$ implies $\Gamma \cup W(\Gamma) \subseteq W(\Sigma)$. Let $\Gamma \subseteq \mathcal{L}$. Since $\Gamma \subseteq \Gamma$, $\Gamma \cup W(\Gamma) \subseteq W(\Gamma)$, which implies that $\Gamma \subseteq W(\Gamma)$. Thus, W is reflexive.

Let $\Gamma \cup \Sigma \subseteq \mathcal{L}$ such that $\Gamma \subseteq \Sigma$. Then, $\Gamma \cup W(\Gamma) \subseteq W(\Sigma)$ which implies that $W(\Gamma) \subseteq W(\Sigma)$. So W is monotonic as well.

Hence, W is a p -consequence operator. $\square$

Theorem 4.2.13 (REPRESENTATION THEOREM FOR p -TYPE (PART I)). *Let $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S_p, \mathcal{P}(\mathcal{L}))$ be a strongly granular semantics for $\mathcal{L}$ such that $S_p = \{(\models_1, \models_2)\}$ and $\models_1 \subseteq \models_2$. Then, $W_{\mathfrak{S}}$ is a p -consequence operator.*

Proof. Let $\Gamma \cup \Sigma \subseteq \mathcal{L}$ such that $\Gamma \subseteq \Sigma$ but $\Gamma \cup W_{\mathfrak{S}}(\Gamma) \not\subseteq W_{\mathfrak{S}}(\Sigma)$. This implies that there exists $\alpha \in \Gamma \cup W_{\mathfrak{S}}(\Gamma)$ such that $\alpha \notin W_{\mathfrak{S}}(\Sigma)$.

Since $\alpha \notin W_{\mathfrak{S}}(\Sigma)$, there exists $m \in \mathbf{M}$ such that $m \models_1 \Sigma$ but $m \not\models_2 \{\alpha\}$. Since $\mathfrak{S}$ is strongly granular (and hence granular at the 1st component), $\Gamma \subseteq \Sigma$ and $m \models_1 \Sigma$, it follows that $m \models_1 \Gamma$. Since $\alpha \in \Gamma \cup W_{\mathfrak{S}}(\Gamma)$, either $\alpha \in \Gamma$ or $\alpha \in W_{\mathfrak{S}}(\Gamma)$.

If $\alpha \in \Gamma$, then as $m \models_1 \Gamma$ and $\mathfrak{S}$ is strongly granular, $m \models_1 \{\alpha\}$ as well. Since $\models_1 \subseteq \models_2$, this implies $m \models_2 \{\alpha\}$ – a contradiction. If, on the other hand, $\alpha \in W_{\mathfrak{S}}(\Gamma)$, then since $m \models_1 \Gamma$, we have $m \models_2 \{\alpha\}$ – a contradiction again. Thus $\Gamma \cup W_{\mathfrak{S}}(\Gamma) \subseteq W_{\mathfrak{S}}(\Sigma)$. Therefore, by the previous theorem, W is a p -consequence operator. $\square$

It has been proved in [56, Theorem 5] that every p -consequence operator can be given a 3-valued description. In our language this is equivalent to stating that every p -type logical structure has an adequate functionally 3-valued semantics. Below we give a proof of the same result mainly for completeness.

Theorem 4.2.14 (REPRESENTATION THEOREM FOR p -TYPE (PART II)). *Let $(\mathcal{L}, W)$ be a p -type logical structure and $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S_p, \mathcal{P}(\mathcal{L}))$ a semantics for $\mathcal{L}$ satisfying the following properties.*

- $\mathbf{M} = \{\mu_{\Sigma} \in \{0, 1, 2\}^{\mathcal{L}} : \Sigma \subseteq \mathcal{L}\}$ where

$$\mu_{\Sigma}(\beta) = \begin{cases} 0 & \text{if } \beta \notin W(\Sigma) \\ 1 & \text{if } \beta \in \Sigma \\ 2 & \text{if } \beta \in W(\Sigma) \setminus \Sigma \end{cases}$$

- For all $m \in \mathbf{M}, \Gamma \subseteq \mathcal{L}$,

$$m \models_1 \Gamma \text{ iff } m(\Gamma) \subseteq \{1\}$$

$$m \models_2 \Gamma \text{ iff } m(\Gamma) \subseteq \{1, 2\}$$

- S_p is as defined in the previous theorem.

Then, $W = W_{\mathfrak{S}}$.

Proof. Let for some $\Gamma \subseteq \mathcal{L}$, $W(\Gamma) \not\subseteq W_{\mathfrak{S}}(\Gamma)$. Then, there exists $\alpha \in W(\Gamma)$ such that $\alpha \notin W_{\mathfrak{S}}(\Gamma)$. So, there exists $m \in \mathbf{M}$ such that $m \models_1 \Gamma$ but $m \not\models_2 \{\alpha\}$. Since $m \in \mathbf{M}$, there exists $\Sigma \subseteq \mathcal{L}$ such that $m = \mu_{\Sigma}$. Now, $\mu_{\Sigma} \models_1 \Gamma$ implies that $\mu_{\Sigma}(\Gamma) \subseteq \{1\}$, i.e., $\Gamma \subseteq \Sigma$. Since W is monotonic, $W(\Gamma) \subseteq W(\Sigma)$. So, $\alpha \in W(\Sigma)$, and hence $\mu_{\Sigma} \models_2 \{\alpha\}$, i.e., $m \models_2 \{\alpha\}$ – a contradiction. Consequently, $W(\Gamma) \subseteq W_{\mathfrak{S}}(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$.

Next, let $\Gamma \subseteq \mathcal{L}$ and $\alpha \in W_{\mathfrak{S}}(\Gamma)$. Now, $\mu_{\Gamma}(\Gamma) \subseteq \{1\}$ and so $\mu_{\Gamma} \models_1 \Gamma$. As $\alpha \in W_{\mathfrak{S}}(\Gamma)$, $\mu_{\Gamma} \models_2 \{\alpha\}$. Thus, $\mu_{\Gamma}(\{\alpha\}) \subseteq \{1, 2\}$. If $\mu_{\Gamma}(\alpha) = 1$ then $\alpha \in \Gamma$, and since W is reflexive, $\alpha \in W(\Gamma)$ as well. If, on the other hand, $\mu_{\Gamma}(\alpha) = 2$ then also $\alpha \in W(\Gamma)$. Thus, in either case, $\alpha \in W(\Gamma)$. Hence, $W_{\mathfrak{S}}(\Gamma) \subseteq W(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$. This completes the proof. $\square$

Theorem 4.2.15 (REPRESENTATION THEOREM FOR p -TYPE (PART III)). *Let $(\mathcal{L}, W)$ be a p -type logical structure and $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S, \mathcal{P}(\mathcal{L}))$ a strongly granular semantics for $\mathcal{L}$ such that $S = \{(\models_1, \models_2)\}$ and $W = W_{\mathfrak{S}}$. Then $\models_1 \subseteq \models_2$.*

Proof. Let $\alpha \in \mathcal{L}$ and $m \in \mathbf{M}$ be such that $m \models_1 \{\alpha\}$. If possible, let $m \not\models_2 \{\alpha\}$. But then $\alpha \notin W_{\mathfrak{S}}(\{\alpha\})$ which implies that $\alpha \notin W(\{\alpha\})$ as $W = W_{\mathfrak{S}}$. This, however, is a contradiction since W is reflexive. Thus, for all $\alpha \in \mathcal{L}, m \in \mathbf{M}$, $m \models_1 \{\alpha\}$ implies that $m \models_2 \{\alpha\}$.

Now, let $\Gamma \subseteq \mathcal{L}$ be such that for some $n \in \mathbf{M}$, $n \models_1 \Gamma$. Then, as $\mathfrak{S}$ is strongly granular (and hence granular at the first component) $n \models_1 \{\gamma\}$ for all $\gamma \in \Gamma$. So, by the above argument $n \models_2 \{\gamma\}$ for all $\gamma \in \Gamma$. Then again, as $\mathfrak{S}$ is strongly granular (and hence granular at the second component) $n \models_2 \Gamma$. Hence, $\models_1 \subseteq \models_2$. $\square$

By Theorem 4.2.15 every p -consequence operator has an adequate functionally 3-valued semantics. However, not every p -type logical structure is inferentially 3-valued.

Given a logical structure $(\mathcal{L}, W)$, W is said to be *idempotent* if $W = W \circ W$. We now show that every logical structure induced by a non-idempotent p -consequence operator is inferentially 3-valued.

Theorem 4.2.16 (INFERENTIAL IRREDUCIBILITY THEOREM FOR p -TYPE). *Every logical structure induced by a non-idempotent p -consequence operator is inferentially 3-valued.*

Proof. Let $(\mathcal{L}, W)$ be a logical structure induced by a non-idempotent p -consequence operator. Suppose that it is not inferentially 3-valued. Then, there exists some functionally 2-valued semantics for $\mathcal{L}$, say $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S_p, \mathcal{P}(\mathcal{L}))$ where $S_p = \{(\models_1, \models_2)\}$ and $\models_1 \subseteq \models_2$, such that $W = W_{\mathfrak{S}}$.

Using an argument similar to that in Theorem 4.2.10, we can show that for all $m \in \mathbf{M}, \Gamma \subseteq \mathcal{L}$,

$$m \models_1 \Gamma \iff m(\Gamma) \subseteq \{1\} \iff m \models_2 \Gamma$$

In other words, $\models_1 = \models_2$. Let $\models_1 = \models_2 = \models$.

By Remark 4.1.15, every functionally 2-valued semantics is strongly granular. Now, let $\Sigma \subseteq \mathcal{L}$ such that $W_{\mathfrak{S}}(W(\Sigma)) \not\subseteq W_{\mathfrak{S}}(\Sigma)$. Then, there exists $\alpha \in W_{\mathfrak{S}}(W(\Sigma))$ such that $\alpha \notin W_{\mathfrak{S}}(\Sigma)$. So, there exists $m \in \mathbf{M}$ such that $m \models \Sigma$ such that $m \not\models \{\alpha\}$. Observe that,

for all $\beta \in W(\Sigma)$, $m \models \{\beta\}$. Consequently, $m \models W(\Sigma)$. Now, as $m \not\models \{\alpha\}$, $\alpha \notin W_{\mathfrak{S}}(W(\Sigma))$ – a contradiction. Hence $W_{\mathfrak{S}}(W(\Sigma)) \subseteq W_{\mathfrak{S}}(\Sigma)$.

Since $W = W_{\mathfrak{S}}$, this shows that $W(W(\Sigma)) \subseteq W(\Sigma)$. Now, by reflexivity of W , we have $W(\Sigma) \subseteq W(W(\Sigma))$. This shows that $W(W(\Sigma)) = W(\Sigma)$ for all $\Sigma \subseteq \mathcal{L}$. Thus, W is idempotent – a contradiction. This completes the proof. $\square$

4.2.3 s_{κ} -consequence Operators

Let $(\mathcal{L}, W)$ be a logical structure such that W is reflexive. We now discuss a family of non-reflexive monotonic consequence operators. Given a logical structure $(\mathcal{L}, W)$, a nonempty set $\Gamma \subseteq \mathcal{L}$, is said to satisfy *anti-reflexivity* if $\Gamma \subseteq \mathcal{L} \setminus W(\Gamma)$, i.e., $\Gamma \cap W(\Gamma) = \emptyset$. We will say that $(\mathcal{L}, W)$ is *anti-reflexive* if for all $\Gamma \subseteq \mathcal{L}$, Γ satisfies anti-reflexivity. We note that there exists only one monotonic logical structure $(\mathcal{L}, W)$ such that Γ is anti-reflexive for all $\Gamma \subseteq \mathcal{L}$. More precisely, we have the following theorem.

Theorem 4.2.17. *A monotonic logical structure $(\mathcal{L}, W)$ with $\mathcal{L} \neq \emptyset$ is anti-reflexive iff for all nonempty $\Gamma \subseteq \mathcal{L}$, $W(\Gamma) = \emptyset$.*

Proof. Let $(\mathcal{L}, W)$ be a monotonic logical structure such that for all nonempty $\Gamma \subseteq \mathcal{L}$, $W(\Gamma) = \emptyset$. Then, in particular, $W(\mathcal{L}) = \emptyset$. Since W is monotonic, for any $\Gamma \subseteq \mathcal{L}$, $W(\Gamma) = \emptyset \subseteq \emptyset = W(\mathcal{L})$. Therefore, for any nonempty $\Gamma \subseteq \mathcal{L}$, $\Gamma \subseteq \mathcal{L} \setminus W(\Gamma) = \mathcal{L} \setminus \emptyset = \mathcal{L}$. Consequently, W is anti-reflexive as well.

Conversely, suppose that $(\mathcal{L}, W)$ is anti-reflexive. Now, as $\mathcal{L} \neq \emptyset$, by anti-reflexivity, $\mathcal{L} \cap W(\mathcal{L}) = \emptyset$, which implies $W(\mathcal{L}) = \emptyset$. Then, for any nonempty $\Gamma \subseteq \mathcal{L}$, $\emptyset \subseteq W(\Gamma) \subseteq W(\mathcal{L}) = \emptyset$, i.e., $W(\Gamma) = \emptyset$ by monotonicity, and we are done. $\square$

The above theorem shows that the concept of ‘global’ anti-reflexivity in the presence of monotonicity is too strong. This, however, is not the case for non-monotonic logical structures (see, e.g., [13, 24]). This is also not the case if one considers certain ‘relativised’ versions of anti-reflexivity.

Definition 4.2.18 (INTERNALLY κ SET). Given a set $\mathcal{L}$, a cardinal κ , and $\mathbf{K} \subseteq \mathcal{P}(\mathcal{L})$, we say that $\mathbf{K}$ is *internally κ* if for all $\Gamma \in \mathbf{K}$, $|\Gamma| \geq \kappa$ and there exists $\Sigma \in \mathbf{K}$ such that $|\Sigma| = \kappa$.

Definition 4.2.19 (ANTI-REFLEXIVITY $_{\kappa}^{\mathbf{K}}$). Given a logical structure $(\mathcal{L}, W)$, a cardinal κ and $\mathbf{K} \subseteq \mathcal{P}(\mathcal{L})$, W is said to be *anti-reflexive $_{\kappa}^{\mathbf{K}}$* (anti-reflexive relative to $\mathbf{K}$ that is internally κ) or to satisfy *anti-reflexivity $_{\kappa}^{\mathbf{K}}$* if for every $\Gamma \in \mathbf{K}$, Γ satisfies anti-reflexivity, i.e., $\Gamma \subseteq \mathcal{L} \setminus W(\Gamma)$ for all $\Gamma \in \mathbf{K}$. In this case, $(\mathcal{L}, W)$ is also said to be *anti-reflexive $_{\kappa}^{\mathbf{K}}$* .

Definition 4.2.20 ($s_{\kappa}^{\mathbf{K}}$ -CONSEQUENCE OPERATOR). Let $\mathcal{L}$ be a set, $W : \mathcal{P}(\mathcal{L}) \rightarrow \mathcal{P}(\mathcal{L})$ and κ a cardinal. Then W is said to be an *$s_{\kappa}^{\mathbf{K}}$ -consequence operator* on $\mathcal{L}$ if it is monotonic and anti-reflexive $_{\kappa}^{\mathbf{K}}$. A logical structure $(\mathcal{L}, W)$ is said to be of *$s_{\kappa}^{\mathbf{K}}$ -type* if W is an $s_{\kappa}^{\mathbf{K}}$ -consequence on $\mathcal{L}$.

W is said to be an *s_{κ} -consequence operator on $\mathcal{L}$* if W is an $s_{\kappa}^{\mathbf{K}}$ -consequence operator on $\mathcal{L}$ for some $\mathbf{K} \subseteq \mathcal{P}(\mathcal{L})$ that is internally κ . A logical structure $(\mathcal{L}, W)$ is said to be of *s_{κ} -type* if W is an s_{κ} -consequence operator on $\mathcal{L}$.

We now list a few properties of s_κ -consequence operators.

Theorem 4.2.21. *Let $(\mathcal{L}, W)$ be an s_κ^K -type logical structure where $\mathcal{L} \neq \emptyset$. Then, for all $\Gamma \in \mathbf{K}$, the following statements hold.*

- (1) $W(\Gamma) \subseteq W(\mathcal{L} \setminus W(\Gamma))$. The reverse inclusion holds if $\mathcal{L} \setminus W(\Gamma) \in \mathbf{K}$ as well.
- (2) If $\mathcal{L} \setminus \Gamma \in \mathbf{K}$, then $W(\Gamma) \cap W(\mathcal{L} \setminus \Gamma) = \emptyset$.
- (3) If $\Sigma \in \mathbf{K}$ such that $\Gamma \cap \Sigma \neq \emptyset$, then $W(\Sigma) \neq \Gamma$

Proof. (1) Let $\Gamma \in \mathbf{K}$. Then $\Gamma \subseteq \mathcal{L} \setminus W(\Gamma)$ by anti-reflexivity $_\kappa^K$ of W . So, by monotonicity, $W(\Gamma) \subseteq W(\mathcal{L} \setminus W(\Gamma))$.

Now, suppose that $\mathcal{L} \setminus W(\Gamma) \in \mathbf{K}$ as well. Then, for any $\alpha \notin W(\Gamma)$, $\alpha \in \mathcal{L} \setminus W(\Gamma)$, which implies that $\alpha \notin W(\mathcal{L} \setminus W(\Gamma))$ as W is anti-reflexive $_\kappa^K$. Thus, $W(\mathcal{L} \setminus W(\Gamma)) \subseteq W(\Gamma)$.

(2) Let $\Gamma \in \mathbf{K}$ such that $\mathcal{L} \setminus \Gamma \in \mathbf{K}$. Then, for any $\alpha \in W(\Gamma)$,

$$\begin{aligned}
 \alpha \notin \mathcal{L} \setminus W(\Gamma) &\implies \alpha \notin \Gamma && \text{(since } W \text{ is anti-reflexive}_\kappa^K) \\
 &\implies \alpha \in \mathcal{L} \setminus \Gamma \\
 &\implies \alpha \in \mathcal{L} \setminus W(\mathcal{L} \setminus \Gamma) && \text{(since } \mathcal{L} \setminus \Gamma \in \mathbf{K} \text{ and } W \text{ is anti-reflexive}_\kappa^K) \\
 &\implies \alpha \notin W(\mathcal{L} \setminus \Gamma)
 \end{aligned}$$

Thus, $W(\Gamma) \cap W(\mathcal{L} \setminus \Gamma) = \emptyset$.

(3) Suppose there exists $\Gamma \subseteq \mathcal{L}$ and $\Sigma \in \mathbf{K}$ with $\Gamma \cap \Sigma \neq \emptyset$ such that $W(\Sigma) = \Gamma$. Now, since W is anti-reflexive $_\kappa^K$, we have $\Gamma = W(\Sigma) \subseteq \mathcal{L} \setminus \Sigma$, which implies that $\Gamma \cap \Sigma = \emptyset$ – a contradiction. Hence $W(\Sigma) \neq \Gamma$. $\square$

Example 4.2.22. Let, $(\mathcal{L}, W)$ be a logical structure, where $\mathcal{L} = \mathbb{N} \times \mathbb{N}$ and $W : \mathcal{P}(\mathcal{L}) \rightarrow \mathcal{P}(\mathcal{L})$ is defined as follows.

$$W(\Gamma) = \begin{cases} \{(n, m) : (m, n) \in \Gamma\} & \text{if } \Gamma \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

for any $\Gamma \subseteq \mathcal{L}$.

Let $\Phi = \{(2n+1, 2n+2) : n \in \mathbb{N}\}$ and $\mathbf{K} = \mathcal{P}(\Phi) \setminus \{\emptyset\}$. It is easy to see that $\mathbf{K}$ is internally 1. We now claim that for all $\Gamma \in \mathbf{K}$, $\Gamma \subseteq \mathcal{L} \setminus W(\Gamma)$.

Suppose the contrary, i.e., there exists $\Gamma \in \mathbf{K}$, $\Gamma \cap \mathcal{L} \setminus W(\Gamma) \neq \emptyset$. Let $(a, b) \in \Gamma \cap W(\Gamma)$. Now, $(a, b) \in W(\Gamma)$ implies that $(b, a) \in \Gamma \subseteq \Phi$ – an impossibility! Thus, $\Gamma \cap \mathcal{L} \setminus W(\Gamma) = \emptyset$ for all $\Gamma \in \mathbf{K}$.

Hence $(\mathcal{L}, W)$ is anti-reflexive $_\kappa^K$.

Next, let $\Gamma \cup \Sigma \subseteq \mathcal{L}$ such that $\Gamma \subseteq \Sigma$. If $\Gamma = \emptyset$, then $W(\Gamma) = \emptyset$, and hence $W(\Gamma) \subseteq W(\Sigma)$.

Suppose that $\Gamma \neq \emptyset$. So $W(\Gamma) \neq \emptyset$. Then,

$$\begin{aligned} (n, m) \in W(\Gamma) &\implies (m, n) \in \Gamma && \text{(by the definition of } W) \\ &\implies (m, n) \in \Sigma && \text{(since } \Gamma \subseteq \Sigma) \\ &\implies (n, m) \in W(\Sigma) && \text{(by the definition of } W) \end{aligned}$$

So W is monotonic as well. Thus, W is an s_1^K -consequence operator.

We now show that given any infinite set $\mathcal{L}$ and $0 < \lambda < |\mathcal{L}|$, there exists a logical structure $(\mathcal{L}, W)$ and $\emptyset \subsetneq K \subseteq \mathcal{P}(\mathcal{L})$ such that W is an s_κ^K -consequence operator. But for this, we need the following result about cardinal numbers.

Theorem 4.2.23. *Given a set X , and a cardinal κ such that $|X| = \kappa$, there exists $Y \subseteq X$ such that $|Y| = \lambda$ for all $\lambda \leq \kappa$.*

Example 4.2.24. Let κ be an infinite cardinal and $\mathcal{L}$ a set such that $|\mathcal{L}| = \kappa$. Then, $|\mathcal{L} \times \{0\}| = |\mathcal{L} \times \{1\}| = \kappa$. So, there exist a bijection $\sigma : \mathcal{L} \longrightarrow (\mathcal{L} \times \{0\}) \cup (\mathcal{L} \times \{1\})$. Let $X = \sigma^{-1}(\mathcal{L} \times \{0\})$ and $Y = \sigma^{-1}(\mathcal{L} \times \{1\})$. Now,

$$X \cap Y = \sigma^{-1}(\mathcal{L} \times \{0\}) \cap \sigma^{-1}(\mathcal{L} \times \{1\}) = \sigma^{-1}((\mathcal{L} \times \{0\}) \cap (\mathcal{L} \times \{1\})) = \sigma^{-1}(\emptyset) = \emptyset$$

and

$$X \cup Y = \sigma^{-1}(\mathcal{L} \times \{0\}) \cup \sigma^{-1}(\mathcal{L} \times \{1\}) = \sigma^{-1}((\mathcal{L} \times \{0\}) \cup (\mathcal{L} \times \{1\})) = \mathcal{L}.$$

Moreover, $|X| = |Y| = \kappa$.

We now define $f : \mathcal{L} \longrightarrow \mathcal{L}$ as follows.

$$f(z) = \begin{cases} \sigma^{-1}(a, 1) & \text{if } \sigma(z) = (a, 0) \\ \sigma^{-1}(a, 0) & \text{if } \sigma(z) = (a, 1) \end{cases}$$

Well-definedness of f follows from the fact that σ is a bijection.

Let us now choose any $0 < \lambda < \kappa$ and define, $W_\lambda : \mathcal{P}(\mathcal{L}) \longrightarrow \mathcal{P}(\mathcal{L})$ as follows.

$$W_\lambda(\Gamma) = \begin{cases} f(\Gamma) & \text{if } |\Gamma| \geq \lambda \\ \emptyset & \text{otherwise} \end{cases}$$

Finally, let $K = \{\Gamma \subseteq \mathcal{L} : |\Gamma| \geq \lambda \text{ and } \Gamma \subseteq X \text{ or } \Gamma \subseteq Y\}$ and consider the logical structure $(\mathcal{L}, W_\lambda)$.

We note that since $\lambda < \kappa$, by [Theorem 4.2.23](#), there exists $\Delta \subseteq \mathcal{L}$ such that $|\Delta| = \lambda$. Moreover, for every $\Gamma \in K$, $|\Gamma| \geq \lambda$. Therefore, K is internally λ .

We now claim that for all $\Gamma \in K$, $\Gamma \subseteq \mathcal{L} \setminus W_\lambda(\Gamma)$. Let $\Gamma \in K$. Then, $|\Gamma| \geq \lambda$ and $\Gamma \subseteq X$ or $\Gamma \subseteq Y$. Without loss of generality, suppose that $\Gamma \subseteq X$. Then, $W_\lambda(\Gamma) = f(\Gamma) \subseteq Y$. Now, as $X \cap Y = \emptyset$, $\Gamma \cap W_\lambda(\Gamma) = \emptyset$, i.e., $\Gamma \subseteq \mathcal{L} \setminus W_\lambda(\Gamma)$. Thus W_κ is anti-reflexive $_\lambda^K$.

Next, let $\Gamma \cup \Sigma \subseteq \mathcal{L}$ such that $\Gamma \subseteq \Sigma$. If $|\Gamma| < \lambda$ then $W_\lambda(\Gamma) = \emptyset$ which implies $W_\lambda(\Gamma) \subseteq W_\lambda(\Sigma)$. If, on the other hand, $|\Gamma| \geq \lambda$, then as $\Gamma \subseteq \Sigma$, $|\Sigma| \geq \lambda$ as well. So,

$$W_\lambda(\Gamma) = f(\Gamma) \subseteq f(\Sigma) = W_\kappa(\Sigma)$$

Therefore W_λ is monotonic. Consequently, W_λ is an s_λ^K -consequence operator.

Remark 4.2.25. Given a logical structure $(\mathcal{L}, W)$, where W is an s_κ^K -consequence operator, it can never be a p -consequence operator unless $K = \{\emptyset\}$ and $\kappa = 0$. Consequently, it follows that, the class of p -type logical structures is *disjoint* from the class of s_κ -type logical structures for all cardinal $\kappa > 0$.

However, every logical structure $(\mathcal{L}, W)$, induced by a non-reflexive monotonic consequence operator is of s_1 -type, as shown below.

Theorem 4.2.26. *Every non-reflexive and monotonic logical structure is of s_1 -type.*

Proof. Let $(\mathcal{L}, W)$ be a non-reflexive and monotonic logical structure. Then, by non-reflexivity, there exists $\Gamma \subseteq \mathcal{L}$ such that $\Gamma \not\subseteq W(\Gamma)$. Let $\alpha \in \Gamma$ such that $\alpha \notin W(\Gamma)$. Consequently, since W is monotonic, $\alpha \notin W(\{\alpha\})$ as well. It is now easy to see that W is anti-reflexive $_1^{\{\alpha\}}$. Since W is monotonic as well, W is an $s_1^{\{\alpha\}}$ -type logical structure, and hence $(\mathcal{L}, W)$ is of s_1 -type. $\square$

Remark 4.2.27. For each cardinal κ , the class of s_κ -type logical structures is *incomparable* to that of the class of q -type logical structures. In other words, some s_κ -type logical structures are not of q -type, and vice versa. For example, the s_κ -consequence operator discussed in [Example 4.2.24](#) is not a q -consequence operator. To see this, note that

$$W_\kappa(X \cup W_\kappa(X)) = W_\kappa(X \cup Y) = W_\kappa(\mathcal{L}) = f(\mathcal{L}) = \mathcal{L} \neq Y = f(X) = W_\kappa(X)$$

However, the s_κ -consequence operator discussed in [Theorem 4.2.17](#) is a q -consequence operator.

Theorem 4.2.28 (REPRESENTATION THEOREM FOR s_κ^K -TYPE (PART I)). *Let $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S_{s_\kappa^K}, \mathcal{P}(\mathcal{L}))$ be a strongly granular semantics for $\mathcal{L}$ and $K \subseteq \mathcal{P}(\mathcal{L})$ such that the following properties hold.*

- (1) K is internally κ .
- (2) $S_{s_\kappa^K} = \{(\models_1, \models_2)\}$ and for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\Gamma \in K$ and $\alpha \in \Gamma$ there exists $m \in \mathbf{M}$ such that $m \models_1 \Gamma$ but $m \not\models_2 \{\alpha\}$; in other words,

$$\pi_1(\models_1 \cap (\mathbf{M} \times \{\Gamma\})) \not\subseteq \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\alpha\}\}))$$

for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\Gamma \in K$ and $\alpha \in \Gamma$. Here $\pi_1 : \mathbf{M} \times \mathcal{P}(\mathcal{L}) \rightarrow \mathbf{M}$ is defined as follows: $\pi_1(m, \Delta) = m$ for all $(m, \Delta) \in \mathbf{M} \times \mathcal{P}(\mathcal{L})$.

Then, $W_{\mathfrak{S}}$ is an s_κ^K -consequence operator.

Proof. First, we show that $W_{\mathfrak{S}}$ is anti-reflexive $_K^K$. Suppose the contrary, and let $\Gamma \in K$ such that $\Gamma \cap W_{\mathfrak{S}}(\Gamma) \neq \emptyset$. Let $\alpha \in \Gamma$ such that $\alpha \in W_{\mathfrak{S}}(\Gamma)$. Now, since $\alpha \in \Gamma$ and $\Gamma \in K$, by (2), there exists $m \in \mathbf{M}$ such that $m \models_1 \Gamma$ but $m \not\models_2 \{\alpha\}$. This, however, implies that $\alpha \notin W_{\mathfrak{S}}(\Gamma)$ – a contradiction. Hence, $W_{\mathfrak{S}}$ is anti-reflexive $_K^K$. Since $\mathfrak{S}$ is strongly granular, $W_{\mathfrak{S}}$ is monotonic, by [Theorem 4.2.1](#). Thus, $W_{\mathfrak{S}}$ is an s_κ^K -consequence operator. $\square$

We now show that every monotonic logical structure induced by an s_κ^K -consequence operator is functionally 3-valued.

Theorem 4.2.29 (REPRESENTATION THEOREM FOR s_κ^K -TYPE (PART II)). *Let $(\mathcal{L}, W)$ be an s_κ^K -type logical structure for some $K \subseteq \mathcal{L}$ that is internally κ . Let $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S_{s_\kappa^K}, \mathcal{P}(\mathcal{L}))$ be a semantics for $\mathcal{L}$ satisfying the following properties.*

- $\mathbf{M} = \{\mu_\Sigma \in \{0, 1, 2\}^\mathcal{L} : \Sigma \subseteq \mathcal{L}\}$ where

$$\mu_\Sigma(\beta) = \begin{cases} 0 & \text{if } \beta \notin W(\Sigma) \cup \Sigma \\ 1 & \text{if } \beta \in \Sigma \\ 2 & \text{if } \beta \in W(\Sigma) \end{cases}$$

- For all $m \in \mathbf{M}, \Gamma \subseteq \mathcal{L}$,

$$m \models_1 \Gamma \text{ iff } m(\Gamma) \subseteq \{1\}$$

$$m \models_2 \Gamma \text{ iff } m(\Gamma) \subseteq \{2\}$$

- $S_{s_\kappa^K} = \{(\models_1, \models_2)\}$ and for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\Gamma \in K$ and $\alpha \in \Gamma$,

$$\pi_1(\models_1 \cap (\mathbf{M} \times \{\Gamma\})) \not\subseteq \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\alpha\}\}))$$

where $\pi_1 : \mathbf{M} \times \mathcal{P}(\mathcal{L}) \rightarrow \mathbf{M}$ is as defined in the previous theorem.

Then, $W = W_{\mathfrak{S}}$.

Proof. Suppose, there exists $\Gamma \subseteq \mathcal{L}$ such that $W(\Gamma) \not\subseteq W_{\mathfrak{S}}(\Gamma)$. Then, there exists $\alpha \in W(\Gamma)$ such that $\alpha \notin W_{\mathfrak{S}}(\Gamma)$. Now, $\alpha \notin W_{\mathfrak{S}}(\Gamma)$ implies that there exists $m \in \mathbf{M}$ such that $m \models_1 \Gamma$ but $m \not\models_2 \{\alpha\}$. Since $m \in \mathbf{M}$, $m = \mu_\Sigma$ for some $\Sigma \subseteq \mathcal{L}$.

Now,

$$\begin{aligned} m \models_1 \Gamma &\implies \mu_\Sigma \models_1 \Gamma \\ &\implies \mu_\Sigma(\Gamma) \subseteq \{1\} \\ &\implies \Gamma \subseteq \Sigma \\ &\implies W(\Gamma) \subseteq W(\Sigma) && \text{(since } W \text{ is monotonic)} \\ &\implies \alpha \in W(\Sigma) && \text{(since } \alpha \in W(\Gamma)) \end{aligned}$$

On the other hand,

$$\begin{aligned} m \not\models_2 \{\alpha\} &\implies \mu_\Sigma \not\models_2 \{\alpha\} \\ &\implies \mu_\Sigma(\alpha) \neq 2 \\ &\implies \alpha \notin W(\Sigma) \end{aligned}$$

This is a contradiction. Thus $W(\Gamma) \subseteq W_{\mathfrak{S}}(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$.

Next, suppose that $W_{\mathfrak{S}}(\Gamma) \not\subseteq W(\Gamma)$. Then, there exists $\alpha \in W_{\mathfrak{S}}(\Gamma)$ such that $\alpha \notin W(\Gamma)$. Now, as $\Gamma \subseteq \Gamma$, $\mu_\Gamma(\Gamma) \subseteq \{1\}$, i.e., $\mu_\Gamma \models_1 \Gamma$. However, since $\alpha \notin W(\Gamma)$, $\mu_\Gamma(\alpha) \neq 2$, which implies that $\mu_\Gamma \not\models_2 \{\alpha\}$ and hence $\alpha \notin W_{\mathfrak{S}}(\Gamma)$ – a contradiction. Thus $W_{\mathfrak{S}}(\Gamma) \subseteq W(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$. Hence $W = W_{\mathfrak{S}}$. $\square$

Theorem 4.2.30 (REPRESENTATION THEOREM FOR s_κ^K -TYPE (PART III)). *Let $(\mathcal{L}, W)$ be an s_κ^K -type logical for some $K \subseteq \mathcal{L}$ that is internally κ . Let $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S, \mathcal{P}(\mathcal{L}))$ be a strongly granular semantics for $\mathcal{L}$ where $S = \{(\models_1, \models_2)\}$ and $W = W_{\mathfrak{S}}$. Then, for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\Gamma \in K$ and $\alpha \in \Gamma$, we have,*

$$\pi_1(\models_1 \cap (\mathbf{M} \times \{\Gamma\})) \not\subseteq \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\alpha\}\}))$$

where $\pi_1 : \mathbf{M} \times \mathcal{P}(\mathcal{L}) \longrightarrow \mathbf{M}$ is as defined in the above theorem.

Proof. Suppose the contrary and let $\Gamma \in K$ and $\alpha \in \Gamma$ such that $\pi_1(\models_1 \cap (\mathbf{M} \times \{\Gamma\})) \subseteq \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\alpha\}\}))$. Now, since W is anti-reflexive $_\kappa^K$, $\Gamma \in K$ and $\alpha \in \Gamma$, $\alpha \notin W(\Gamma)$. Then, as $W = W_{\mathfrak{S}}$, $\alpha \notin W_{\mathfrak{S}}(\Gamma)$ as well. So, there exists $m \in \mathbf{M}$ such that $m \models_1 \Gamma$ but $m \not\models_2 \{\alpha\}$. Now, $m \not\models_2 \{\alpha\}$ implies that $m \notin \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\alpha\}\}))$, and since $\pi_1(\models_1 \cap (\mathbf{M} \times \{\Gamma\})) \subseteq \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\alpha\}\}))$, it follows that $m \not\models_1 \Gamma$ – a contradiction. Consequently, $\pi_1(\models_1 \cap (\mathbf{M} \times \{\Gamma\})) \not\subseteq \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\alpha\}\}))$ for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\Gamma \in K$ and $\alpha \in \Gamma$. $\square$

The following theorem establishes the inferential 3-valuedness of s_κ^K -consequence operators. The proof is similar to [Theorem 4.2.10](#) and hence we omit it.

Theorem 4.2.31 (INFERENTIAL IRREDUCIBILITY THEOREM FOR s_κ^K -TYPE). *Every s_κ -type logical structure is inferentially 3-valued.*

4.2.4 r_κ -consequence Operators

We know from [Theorem 4.2.2](#) that monotonic logical structures, in general, are functionally at most 4-valued. However, the monotonic logical structures discussed so far in the previous subsections are all functionally at most 3-valued. In this section, we identify the class of inferentially 4-valued monotonic logical structures.

Suppose $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S, \mathcal{P}(\mathcal{L}))$ where $S = \{(\models_1, \models_2)\}$ is a strongly granular semantics for $\mathcal{L}$ such that $(\mathcal{L}, W_{\mathfrak{S}})$ is an inferentially 4-valued monotonic logical structure. We know from [Theorem 4.2.8](#), [Theorem 4.2.14](#) and [Theorem 4.2.29](#) that every q -, p - and s_κ -type logical structure is functionally 3-valued, and hence cannot be inferentially 4-valued. Therefore, $(\mathcal{L}, W_{\mathfrak{S}})$ cannot be a q -, p - or s_κ -type logical structure. This implies, in particular, that $W_{\mathfrak{S}}$ cannot be a q -, p - or s_κ -consequence operator.

Now, by [Theorem 4.2.7](#), if $\models_2 \subseteq \models_1$, then $W_{\mathfrak{S}}$ is a q -consequence operator. Similarly, if $\models_2 \supseteq \models_1$, then by [Theorem 4.2.13](#) $W_{\mathfrak{S}}$ is a q -consequence operator. Therefore, since $W_{\mathfrak{S}}$ is not a q - or p -consequence operator, $\models_1$ and $\models_2$ must be incomparable.

Moreover, if there exists $K \subseteq \mathcal{P}(\mathcal{L})$ that is internally κ for some cardinal κ such that for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\Gamma \in K$ and $\alpha \in \Gamma$, $\pi_1(\models_1 \cap (\mathbf{M} \times \{\Gamma\})) \not\subseteq \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\alpha\}\}))$, then by [Theorem 4.2.28](#), $W_{\mathfrak{S}}$ would be an s_κ -consequence operator. Therefore, since $W_{\mathfrak{S}}$ is not an s_κ -consequence operator, for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $|\Gamma| \geq \kappa$ there exists $\alpha \in \Gamma$ such that $\pi_1(\models_1 \cap (\mathbf{M} \times \{\Gamma\})) \subseteq \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\alpha\}\}))$.

Now, let $(\mathcal{L}, W)$ be a monotonic logical structure which is inferentially 4-valued. Then, by [Theorem 4.2.2](#) there exists a strongly granular semantics $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S, \mathcal{P}(\mathcal{L}))$ for $\mathcal{L}$ such that $W = W_{\mathfrak{S}}$. So, by the above discussion, W is not a q -, p - or s_{κ} -consequence operator.

- (a) Since W is not a q -consequence operator, by [Theorem 4.2.6\(5\)](#) there exists $\Gamma \subseteq \mathcal{L}$ such that $W(\Gamma) \not\subseteq \Gamma$. Otherwise, for any $\Gamma \subseteq \mathcal{L}$, $W(\Gamma) \cup \Gamma \subseteq \Gamma$, and hence $W(\Gamma) \cup \Gamma = \Gamma$ which implies that $W(W(\Gamma) \cup \Gamma) = W(\Gamma)$. This, however, would imply that W satisfies quasi-closure, and so is a q -consequence operator – a contradiction.
- (b) Since W is monotonic, but not a p -consequence operator, it is not reflexive. So, there exists $\Gamma \subseteq \mathcal{L}$ such that $\Gamma \not\subseteq W(\Gamma)$.
- (c) Since W is monotonic, but not an s_{κ} -consequence operator, it is not anti-reflexive $_{\kappa}^K$ for any $K \subseteq \mathcal{P}(\mathcal{L})$ that is internally κ . This is the same as saying that for all $\Gamma \subseteq \mathcal{L}$ with $|\Gamma| \geq \kappa$, $\Gamma \not\subseteq \mathcal{L} \setminus W(\Gamma)$.

The above discussion motivates the following definition of r_{κ} -consequence operators.

Definition 4.2.32 (r_{κ} -CONSEQUENCE OPERATOR). Given a cardinal κ , an r_{κ} -consequence operator on $\mathcal{L}$ is a function $W : \mathcal{P}(\mathcal{L}) \longrightarrow \mathcal{P}(\mathcal{L})$ satisfying the following properties.

- (i) There exists $\Gamma \subseteq \mathcal{L}$ such that $\Gamma \not\subseteq W(\Gamma)$. (Non-reflexivity)
- (ii) There exists $\Gamma \subseteq \mathcal{L}$ such that $W(\Gamma) \not\subseteq \Gamma$. (Non-quasi-closure)
- (iii) For all $\Gamma \subseteq \mathcal{L}$, with $|\Gamma| \geq \kappa$, $\Gamma \not\subseteq \mathcal{L} \setminus W(\Gamma)$. (Non-anti-reflexivity $_{\kappa}$)
- (iv) For all $\Gamma \cup \Sigma \subseteq \mathcal{L}$, if $\Gamma \subseteq \Sigma$ then $W(\Gamma) \subseteq W(\Sigma)$. (Monotonicity)

A logical structure $(\mathcal{L}, W)$ is said to be of r_{κ} -type if W is an r_{κ} -consequence operator on $\mathcal{L}$.

Theorem 4.2.33. *There exists no logical structure $(\mathcal{L}, W)$ such that W is an r_1 -consequence operator.*

Proof. Suppose the contrary. Let $(\mathcal{L}, W)$ be a logical structure such that W is an r_1 -consequence operator. Since W is non-reflexive, there exists $\Gamma \subseteq \mathcal{L}$ such that $\Gamma \not\subseteq W(\Gamma)$. Therefore, there exists $\alpha \in \Gamma$ such that $\alpha \notin W(\Gamma)$. Since W is monotonic, this implies that $\alpha \notin W(\{\alpha\})$ as well. However, $\alpha \notin W(\{\alpha\})$ implies that $\alpha \in \mathcal{L} \setminus W(\{\alpha\})$, i.e., $\{\alpha\} \subseteq \mathcal{L} \setminus W(\{\alpha\})$, contradicting the fact that W satisfies non-anti-reflexivity $_1$. Thus $(\mathcal{L}, W)$ cannot be of r_1 -type. Since $(\mathcal{L}, W)$ was chosen arbitrarily, this completes the proof. $\square$

However, there exists an r_{κ} -consequence operator for any $\kappa \geq 2$, as shown in the following example.

Example 4.2.34. Suppose $\mathcal{L}$ is an infinite set, and $\kappa \geq 2$ be a cardinal such that $|\mathcal{L}| > \kappa$. Let $\alpha \in \mathcal{L}$ and $W : \mathcal{P}(\mathcal{L}) \rightarrow \mathcal{P}(\mathcal{L})$ be defined as follows.

$$W(\Gamma) = \begin{cases} \{\alpha\} & \text{if } |\Gamma| < \kappa \\ \Gamma \cup \{\alpha\} & \text{otherwise} \end{cases}$$

We note that, since $\mathcal{L}$ is infinite, there exists $\beta \in \mathcal{L}$ such that $\alpha \neq \beta$. Now, since $|\{\beta\}| = 1 < \kappa$, $W(\{\beta\}) = \{\alpha\}$. Then, as $\{\beta\} \not\subseteq W(\{\beta\}) = \{\alpha\}$, W is non-reflexive. Since $|\emptyset| = 0 < \kappa$, $W(\emptyset) = \{\alpha\} \not\subseteq \emptyset$. So, W satisfies non-quasi-closure as well.

Next, let $\Gamma \subseteq \mathcal{L}$ such that $|\Gamma| \geq \kappa$. Then, $W(\Gamma) = \Gamma \cup \{\alpha\}$, and hence $\Gamma \not\subseteq \mathcal{L} \setminus W(\Gamma)$. Thus W satisfies non-anti-reflexivity $_{\kappa}$.

Finally, let $\Gamma \cup \Sigma \subseteq \mathcal{L}$ such that $\Gamma \subseteq \Sigma$. If $|\Gamma| \geq \kappa$, then as $\Sigma \supseteq \Gamma$, $|\Sigma| \geq \kappa$ as well. So, $W(\Gamma) = \Gamma \cup \{\alpha\} \subseteq \Sigma \cup \{\alpha\} = W(\Sigma)$. If, on the other hand, $|\Gamma| < \kappa$, then $W(\Gamma) = \{\alpha\}$. Now, irrespective of Σ , $\{\alpha\} \subseteq W(\Sigma)$. Thus, in all cases $W(\Gamma) \subseteq W(\Sigma)$, and so W is an r_{κ} -consequence operator.

Remark 4.2.35. It is worth pointing out that the r_{κ} -consequence operator W in the above example is *not* an r_{λ} -consequence operator for any cardinal λ such that $2 \leq \lambda < \kappa$. To see this, let λ be a cardinal such that $2 \leq \lambda < \kappa$ and Γ_{λ} a set of cardinality λ such that $\alpha \notin \Gamma_{\lambda}$. The existence of such a Γ_{λ} can be justified as follows. If λ is infinite, then we choose any set Γ of cardinality λ (which exists by [Theorem 4.2.23](#)), and take $\Gamma \setminus \{\alpha\}$ to be Γ_{λ} . On the other hand, if λ is finite, we consider any set Γ of cardinality λ , $\beta \notin \Gamma$ (such a β exists since $\mathcal{L}$ is infinite but Γ is finite) and define $\Gamma_{\lambda} = (\Gamma \setminus \{\alpha\}) \cup \{\beta\}$. Then, as $|\Gamma_{\lambda}| = \lambda < \kappa$, $W(\Gamma_{\lambda}) = \{\alpha\}$, which implies that $\Gamma_{\lambda} \subseteq \mathcal{L} \setminus W(\Gamma_{\lambda}) = \mathcal{L} \setminus \{\alpha\}$. This shows that W does not satisfy non-anti-reflexivity $_{\lambda}$. Thus, W is not an r_{λ} -consequence operator on $\mathcal{L}$ for any cardinal λ such that $2 \leq \lambda < \kappa$. One might compare this with [Example 4.2.24](#).

Theorem 4.2.36 (REPRESENTATION THEOREM FOR r_{κ} -TYPE). *Let $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S_{r_{\kappa}}, \mathcal{P}(\mathcal{L}))$ be a strongly granular semantics for $\mathcal{L}$. Let $S_{r_{\kappa}} = \{(\models_1, \models_2)\}$ be such that,*

- (1) $\models_1 \not\subseteq \models_2$,
- (2) *there exists $\Sigma \cup \Delta \subseteq \mathcal{L}$ such that $\Sigma \subseteq W_{\mathfrak{S}}(\Delta) \cup \Delta$ but $W_{\mathfrak{S}}(\Sigma) \not\subseteq W_{\mathfrak{S}}(\Delta)$,*
- (3) *for all $\Lambda \subseteq \mathcal{L}$ with $|\Lambda| \geq \kappa$ there exists $\lambda \in \Lambda$ such that $\pi_1(\models_1 \cap \mathbf{M} \times \{\lambda\}) \subseteq \pi_1(\models_2 \cap \mathbf{M} \times \{\{\lambda\}\})$.*

Then, $W_{\mathfrak{S}}$ is an r_{κ} -consequence operator.

Proof. Since $\mathfrak{S}$ is strongly granular, by [Theorem 4.2.1](#), $W_{\mathfrak{S}}$ is monotonic.

Since $\models_1 \not\subseteq \models_2$, there exists $m \in \mathbf{M}$ and $\Gamma \subseteq \mathcal{L}$ such that $m \models_1 \Gamma$ but $m \not\models_2 \Gamma$. Since $\mathfrak{S}$ is strongly granular, hence granular the 2nd component, there exists $\beta \in \Gamma$ such that $m_{\Gamma} \not\models_2 \{\beta\}$. So, as $m_{\Gamma} \models_1 \Gamma$, $\beta \notin W_{\mathfrak{S}}(\Gamma)$, i.e., $\Gamma \not\subseteq W_{\mathfrak{S}}(\Gamma)$. Therefore, $W_{\mathfrak{S}}$ is non-reflexive.

Next, let us choose any $\Sigma \cup \Delta \subseteq \mathcal{L}$ satisfying (2). Then, $\Sigma \subseteq W_{\mathfrak{S}}(\Delta) \cup \Delta$ but $W_{\mathfrak{S}}(\Sigma) \not\subseteq W_{\mathfrak{S}}(\Delta)$. So, there exists $\alpha \in \mathcal{L}$ such that $\alpha \in W_{\mathfrak{S}}(\Sigma)$ but $\alpha \notin W_{\mathfrak{S}}(\Delta)$. Thus, there

exists $n \in \mathbf{M}$ such that $n \models_1 \Delta$ and $n \not\models_2 \{\alpha\}$; but, for all $m \in \mathbf{M}$ either $m \not\models_1 \Sigma$ or $m \models_2 \{\alpha\}$. We note that $\Sigma \not\subseteq \Delta$ because otherwise, since $n \models_1 \Delta$, we would have $n \models_1 \Sigma$. But then $n \models_2 \{\alpha\}$ – a contradiction. Now, as $\Sigma \not\subseteq \Delta$, there exists $\sigma \in \Sigma$ such that $\sigma \notin \Delta$. However, since $\Sigma \subseteq W_{\mathfrak{S}}(\Delta) \cup \Delta$, $\sigma \in W_{\mathfrak{S}}(\Delta)$. Hence, $W_{\mathfrak{S}}(\Delta) \not\subseteq \Delta$. Therefore, $W_{\mathfrak{S}}$ satisfies non-quasi-closure as well.

Finally, let $\Lambda \subseteq \mathcal{L}$ such that $|\Lambda| \geq \kappa$. Then, by (3), there exists $\lambda \in \Lambda$ such that $\pi_1(\models_1 \cap (\mathbf{M} \times \{\Lambda\})) \subseteq \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\lambda\}\}))$. We choose any such λ and $m \in \mathbf{M}$ such that $m \models_1 \Lambda$. Then, since $\pi_1(\models_1 \cap (\mathbf{M} \times \{\Lambda\})) \subseteq \pi_1(\models_2 \cap (\mathbf{M} \times \{\{\lambda\}\}))$, $m \models_2 \{\lambda\}$. Thus, $\lambda \in \Lambda \cap W_{\mathfrak{S}}(\Lambda)$ and hence $\Lambda \not\subseteq \mathcal{L} \setminus W_{\mathfrak{S}}(\Lambda)$. Therefore, $W_{\mathfrak{S}}$ satisfies non-anti-reflexivity $_{\kappa}$ as well.

Consequently, $W_{\mathfrak{S}}$ is an r_{κ} -consequence operator. $\square$

The functionally 4-valued characterisation, which was mentioned in [Theorem 4.2.2](#), works here as well. But, in this case, we can also show that the logical structures induced by r_{κ} -consequence operators are inferentially 4-valued.

Theorem 4.2.37 (INFERENCE IRREDUCIBILITY THEOREM FOR r_{κ} -TYPE). *Every logical structure induced by an r_{κ} -consequence operator is inferentially 4-valued.*

Proof. We first note that every r_{κ} -consequence operator has a functionally 4-valued characterisation by [Theorem 4.2.2](#). Let $(\mathcal{L}, W)$ be a logical structure induced by an r_{κ} -consequence operator. Suppose that it is not inferentially 4-valued. So, there exists some functionally 2-valued or 3-valued semantics for $\mathcal{L}$, say $\mathfrak{S} = (\mathbf{M}, \models_1, \models_2, S_{r_{\kappa}}, \mathcal{P}(\mathcal{L}))$ such that $W = W_{\mathfrak{S}}$.

If $\mathfrak{S}$ is functionally 2-valued, then along the same lines as in the proof of [Theorem 4.2.10](#), we are done. We skip this, and move on to where $\mathfrak{S}$ is a functionally 3-valued functional semantics.

Then, for $i \in \{1, 2\}$, there exists $\emptyset \subsetneq D_i \subsetneq \{0, 1, 2\}$ such that $D_1 \cup D_2 \subsetneq \{0, 1, 2\}$ and for all $m \in \mathbf{M}$, $\Gamma \subseteq \mathcal{L}$, $m \models_i \Gamma$ iff $m(\Gamma) \subseteq D_i$. If $D_1 \cup D_2$ is a singleton set, then using arguments similar to the one in the proof [Theorem 4.2.10](#), we are done. Without loss of generality, let $D_1 \cup D_2 = \{1, 2\}$. Now, $D_1 \subseteq D_2$ implies that $\models_1 \subseteq \models_2$, which by [Theorem 4.2.13](#) implies that W is a p -consequence operator. Similarly, $D_2 \subseteq D_1$, implies that $\models_1 \subseteq \models_2$, which by [Theorem 4.2.7](#) implies that W is a q -consequence operator. Therefore, $D_1 \not\subseteq D_2$ and $D_2 \not\subseteq D_1$ as W is neither a p - nor q -consequence operator.

Either $D_1 \cap D_2 = \emptyset$ or $D_1 \cap D_2 \neq \emptyset$. Suppose $D_1 \cap D_2 = \emptyset$. Let $\Gamma \subseteq \mathcal{L}$ such that $|\Gamma| \geq \kappa$ and $\alpha \in \Gamma$. Moreover, let $m \in \mathbf{M}$ such that $m \models_1 \Gamma$, i.e., $m(\Gamma) \subseteq D_1$. Then, as $D_1 \cap D_2 = \emptyset$, $m(\alpha) \notin D_2$, i.e., $m \not\models_2 \{\alpha\}$. In other words, $\alpha \notin W_{\mathfrak{S}}(\Gamma)$. Thus $\Gamma \subseteq \mathcal{L} \setminus W_{\mathfrak{S}}(\Gamma)$ for all $|\Gamma| \geq \kappa$, which implies that W is anti-reflexive $_{\kappa}^K$. This is a contradiction to the assumption that W is a r_{κ} -consequence operator. So, $D_1 \cap D_2 \neq \emptyset$.

If $D_1 \cap D_2 = \{1, 2\} = D_1 \cup D_2$, then $D_1 = D_2$, contrary to our earlier conclusions that $D_1 \not\subseteq D_2$ and $D_2 \not\subseteq D_1$. Therefore, $D_1 \cap D_2 \subsetneq \{1, 2\}$. Without loss of generality, let us assume that $D_1 \cap D_2 = \{1\}$. Now, as $D_1 \not\subseteq D_2$, there exists $a \in D_1$ such that $a \notin D_2$. Again, without loss of generality, let $a = 2$. So, $D_1 = \{1, 2\}$. Then, as $D_1 \cup D_2 \subsetneq \{0, 1, 2\}$, $D_2 = \{1\}$ – a contradiction, since $D_2 \not\subseteq D_1$. Thus $W \neq W_{\mathfrak{S}}$. Hence W is inferentially 4-valued. $\square$

4.3 Suszko Reduction: A Generalised Perspective

In this section, we propose a different notion of semantics than the one in [Definition 4.1.1](#) and show that every monotonic logical structure has an adequate 2-valued semantics of this kind.

Definition 4.3.1 (*S-SEMANTICS*). Given a set $\mathcal{L}$, a *Suszko semantics for $\mathcal{L}$* or *S-semantics for $\mathcal{L}$* , say $\mathbf{S}$, is a tuple $(\mathbf{B}, R, \models, \mathcal{P}(\mathcal{L}))$ satisfying the following properties.

- (i) $\mathbf{B} \subseteq \{0, 1\}^{\mathcal{L}}$.
- (ii) $R \subseteq \mathbf{B} \times \mathbf{B}$.
- (iii) $\models \subseteq \mathbf{B} \times \mathcal{P}(\mathcal{L})$.

Given an *S-semantics* $\mathbf{S} = (\mathbf{B}, R, \models, \mathcal{P}(\mathcal{L}))$, the *type-I S-entailment relation on $\mathcal{L}$* , denoted by $\vdash_I^{\mathbf{S}}$ is defined as follows:

$$\Gamma \vdash_I^{\mathbf{S}} \alpha \iff \text{for all } (v, w) \in R, v \models \Gamma \text{ implies that } w \models \{\alpha\}$$

The corresponding consequence operator is denoted by $W_I^{\mathbf{S}}$. The *type-II S-entailment relation on $\mathcal{L}$* , denoted by $\vdash_{II}^{\mathbf{S}}$, is defined as follows.

$$\Gamma \vdash_{II}^{\mathbf{S}} \alpha \iff \text{there exists } U \subseteq R \text{ with } (\chi_{\Gamma}, \chi_{W(\Gamma)}) \in U \text{ such that} \\ \text{for all } (v, w) \in U, v \models \Gamma \text{ implies that } w \models \{\alpha\}$$

The corresponding consequence operator is denoted by $W_{II}^{\mathbf{S}}$.

Given a logical structure $(\mathcal{L}, W)$ and an *S-semantics* $\mathbf{S}$ for $\mathcal{L}$, $(\mathcal{L}, W)$ is said to have an *adequate type-I S-semantics* (respectively, *adequate type-II S-semantics*) if $W = W_I^{\mathbf{S}}$ (respectively, if $W = W_{II}^{\mathbf{S}}$).

Remark 4.3.2. The name ‘Suszko semantics’ has already been used in [\[53\]](#). We quote the relevant passage below. Our definition, however, is more general than the one given below.

Suszko and his followers admit as such an *arbitrary* family of *arbitrary* functions from the set of propositional formulas to the set of logical values. This last set can be identified with the set $2 = \{0, 1\}$, where 0 represents the value **false** and 1 represents the value **true**. Such a semantics can be called a *Suszko semantics*...

Remark 4.3.3. Let $\mathbf{S} = (\mathbf{B}, R, \models, \mathcal{P}(\mathcal{L}))$ an *S-semantics* for a set $\mathcal{L}$ such that $\mathbf{B} = \emptyset$. If there exists $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ such that $\alpha \notin W^{\mathbf{S}}(\Gamma)$ then there exists $(v, w) \in R$ such that $v \models \Gamma$ but $w \not\models \{\alpha\}$. This, however, is a contradiction since $R = \emptyset$ as $\emptyset \subseteq R \subseteq \mathbf{B} \times \mathbf{B} = \emptyset \times \emptyset = \emptyset$. Hence, $W_{\mathbf{S}}(\Gamma) = \mathcal{L}$ for all $\Gamma \subseteq \mathcal{L}$.

The following theorem shows that every logical structure induced by a semantics (in the sense of [Definition 4.1.1](#)) has an adequate type-I *S-semantics*.

Theorem 4.3.4 (SUSZKO REDUCTION FOR LOGICAL STRUCTURES INDUCED BY A SEMANTICS). *If a logical structure has an adequate semantics, it has an adequate type-I S-semantics as well.*

Proof. Let $(\mathcal{L}, W)$ be a logical structure that is adequate with respect to a semantics $\mathfrak{S} = (\mathbf{M}, \{\models_i\}_{i \in I}, S, \mathcal{P}(\mathcal{L}))$ for $\mathcal{L}$. So, $W = W_{\mathfrak{S}}$.

If $\mathbf{M} = \emptyset$ then we consider any S-semantics $\mathbf{S} = (\mathbf{B}, R, \models, \mathcal{P}(\mathcal{L}))$ for $\mathcal{L}$ such that $\mathbf{B} = \emptyset$. Then by, [Remark 4.1.2](#) and [Remark 4.3.3](#) $W_{\mathfrak{S}}(\Gamma) = W^{\mathbf{S}}(\Gamma) = \mathcal{L}$ for all $\Gamma \subseteq \mathcal{L}$. Therefore, in this case, $W_{\mathfrak{S}} = W^{\mathbf{S}}$.

If, on the other hand, $\mathbf{M} \neq \emptyset$, then for each $(m, i) \in \mathbf{M} \times I$, we choose $v_i^m \in \{0, 1\}^{\mathcal{L}}$ and define a S-semantics for $\mathcal{L}$, $\mathbf{S}^{\mathfrak{S}} = (\mathbf{B}^{\mathfrak{S}}, R^{\mathfrak{S}}, \models, \mathcal{P}(\mathcal{L}))$, as follows.

- (1) $\mathbf{B}^{\mathfrak{S}} = \{v_i^m \in \{0, 1\}^{\mathcal{L}} : (m, i) \in \mathbf{M} \times I\}$.
- (2) $R^{\mathfrak{S}} = \{(v_i^m, v_j^m) : (\models_i, \models_j) \in S \text{ and } m \in \mathbf{M}\}$.
- (3) For all $(m, i) \in \mathbf{M} \times I$, $v_i^m \models \Gamma$ iff $m \models_i \Gamma$.

We now show that $W_{\mathfrak{S}} = W_I^{\mathbf{S}^{\mathfrak{S}}}$.

Let $\Gamma \subseteq \mathcal{L}$ such that $W_{\mathfrak{S}}(\Gamma) \not\subseteq W_I^{\mathbf{S}^{\mathfrak{S}}}(\Gamma)$. Then there exists $\alpha \in W_{\mathfrak{S}}(\Gamma)$ such that $\alpha \notin W_I^{\mathbf{S}^{\mathfrak{S}}}(\Gamma)$. So, there exists $(v_i^m, v_j^m) \in R^{\mathfrak{S}}$ such that $v_i^m \models \Gamma$, but $v_j^m \not\models \{\alpha\}$. Now, $v_i^m \models \Gamma$ implies that $m \models_i \Gamma$ and $v_j^m \not\models \{\alpha\}$ implies that $m \not\models_j \{\alpha\}$. In other words, $\alpha \notin W_{\mathfrak{S}}(\Gamma)$ – a contradiction. Thus $W_{\mathfrak{S}}(\Gamma) \subseteq W_I^{\mathbf{S}^{\mathfrak{S}}}(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$.

Next, let $\Gamma \subseteq \mathcal{L}$ such that $W_I^{\mathbf{S}^{\mathfrak{S}}}(\Gamma) \not\subseteq W_{\mathfrak{S}}(\Gamma)$. Then there exists $\alpha \in W_I^{\mathbf{S}^{\mathfrak{S}}}(\Gamma)$ such that $\alpha \notin W_{\mathfrak{S}}(\Gamma)$. So, there exists $m \in \mathbf{M}$ and $(\models_i, \models_j) \in S$ such that $m \models_i \Gamma$, but $m \not\models_j \{\alpha\}$. Now, $m \models_i \Gamma$ implies that $v_i^m \models \Gamma$, and $m \not\models_j \{\alpha\}$ implies that $v_j^m \not\models \{\alpha\}$. In other words, $\alpha \notin W_I^{\mathbf{S}^{\mathfrak{S}}}(\Gamma)$ – a contradiction. Thus $W_I^{\mathbf{S}^{\mathfrak{S}}}(\Gamma) \subseteq W_{\mathfrak{S}}(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$. Therefore, $W_{\mathfrak{S}} = W_I^{\mathbf{S}^{\mathfrak{S}}}$. $\square$

The converse of the above theorem is established below.

Theorem 4.3.5 (SEMANTIC COUNTERPART OF AN S-SEMANTICS). *If a logical structure has an adequate type-I S-semantics, then it has an adequate semantics as well.*

Proof. Let $(\mathcal{L}, W)$ be a logical structure that has an adequate type-I S-semantics $\mathbf{S} = (\mathbf{B}, R, \models, \mathcal{P}(\mathcal{L}))$ for $\mathcal{L}$. So, $W = W_I^{\mathbf{S}}$.

If $\mathbf{B} = \emptyset$, then we consider any semantics $\mathfrak{S} = (\mathbf{M}, \{\models_i\}_{i \in I}, S, \mathcal{P}(\mathcal{L}))$ for $\mathcal{L}$ such that $\mathbf{M} = \emptyset$. Then by, [Remark 4.3.3](#) and [Remark 4.1.2](#) $W_I^{\mathbf{S}}(\Gamma) = \mathcal{L} = W_{\mathfrak{S}}(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$. Therefore, in this case, $W_{\mathfrak{S}} = W_I^{\mathbf{S}}$.

If, on the other hand, $\mathbf{B} \neq \emptyset$, then we define a semantics for $\mathcal{L}$, $\mathfrak{S}_{\mathbf{S}} = (\mathbf{M}_{\mathbf{S}}, \{\models_v\}_{v \in \mathbf{B}}, S_{\mathbf{S}}, \mathcal{P}(\mathcal{L}))$ as follows.

- (1) $\mathbf{M}_{\mathbf{S}} = \{*\}$ where $* \notin \mathbf{B}$.

(2) For each $v \in \mathbf{B}$, $* \models_v \Gamma$ iff $v \models \Gamma$.

(3) $S_S = \{(\models_v, \models_w) : (v, w) \in R\}$.

We now claim that $W_{\mathfrak{S}_S} = W_I^S$.

Let $\Gamma \subseteq \mathcal{L}$ such that $W_{\mathfrak{S}_S}(\Gamma) \not\subseteq W_I^S(\Gamma)$. Then there exists $\alpha \in W_{\mathfrak{S}_S}(\Gamma)$ such that $\alpha \notin W_I^S(\Gamma)$. So, there exists $(v, w) \in R$ such that $v \models \Gamma$, but $w \not\models \{\alpha\}$. Now, $v \models \Gamma$ implies that $* \models_v \Gamma$ and $w \not\models \{\alpha\}$ implies that $* \not\models_w \{\alpha\}$. In other words, $\alpha \notin W_{\mathfrak{S}_S}(\Gamma)$ – a contradiction. Thus $W_{\mathfrak{S}_S}(\Gamma) \subseteq W_I^S(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$.

Next, let $\Gamma \subseteq \mathcal{L}$ such that $W_I^S(\Gamma) \not\subseteq W_{\mathfrak{S}_S}(\Gamma)$. Then there exists $\alpha \in W_I^S(\Gamma)$ such that $\alpha \notin W_{\mathfrak{S}_S}(\Gamma)$. So, there exists $(\models_v, \models_w) \in S$ such that $* \models_v \Gamma$, but $* \not\models_w \{\alpha\}$. Now, $* \models_v \Gamma$ implies that $v \models \Gamma$, and $* \not\models_w \{\alpha\}$ implies that $w \not\models \{\alpha\}$. In other words, $\alpha \notin W_I^S(\Gamma)$ – a contradiction. Thus $W_I^S(\Gamma) \subseteq W_{\mathfrak{S}_S}(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$. Therefore, $W_{\mathfrak{S}_S} = W_I^S$. $\square$

4.3.1 Suszko Reduction for Monotonic Logical Structures

Definition 4.3.6 (ATOMIC S -SEMANTICS, BIVALENT LOGICAL STRUCTURE). Given a set $\mathcal{L}$ and an S -semantics $\mathbf{S} = (\mathbf{B}, R, \models, \mathcal{P}(\mathcal{L}))$ for $\mathcal{L}$, it is said to be *atomic* if for all $\Gamma \subseteq \mathcal{L}$ and $v \in \mathbf{B}$, $v \models \Gamma$ iff $v \models \{\beta\}$ for all $\beta \in \Gamma$. We call a logical structure *bivalent* if it is adequate with respect to type-I (or type II) S -semantics.

Remark 4.3.7. Since, by [Theorem 4.1.6](#) and [Theorem 4.3.4](#), every logical structure has an adequate type-I S -semantics, according to the definition above every logical structure is bivalent. We will come back to this, probably surprising observation, later in [Section 4.4](#).

Remark 4.3.8. One may compare the notion of atomic S -semantics with that of granular semantics defined in [Definition 4.1.14](#).

Given a set $\mathcal{L}$ and an atomic S -semantics $\mathbf{S} = (\mathbf{B}, R, \models, \mathcal{P}(\mathcal{L}))$ for $\mathcal{L}$, one can construct the following S -semantics $\mathbf{S}_2 = (\mathbf{B}_2, R_2, \models_2, \mathcal{P}(\mathcal{L}))$ for $\mathcal{L}$ such that the following conditions hold.

- $\mathbf{B}_2 = \{\nu_v : v \in \mathbf{B}\}$ where $\nu_v : \mathcal{L} \longrightarrow \{0, 1\}$ is defined as follows.

$$\nu_v(\beta) = \begin{cases} 1 & \text{if } v \models \{\beta\} \\ 0 & \text{otherwise} \end{cases}$$

- $R_2 = \{(\nu_v, \nu_w) : (v, w) \in R\}$.
- For all $\nu_v \in \mathbf{B}_2$, $\nu_v \models_2 \Gamma$ iff $\nu_v(\Gamma) \subseteq \{1\}$.

It is easy to show that $W^S = W^{\mathbf{S}_2}$. Due to this, we will henceforth consider only those atomic S -semantics for which $v \models \Gamma$ iff $v(\Gamma) \subseteq \{1\}$. This gives rise to the following definition.

Definition 4.3.9 (NORMAL S -SEMANTICS). An atomic S -semantics $\mathbf{S} = (\mathbf{B}, R, \models, \mathcal{P}(\mathcal{L}))$ for a set $\mathcal{L}$ is said to be *normal* if for all $v \in \mathbf{B}$ and $\Gamma \subseteq \mathcal{L}$, $v \models \Gamma$ iff $v(\Gamma) \subseteq \{1\}$

In the following two theorems, we show that every monotonic logical structure is bivalent.

Theorem 4.3.10 (SUSZKO REDUCTION FOR MONOTONIC LOGICAL STRUCTURES (PART I)). *Let $\mathcal{L}$ be a set and $\mathbf{S} = (\mathbf{B}, R, \models, \mathcal{P}(\mathcal{L}))$ a normal S -semantics for $\mathcal{L}$. Then $W_I^{\mathbf{S}}$ is monotonic.*

Proof. Suppose the contrary. Let $\Gamma \subseteq \Sigma \subseteq \mathcal{L}$ such that $W_I^{\mathbf{S}}(\Gamma) \not\subseteq W_I^{\mathbf{S}}(\Sigma)$. So, there exists $\alpha \in W_I^{\mathbf{S}}(\Gamma)$ such that $\alpha \notin W_I^{\mathbf{S}}(\Sigma)$. Then, there exists $(v, w) \in R$ such that $v \models \Sigma$ but $w \not\models \{\alpha\}$, i.e., $v(\Sigma) \subseteq \{1\}$ but $w(\alpha) \neq 1$. Now, as $\Gamma \subseteq \Sigma$, $v(\Sigma) \subseteq \{1\}$ implies that $v(\Gamma) \subseteq \{1\}$. Then, since $w(\alpha) \neq 1$, $\alpha \notin W_I^{\mathbf{S}}(\Gamma)$ – a contradiction. Hence, $W_I^{\mathbf{S}}$ is monotonic. $\square$

Theorem 4.3.11 (SUSZKO REDUCTION FOR MONOTONIC LOGICAL STRUCTURES (PART II)). *Let $(\mathcal{L}, W)$ be a monotonic logical structure and $\mathbf{S} = (\mathbf{B}, R_m, \models, \mathcal{P}(\mathcal{L}))$ be a normal S -semantics for $\mathcal{L}$ where*

$$\mathbf{B} = \{\chi_\Gamma : \Gamma \subseteq \mathcal{L}\}$$

and,

$$R_m = \{(\chi_\Gamma, \chi_{W(\Gamma)}) : \Gamma \subseteq \mathcal{L}\}$$

Then, $W = W_I^{\mathbf{S}}$. Here $\chi_\Gamma : \mathcal{L} \longrightarrow \{0, 1\}$ defined as follows.

$$\chi_\Gamma(\beta) = \begin{cases} 1 & \text{if } \beta \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $\Gamma \subseteq \mathcal{L}$ such that $W(\Gamma) \not\subseteq W_I^{\mathbf{S}}(\Gamma)$. Then, there exists $\alpha \in W(\Gamma)$ such that $\alpha \notin W_I^{\mathbf{S}}(\Gamma)$. So, there exists $(\chi_\Sigma, \chi_{W(\Sigma)}) \in R_m$ such that $\chi_\Sigma \models \Gamma$ but $\chi_{W(\Sigma)} \not\models \{\alpha\}$, i.e., $\chi_\Sigma(\Gamma) \subseteq \{1\}$ but $\chi_{W(\Sigma)}(\alpha) \neq 1$. Now, $\chi_\Sigma(\Gamma) \subseteq \{1\}$ implies that $\Gamma \subseteq \Sigma$ and $\chi_{W(\Sigma)}(\alpha) \neq 1$ implies that $\alpha \notin W(\Sigma)$. But, since $\Gamma \subseteq \Sigma$, and W is monotonic, $W(\Gamma) \subseteq W(\Sigma)$. Then, as $\alpha \in W(\Gamma)$, $\alpha \in W(\Sigma)$ – a contradiction. Thus $W(\Gamma) \subseteq W_I^{\mathbf{S}}(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$.

Next, let $\Gamma \subseteq \mathcal{L}$ such that $W_I^{\mathbf{S}}(\Gamma) \not\subseteq W(\Gamma)$. Then, there exists $\alpha \in W_I^{\mathbf{S}}(\Gamma)$ such that $\alpha \notin W(\Gamma)$. Now $(\chi_\Gamma, \chi_{W(\Gamma)}) \in R_m$, $\chi_\Gamma(\Gamma) \subseteq \{1\}$ but $\chi_{W(\Gamma)}(\alpha) \neq 1$. So, $\alpha \notin W_I^{\mathbf{S}}(\Gamma)$ – a contradiction. Consequently, $W_I^{\mathbf{S}}(\Gamma) \subseteq W(\Sigma)$ for all $\Gamma \subseteq \mathcal{L}$. Hence $W = W_I^{\mathbf{S}}$. $\square$

Since every monotonic logical structure has an adequate type-I normal S -semantics, any q -, p -, s_κ - or r_κ -type logical structure has an adequate type-I normal S -semantics. For q -, p - and s_κ -type logical structures, however, the elements of $\mathbf{B}$ and R satisfies some additional properties, as demonstrated in the following theorems.

Theorem 4.3.12 (SUSZKO REDUCTION FOR q -TYPE (PART I)). *Let $\mathcal{L}$ be a set and $\mathbf{S} = (\mathbf{B}, R, \models, \mathcal{P}(\mathcal{L}))$ be a normal S -semantics such that,*

$$(v, w) \in R \text{ iff for all } \alpha \in \mathcal{L}, w(\alpha) = 1 \text{ implies that } v(\alpha) = 1$$

Then, $W_I^{\mathbf{S}}$ is a q -consequence operator.

Proof. Let $\Gamma \cup \Sigma \subseteq \mathcal{L}$ such that $\Gamma \subseteq W_I^S(\Sigma) \cup \Sigma$ but $W_I^S(\Gamma) \not\subseteq W_I^S(\Sigma)$. So, there exists $\alpha \in W_I^S(\Gamma)$ such that $\alpha \notin W_I^S(\Sigma)$. Then, there exists $(v, w) \in R$ such that $v(\Sigma) \subseteq \{1\}$ but $w(\alpha) \neq 1$.

Let $\beta \in W_I^S(\Sigma)$. Then, as $v(\Sigma) \subseteq \{1\}$ and $(v, w) \in R$, $w(\beta) = 1$. Now, by the definition of R , since $(v, w) \in R$, and $w(\beta) = 1$, it follows that $v(\beta) = 1$ as well. Consequently, $v(W_I^S(\Sigma) \cup \Sigma) \subseteq \{1\}$. Since $\Gamma \subseteq W_I^S(\Sigma) \cup \Sigma$, $v(\Gamma) \subseteq \{1\}$ as well. Now, as $w(\alpha) \neq 1$, $\alpha \notin W_I^S(\Gamma)$ – a contradiction. Thus $W_I^S(\Gamma) \subseteq W_I^S(\Sigma)$. Hence for all $\Gamma \cup \Sigma \subseteq \mathcal{L}$, $\Gamma \subseteq W(\Sigma) \cup \Sigma$ implies that $W(\Gamma) \subseteq W(\Sigma)$. So, by [Theorem 4.2.6\(5\)](#), W_I^S is a q -consequence operator. $\square$

Theorem 4.3.13 (SUSZKO REDUCTION FOR q -TYPE (PART II)). *Let $(\mathcal{L}, W)$ be q -type logical structure and $\mathbf{S} = (\mathbf{B}, R^q, \models, \mathcal{P}(\mathcal{L}))$ be a normal S -semantics for $\mathcal{L}$, where*

$$\mathbf{B} = \{\chi_{W(\Gamma) \cup \Gamma} : \Gamma \subseteq \mathcal{L}\} \cup \{\chi_{W(\Gamma)} : \Gamma \subseteq \mathcal{L}\}$$

and,

$$R^q = \{(\chi_{W(\Gamma) \cup \Gamma}, \chi_{W(\Gamma)}) : \Gamma \subseteq \mathcal{L}\}$$

Then, $W = W_I^S$. χ_Γ 's are defined as in [Theorem 4.3.11](#).

Proof. Suppose $\Gamma \subseteq \mathcal{L}$ such that $W(\Gamma) \not\subseteq W_I^S(\Gamma)$. So, $\alpha \in W(\Gamma)$ such that $\alpha \notin W_I^S(\Gamma)$. Now, as $\alpha \notin W_I^S(\Gamma)$, there exists $(\chi_{W(\Sigma) \cup \Sigma}, \chi_{W(\Sigma)}) \in R^q$ such that $\chi_{W(\Sigma) \cup \Sigma}(\Gamma) \subseteq \{1\}$ but $\chi_{W(\Sigma)}(\alpha) \neq 1$. Now, $\chi_{W(\Sigma) \cup \Sigma}(\Gamma) \subseteq \{1\}$ implies that $\Gamma \subseteq W(\Sigma) \cup \Sigma$, while $\chi_{W(\Sigma)}(\alpha) \neq 1$ implies that $\alpha \notin W(\Sigma)$. Since $\Gamma \subseteq W(\Sigma) \cup \Sigma$ and W is a q -consequence operator, by [Theorem 4.2.6\(5\)](#), $W(\Gamma) \subseteq W(\Sigma)$. Then, as $\alpha \in W(\Gamma)$, $\alpha \in W(\Sigma)$ – a contradiction. Thus, $W(\Gamma) \subseteq W_I^S(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$.

Next, let $\Gamma \subseteq \mathcal{L}$ and $\alpha \in W_I^S(\Gamma)$. Then, $(\chi_{W(\Gamma) \cup \Gamma}, \chi_{W(\Gamma)}) \in R^q$ and $\chi_{W(\Gamma) \cup \Gamma}(\Gamma) \subseteq \{1\}$. Since $\alpha \in W_I^S(\Gamma)$, $\chi_{W(\Gamma)}(\alpha) = 1$, i.e., $\alpha \in W(\Gamma)$. Thus $W_I^S(\Gamma) \subseteq W(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$. Hence $W = W_I^S$. $\square$

Theorem 4.3.14 (SUSZKO REDUCTION FOR p -TYPE (PART I)). *Let $\mathcal{L}$ be a set and $\mathbf{S} = (\mathbf{B}, R, \models, \mathcal{P}(\mathcal{L}))$ a normal S -semantics such that,*

$$(v, w) \in R \text{ iff for all } \alpha \in \mathcal{L}, v(\alpha) = 1 \text{ implies that } w(\alpha) = 1$$

Then, W_I^S is a p -consequence operator.

Proof. Let there exists $\Gamma \cup \Sigma \subseteq \mathcal{L}$ such that $\Gamma \subseteq \Sigma$ but $\Gamma \cup W_I^S(\Gamma) \not\subseteq W_I^S(\Sigma)$. So, there exists $\alpha \in \Gamma \cup W_I^S(\Gamma)$ such that $\alpha \notin W_I^S(\Sigma)$. Then, there exists $(v, w) \in R$ such that $v(\Sigma) \subseteq \{1\}$ but $w(\alpha) \neq 1$.

Since $\Gamma \subseteq \Sigma$ and $v(\Sigma) \subseteq \{1\}$, $v(\Gamma) \subseteq \{1\}$ as well. Also, since $w(\alpha) \neq 1$, $\alpha \notin W_I^S(\Gamma)$. So, $\alpha \in \Gamma$. Then $v(\alpha) = 1$, since $v(\Gamma) \subseteq \{1\}$. Now, by the definition of R , since $(v, w) \in R$, $v(\alpha) = 1$ implies that $w(\alpha) = 1$ – a contradiction. Hence for all $\Gamma \cup \Sigma \subseteq \mathcal{L}$, $\Gamma \subseteq \Sigma$ implies that $\Gamma \cup W_I^S(\Gamma) \subseteq W_I^S(\Sigma)$. So, by [Theorem 4.2.12](#), W_I^S is a p -consequence operator. $\square$

Theorem 4.3.15 (SUSZKO REDUCTION FOR p -TYPE (PART II)). *Let $(\mathcal{L}, W)$ be a p -type logical structure and $\mathbf{S} = (\mathbf{B}, R^p, \models, \mathcal{P}(\mathcal{L}))$ a normal S -semantics for $\mathcal{L}$, where*

$$\mathbf{B} = \{\chi_\Gamma : \Gamma \subseteq \mathcal{L}\} \cup \{\chi_{\Gamma \cup W(\Gamma)} : \Gamma \subseteq \mathcal{L}\}$$

and

$$R^p = \{(\chi_\Gamma, \chi_{\Gamma \cup W(\Gamma)}) : \Gamma \subseteq \mathcal{L}\}$$

Then, $W = W_I^S$.

Proof. Let $\Gamma \subseteq \mathcal{L}$ such that $W(\Gamma) \not\subseteq W_I^S(\Gamma)$. Then, there exists $\alpha \in W(\Gamma)$ such that $\alpha \notin W_I^S(\Gamma)$. So, there exists $(\chi_\Sigma, \chi_{\Sigma \cup W(\Sigma)}) \in R^p$ such that $\chi_\Sigma(\Gamma) \subseteq \{1\}$ but $\chi_{\Sigma \cup W(\Sigma)}(\alpha) \neq 1$. But, $\chi_\Sigma(\Gamma) \subseteq \{1\}$ implies that $\Gamma \subseteq \Sigma$, while $\chi_{\Sigma \cup W(\Sigma)}(\alpha) \neq 1$ implies that $\alpha \notin \Sigma \cup W(\Sigma)$, and so $\alpha \notin W(\Sigma)$. Since $\Gamma \subseteq \Sigma$, and W is a p -consequence operator, by [Theorem 4.2.12](#), $\Gamma \cup W(\Gamma) \subseteq W(\Sigma)$. Then, as $\alpha \notin W(\Sigma)$, $\alpha \notin \Gamma \cup W(\Gamma)$, which implies that $\alpha \notin W(\Gamma)$ – a contradiction. Thus, $W(\Gamma) \subseteq W_I^S(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$.

Next, let $\Gamma \subseteq \mathcal{L}$ such that $W_I^S(\Gamma) \not\subseteq W(\Gamma)$. Then, there exists $\alpha \in W_I^S(\Gamma)$ such that $\alpha \notin W(\Gamma)$. Since W is a p -consequence operator, and hence reflexive, $\alpha \notin W(\Gamma)$ implies that $\alpha \notin \Gamma$ as well. Thus, $\alpha \notin \Gamma \cup W(\Gamma)$.

Now, $(\chi_\Gamma, \chi_{\Gamma \cup W(\Gamma)}) \in R^p$, $\chi_\Gamma(\Gamma) \subseteq \{1\}$ but $\chi_{\Gamma \cup W(\Gamma)}(\alpha) \neq 1$. Hence, $\alpha \notin W_I^S(\Gamma)$ – a contradiction. Thus, $W_I^S(\Gamma) \subseteq W(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$. Hence $W = W_I^S$. $\square$

Theorem 4.3.16 (SUSZKO REDUCTION FOR s_κ^K -TYPE (PART I)). *Let $\mathcal{L}$ be a set and $\mathbf{K} \subseteq \mathcal{P}(\mathcal{L})$ be internally κ for some cardinal κ . Let $\mathbf{S} = (\mathbf{B}, R, \models, \mathcal{P}(\mathcal{L}))$ be a normal S -semantics such that for each $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$ with $\alpha \in \Gamma$ and $\Gamma \in \mathbf{K}$, there exists $(v, w) \in R$ such that $v(\Gamma) \subseteq \{1\}$ but $w(\alpha) \notin \{1\}$. Then W_I^S is an s_κ^K -consequence operator.*

Proof. We first note that W_I^S is monotonic, by [Theorem 4.3.10](#).

Now, let $\Gamma \in \mathbf{K}$, $\alpha \in \Gamma$ and $(v, w) \in R$ such that $v(\Gamma) \subseteq \{1\}$ and $w(\alpha) \neq 1$ (such a (v, w) exists by hypothesis). Thus, $\alpha \notin W_I^S(\Gamma)$, which implies that $\Gamma \subseteq \mathcal{L} \setminus W_I^S(\Gamma)$. Hence, $\Gamma \subseteq \mathcal{L} \setminus W_I^S(\Gamma)$ for all $\Gamma \in \mathbf{K}$. Thus, W_I^S is anti-reflexive $_\kappa^K$ and hence an s_κ^K -consequence operator. $\square$

Theorem 4.3.17 (SUSZKO REDUCTION FOR s_κ^K -TYPE (PART II)). *Let $(\mathcal{L}, W)$ be an s_κ^K -type logical structure and $\mathbf{S} = (\mathbf{B}, R_{s_\kappa^K}, \models, \mathcal{P}(\mathcal{L}))$ a normal S -semantics for $\mathcal{L}$ satisfying the following conditions.*

- $\mathbf{B} = \mathbf{B}_1 \cup \mathbf{B}_2 \cup \mathbf{B}_3$, where

$$\mathbf{B}_1 = \{\chi_{\mathcal{L} \setminus W(\Gamma)} : \Gamma \in \mathbf{K} \text{ and } \mathcal{L} \setminus W(\Gamma) \in \mathbf{K}\}$$

$$\mathbf{B}_2 = \{\chi_{W(\Gamma)} : \Gamma \in \mathbf{K} \text{ and } \mathcal{L} \setminus W(\Gamma) \in \mathbf{K}\}$$

$$\mathbf{B}_3 = \{\chi_\Gamma : \Gamma \notin \mathbf{K} \text{ or } \mathcal{L} \setminus W(\Gamma) \notin \mathbf{K}\}$$

- $R_{s_\kappa^K} = R_1 \cup R_2$, where

$$R_1 = \{(\chi_{\mathcal{L} \setminus W(\Gamma)}, \chi_{W(\Gamma)}) : \Gamma \in \mathbf{K} \text{ and } \mathcal{L} \setminus W(\Gamma) \in \mathbf{K}\}$$

$$R_2 = \{(\chi_\Gamma, \chi_{W(\Gamma)}) : \Gamma \notin \mathbf{K} \text{ or } \mathcal{L} \setminus W(\Gamma) \notin \mathbf{K}\}$$

Then, $W = W_I^S$.

Proof. Let for some $\Gamma \subseteq \mathcal{L}$, $W(\Gamma) \not\subseteq W_I^S(\Gamma)$. So, there exists $\alpha \in W(\Gamma)$ such that $\alpha \notin W_I^S(\Gamma)$. Then, the following cases arise.

- (a) There exists $(\chi_{\mathcal{L} \setminus W(\Sigma)}, \chi_{W(\Sigma)}) \in R_1$ such that $\chi_{\mathcal{L} \setminus W(\Sigma)}(\Gamma) \subseteq \{1\}$ but $\chi_{W(\Sigma)}(\alpha) \neq 1$.
- (b) There exists $(\chi_\Sigma, \chi_{W(\Sigma)}) \in R_2$ such that $\chi_\Sigma(\Gamma) \subseteq \{1\}$ but $\chi_{W(\Sigma)}(\alpha) \neq 1$.

We discuss each case one by one.

Case 4.3.17.1. $\chi_{\mathcal{L} \setminus W(\Sigma)}(\Gamma) \subseteq \{1\}$ implies that $\Gamma \subseteq \mathcal{L} \setminus W(\Sigma)$, while $\chi_{W(\Sigma)}(\alpha) \neq 1$ implies that $\alpha \notin W(\Sigma)$. Now,

$$\begin{aligned} \Gamma \subseteq \mathcal{L} \setminus W(\Sigma) &\implies W(\Gamma) \subseteq W(\mathcal{L} \setminus W(\Sigma)) && \text{(by monotonicity of } W) \\ &\implies W(\Gamma) \subseteq W(\Sigma) && \text{(by Theorem 4.2.21(1), since } \mathcal{L} \setminus W(\Sigma) \in \mathbf{K}) \\ &\implies \alpha \in W(\Sigma) && \text{(since } \alpha \in W(\Gamma)) \end{aligned}$$

This is a contradiction.

Case 4.3.17.2. Since, $\chi_\Sigma(\Gamma) \subseteq \{1\}$, $\Gamma \subseteq \Sigma$. Then, by Theorem 4.3.10, since W is monotone, $W(\Gamma) \subseteq W(\Sigma)$. So, $\alpha \in W(\Gamma)$ implies that $\alpha \in W(\Sigma)$, i.e. $\chi_{W(\Sigma)}(\alpha) \neq 1$ – a contradiction.

Therefore, $W(\Gamma) \subseteq W_I^S(\Gamma)$, for all $\Gamma \subseteq \mathcal{L}$.

Next, let $\Gamma \subseteq \mathcal{L}$ and $\alpha \in W_I^S(\Gamma)$. If $\Gamma \in \mathbf{K}$ and $\mathcal{L} \setminus W(\Gamma) \in \mathbf{K}$, then $(\chi_{\mathcal{L} \setminus W(\Gamma)}, \chi_{W(\Gamma)}) \in R_{s_\kappa^\kappa}$. Since W is anti-reflexive $_\kappa^\kappa$, and $\Gamma \in \mathbf{K}$, $\Gamma \subseteq \mathcal{L} \setminus W(\Gamma)$, which implies that $\chi_{\mathcal{L} \setminus W(\Gamma)}(\Gamma) \subseteq \{1\}$. Therefore, since $\alpha \in W_I^S(\Gamma)$, we have $\chi_{W(\Gamma)}(\alpha) = 1$, i.e., $\alpha \in W(\Gamma)$. If, on the other hand, $\Gamma \notin \mathbf{K}$ or $\mathcal{L} \setminus W(\Gamma) \notin \mathbf{K}$, then $(\chi_\Gamma, \chi_{W(\Gamma)}) \in R_{s_\kappa^\kappa}$. Since $\chi_\Gamma(\Gamma) \subseteq \{1\}$ and $\alpha \in W_I^S(\Gamma)$, $\chi_{W(\Gamma)}(\alpha) = 1$, i.e., $\alpha \in W(\Gamma)$.

Thus, $W_I^S(\Gamma) \subseteq W(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$. Hence, $W = W_I^S$. $\square$

4.3.2 Suszko Reduction for Non-monotonic Logical Structures

We know from Theorem 4.1.6 and Theorem 4.3.4 that every logical structure, and hence any non-monotonic logical structure, has an adequate type-I S -semantics.

Theorem 4.3.18. *Let $(\mathcal{L}, W)$ be a logical structure and $\mathbf{S} = (\mathbf{B}, R, \models, \mathcal{P}(\mathcal{L}))$ an atomic S -semantics for $\mathcal{L}$ such that $W = W_I^S$. Then, for all $\Gamma \subseteq \mathcal{L}$, $W(\Gamma) = \mathcal{L}$ or $\bigcup_{\substack{\Gamma_0 \subseteq \Gamma \\ \Gamma_0 \text{ is finite}}} W(\Gamma_0) \subseteq W(\Gamma)$.*

Proof. Let $\Gamma \subseteq \mathcal{L}$ such that $W(\Gamma) \neq \mathcal{L}$ and $\alpha \notin W(\Gamma)$. Then, since $W = W_I^S$, $\alpha \notin W_I^S(\Gamma)$. So, there exists $(v, w) \in R$ such that $v \models \Gamma$ but $w \not\models \{\alpha\}$. Since $\mathbf{S}$ is atomic, $v \models \Gamma$ implies that $v \models \Gamma_0$ for all finite $\Gamma_0 \subseteq \Gamma$. Now, as $w \not\models \{\alpha\}$, $\alpha \notin W_I^S(\Gamma_0)$ for all finite $\Gamma_0 \subseteq \Gamma$, i.e., $\alpha \notin \bigcup_{\substack{\Gamma_0 \subseteq \Gamma \\ \Gamma_0 \text{ is finite}}} W_I^S(\Gamma_0)$. Since $W = W_I^S$, this implies that $\alpha \notin \bigcup_{\substack{\Gamma_0 \subseteq \Gamma \\ \Gamma_0 \text{ is finite}}} W(\Gamma_0)$. Thus,

$$\bigcup_{\substack{\Gamma_0 \subseteq \Gamma \\ \Gamma_0 \text{ is finite}}} W(\Gamma_0) \subseteq W(\Gamma). \text{ Hence, for all } \Gamma \subseteq \mathcal{L}, W(\Gamma) = \mathcal{L} \text{ or } \bigcup_{\substack{\Gamma_0 \subseteq \Gamma \\ \Gamma_0 \text{ is finite}}} W(\Gamma_0) \subseteq W(\Gamma). \quad \square$$

However, there exists non-monotonic logical structure $(\mathcal{L}, W)$ such that W does not satisfy the property mentioned in the above theorem. The following gives an example of such a logical structure.

Example 4.3.19. Let $(\mathcal{L}, W)$ be a logical structure where $\mathcal{L} = \mathbb{Q}$ (the set of rational numbers) and $W : \mathcal{P}(\mathcal{L}) \rightarrow \mathcal{P}(\mathcal{L})$ is defined as follows.

$$W(\Gamma) = \begin{cases} \mathbb{N} & \text{if } \Gamma \text{ is finite} \\ \mathcal{L} \setminus \Gamma & \text{if } \Gamma \text{ is infinite and } \Gamma \subsetneq \mathcal{L} \\ \emptyset & \text{otherwise (i.e., if } \Gamma = \mathcal{L}) \end{cases}$$

Clearly, $W(\Gamma) \neq \mathcal{L}$ for all $\Gamma \subseteq \mathcal{L}$. Choose $\mathbb{Q}$ and note that,

$$\bigcup_{\substack{\Gamma_0 \subseteq \mathbb{Q} \\ \Gamma_0 \text{ is finite}}} W(\Gamma_0) = \mathbb{N} \not\subseteq \emptyset = W(\mathbb{Q})$$

However, $(\mathcal{L}, W)$ is non-monotonic since $\mathbb{N} \subseteq \mathbb{N} \cup \{\frac{1}{2}\}$, but $W(\mathbb{N}) = \mathcal{L} \setminus \mathbb{N} \not\subseteq \mathcal{L} \setminus (\mathbb{N} \cup \{\frac{1}{2}\}) = W(\mathbb{N} \cup \{\frac{1}{2}\})$.

Every logical structure that has an adequate type-I normal S -semantics is monotonic by [Theorem 4.3.10](#). We now show that $(\mathcal{L}, W)$, viz., the ones where W satisfies *cautious monotonicity* or *weak cumulative transitivity* have adequate type-II normal S -semantics.

Definition 4.3.20 (CAUTIOUS MONOTONICITY, WEAK CUMULATIVE TRANSITIVITY). Suppose $(\mathcal{L}, W)$ is a logical structure.

- (i) W is said to satisfy *cautious monotonicity* if for all $\Gamma \cup \Sigma \subseteq \mathcal{L}$, $\Gamma \subseteq \Sigma \subseteq W(\Gamma)$ implies that $W(\Gamma) \subseteq W(\Sigma)$. In this case, $(\mathcal{L}, W)$ is said to be of *cm-type*.
- (ii) W is said to satisfy *weak cumulative transitivity* if for all $\Gamma \cup \Sigma \subseteq \mathcal{L}$, $\Gamma \subseteq \Sigma \subseteq W(\Gamma)$ implies that $W(\Sigma) \subseteq W(\Gamma)$. In this case, $(\mathcal{L}, W)$ is said to be of *wct-type*.

Remark 4.3.21. ‘Cautious monotonicity’ was introduced by Gabbay in 1985 under the name ‘restricted monotonicity’ (see, e.g., [\[57\]](#)). In the above definition, we have used Makinson’s version of ‘cautious monotonicity’ instead (see, e.g., [\[73\]](#)). The name ‘weak cumulative transitivity’ is taken from [\[86\]](#).

Theorem 4.3.22 (SUSZKO REDUCTION FOR *cm*-TYPE). *Let $(\mathcal{L}, W)$ be a *cm-type* logical structure and $S = (\mathbf{B}, R_{cm}, \models, \mathcal{P}(\mathcal{L}))$ be a normal S -semantics for $\mathcal{L}$ satisfying the following properties.*

- $\mathbf{B} = \{\chi_\Gamma : \Gamma \subseteq \mathcal{L}\} \cup \{\chi_{W(\Gamma)} : \Gamma \subseteq \mathcal{L}\}$.
- $R_{cm} = \bigcup_{\Gamma \subseteq \mathcal{L}} R_\Gamma$, where for each $\Gamma \subseteq \mathcal{L}$, R_Γ is such that

$$(\chi_\Sigma, \chi_{W(\Sigma)}) \in R_\Gamma \text{ iff } \Sigma = \Gamma \text{ or } \Sigma \subseteq W(\Gamma)$$

Then, $W = W_{II}^S$.

Proof. Let $\Gamma \subseteq \mathcal{L}$ such that $W(\Gamma) \not\subseteq W_{II}^S(\Gamma)$. Then, there exists $\alpha \in W(\Gamma)$ such that $\alpha \notin W_{II}^S(\Gamma)$. Then, for all $U \subseteq R_{cm}$, $(\chi_\Gamma, \chi_{W(\Gamma)}) \notin U$ or there exists $(\chi_\Sigma, \chi_{W(\Sigma)}) \in U$ such that $\chi_\Sigma \models \Gamma$ but $\chi_{W(\Sigma)} \not\models \{\alpha\}$. Now, $(\chi_\Gamma, \chi_{W(\Gamma)}) \in R_\Gamma$. So, there exists $\Sigma \subseteq \mathcal{L}$ with $(\chi_\Sigma, \chi_{W(\Sigma)}) \in R_\Gamma$ such that $\chi_\Sigma \models \Gamma$ but $\chi_{W(\Sigma)} \not\models \{\alpha\}$.

Now, $\chi_\Sigma \models \Gamma$ implies that $\Gamma \subseteq \Sigma$, and $\chi_{W(\Sigma)} \not\models \{\alpha\}$ implies that $\alpha \notin W(\Sigma)$. Since, $\alpha \in W(\Gamma)$, $\Sigma \neq \Gamma$. Then, as $(\chi_\Sigma, \chi_{W(\Sigma)}) \in R_\Gamma$, $\Sigma \subseteq W(\Gamma)$. Thus, $\Gamma \subseteq \Sigma \subseteq W(\Gamma)$, and hence by cautious monotonicity, $W(\Gamma) \subseteq W(\Sigma)$. Then, since $\alpha \notin W(\Sigma)$, $\alpha \notin W(\Gamma)$ – a contradiction. Thus, $W(\Gamma) \subseteq W_{II}^S(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$.

Next, let $\Gamma \subseteq \mathcal{L}$ be such that $W_{II}^S(\Gamma) \not\subseteq W(\Gamma)$. So, there exists $\alpha \in W_{II}^S(\Gamma)$ such that $\alpha \notin W(\Gamma)$. Now $R_\Gamma \subseteq R_{cm}$ and $(\chi_\Gamma, \chi_{W(\Gamma)}) \in R_\Gamma$. Then, since $\chi_\Gamma \models \Gamma$ and $\chi_{W(\Gamma)} \not\models \{\alpha\}$, $\alpha \notin W_{II}^S(\Gamma)$ – a contradiction. Thus, $W_{II}^S(\Gamma) \subseteq W(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$. Hence $W = W_{II}^S$. $\square$

Theorem 4.3.23 (SUSZKO REDUCTION FOR WEAK CUMULATIVE TRANSITIVE LOGICAL STRUCTURES). *Let $(\mathcal{L}, W)$ be a wct-type logical structure and $\mathbf{S} = (\mathbf{B}, R_{wct}, \models, \mathcal{P}(\mathcal{L}))$ be a normal S -semantics for $\mathcal{L}$ satisfying the following properties.*

- $\mathbf{B} = \{\chi_\Gamma : \Gamma \subseteq \mathcal{L}\} \cup \{\chi_{W(\Gamma)} : \Gamma \subseteq \mathcal{L}\}$.
- $R_{wct} = \bigcup_{\Gamma \subseteq \mathcal{L}} R_\Gamma$ where for each $\Gamma \subseteq \mathcal{L}$,

$$R_\Gamma = \{(\chi_\Gamma, \chi_{W(\Gamma)})\} \cup \{(\chi_{W(\Sigma)}, \chi_{W(\Sigma)}) : \Sigma \subseteq \Gamma\}$$

Then, $W = W_{II}^S$.

Proof. Let $\Gamma \subseteq \mathcal{L}$ such that $W(\Gamma) \not\subseteq W_{II}^S(\Gamma)$. Then, there exists $\alpha \in W(\Gamma)$ such that $\alpha \notin W_{II}^S(\Gamma)$. Then, for all $U \subseteq R_{wct}$, $(\chi_\Gamma, \chi_{W(\Gamma)}) \notin U$ or there exists $(u, v) \in U$ such that $u \models \Gamma$ but $v \not\models \{\alpha\}$. Since $(\chi_\Gamma, \chi_{W(\Gamma)}) \in R_\Gamma$, there exists $(u', v') \in R_\Gamma$ such that $u' \models \Gamma$ but $v' \not\models \{\alpha\}$. Moreover, $v' \not\models \{\alpha\}$ implies that $v' \neq \chi_{W(\Gamma)}$, and so $(u', v') \neq (\chi_\Gamma, \chi_{W(\Gamma)})$. Therefore, $u' = v' = W(\Delta)$ for some $\Delta \subseteq \mathcal{L}$. Now, $\chi_{W(\Delta)} \models \Gamma$ implies that $\Gamma \subseteq W(\Delta)$, and $\chi_{W(\Delta)} \not\models \{\alpha\}$ implies that $\alpha \notin W(\Delta)$. Furthermore, since $(\chi_{W(\Delta)}, \chi_{W(\Delta)}) \in R_\Gamma$, $\Delta \subseteq \Gamma$ which implies that $\Delta \subseteq \Gamma \subseteq W(\Delta)$. Therefore, by weak cumulative transitivity, $W(\Gamma) \subseteq W(\Delta)$. Then, as $\alpha \notin W(\Delta)$, $\alpha \notin W(\Gamma)$ – a contradiction. Thus, $W(\Gamma) \subseteq W_{II}^S(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$.

Next, let $\Gamma \subseteq \mathcal{L}$ such that $W_{II}^S(\Gamma) \not\subseteq W(\Gamma)$. So, there exists $\alpha \in W_{II}^S(\Gamma)$ such that $\alpha \notin W(\Gamma)$. Now $R_\Gamma \subseteq R_{cm}$ and $(\chi_\Gamma, \chi_{W(\Gamma)}) \in R_\Gamma$. Then, since $\chi_\Gamma \models \Gamma$ and $\chi_{W(\Gamma)} \not\models \{\alpha\}$, $\alpha \notin W_{II}^S(\Gamma)$ – a contradiction. Thus, $W_{II}^S(\Gamma) \subseteq W(\Gamma)$ for all $\Gamma \subseteq \mathcal{L}$. Hence $W = W_{II}^S$. $\square$

4.4 What is a Many-valued Logical Structure?

A *many-valued logic* is usually thought to be a logic which does not have an adequate bivalent semantics. The term ‘bivalent semantics’ is generally understood via the notion of logical matrix (see, e.g., [76]). However, as we have pointed out in [Section 4.1](#), there is no *formal*

definition of bivalent semantics. The definition based on logical matrices is problematic, as pointed out in [48].

[T]his characterization is purely matrical and there are several reasons to wish not to be blocked within matrix theory. Firstly, because the intuitive idea of many-valuedness does not necessarily depend on matrix theory, and secondly, because the rise of two-valued non-matrical semantics has shed a new light on the problem. . . .

This problem is perhaps of even greater significance while attempting to define many-valued *logical structures*. In fact, one might argue that there is no *a priori* reason to be ‘confined’ by *any* particular definition of many-valued logical structures based on a specific notion of semantics, including the ones proposed in Definition 4.1.1 or Definition 4.3.1.

However, if we are willing to maintain such a relativistic perspective, we have to accept that different definitions of semantics will lead to different conclusions regarding **ST** (cf. e.g., Theorem 4.2.10 and Theorem 4.3.13), and hence, potentially different, mutually incomparable definitions of many-valued logical structures.

Although this might be mathematically enriching, it is perhaps unsatisfactory from a philosophical perspective because such an approach fails to elucidate the intuition behind ‘many-valuedness’. Consequently, one cannot help but wonder whether something more fundamental is hidden underneath this potential menagerie of notions. The issue of many-valuedness is perhaps intimately connected with the *principle of bivalence*. We expound on this connection in the rest of this section.

The following statements are two very general formulations of the principle of bivalence in the context of logical structures.

PoB₁^P : There is a property P such that for every logical structure $(\mathcal{L}, W)$ exactly one of the following holds:

- (a) $(\mathcal{L}, W)$ satisfies P .
- (b) $(\mathcal{L}, W)$ does not satisfy P .

PoB₂^P : For every logical structure $(\mathcal{L}, W)$, there is a property P such that exactly one of the following statements holds:

- (a) $(\mathcal{L}, W)$ satisfies P .
- (b) $(\mathcal{L}, W)$ does not satisfy P .

One needs to be a bit careful regarding the precise notion of ‘property’ here. One way to interpret **ST** is to view it as a statement about properties preserved in a valid logical inference. This can be phrased as follows: *there is a property Q that is preserved from the premise set(s) to the conclusion(s) in a valid logical inference*. A more precise version of this statement in the context of logical structures would be the following: *there is a property Q such for every logical structure $(\mathcal{L}, W)$ and $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, the following statement holds: if*

$\alpha \in W(\Gamma)$ and Γ satisfies Q then so does α . In other words, every logically valid inference is Q -preserving; the importance of **ST** lies in the fact that it declares Q to be *truth*.⁵ Notice that any such property Q naturally gives rise to a property $\overline{Q}$ such that the following holds for every logical structure $(\mathcal{L}, W)$.

$(\mathcal{L}, W)$ satisfies $\overline{Q}$ iff $(\mathcal{L}, W)$ is Q -preserving.

Henceforth, whenever we refer to $\mathbf{PoB}_1^P$ or $\mathbf{PoB}_2^P$, we will understand P to be of the form $\overline{Q}$ for some property Q .

Now, if one subscribes to the *truth-theoretic conception of the notion of logical validity* – the view that in logical inferences, ‘truth’ is necessarily preserved from the premise(s) to the conclusion(s) – and maintains that there is a generic notion of ‘truth’, then **ST** can be seen as a principle of bivalence of the form $\mathbf{PoB}_1^P$. To see this, let P be defined as follows: a logical structure $(\mathcal{L}, W)$ satisfies P iff it has an adequate bivalent semantics; the two values representing, respectively, *designation* and *anti-designation*. In other words, if a logical structure $(\mathcal{L}, W)$ has an adequate bivalent semantics, then every element of $\mathcal{L}$ is either designated or anti-designated (but not both). Now, if we identify the (meta-logical) property of ‘designation’ with ‘truth’ (as Suszko also does), we obtain **ST**. This is explained in more detail in the following passage from [101] (emphasis ours).

[T]o identify a property with truth, at the bare minimum, it must be possible to treat such a property as a semantic value of sentences. This is a reasonable assumption, since truth and falsehood are not just semantic properties, but the semantic properties par excellence of sentences. If a property could not be treated as a value of sentences in a suitable semantics, then an identification with truth would be prevented. Suszko’s theorem fills this possible gap between designation and truth by showing that designation can actually be turned into a semantic (namely algebraic) value. Clearly, the mere possibility of treating a property as a semantic referent of a suitable semantic is not sufficient to conclude that it represents truth. After all, a semantic value could also stand for other semantic properties, like, for instance, falsehood. Suszko’s theorem alone only shows that designation can be identified with a semantic value, but it does not show that it is a semantic value for truth. This, however, is a little step easily covered once it is accepted that validity consists in necessary truth preservation. If valid inferences are those necessarily preserving truth, and designation is the value preserved in valid inferences, then designation and truth must be identical. That validity consists in truth preservation is a reasonable philosophical assumption whose legitimacy Suszko accepts without further inquiry, and that it is also assumed in the truth pluralist debate. The crucial and critical step, thus, is exactly the one handled by Suszko’s theorem, which shows that designation can be turned into a semantic (algebraic) value of an adequate semantics.

⁵This is more or less the analysis of [106]. However, whereas Wansing and Shramko argue for the *falsity* of **ST**, we argue for the *truth* of **ST** (cf. [Remark 4.3.7](#)).

This identification, however, is not maintained by every *truth pluralist*⁶ even if they believe in the truth-theoretic account of logical validity. The reason behind this is explained succinctly in the following passage from [101] (emphasis ours).

The reason is that the notion of designation, unproblematic as it might look, is troublesome in the context of truth pluralism. It is not hard to see why. The definition of validity is made possible by having at disposal a set of designated values that allows to speak of a sentence having an unspecified designated value. *...It thus follows that having a designated value is equivalent to [...] being true in some way. Since this seems to amount to a generic notion of truth, designation arguably carries with it a commitment to a generic property of truth. Consequently, if truth pluralists reject generic truth, the notion of designation is anathema to them.*

However, not committing to a truth-theoretic account of validity is not going to help if one is a pluralist relative to some other property P as well. Moreover, not being a pluralist relative to P will require further justification.

Nevertheless, one can maintain $\mathbf{PoB}_2^P$ without any qualms since in this case the property P depends on the logical structure under consideration. More precisely, given two properties, P and Q , it is possible that a logical structure satisfies $\mathbf{PoB}_2^Q$ but not $\mathbf{PoB}_2^P$. Then, it may be referred to as ‘many-valued relative to P ’ but ‘two-valued relative to Q ’.

We note the following passage from [101] (emphasis ours).

To make the point more vivid, and clearly see why the theorem is crucial to defend the philosophical thesis, suppose that Suszko’s theorem did not go through, and designation could not be turned into an algebraic value of a suitable semantics. In this case, two ingredients would be disentangled in the formulation of a many-valued logic: a level of semantic properties (like truth, falsehood, and possibly others) represented by algebraic values and a level of logical notions employed to define validity (like designation and non-designation). Algebraic values would correspond to properties of sentences that characterise their semantic status. Logical notions, by contrast, would not characterise any semantic feature of sentences, but just display their logical relations. While certainly connected, logic and semantics would be taken apart. In particular, logical notions like designation would not be, even implicitly, semantic values. *Designation, far from being a value standing for a semantic property on its own, would just be a meta-logical notion employed to characterise what relations among semantic values take place in valid inferences.* In such a case, the philosophical claim that every logic is logically two-valued would vanish. First of all, with the reduction neutralised, the semantic values transmitted in valid inferences could only be those in the many-valued semantics, because there would not be other values at all. Strictly speaking, designation could not be a preserved value, because it

⁶A *truth pluralist* is person who believes ‘that sentences belonging to different areas of discourse can be, and in some cases are, true in different ways’ [101]

could not even be a value. Secondly, given that designation could not be a value, let alone the preserved value, it could not be identified with truth. Thirdly, given that designation would not stand for a value, *logical bivalence* (reflected in the bivalent distinction between designated and non-designated) would not collapse into semantic *two-valuedness*. Many-valued logic would be authentically semantically many-valued and truth one of the semantic values—and indeed the preserved value—despite logical validity being defined in bivalent terms by means of bivalent meta-theoretical notions. In this way, the threat to many-valued logic would be thwarted.

Suszko’s Theorem (Suszko Reduction) thus shows us that the metalogical property of designation can be ‘expressed’ in the object language. The fact that there can only be two possibilities, i.e., designation and non-designation, is thus crucial to Suszko Reduction. However, this dichotomy is nothing but a meta-linguistic assumption about designation. This point has been made, e.g., in [53].

One of the ways to understand Suszko’s distinction between “algebraic values” and “logical values” is to interpret the latter as *the truth values of the metatheory*, while the former would be *the truth values of the theory*.

In other words, **ST** can be seen as a pointer to the bivalence of the metatheory. Now, what happens if the metatheory is *not* two-valued, or is it even possible for the metatheory to *not* be two-valued?

Caleiro et al. in [34] say that “there is some metalinguistic bivalence that one will not easily get rid of: *either* an inference obtains *or* it does not, but *not both*”. However, many-valued consequence relations at the meta-level have been discussed and investigated thoroughly in [38]. A quotation from [38] would be relevant here (emphasis ours).

[T]he objective of the theory of graded consequence (GCT) is to *introduce many-valuedness to the consequence relation* and other related notions such as consistency, tautologihood, etc. *This is in the same direction as had been pursued during the origination and development of many-valued logics*. While in many-valued logics the object-level formulae are considered to be many-valued, meta-level notions like consequence are taken as two-valued; in GCT, items of both the levels are treated as many-valued. Various reasons may be ascribed for lifting the meta-level notions to many-valued ones.

We think that it is possible to give a precise definition of many-valued logical structures with the help of GCT. The rest of this section is devoted to realising one such possibility.

We begin by observing that the crispness of $\models$ in $S_{\mathcal{E}}$ relies on the crispness of $\models_i$ ’s (see Theorem 4.3.4 and Theorem 4.3.5). We, therefore, propose a notion of many-valued logical structures via many valued $\models_i$ ’s, similar to what has been done in GCT. More specifically, we define the following.

Definition 4.4.1 (κ -VALUED SEMANTICS OF ORDER λ). Given a set $\mathcal{L}$, a cardinal κ and an ordinal $\lambda > 0$, a κ -valued semantics of order λ for a family of sets $\{\mathcal{L}_i\}_{i \in \lambda}$ is a tuple of the following form $\mathfrak{S} = (\{\mathbf{M}_i\}_{i \in \lambda}, \{\models_i\}_{i \in \lambda}, \{A_i\}_{i \in \lambda}, S, \{\mathcal{P}(\mathcal{L}_i)\}_{i \in \lambda})$ satisfying the following conditions.

- (i) $\mathcal{L}_0 = \mathcal{L}$.
- (ii) For each $i \in \lambda$, $\mathbf{M}_i, A_i$ are sets, $|A_i| \geq \kappa$, and $\models_i: \mathbf{M}_i \times \mathcal{P}(\mathcal{L}_i) \longrightarrow A_i$ for each $i \in \lambda$.
- (iii) There exists $i \in \lambda$ such that $|A_i| = \kappa$.
- (iv) $\emptyset \subsetneq S \subseteq \{(\models_i, a_i), (\models_j, a_j) : (a_i, a_j) \in A_i \times A_j\}$.

Given a κ -valued semantics of order λ for a family of sets $\{\mathcal{L}_i\}_{i \in \lambda}$, the entailment relation induced by $\mathfrak{S}$, denoted by $\vdash_{\mathfrak{S}}^{(\kappa, \lambda)}$ is defined as follows. For all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$,

$$\Gamma \vdash_{\mathfrak{S}}^{(\kappa, \lambda)} \alpha \iff \text{for all } ((\models_i, a_i), (\models_j, a_j)) \in S \text{ and } (m_i, m_j) \in \mathbf{M}_i \times \mathbf{M}_j, \\ \models_i(m_i, \Gamma) = a_i \text{ implies that } \models_j(m_j, \{\alpha\}) = a_j$$

We now define a many-valued logical structure as follows.

Definition 4.4.2 (κ -VALUED LOGICAL STRUCTURE OF ORDER λ). Suppose κ is a cardinal, $\lambda > 0$ is an ordinal, $\{(\mathcal{L}_i, \models_i)\}_{i \in \lambda}$ is a family of logical structures and $\mathfrak{S} = (\{\mathbf{M}_i\}_{i \in \lambda}, \{\models_i\}_{i \in \lambda}, \{A_i\}_{i \in \lambda}, S, \{\mathcal{P}(\mathcal{L}_i)\}_{i \in \lambda})$ is a κ -valued semantics of order λ . A logical structure $(\mathcal{L}, \vdash)$ is said to be κ -valued of order λ for $\{(\mathcal{L}_i, \models_i)\}_{i \in \lambda}$ relative to $\mathfrak{S}$ if the following conditions hold.

- (i) For all $i, j \in \lambda$, with $i \in j$, there exists an injective function from $\mathcal{L}_j$ to $\mathcal{L}_i$.
- (ii) $\vdash = \vdash_{\mathfrak{S}}^{(\kappa, \lambda)}$.
- (iii) For every μ -valued semantics $\mathfrak{T} = (\{\mathbf{N}_i\}_{i \in \lambda}, \{\models_i\}_{i \in \lambda}, \{A_i\}_{i \in \lambda}, S, \{\mathcal{P}(\mathcal{L}_i)\}_{i \in \lambda})$ of order λ for $\{\mathcal{L}_i\}_{i \in \lambda}$, with $\mu < \kappa$, $\vdash \neq \vdash_{\mathfrak{T}}^{(\mu, \lambda)}$.

A logical structure $(\mathcal{L}, \vdash)$ is said to be κ -valued of order λ , if there exists a semantics $\mathfrak{S}$ and a family of sets $\{\mathcal{L}_i\}_{i \in \lambda}$ such that $(\mathcal{L}, \vdash)$ is κ -valued of order λ for $\{\mathcal{L}_i\}_{i \in \lambda}$ relative to $\mathfrak{S}$.

Remark 4.4.3. It is imperative to emphasise that even though we defined $\vdash_{\mathfrak{S}}^{(\kappa, \lambda)}$ in a specific manner, other definitions are possible. Thus, a definition of κ -valued logical structure depends on how $\vdash_{\mathfrak{S}}^{(\kappa, \lambda)}$ is defined.

Remark 4.4.4. (a) The key idea behind the above definition is the observation of the distinction between object and metalanguage and the resulting hierarchy of languages (see, e.g., [61]). This is the main reason for considering a family of logical structures $\{(\mathcal{L}_i, \models_i)\}_{i \in \lambda}$. This family is indexed via ordinals so as to introduce an ordering among the $\mathcal{L}_i$'s in such a way that for all $i \in \lambda$, $\mathcal{L}_{i+1}$ can be thought of as a metalanguage of $\mathcal{L}_i$.

- (b) To distinguish between the *use* and *mention* of a word or symbol, we sometimes put quotation marks around that symbol whenever we mention it. In the context of (formal) object language and (formal) metalanguage, this means that for every formula α of the object language, ' α ' is a constant symbol of the metalanguage (see, [39, Footnote 136] and [38, Section 6.2]). This gives rise to a natural injective function from the metalanguage to the object language under consideration. Property (i) of Definition 4.4.2 has been formulated with this issue in mind.
- (c) Each $\mathbf{M}_i$ can be thought of as a 'class of models' of the language $\mathcal{L}_i$. Similarly, $\models_i$'s can be thought of as providing the 'satisfaction relation' between the models and the corresponding languages.
- (d) The motivation for (iii) and (iv) are similar to that of Remark 4.1.10.

Since this hierarchy of languages is not unique for a given (object) language, the crux of all these discussions is that *many-valuedness of a logical structure is not dependent on the logical structure alone but also on the language/meta-language hierarchy*. Hence, interpreting 'logical value' as truth-values of the metatheory/metalogic makes us wonder about the reason for ignoring the truth-values of the higher order metatheories/metalogics. A justification for this, however, can be provided by observing that in most cases, it suffices to consider the meta-metalogic to be a fragment of classical propositional logic, as e.g., in [38]. Nevertheless, there is no *a priori* reason for not considering a many-valued meta-metalogic, and continuing in this line of argument, we can conclude that there is no reason to consider *any one* of the higher metalogic to be two-valued.

This motivates us to propose the following generalisations of **ST**.

- (a) Let $(\mathcal{L}, \vdash)$ be a logical structure which is κ -valued of order λ for $\{(\mathcal{L}_i, \vdash_i)\}_{i \in \lambda}$ relative to a κ -valued semantics of order λ . Then, there exists $i \in \lambda$ and an ordinal γ such that $(\mathcal{L}_i, \vdash_i)$ is 2-valued of order γ .
- (b) Let $(\mathcal{L}, \vdash)$ be a logical structure which is κ -valued of order λ for $\{(\mathcal{L}_i, \vdash_i)\}_{i \in \lambda}$ relative to a semantics κ -valued semantics of order λ . Then, for all $i \in \lambda$ there exists a cardinal σ_i and an ordinal γ_i with $\sigma_i \leq \kappa$ such that $(\mathcal{L}_i, \vdash_i)$ is σ_i -valued of order γ_i .

4.5 Future Works

- Is it possible to obtain an adequate S -semantics for *every* non-monotonic logical structure, perhaps by changing the notion of S -entailment?
- Given an ordinal λ and $n \in \mathbb{N}$, what exactly are the distinguishing properties between a n - and a $n + 1$ -valued logical structure of order λ ?
- Given a set $\mathcal{L}$, is it possible to characterise $\mathcal{F}_\kappa(\mathcal{L})$ for each cardinal $\kappa \geq 1$?
- Is it possible to find a minimal adequate S -semantics for every logical structure which admits an adequate S -semantics?

- Is it possible to give example(s) of logic(s) for which both the generalisation of Suszko's Thesis, as proposed above, fails?

Chapter 5

Generalised Explosion Principles¹

5.1 Introduction

A *principle of explosion* can be described broadly as a rule that allows one to entail every $\beta \in \mathcal{L}$, where $(\mathcal{L}, \vdash)$ is a logical structure, from a $\Gamma \subsetneq \mathcal{L}$, i.e., $\Gamma \vdash \beta$ for all $\beta \in \mathcal{L}$. The set Γ is then said to be *trivial* or to *explode* (see [Definition 2.4.17](#)). Perhaps the most discussed principle of explosion is the *ex contradictione sequitur quodlibet* (*ECQ*) which asserts that any $\Gamma \subseteq \mathcal{L}$ that contains, for some $\alpha \in \mathcal{L}$, the pair $\alpha, \neg\alpha$, where $\neg$ is a negation operator, explodes. Assuming monotonicity, this is then expressed symbolically as follows. For any $\alpha \in \mathcal{L}$,

$$\{\alpha, \neg\alpha\} \vdash \beta \quad \text{for all } \beta \in \mathcal{L}.$$

However, principles of explosion come in various other forms, as discussed in [\[93\]](#). These different forms of explosion are all equivalent in classical logic but can differ in case of weaker logics.

Paraconsistency is characterised by the failure of a principle of explosion. Thus, failure of different principles of explosion lead to different notions of paraconsistency. Hence it is possible to have logics paraconsistent in one sense but not in the other. We have discussed this phenomenon in more detail in [\[10\]](#). An observation we made there is the dependence of all but one principle of explosion on a negation operator. This is not surprising considering that a contradiction can be described as the simultaneous assertion and denial of a statement, the latter of which is often regarded as equivalent to accepting the negation of the statement. However, it is not hard to see that the denial of a statement may not amount to the acceptance of its negation and that a statement can oppose another without being the negation of it (see, e.g., the discussion on this in [\[36, Chapter 1\]](#)).

There is, however, one principle of explosion, and hence a corresponding notion of paraconsistency, that is slightly different as it does not explicitly depend on negation. Instead, it depends on the presence of a nullary operator or constant and its properties. Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure and $\perp \in \mathcal{L}$ is a nullary operator or constant such that

¹A version of this chapter forms part of [\[11\]](#), and a preliminary version of it is published in [\[10\]](#).

$\{\perp\} \vdash \beta$ for all $\beta \in \mathcal{L}$. This is a principle of explosion that we can call $\perp$ -ECQ², and such a constant, a *falsity constant* for the logical structure $\mathcal{S}$. A logical structure $\mathcal{S} = (\mathcal{L}, \vdash)$ is said to be $\perp$ -*paraconsistent* if there is no falsity constant in $\mathcal{L}$ for the logical structure.

Now, as discussed in [93], a falsity constant $\perp$ (referred to as “Das Absurde” there) can be understood in multiple ways. Under certain understandings, $\perp$ -paraconsistency is essentially the same as paraconsistency, while in others, they fork. Moreover, there are understandings of $\perp$ that make the notion of $\perp$ -paraconsistency inherently dependent on negation, while in others where $\perp$ is not definitionally eliminable (see [94]), that is not the case. In the latter case, although this notion of paraconsistency is different from that of paraconsistency and independent of negation, it requires the presence of another logical operator, viz., $\perp$.

The broad question we dealt with in [10] is the following. Is it possible to describe paraconsistency (and thus, also explosion) independently of the language or connectives, especially negation? One motivation for this question comes from universal logic [14, 19]. The fundamental notion in universal logic is that of a logical structure, which is a pair of the form $(\mathcal{L}, \vdash)$, where $\mathcal{L}$ is a set and $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$. A logical structure is thus defined without regard to connectives, and the set $\mathcal{L}$ need not be a formula algebra as in a logic. The above question can therefore be rephrased as follows. Can paraconsistency be defined in the context of universal logic, i.e., is it possible to define something like a *paraconsistent logical structure*? In some cases, where $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logic such that $\mathcal{L}$ is an algebra, and $\neg$ is a unary operator in $\mathcal{S}$, with respect to which we discuss explosion or the failure of it, one might think of these as negation operators. Keeping in mind that there might be more than one such unary operator in $\mathcal{S}$, we will refer to ECQ written in terms of $\neg$ as $\neg$ -ECQ, dropping the prefix only when there is no chance of confusion or when commenting in general about $\neg$ -ECQ for some unary operator $\neg$. If $\neg$ -ECQ fails in $\mathcal{S}$, then the logic $\mathcal{S}$ will be called paraconsistent with respect to $\neg$.

We have freely moved back and forth between the relation $\vdash$ and its corresponding operator $W_{\vdash} : \mathcal{P}(\mathcal{L}) \longrightarrow \mathcal{P}(\mathcal{L})$ (see Remark 2.2.3 for details).

Section 5.2 discusses and extends one of the answers provided in [10] to the broad question mentioned above. This is achieved via a generalisation of ECQ. Here we include the discussion on a *quasi-negation*, also introduced in [10]. This, in a way, points to the extent to which negation is fundamental to the notion of paraconsistency.

In the remainder of the chapter, our focus rests primarily on the notion of explosion. The main question thus becomes the following. What is a principle of explosion? We discuss various possible alternatives to the already-known principles of explosion and their interconnections. Thus, in a way, this generalises and extends the discussion started in [10]. However, we neither claim nor believe that the list of principles of explosion provided here is exhaustive.

²This is also referred to as the principle of *ex falso sequitur quodlibet*, as in [36, 37].

5.2 Negation-free Paraconsistency

The material in this section is an extension of [10, Section 2]. We have included the results from there, along with the new developments, for context and continuity.

The standard principle of explosion, ECQ, can be generalised as follows. For each $\alpha \in \mathcal{L}$ in a logical structure $\mathcal{S} = (\mathcal{L}, \vdash)$, there exists $\beta \in \mathcal{L}$ such that

$$\{\alpha, \beta\} \vdash \gamma, \text{ where } \gamma \in \mathcal{L} \text{ is arbitrary.}$$

We call this rule gECQ (the letter ‘g’ denoting ‘generalised’). In other words, gECQ says that for each $\alpha \in \mathcal{L}$, there exists $\beta \in \mathcal{L}$ such that $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$. Then a type of paraconsistency can be defined without referring to a negation operator as follows.

Definition 5.2.1 (NF-PARACONSISTENT LOGICAL STRUCTURE). A logical structure $\mathcal{S} = (\mathcal{L}, \vdash)$ is *NF-paraconsistent* (NF stands for negation-free) if gECQ fails in it, i.e., there exists an $\alpha \in \mathcal{L}$ such that for every $\beta \in \mathcal{L}$, $W_{\vdash}(\{\alpha, \beta\}) \neq \mathcal{L}$.

Theorem 5.2.2. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is an NF-paraconsistent logic with a unary operator $\neg$. Then $\mathcal{S}$ is also paraconsistent with respect to $\neg$.*

Proof. Suppose $(\mathcal{L}, \vdash)$ is NF-paraconsistent. Then there exists $\alpha \in \mathcal{L}$ such that for all $\beta \in \mathcal{L}$, $W_{\vdash}(\{\alpha, \beta\}) \neq \mathcal{L}$. Then, in particular, $W_{\vdash}(\{\alpha, \neg\alpha\}) \neq \mathcal{L}$. Hence $(\mathcal{L}, \vdash)$ is paraconsistent. $\square$

It is, however, possible for a logic to be paraconsistent but not NF-paraconsistent (see [Example 5.2.14](#) below).

Theorem 5.2.3. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a monotonic logical structure. If $\mathcal{S}$ is NF-paraconsistent, then there does not exist any $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$.*

Proof. Suppose there exists $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$. Now, let $\alpha \in \mathcal{L}$. Then $W_{\vdash}(\{\alpha, \varphi\}) = \mathcal{L}$ by monotonicity. Thus, for every $\alpha \in \mathcal{L}$, there exists $\beta \in \mathcal{L}$, viz., φ , such that $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$. Hence $(\mathcal{L}, \vdash)$ is not NF-paraconsistent. Thus, if $\mathcal{S}$ is a monotonic NF-paraconsistent logical structure, then there cannot be a $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$. $\square$

Remark 5.2.4. It is well-known that the reflexivity and transitivity conditions together imply monotonicity. Thus, the above result also holds for logical structures that are simultaneously reflexive and transitive.

We now obtain the following result of [10] as a corollary to the above theorem.

Corollary 5.2.5. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a monotonic logical structure. If there is a falsity constant for the logical structure in $\mathcal{L}$, then $\mathcal{S}$ is not NF-paraconsistent, i.e., if $\mathcal{S}$ is NF-paraconsistent, then it must be $\perp$ -paraconsistent.*

Proof. Clear, since if there is a falsity constant for the logical structure $\perp \in \mathcal{L}$, then $W_{\vdash}(\{\perp\}) = \mathcal{L}$. $\square$

We now turn to some examples of NF-paraconsistent logical structures.

Example 5.2.6. Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure, where $\vdash$ is defined as follows. For any $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, $\Gamma \vdash \alpha$ iff $\alpha \in \Gamma$. Such a logical structure, which may be called a *purely reflexive* logical structure, is clearly NF-paraconsistent as long as $\mathcal{L}$ has at least three distinct elements.

Such a purely reflexive logical structure is also monotonic and transitive, and hence of Tarski-type.

Example 5.2.7. Suppose $(\mathcal{L}, \leq)$ is a poset with two incomparable elements α, β (i.e., $\alpha \not\leq \beta$ and $\beta \not\leq \alpha$). We now define a logical structure $\mathcal{S} = (\mathcal{L}, \vdash)$ as follows. For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$, $\Gamma \vdash \varphi$ iff for all $\gamma \in \Gamma$, $\gamma \leq \varphi$. Clearly, for all $\gamma \in \mathcal{L}$, $\{\alpha, \gamma\} \not\vdash \beta$ and $\{\beta, \gamma\} \not\vdash \alpha$. Thus, for all $\gamma \in \mathcal{L}$, $W_{\vdash}(\{\alpha, \gamma\}) \neq \mathcal{L}$ and $W_{\vdash}(\{\beta, \gamma\}) \neq \mathcal{L}$. Hence $\mathcal{S}$ is NF-paraconsistent.

We can define a dual logical structure, $\mathcal{S}' = (\mathcal{L}, \vdash')$, of $\mathcal{S}$ as follows. For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$, $\Gamma \vdash' \varphi$ iff for all $\gamma \in \Gamma$, $\varphi \leq \gamma$. $\mathcal{S}'$ is also clearly NF-paraconsistent for the same reasons as above.

The logical structures described above are all non-reflexive and non-monotonic, and hence of non-Tarski-type.

The above example is mentioned in [10, Remark 2.10]. See also [10, Examples 2.8, 2.9] for special cases of the general ideas described in [Example 5.2.7](#).

Continuing along the same lines, let $\mathcal{L}$ be any set, $(P, \leq)$ a poset, and $\emptyset \subsetneq \mathcal{V} \subseteq P^{\mathcal{L}}$ a set of functions (these may be thought of as *valuation* functions). We can now define a relation $\models_{\mathcal{V}} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ (similar to a *semantic consequence* relation) as follows. For all $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$,

$$\Gamma \models_{\mathcal{V}} \alpha \quad \text{iff} \quad \text{for all } v \in \mathcal{V}, v(\beta) \leq v(\alpha) \text{ for every } \beta \in \Gamma.$$

We then have the following theorem.

Theorem 5.2.8. *Let $\mathcal{L}$ be a non-empty set, $(P, \leq)$ a poset with at least two incomparable elements, and $\mathcal{V}, \models_{\mathcal{V}}$ is defined as above. If there exists $\hat{v} \in \mathcal{V}$ such that $\hat{v}(\gamma)$ and $\hat{v}(\delta)$ are incomparable for some $\gamma, \delta \in \mathcal{L}$, then $(\mathcal{L}, \models_{\mathcal{V}})$ is NF-paraconsistent.*

Proof. Suppose $\hat{v} \in \mathcal{V}$ and $\gamma, \delta \in \mathcal{L}$ such that $\hat{v}(\gamma)$ and $\hat{v}(\delta)$ are incomparable. Let $\beta \in \mathcal{L}$. Then we note that $\{\gamma, \beta\} \not\models_{\mathcal{V}} \delta$ since $\hat{v}(\gamma) \neq \hat{v}(\delta)$. Thus $W_{\models_{\mathcal{V}}}(\{\gamma, \beta\}) \neq \mathcal{L}$ for all $\beta \in \mathcal{L}$. Hence $(\mathcal{L}, \models_{\mathcal{V}})$ is NF-paraconsistent. $\square$

Our next example is taken, with some change of notation, from [37], where this is used in a different context.

Example 5.2.9. Let $\mathcal{S} = (\mathcal{L}, \vdash)$ be the logical structure, where $\mathcal{L} = \mathbb{R}$, the set of real numbers, and $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ is defined as follows. For any $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$,

$$\Gamma \vdash \alpha \quad \text{iff} \quad \begin{cases} \alpha \in \Gamma, \text{ or} \\ \alpha = \frac{1}{n} \text{ for some natural number } n \in \mathbb{N}, n \geq 1, \text{ or} \\ \text{there is a sequence } (\alpha_n)_{n \in \mathbb{N}} \text{ in } \Gamma \text{ that converges to } \alpha. \end{cases}$$

It is straightforward to see that for any $\alpha, \beta, \gamma \in \mathcal{L}$ such that $\gamma \neq \alpha, \beta$ and $\gamma \neq \frac{1}{n}$ for any $n \in \mathbb{N}$, $\{\alpha, \beta\} \not\vdash \gamma$. Thus, $\mathcal{S}$ is NF-paraconsistent.

The logical structure $\mathcal{S}$ is reflexive and monotonic but not transitive. To see the latter, one may note that for any $n \in \mathbb{N}$, $\vdash \frac{1}{n}$ and $\{\frac{1}{n} \mid n \in \mathbb{N}\} \vdash 0$ but $\not\vdash 0$. This phenomenon is also noted in [37].

Theorem 5.2.10. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is the logical structure induced by a q -consequence operator³ W on the set $\mathcal{L}$. If $W(\mathcal{L}) \subsetneq \mathcal{L}$, then $\mathcal{S}$ is NF-paraconsistent.*

Proof. We note that since $W(\mathcal{L}) \subsetneq \mathcal{L}$, there exists $\alpha \in \mathcal{L}$ such that $\alpha \notin W(\mathcal{L})$, i.e. $\mathcal{L} \not\vdash \alpha$, i.e., $\mathcal{L}$ is nontrivial. Then, by monotonicity, every $\Gamma \subseteq \mathcal{L}$ is nontrivial. Thus in particular, for any $\beta, \gamma \in \mathcal{L}$, $W_\vdash(\{\beta, \gamma\}) \neq \mathcal{L}$. Hence $(\mathcal{L}, \vdash)$ is NF-paraconsistent. $\square$

5.2.1 NF-paraconsistency and Logics of Variable Inclusion

The *logics of variable inclusion* have recently been rigorously studied, e.g., in [28, 29, 88]. The following definitions can be found there. For these logics, we assume that $\mathcal{L}$ is the set of formulas defined inductively in the usual way over a denumerable set of variables V using a finite set of connectives/operators called the *signature/type*. In other words, $\mathcal{L}$ is the formula algebra over V of some type. The formula algebra has the universal mapping property for the class of all algebras of the same type as $\mathcal{L}$ over V , i.e., any function $f : V \rightarrow A$, where A is the universe of an algebra $\mathbf{A}$ of the same type as $\mathcal{L}$, can be uniquely extended to a homomorphism from $\mathcal{L}$ to $\mathbf{A}$ (see [54, 52] for more details). For the discussion in this subsection, we will assume that for any logic $(\mathcal{L}, \vdash)$, $\mathcal{L}$ is as described above. We will denote the universe of $\mathcal{L}$ as $\mathcal{L}$.

We first deal with the *left variable inclusion companions*. These are described below.

Definition 5.2.11 (LEFT VARIABLE/PURE LEFT VARIABLE INCLUSION LOGICS). Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logic. For any $\alpha \in \mathcal{L}$, let $\text{var}(\alpha)$ denote the set of all the variables occurring in α , and $\text{var}(\Delta) = \bigcup_{\alpha \in \Delta} \text{var}(\alpha)$.

- (i) The *left variable inclusion companion* of $\mathcal{S}$, denoted by $\mathcal{S}^l = (\mathcal{L}, \vdash^l)$, is defined as follows. For any $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$,

$$\Gamma \vdash^l \alpha \text{ iff there is a } \Delta \subseteq \Gamma \text{ such that } \text{var}(\Delta) \subseteq \text{var}(\alpha) \text{ and } \Delta \vdash \alpha.$$

- (ii) The *pure left variable inclusion companion* of $\mathcal{S}$, denoted by $\mathcal{S}^{pl} = (\mathcal{L}, \vdash^{pl})$, is defined as follows. For any $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$,

$$\Gamma \vdash^{pl} \alpha \text{ iff there is a } \Delta \subseteq \Gamma \text{ such that } \Delta \neq \emptyset, \text{var}(\Delta) \subseteq \text{var}(\alpha), \text{ and } \Delta \vdash \alpha.$$

Theorem 5.2.12. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logic with $\perp \in \mathcal{L}$, a falsity constant for $\mathcal{S}$ (i.e., $\mathcal{S}$ is not $\perp$ -paraconsistent). Then $\mathcal{S}^l = (\mathcal{L}, \vdash^l)$ and $\mathcal{S}^{pl} = (\mathcal{L}, \vdash^{pl})$ are not NF-paraconsistent.*

³See Definition 4.2.4.

Proof. Since $\perp$ is a falsity constant for $\mathcal{S}$, $\perp \vdash \alpha$ for all $\alpha \in \mathcal{L}$.

We note that $\{\perp\}$ is a non-empty subset of itself, and $\text{var}(\perp) = \emptyset \subseteq \text{var}(\alpha)$ for all $\alpha \in \mathcal{L}$. Thus, $\perp \vdash^l \alpha$ and $\perp \vdash^{pl} \alpha$ for all $\alpha \in \mathcal{L}$. Hence, $\perp$ is also a falsity constant for $\mathcal{S}^l$ and $\mathcal{S}^{pl}$.

Now, suppose $\Gamma \vdash^l \alpha$ ($\Gamma \vdash^{pl} \alpha$). Then there exists $\Delta \subseteq \Gamma$ ($\emptyset \neq \Delta \subseteq \Gamma$) such that $\text{var}(\Delta) \subseteq \text{var}(\alpha)$ and $\Delta \vdash \alpha$. Then for any $\Gamma' \supseteq \Gamma$, $\Delta \subseteq \Gamma'$, and hence $\Gamma' \vdash^l \alpha$ (respectively, $\Gamma' \vdash^{pl} \alpha$). Thus, $\mathcal{S}^l$ and $\mathcal{S}^{pl}$ are both monotonic logics (regardless of whether $\mathcal{S}$ is monotonic or not).

Hence, by [Corollary 5.2.5](#), $\mathcal{S}^l$ and $\mathcal{S}^{pl}$ are not NF-paraconsistent. $\square$

Remark 5.2.13. The above theorem shows that if the (pure) left variable inclusion companion of a logic $\mathcal{S} = (\mathcal{L}, \vdash)$ is NF-paraconsistent, then there cannot be a falsity constant for $\mathcal{S}$ in $\mathcal{L}$, and $\mathcal{S}, \mathcal{S}^l, \mathcal{S}^{pl}$ are all $\perp$ -paraconsistent.

Example 5.2.14. Paraconsistent weak Kleene logic (PWK) (described in [27]) and intuitionistic paraconsistent weak Kleene logic (IPWK) (described in [9]) are the left variable inclusion companions of classical and intuitionistic propositional logics, respectively. We know that the classical and intuitionistic propositional logics (described in a language with a falsity constant) are not $\perp$ -paraconsistent. Thus, by [Theorem 5.2.12](#), PWK and IPWK are not NF-paraconsistent, although they are paraconsistent.

We first prove the following result before proceeding to a partial converse to [Theorem 5.2.12](#).

Theorem 5.2.15. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logic. If $\alpha, \beta \in \mathcal{L}$ such that $W_{\vdash^{pl}}(\{\alpha, \beta\}) = \mathcal{L}$, then either $\text{var}(\alpha) = \emptyset$ or $\text{var}(\beta) = \emptyset$ (i.e., either α or β is a constant). Moreover, if $\not\vdash q$ for all $q \in V$, the same conclusion holds for $\mathcal{S}^l$, i.e., if $\alpha, \beta \in \mathcal{L}$ such that $W_{\vdash^l}(\{\alpha, \beta\}) = \mathcal{L}$, then either $\text{var}(\alpha) = \emptyset$ or $\text{var}(\beta) = \emptyset$.*

Proof. Suppose $\alpha, \beta \in \mathcal{L}$ such that $W_{\vdash^{pl}}(\{\alpha, \beta\}) = \mathcal{L}$. Suppose further that $\text{var}(\alpha) \neq \emptyset$.

Let $q \in V \setminus \text{var}(\{\alpha, \beta\})$. Such a variable q exists as V is denumerable, while $\text{var}(\{\alpha, \beta\})$ is finite. Now, $\{\alpha, \beta\} \vdash^{pl} q$. So, there exists $\Delta \subseteq \{\alpha, \beta\}$ such that $\Delta \neq \emptyset$, $\text{var}(\Delta) \subseteq \text{var}(q) = \{q\}$, and $\Delta \vdash q$. Clearly, $\text{var}(\Delta) \subseteq \text{var}(\{\alpha, \beta\})$. However, $q \notin \text{var}(\{\alpha, \beta\})$. So, $\text{var}(\Delta) = \emptyset$. Since $\text{var}(\alpha) \neq \emptyset$, $\alpha \notin \Delta$. Then as $\Delta \neq \emptyset$, $\Delta = \{\beta\}$. So, $\text{var}(\beta) = \text{var}(\Delta) = \emptyset$.

The argument for $\mathcal{S}^l$ proceeds in an identical manner with $\vdash^{pl}$ replaced by $\vdash^l$. The only other difference is that, in this case, the set $\Delta \subseteq \{\alpha, \beta\}$ can be empty. However, if $\Delta = \emptyset$, then $\vdash q$, which contradicts the additional assumption in this case that $\not\vdash q$ for all $q \in V$. Thus, it must be that $\text{var}(\beta) = \emptyset$ if $\text{var}(\alpha) \neq \emptyset$. $\square$

Theorem 5.2.16. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logic. If $\text{var}(\varphi) \neq \emptyset$ for all $\varphi \in \mathcal{L}$, then $\mathcal{S}^{pl}$ is NF-paraconsistent. Moreover, if $\not\vdash q$ for all $q \in V$, then $\mathcal{S}^l$ is also NF-paraconsistent.*

Proof. Suppose $\text{var}(\varphi) \neq \emptyset$ for all $\varphi \in \mathcal{L}$, but $\mathcal{S}^{pl}$ is not NF-paraconsistent. Let $p \in V$. So, by gECQ, there exists $\beta \in \mathcal{L}$ such that $W_{\vdash^{pl}}(\{p, \beta\}) = \mathcal{L}$. Then by [Theorem 5.2.15](#),

$\text{var}(\beta) = \emptyset$ since $\text{var}(p) = \{p\}$. This contradicts our assumption that $\text{var}(\varphi) \neq \emptyset$ for all $\varphi \in \mathcal{L}$. Thus, $\mathcal{S}^{pl}$ must be NF-paraconsistent.

In the case of $\mathcal{S}^l$, we use the additional hypothesis that $\not\vdash q$ for all $q \in V$ and arrive at a similar contradiction using [Theorem 5.2.15](#). Thus, under this additional hypothesis, $\mathcal{S}^l$ is also NF-paraconsistent. $\square$

Remark 5.2.17. We note that [Theorem 5.2.16](#) is not the exact converse of [Theorem 5.2.12](#). While $\text{var}(\varphi) \neq \emptyset$ for all $\varphi \in \mathcal{L}$ (i.e., there are no logical constants) implies that there is no falsity constant, the former is clearly a stronger assumption. We can, however, achieve the full converse if the logic $\mathcal{S}$ is *substitution invariant* (described below). The full converse is also obtained, in the case of $\mathcal{S}^{pl}$, if $\mathcal{S}$ is a transitive logical structure, and, in the case of $\mathcal{S}^l$, if $\mathcal{S}$ is monotonic and transitive.

Suppose $\mathcal{L}$ is a formula algebra over a set of variables V . A *substitution* is any function $\sigma : V \rightarrow \mathcal{L}$ that extends to a unique endomorphism (also denoted by σ) from $\mathcal{L}$ to itself via the universal mapping property. The logic $\mathcal{S} = (\mathcal{L}, \vdash)$ (and the relation $\vdash$) is said to be *substitution invariant* or *structural* if the following property holds. For any substitution σ and for any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$, if $\Gamma \vdash \varphi$, then $\sigma(\Gamma) \vdash \sigma(\varphi)$. The property is referred to as *structurality*. For more details and context, see [\[54, 52\]](#).

Theorem 5.2.18. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a structural logic. If there is no falsity constant for $\mathcal{S}$ in $\mathcal{L}$, then $\mathcal{S}^{pl}$ is NF-paraconsistent. Moreover, if there exists $\varphi \in \mathcal{L}$ such that $\not\vdash \varphi$, then $\mathcal{S}^l$ is also NF-paraconsistent.*

Proof. Suppose there is no falsity constant for $\mathcal{S}$ in $\mathcal{L}$, but $\mathcal{S}^{pl}$ is not NF-paraconsistent. Let $p \in V$. So, by gECQ, there exists $\beta \in \mathcal{L}$ such that $W_{\vdash^{pl}}(\{p, \beta\}) = \mathcal{L}$. Then, as in the proof of [Theorem 5.2.15](#), let $q \in V \setminus \text{var}(\{p, \beta\})$. So, $\{p, \beta\} \vdash^{pl} q$, which implies that there exists $\Delta \subseteq \{p, \beta\}$ such that $\Delta \neq \emptyset$, $\text{var}(\Delta) \subseteq \{q\}$, and $\Delta \vdash q$. So, by the same arguments as in the proof of [Theorem 5.2.15](#), $\Delta = \{\beta\}$ and $\text{var}(\beta) = \text{var}(\Delta) = \emptyset$. Thus, β is a constant, and $\{\beta\} \vdash q$. So, by structurality, $\{\beta\} \vdash \alpha$ for all $\alpha \in \mathcal{L}$, which implies that β is a falsity constant for $\mathcal{S}$. This is a contradiction. Hence $\mathcal{S}^{pl}$ is NF-paraconsistent.

The argument for $\mathcal{S}^l$ proceeds similarly with $\vdash^{pl}$ replaced by $\vdash^l$. The extra case we need to deal with now is where $\Delta = \emptyset$. However, that implies $\vdash q$, and hence by structurality, $\vdash \alpha$ for all $\alpha \in \mathcal{L}$. This contradicts the additional assumption that there exists $\varphi \in \mathcal{L}$ such that $\not\vdash \varphi$. Thus, $\mathcal{S}^l$ is also NF-paraconsistent. $\square$

Theorem 5.2.19. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a transitive logic. If there is no falsity constant for $\mathcal{S}$ in $\mathcal{L}$, then $\mathcal{S}^{pl}$ is NF-paraconsistent.*

Proof. Suppose $\mathcal{S}^{pl}$ is not NF-paraconsistent. Let $p \in V$. So, by gECQ, there exists $\beta \in \mathcal{L}$ such that $W_{\vdash^{pl}}(\{p, \beta\}) = \mathcal{L}$. Then, by [Theorem 5.2.15](#), $\text{var}(\beta) = \emptyset$ since $\text{var}(p) \neq \emptyset$. Thus, β is a constant.

Let $\gamma \in \mathcal{L}$ such that $p \notin \text{var}(\gamma)$. Now, $\{p, \beta\} \vdash^{pl} \gamma$. So, there exists $\Delta \subseteq \{p, \beta\}$ such that $\Delta \neq \emptyset$, $\text{var}(\Delta) \subseteq \text{var}(\gamma)$, and $\Delta \vdash \gamma$. Since $p \notin \text{var}(\gamma)$, $p \notin \Delta$. Thus, $\Delta = \{\beta\}$, and

$\{\beta\} \vdash \gamma$. Hence we can conclude that β is a constant such that $\{\beta\} \vdash \gamma$ for all $\gamma \in \mathcal{L}$ with $p \notin \text{var}(\gamma)$.

Suppose β is not a falsity constant. Then there exists $\varphi \in \mathcal{L}$ such that $\{\beta\} \not\vdash \varphi$. So, by the above arguments, $p \in \text{var}(\varphi)$. Now, let $p' \in V \setminus \text{var}(\varphi)$. Again by gECQ, there exists $\psi \in \mathcal{L}$ such that $W_{\vdash^{pl}}(\{p', \psi\}) = \mathcal{L}$. Then, again by Theorem 5.2.15, $\text{var}(\psi) = \emptyset$ since $\text{var}(p') \neq \emptyset$. This, in particular, implies that $p \notin \text{var}(\psi)$, and thus, $\{\beta\} \vdash \psi$.

Now, since $W_{\vdash^{pl}}(\{p', \psi\}) = \mathcal{L}$, $\{p', \psi\} \vdash^{pl} \varphi$. So, there exists $\Delta \subseteq \{p', \psi\}$ such that $\Delta \neq \emptyset$, $\text{var}(\Delta) \subseteq \text{var}(\varphi)$, and $\Delta \vdash \varphi$. Since $p' \notin \text{var}(\varphi)$, $p' \notin \Delta$. Thus, $\Delta = \{\psi\}$ and so, $\{\psi\} \vdash \varphi$. Since $\{\beta\} \vdash \psi$ and $\{\psi\} \vdash \varphi$, by transitivity, we have $\{\beta\} \vdash \varphi$. This is a contradiction to our earlier assumption. Hence $\mathcal{S}^{pl}$ must be NF-paraconsistent. $\square$

Theorem 5.2.20. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a monotonic and transitive logic such that $\not\vdash q$ for all $q \in V$. If there is no falsity constant for $\mathcal{S}$ in $\mathcal{L}$, then $\mathcal{S}^l$ is NF-paraconsistent.*

Proof. Suppose $\mathcal{S}^l$ is not NF-paraconsistent. Let $p \in V$. So, by gECQ, there exists $\beta \in \mathcal{L}$ such that $W_{\vdash^l}(\{p, \beta\}) = \mathcal{L}$. Then, by Theorem 5.2.15, $\text{var}(\beta) = \emptyset$ since $\text{var}(p) \neq \emptyset$. Thus, β is a constant.

Let $\gamma \in \mathcal{L}$. If $\vdash \gamma$, then $\{\beta\} \vdash \gamma$ by monotonicity. Now, suppose $\not\vdash \gamma$ and $p \notin \text{var}(\gamma)$. Since V is denumerable and $\not\vdash q$ for all $q \in V$, such a γ exists. Then $\{p, \beta\} \vdash^l \gamma$. This implies that there exists $\Delta \subseteq \{p, \beta\}$ such that $\text{var}(\Delta) \subseteq \text{var}(\gamma)$, and $\Delta \vdash \gamma$. Since $p \notin \text{var}(\gamma)$, $p \notin \Delta$. Then as $\not\vdash \gamma$, $\Delta = \{\beta\}$, and $\{\beta\} \vdash \gamma$. Thus, for any $\gamma \in \mathcal{L}$, if $\vdash \gamma$ or $p \notin \text{var}(\gamma)$, then $\{\beta\} \vdash \gamma$. In other words, β is a constant such that for all $\gamma \in \mathcal{L}$, if $\{\beta\} \not\vdash \gamma$, then $\not\vdash \gamma$ and $p \in \text{var}(\gamma)$. In particular, this implies that $\{\beta\} \vdash \sigma$ for all $\sigma \in \mathcal{L}$ such that $\text{var}(\sigma) = \emptyset$.

Suppose β is not a falsity constant. Then there exists $\varphi \in \mathcal{L}$ such that $\{\beta\} \not\vdash \varphi$. So, by the above arguments, $\not\vdash \varphi$ and $p \in \text{var}(\varphi)$. Now, let $p' \in V \setminus \text{var}(\varphi)$ (such a variable exists since V is denumerable). Again by gECQ, there exists $\psi \in \mathcal{L}$ such that $W_{\vdash^l}(\{p', \psi\}) = \mathcal{L}$. Then, again by Theorem 5.2.15, $\text{var}(\psi) = \emptyset$ since $\text{var}(p') \neq \emptyset$. Thus, $\{\beta\} \vdash \psi$.

Now, since $W_{\vdash^l}(\{p', \psi\}) = \mathcal{L}$, $\{p', \psi\} \vdash^l \varphi$. So, there exists $\Delta \subseteq \{p', \psi\}$ such that $\text{var}(\Delta) \subseteq \text{var}(\varphi)$, and $\Delta \vdash \varphi$. Since $p' \notin \text{var}(\varphi)$, $p' \notin \Delta$. Moreover, $\not\vdash \varphi$, which implies that $\Delta \neq \emptyset$. Thus, $\Delta = \{\psi\}$ and so, $\{\psi\} \vdash \varphi$. Since $\{\beta\} \vdash \psi$ and $\{\psi\} \vdash \varphi$, by transitivity, we have $\{\beta\} \vdash \varphi$. This is a contradiction to our earlier assumption. Hence $\mathcal{S}^l$ must be NF-paraconsistent. $\square$

Remark 5.2.21. We note that we do not require the full strength of the monotonicity condition in the above theorem. In fact, we only need the following weaker condition. For any $\gamma \in \mathcal{L}$, if $\vdash \gamma$, then $\{\beta\} \vdash \gamma$ for all $\beta \in \mathcal{L}$.

One may also note that Theorem 5.2.19 and Theorem 5.2.20 do not require the full strength of the transitivity condition. The following weaker “pointwise transitivity” is enough. For any $\alpha, \beta, \gamma \in \mathcal{L}$, if $\{\alpha\} \vdash \beta$ and $\{\beta\} \vdash \gamma$, then $\{\alpha\} \vdash \gamma$.

Next, we explore the *right variable inclusion companions*. These are described below.

Definition 5.2.22 (RIGHT VARIABLE/PURE RIGHT VARIABLE INCLUSION COMPANION OF A LOGIC). Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logic.

- (i) The *right variable inclusion companion* of $\mathcal{S}$, denoted by $\mathcal{S}^r = (\mathcal{L}, \vdash^r)$, is defined as follows. For any $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$,

$$\Gamma \vdash^r \alpha \text{ iff either } \Gamma \text{ contains an } \mathcal{S}\text{-antitheorem, or } \Gamma \vdash \alpha \text{ and } \text{var}(\alpha) \subseteq \text{var}(\Gamma).$$

An $\mathcal{S}$ -antitheorem is a set $\Sigma \subseteq \mathcal{L}$ such that for every substitution σ , $\sigma(\Sigma) \vdash \varphi$ for all $\varphi \in \mathcal{L}$, i.e., $\sigma(\Sigma)$ *explodes* in $\mathcal{S}$ for every σ .

- (ii) The *pure right variable inclusion companion* of $\mathcal{S}$, denoted by $\mathcal{S}^{pr} = (\mathcal{L}, \vdash^{pr})$, is defined as follows. For any $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$,

$$\Gamma \vdash^{pr} \alpha \text{ iff } \Gamma \vdash \alpha \text{ and } \text{var}(\alpha) \subseteq \text{var}(\Gamma).$$

Remark 5.2.23. The above definition of a right variable inclusion companion agrees with the ones in [88, 90]. However, in [29], these are defined with the following slight variation. For any $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$,

$$\Gamma \vdash^r \alpha \text{ iff either } \Gamma \text{ is an } \mathcal{S}\text{-antitheorem, or } \Gamma \vdash \alpha \text{ and } \text{var}(\alpha) \subseteq \text{var}(\Gamma).$$

The same has been indicated in [44]. This difference is, of course, inconsequential in the presence of monotonicity. We choose to work with the weaker definition here as we aim to be frugal with our assumptions.

It is easy to see that the pure right variable inclusion companion of a logic $\mathcal{S}$ is paraconsistent with respect to a unary (negation) operator $\neg$, provided there are at least two distinct variables. The following result tells us that they are also NF-paraconsistent provided there is an ample supply of variables.

Theorem 5.2.24. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logic. Then $\mathcal{S}^{pr} = (\mathcal{L}, \vdash^{pr})$ is NF-paraconsistent.*

Proof. Suppose $\mathcal{S}^{pr}$ is not NF-paraconsistent. Let $p \in V$. So, by gECQ, there exists $\beta \in \mathcal{L}$ such that $W_{\vdash^{pr}}(\{p, \beta\}) = \mathcal{L}$.

Let $q \in V \setminus \text{var}(\{p, \beta\})$. So, $\{p, \beta\} \vdash^{pr} q$. This implies that $q \in \text{var}(\{p, \beta\})$, which contradicts our assumption about q . Hence $\mathcal{S}^{pr}$ must be NF-paraconsistent. $\square$

The right variable inclusion companion of a logic is not paraconsistent, in general. For example, Bochvar logic, B_3 , which is the right variable inclusion companion of classical propositional logic, is not paraconsistent (see [29, 90]). Hence B_3 is not NF-paraconsistent, by Theorem 5.2.2.

We can, however, show that if a logic $\mathcal{S} = (\mathcal{L}, \vdash)$ is NF-paraconsistent, then its right variable inclusion companion is also NF-paraconsistent, provided $\mathcal{S}$ satisfies a weak form of monotonicity, viz., monotonicity for the trivial subsets of $\mathcal{L}$. This can be described as follows. For any $\Gamma \subseteq \mathcal{L}$, if Γ is $\mathcal{S}$ -trivial, i.e., if $W_{\vdash}(\Gamma) = \mathcal{L}$, then every superset of Γ is also $\mathcal{S}$ -trivial, i.e., $W_{\vdash}(\Gamma') = \mathcal{L}$ for all $\Gamma' \supseteq \Gamma$.

Theorem 5.2.25. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logic. Then every finite $\mathcal{S}^r$ -trivial set contains an $\mathcal{S}$ -trivial set.*

Proof. Suppose Γ is a finite $\mathcal{S}^r$ -trivial set. Let $q \in V \setminus \text{var}(\Gamma)$. Then $\Gamma \vdash^r q$. So, either $q \in \text{var}(\Gamma)$ and $\Gamma \vdash q$, or Γ contains an $\mathcal{S}$ -antitheorem. Since $q \notin \text{var}(\Gamma)$, Γ must contain an $\mathcal{S}$ -antitheorem. Let $\Delta \subseteq \Gamma$ be an $\mathcal{S}$ -antitheorem. This implies that $W_{\vdash}(\sigma(\Delta)) = \mathcal{L}$ for every substitution σ . Thus, in particular, $W_{\vdash}(\Delta) = \mathcal{L}$ (σ is the identity endomorphism from $\mathcal{L}$ to itself in this case). Hence Δ is an $\mathcal{S}$ -trivial set contained in Γ . $\square$

Theorem 5.2.26. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ be a logic that satisfies the property of monotonicity for trivial sets. If $\mathcal{S}$ is NF-paraconsistent, then $\mathcal{S}^r$ is also NF-paraconsistent.*

Proof. Suppose $\mathcal{S}^r$ is not NF-paraconsistent. Let $\alpha \in \mathcal{L}$. Then by gECQ, there exists $\beta \in \mathcal{L}$, such that $W_{\vdash^r}(\{\alpha, \beta\}) = \mathcal{L}$. So, $\{\alpha, \beta\}$ is an $\mathcal{S}^r$ -trivial set. Hence, by Theorem 5.2.25, there must be an $\mathcal{S}$ -trivial $\Delta \subseteq \{\alpha, \beta\}$. Then, by monotonicity for trivial sets, we have that $\{\alpha, \beta\}$ is also $\mathcal{S}$ -trivial, i.e., $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$. This implies that for every $\alpha \in \mathcal{L}$, there exists $\beta \in \mathcal{L}$ such that $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$. Thus, $\mathcal{S}$ satisfies gECQ and, therefore, is not NF-paraconsistent. Hence if $\mathcal{S}$ satisfies monotonicity for trivial sets and is NF-paraconsistent, then $\mathcal{S}^r$ is also NF-paraconsistent. $\square$

5.2.2 A Quasi-negation

Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure. For any $\alpha, \beta \in \mathcal{L}$, if $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$, then β acts like a negation of α . This can be justified by noting that in case $\mathcal{S}$ is not paraconsistent with respect to a unary operator $\neg$ in $\mathcal{S}$, i.e., $\neg$ -ECQ holds, then $\neg\alpha$ is indeed such a β . However, in general, such a β need not be unique. For example, in the case of classical propositional logic (CPC) or intuitionistic propositional logic (IPC), we have $W_{\vdash}(\{\alpha, \alpha \rightarrow \neg\alpha\}) = \mathcal{L}$. Thus, $\alpha \rightarrow \neg\alpha$ is also an instance of such a β . Motivated by this, we propose following definition.

Definition 5.2.27 (QUASI-NEGATION). Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure and $\alpha \in \mathcal{L}$. Then a *quasi-negation* of α is any $\beta \in \mathcal{L}$ such that $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$. The set of all quasi-negations of α is denoted by $QN(\alpha)$.

We now explore some properties of the quasi-negation in the following theorem.

Theorem 5.2.28. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure and $\alpha, \beta \in \mathcal{L}$.*

- (1) $\alpha \in QN(\beta)$ iff $\beta \in QN(\alpha)$. (*Quasi contraposition*).
- (2) If $\mathcal{S}$ is of Tarski-type and $\{\alpha\} \vdash \beta$, then $QN(\beta) \subseteq QN(\alpha)$.

Proof. (1) If $\alpha \in QN(\beta)$, then $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$. Consequently, $\beta \in QN(\alpha)$ as well.

- (2) Let $\gamma \in QN(\beta)$. Then, $W_{\vdash}(\{\beta, \gamma\}) = \mathcal{L}$. Now, $\{\alpha, \gamma\} \vdash \gamma$ by reflexivity and $\{\alpha, \gamma\} \vdash \beta$ by monotonicity. Therefore, by transitivity $W_{\vdash}(\{\alpha, \gamma\}) = \mathcal{L}$. Thus, $QN(\beta) \subseteq QN(\alpha)$. $\square$

The following result establishes some connections between the above defined quasi-negation and the concepts of paraconsistency and NF-paraconsistency.

Theorem 5.2.29. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logic.*

- (1) *Suppose $\mathcal{S}$ has a unary operator $\neg$. Then $\mathcal{S}$ is paraconsistent with respect to $\neg$ iff there exists an $\alpha \in \mathcal{L}$ such that $\neg\alpha \notin QN(\alpha)$.*
- (2) *$\mathcal{S}$ is NF-paraconsistent iff there exists an $\alpha \in \mathcal{L}$ such that $QN(\alpha) = \emptyset$.*
- (3) *If $\mathcal{S}$ is monotonic and there does not exist any nullary operator or constant $\perp \in \mathcal{L}$ such that $QN(\perp) = \mathcal{L}$, then $\mathcal{S}$ is $\perp$ -paraconsistent.*

Proof. The statements (1) and (2) follow directly from the definition of quasi-negation. To prove statement (3), suppose $\mathcal{S}$ is a monotonic logical structure that is not $\perp$ -paraconsistent. Then, $W_{\vdash}(\{\perp\}) = \mathcal{L}$. So, by monotonicity, $W_{\vdash}(\{\vdash, \beta\}) = \mathcal{L}$ for all $\beta \in \mathcal{L}$. Therefore, $\beta \in QN(\perp)$ for all $\beta \in \mathcal{L}$, i.e., $QN(\perp) = \mathcal{L}$. Hence $QN(\perp) \neq \mathcal{L}$ implies that $\mathcal{S}$ is $\perp$ -paraconsistent. $\square$

We end this subsection with a few properties of quasi-negation in the case of CPC = $(\mathcal{L}, \vdash_{\text{CPC}})$, where $\vdash_{\text{CPC}}$ is the usual classical consequence relation, using the kite of negations in [50]. The following observations follow easily using the standard 2-element Boolean algebra interpretation of CPC $\langle \{0, 1\}, \wedge, \vee, \neg \rangle$. For any $\alpha \in \mathcal{L}$, $QN(\alpha) \neq \emptyset$, i.e., there exists a $\beta \in QN(\alpha)$ such that $W_{\vdash_{\text{CPC}}}(\{\alpha, \beta\}) = \mathcal{L}$. This implies that for any valuation $v : \mathcal{L} \rightarrow \{0, 1\}$, if $v(\alpha) = 1$, then $v(\beta) = 0$. Since β was chosen arbitrarily, $v(QN(\alpha)) = \{0\}$. Conversely, suppose $v : \mathcal{L} \rightarrow \{0, 1\}$ is a valuation such that $v(QN(\alpha)) = \{0\}$. Then, in particular, $v(\neg\alpha) = 0$ since $\neg\alpha \in QN(\alpha)$. So, $v(\alpha) = 1$. Thus, we can say that for any $\alpha \in \mathcal{L}$ and any valuation $v : \mathcal{L} \rightarrow \{0, 1\}$, $v(\alpha) = 1$ iff $v(QN(\alpha)) = \{0\}$. A similar argument shows that for any valuation $v : \mathcal{L} \rightarrow \{0, 1\}$, $v(\alpha) = 0$ iff $v(QN(\alpha)) = \{1\}$. Now, for any $\alpha, \beta \in \mathcal{L}$, the following properties are easy to check.

- (1) If $\{\alpha\} \vdash \beta$, then $\{\sigma\} \vdash \tau$ for all $(\sigma, \tau) \in QN(\beta) \times QN(\alpha)$. (Contraposition)
- (2) For all $\gamma \in QN(\alpha)$ and $\delta \in QN(\neg\alpha)$, $\{\alpha\} \vdash \neg\gamma$ and $\{\alpha\} \vdash \delta$ (Galois double negation)
- (3) For all $\gamma \in QN(QN(\alpha))$, $\{\alpha\} \vdash \gamma$ (Constructive double negation)
- (4) For all $\gamma \in QN(QN(\alpha))$, $\{\gamma\} \vdash \alpha$ (Classical double negation)
- (5) For all $\gamma \in QN(\alpha)$, $\{\alpha \wedge \gamma\} \vdash \beta$ (Absurdity)

It is also easy to check that in the case of IPC, all the above properties, except (4), are satisfied by the quasi-negation. Thus, at least for CPC and IPC, the quasi-negations of a formula behave in much the same way as the regular classical and intuitionistic negation operators, respectively.

5.3 Principles of Explosion

In this section, we discuss further generalisations of ECQ. One can develop a notion of paraconsistency via the failure of each of these. We have, however, kept the discussion here restricted to principles of explosion only. Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure. As mentioned in the introductory section, a principle of explosion in $\mathcal{S}$ might be described, in general, as one that allows a set $\Gamma \subsetneq \mathcal{L}$ to blow up to the whole set $\mathcal{L}$, i.e., to result in $W_{\vdash}(\Gamma) = \mathcal{L}$.

The first principle of explosion is called **sECQ** (the ‘s’ here denotes that this is a ‘set-based’ principle of explosion). This can be described as follows.

For each $\alpha \in \mathcal{L}$, there exists a set $\Gamma \subsetneq \mathcal{L}$ such that $\alpha \in \Gamma$ and $W_{\vdash}(\Gamma) = \mathcal{L}$.

Theorem 5.3.1. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logic.*

- (i) *If $\neg$ -ECQ, where $\neg$ is a unary operator in $\mathcal{S}$, holds in $\mathcal{S}$, then sECQ also holds in $\mathcal{S}$, provided $\{\alpha, \neg\alpha\} \subsetneq \mathcal{L}$ for all $\alpha \in \mathcal{L}$.*
- (ii) *If gECQ holds in $\mathcal{S}$, then sECQ also holds in $\mathcal{S}$, provided $\mathcal{L}$ has at least three distinct elements.*

Proof. The proof is straightforward from the formulations of $\neg$ -ECQ, gECQ, and sECQ. $\square$

Remark 5.3.2. The condition that $\mathcal{L}$ has at least three distinct elements is enough to guarantee that if $\neg$ -ECQ holds in $\mathcal{S}$, then sECQ also holds in it. Thus, the above theorem could have been stated more simply. However, the proviso that $\{\alpha, \neg\alpha\} \subsetneq \mathcal{L}$ for all α , in part (i) of the above theorem, is more general as this doesn’t require that $\neg\alpha \neq \alpha$ for all α .

sECQ is clearly a generalisation of ECQ and gECQ. Any paraconsistent logic that is not NF-paraconsistent, such as PWK, and IPWK (considered in [Example 5.2.14](#)), are instances of logics where gECQ holds, and hence by the above theorem, sECQ also holds. However, since these are paraconsistent, ECQ does not hold in these logics.

The following are some further examples of logics, where ECQ fails but gECQ and sECQ hold. See [Example 5.3.9](#) and [Example 5.3.14](#) for logics, where sECQ holds but gECQ fails.

Example 5.3.3. Let $\mathcal{S} = (\mathcal{L}, \vdash)$ be the logic P^1 proposed by Sette in [\[98\]](#). Here $\mathcal{L}$ is the formula algebra over a denumerable set of propositional variables V , of type $\{\neg, \rightarrow\}$, and $\vdash$ is a consequence relation satisfying reflexivity, monotonicity and transitivity. Apart from the original Hilbert-style system presented in [\[98\]](#), alternative axiomatisations for P^1 can be found in [\[41, 43\]](#). These articles also include semantics for P^1 and the soundness and completeness results. P^1 is a paraconsistent logic in the sense that ECQ fails in it but only for $p \in V$. In other words, for all $\alpha \in \mathcal{L} \setminus V$, $W_{\vdash}(\{\alpha, \neg\alpha\}) = \mathcal{L}$.

It can be easily shown that $\vdash \alpha \rightarrow \alpha$. Then, using the semantics and the completeness result provided in the above articles, it is straightforward to derive that $W_{\vdash}(\neg(\alpha \rightarrow \alpha)) = \mathcal{L}$. Now, since P^1 is monotonic, $W_{\vdash}(\{\alpha, \neg(\alpha \rightarrow \alpha)\}) = \mathcal{L}$ for all $\alpha \in \mathcal{L}$. Thus, gECQ holds in P^1 . (Instead of $\neg(\alpha \rightarrow \alpha)$, we can use the negation of any theorem of P^1 to arrive

at the same conclusion.) Since $\mathcal{L}$ is denumerable and hence, has at least three distinct elements, by [Theorem 5.3.1](#), **sECQ** also holds in P^1 .

It is, however, easier to show that **sECQ** holds in P^1 directly as follows. Let $\alpha \in \mathcal{L}$ and $\Gamma = \{\alpha, \neg\alpha, \neg\neg\alpha\}$. Then $\alpha \in \Gamma \subsetneq \mathcal{L}$. Since $W_{\vdash}(\{\neg\alpha, \neg\neg\alpha\}) = \mathcal{L}$, by monotonicity, $W_{\vdash}(\Gamma) = \mathcal{L}$. Hence, **sECQ** holds in P^1 .

Remark 5.3.4. Sette's logic, P^1 , is a member of the class of *gently paraconsistent calculi/logics* discussed in [\[42\]](#). In each of these logics, **ECQ** fails. However, if $(\mathcal{L}, \vdash)$ is a gently paraconsistent logic, then for any $\alpha \in \mathcal{L}$, $W_{\vdash}(\{\alpha, \neg\alpha, \neg\neg\alpha\}) = \mathcal{L}$, which is called the *principle of gentle explosion*. Thus, clearly, **sECQ** holds.

Each of the gently paraconsistent logics contains the positive fragment of classical propositional logic, and hence, as in P^1 , $\alpha \longrightarrow \alpha$ is a theorem in each of them. Thus, by the same arguments as in the previous example, **gECQ** also holds in the gently paraconsistent logics.

Remark 5.3.5. P^1 is also a member of the hierarchies of paraconsistent systems, P^m and P^{*m} , where $m \geq 1$, presented in [\[43\]](#). These logics use the same set of formulas $\mathcal{L}$ as P^1 , and **ECQ** fails in each of these systems.

However, in the logic $P^m = (\mathcal{L}, \vdash)$, where $m \geq 1$, for any $\alpha \in \mathcal{L}$, $W_{\vdash}(\{\alpha, \neg\alpha, \dots, \neg^m\alpha, \neg^{m+1}\alpha\}) = \mathcal{L}$, where for any $n \geq 1$, $\neg^n\alpha$ denotes $\underbrace{\neg \dots \neg}_{n \text{ times}} \alpha$. Thus, **sECQ** holds in P^m for each $m \geq 1$.

As in P^1 , $\alpha \longrightarrow \alpha$ is a theorem of each P^m , and hence, by the same arguments as for P^1 , **gECQ** also holds in these logics.

The same arguments hold for the logics P^{*m} as these logics have the same properties that are used for proving that **gECQ** and **sECQ** hold in P^m .

The next principle of explosion is a slight variation of **sECQ**. We call this **sECQ'**. This can be described as follows.

For each $\alpha \in \mathcal{L}$, there exists a set $\Gamma \subsetneq \mathcal{L}$ such that $\Gamma \cup \{\alpha\} \subsetneq \mathcal{L}$ and $W_{\vdash}(\Gamma \cup \{\alpha\}) = \mathcal{L}$.

Theorem 5.3.6. Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure. **sECQ** holds in $\mathcal{S}$ iff **sECQ'** holds in $\mathcal{S}$, i.e., **sECQ** is equivalent to **sECQ'**.

Proof. Suppose **sECQ** holds in $\mathcal{S}$. Let $\alpha \in \mathcal{L}$. Then there exists $\Gamma \subsetneq \mathcal{L}$ such that $\alpha \in \Gamma$ and $W_{\vdash}(\Gamma) = \mathcal{L}$. Now, let $\Gamma' = \Gamma \setminus \{\alpha\}$. Then $\Gamma' \subsetneq \mathcal{L}$ such that $\Gamma' \cup \{\alpha\} = \Gamma \subsetneq \mathcal{L}$ and $W_{\vdash}(\Gamma' \cup \{\alpha\}) = W_{\vdash}(\Gamma) = \mathcal{L}$. Thus, **sECQ'** holds in $\mathcal{S}$.

Conversely, suppose **sECQ'** holds in $\mathcal{S}$. Let $\alpha \in \mathcal{L}$. Then there exists $\Gamma \subsetneq \mathcal{L}$ such that $\Gamma \cup \{\alpha\} \subsetneq \mathcal{L}$ and $W_{\vdash}(\Gamma \cup \{\alpha\}) = \mathcal{L}$. Now, let $\Gamma' = \Gamma \cup \{\alpha\}$. Then $\Gamma' \subsetneq \mathcal{L}$ such that $\alpha \in \Gamma'$ and $W_{\vdash}(\Gamma') = \mathcal{L}$. Thus, **sECQ** holds in $\mathcal{S}$. $\square$

Remark 5.3.7. In light of the above theorem, **sECQ** will henceforth denote both of the above principles of explosion: **sECQ** and **sECQ'**.

We note that all the principles of explosion that we have discussed so far, viz., **ECQ**, $\perp$ -**ECQ**, **gECQ**, and even **sECQ**, are, in a sense, 'pointwise.' Each of these principles revolves

around elements of the underlying set in the logic/logical structure. In case of ECQ, gECQ, sECQ, for each $\alpha \in \mathcal{L}$, there is some set containing α that explodes, and for $\perp$ -ECQ, $\perp$ belongs to the underlying set. Given a logic $(\mathcal{L}, \vdash)$ and a unary operator $\neg$ on $\mathcal{L}$ we can indeed describe a principle of explosion using a set $\Gamma \subseteq \mathcal{L}$ in the following way.

For all $\Gamma \subseteq \mathcal{L}$ and for all $\alpha \in \mathcal{L}$, if $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$, then $W_{\vdash}(\Gamma) = \mathcal{L}$.

However, this is equivalent to $\neg$ -ECQ if the logic is reflexive and transitive.

We now formulate some principles of explosion that differ from the above-discussed ‘pointwise’ explosions and are not equivalent to ECQ, either. The first among these is obtained by turning sECQ’ around as follows.

For all $\Gamma \subsetneq \mathcal{L}$, there exists $\alpha \in \mathcal{L}$ such that $\Gamma \cup \{\alpha\} \subsetneq \mathcal{L}$ and $W_{\vdash}(\Gamma \cup \{\alpha\}) = \mathcal{L}$.

We call this spECQ (the letter-pair ‘sp’ here denotes set-point in reference to the manner in which the explosion takes place).

Theorem 5.3.8. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure such that $\mathcal{L}$ has at least two distinct elements. If spECQ holds in $\mathcal{S}$, then gECQ holds in it. Moreover, if $\mathcal{L}$ has at least three distinct elements, then sECQ also holds in $\mathcal{S}$.*

Proof. Suppose spECQ holds in $\mathcal{S}$. Let $\alpha \in \mathcal{L}$. Since $\mathcal{L}$ has at least two distinct elements, $\{\alpha\} \subsetneq \mathcal{L}$. Then by spECQ, there exists $\beta \in \mathcal{L}$ such that $\{\alpha\} \cup \{\beta\} \subsetneq \mathcal{L}$ and $W_{\vdash}(\{\alpha\} \cup \{\beta\}) = W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$. Thus, gECQ holds in $\mathcal{S}$. Now, suppose $\mathcal{L}$ has at least three distinct elements. Then, by Theorem 5.3.1, sECQ also holds in $\mathcal{S}$. $\square$

The following example shows that sECQ does not imply spECQ and gECQ.

Example 5.3.9. Let $\mathcal{S} = (\mathcal{L}, \vdash)$ be the logical structure, where $\mathcal{L} = \mathbb{N} \setminus \{0\}$, and $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ is such that, for any $\Gamma \subseteq \mathcal{L}$,

$$W_{\vdash}(\Gamma) = \begin{cases} \mathcal{L} & \text{if } \Gamma = \{n, n+1, \dots, 2n\} \text{ for some } n \in \mathcal{L}, \\ \Gamma & \text{otherwise.} \end{cases}$$

Let $k \in \mathcal{L}$. Then, $\Gamma = \{k, k+1, \dots, 2k\} \subsetneq \mathcal{L}$ is such that $k \in \Gamma$ and $W_{\vdash}(\Gamma) = \mathcal{L}$. Thus, sECQ holds in $\mathcal{S}$.

Now, to see that spECQ does not hold in $\mathcal{S}$, let $\Gamma = \{1, 4\}$. We note that $\Delta = \{1, 2\}$ is the only subset of $\mathcal{L}$ such that $1 \in \Delta$ and $W_{\vdash}(\Delta) = \mathcal{L}$. Thus, $W_{\vdash}(\Gamma \cup \{\alpha\}) \neq \mathcal{L}$ for all $\alpha \in \mathcal{L}$. Hence, spECQ does not hold in $\mathcal{S}$.

Lastly, to see that gECQ does not hold in $\mathcal{S}$, we note that the only two element set that explodes is $\Delta = \{1, 2\}$. Hence, for instance, there does not exist any $\beta \in \mathcal{L}$ such that $W_{\vdash}\{3, \beta\} = \mathcal{L}$. Hence, gECQ cannot hold in $\mathcal{S}$.

The following are some other principles of explosion that are entirely free from any reference to the elements of $\mathcal{L}$.

- For all $\Gamma \subsetneq \mathcal{L}$, there exists $\Delta \subsetneq \mathcal{L}$ such that $\Gamma \subseteq \Delta$ and $W_{\vdash}(\Delta) = \mathcal{L}$. (pfECQ1)
- For all $\Gamma \subsetneq \mathcal{L}$, there exists $\Delta \subsetneq \mathcal{L}$ such that $\Gamma \cup \Delta \subsetneq \mathcal{L}$ and $W_{\vdash}(\Gamma \cup \Delta) = \mathcal{L}$. (pfECQ2)
- For all $\Gamma \subsetneq \mathcal{L}$, there exists $\emptyset \neq \Delta \subsetneq \mathcal{L}$ such that $\Gamma \cup \Delta \subsetneq \mathcal{L}$, and for every $\emptyset \neq \Delta' \subseteq \Delta$, $W_{\vdash}(\Gamma \cup \Delta') = \mathcal{L}$. (pfECQ3)

The letter-pair ‘pf’ in the above principles denotes point-free. We now investigate the relationships between these various principles of explosion.

Theorem 5.3.10. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure. pfECQ1 holds in $\mathcal{S}$ iff pfECQ2 holds in it, i.e., pfECQ1 is equivalent to pfECQ2.*

Proof. Suppose pfECQ1 holds in $\mathcal{S}$. Let $\Gamma \subsetneq \mathcal{L}$. Then, by pfECQ1, there exists $\Delta \subsetneq \mathcal{L}$ such that $\Gamma \subseteq \Delta$ and $W_{\vdash}(\Delta) = \mathcal{L}$. Since $\Gamma \subseteq \Delta$, $\Gamma \cup \Delta = \Delta$. So, $\Gamma \cup \Delta \subsetneq \mathcal{L}$ and $W_{\vdash}(\Gamma \cup \Delta) = \mathcal{L}$. Thus, pfECQ2 holds.

Conversely, suppose pfECQ2 holds in $\mathcal{S}$. Let $\Gamma \subsetneq \mathcal{L}$. Then, by pfECQ2, there exists $\Delta \subsetneq \mathcal{L}$ such that $\Gamma \cup \Delta \subsetneq \mathcal{L}$ and $W_{\vdash}(\Gamma \cup \Delta) = \mathcal{L}$. Now, $\Gamma \subseteq \Gamma \cup \Delta$. Thus, pfECQ1 holds in $\mathcal{S}$. $\square$

Remark 5.3.11. In light of the above theorem, we will use pfECQ to refer to both pfECQ1 and pfECQ2.

Theorem 5.3.12. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure. If spECQ holds in $\mathcal{S}$, then pfECQ also holds in it.*

Proof. Suppose spECQ holds in $\mathcal{S}$. Let $\Gamma \subsetneq \mathcal{L}$. Then, by spECQ, there exists $\alpha \in \mathcal{L}$ such that $\Gamma \cup \{\alpha\} \subsetneq \mathcal{L}$ and $W_{\vdash}(\Gamma \cup \{\alpha\}) = \mathcal{L}$. Using $\Delta = \Gamma \cup \{\alpha\}$, we see that pfECQ holds in $\mathcal{S}$. $\square$

Theorem 5.3.13. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure such that $\mathcal{L}$ has at least two distinct elements. If pfECQ holds in $\mathcal{S}$, then sECQ also holds in it.*

Proof. Suppose pfECQ holds in $\mathcal{S}$. Let $\alpha \in \mathcal{L}$. Since $\mathcal{L}$ has at least two distinct elements, $\{\alpha\} \subsetneq \mathcal{L}$. Then, by pfECQ, there exists $\Delta \subsetneq \mathcal{L}$ such that $\{\alpha\} \subseteq \Delta$, i.e., $\alpha \in \Delta$, and $W_{\vdash}(\Delta) = \mathcal{L}$. Thus, sECQ holds in $\mathcal{S}$. $\square$

The following example shows that the converse of Theorem 5.3.12 does not hold, i.e., pfECQ does not imply spECQ. It also shows that neither pfECQ nor sECQ implies gECQ.

Example 5.3.14. Let $\mathcal{S} = (\mathcal{L}, \vdash)$ be a logical structure, where $\mathcal{L}$ is infinite, and $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ is such that, for any $\Gamma \subseteq \mathcal{L}$,

$$W_{\vdash}(\Gamma) = \begin{cases} \mathcal{L} & \text{if } \Gamma \text{ is infinite,} \\ \Gamma & \text{otherwise.} \end{cases}$$

We first show that **pfECQ** holds in $\mathcal{S}$. Let $\Gamma \subsetneq \mathcal{L}$. If Γ is infinite, then we are done since $W_{\vdash}(\Gamma) = \mathcal{L}$. On the other hand, if Γ is finite, then as $\mathcal{L}$ is infinite, there exists $\alpha \in \mathcal{L} \setminus \Gamma$. Let $\Delta = \mathcal{L} \setminus \{\alpha\}$. Clearly, $\Gamma \subseteq \Delta \subsetneq \mathcal{L}$ and Δ is infinite, which implies that $W_{\vdash}(\Delta) = \mathcal{L}$. Thus, for every $\Gamma \subsetneq \mathcal{L}$, there exists $\Delta \subsetneq \mathcal{L}$ such that $\Gamma \subseteq \Delta$, and $W_{\vdash}(\Delta) = \mathcal{L}$, i.e., **pfECQ** holds in $\mathcal{S}$. Then, as $\mathcal{L}$ is infinite, and so, has at least two distinct elements, by [Theorem 5.3.13](#), **sECQ** also holds in $\mathcal{S}$.

Now, to see that **spECQ** does not hold in $\mathcal{S}$, let Γ be any finite subset of $\mathcal{L}$. Since $\mathcal{L}$ is infinite, $\Gamma \subsetneq \mathcal{L}$. Now, for any $\alpha \in \mathcal{L}$, $\Gamma \cup \{\alpha\}$ is also finite, and hence, $W_{\vdash}(\Gamma \cup \{\alpha\}) = \Gamma \cup \{\alpha\} \neq \mathcal{L}$. Thus, **spECQ** fails in $\mathcal{S}$.

Finally, for any $\alpha, \beta \in \mathcal{L}$, since $\{\alpha, \beta\}$ is finite, $W_{\vdash}(\{\alpha, \beta\}) = \{\alpha, \beta\} \subsetneq \mathcal{L}$. Thus, **gECQ** cannot hold in $\mathcal{S}$.

The following example shows that neither **gECQ** nor **sECQ** implies **pfECQ** or **spECQ**.

Example 5.3.15. Let $\mathcal{S} = (\mathcal{L}, \vdash)$ be the logic, where $\mathcal{L} = \mathbb{N}$, and $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ is such that, for any $\Gamma \subseteq \mathcal{L}$,

$$W_{\vdash}(\Gamma) = \begin{cases} \mathcal{L} & \text{if } \Gamma = \{n, n+1\} \text{ for some } n \in \mathbb{N}, \\ \Gamma & \text{otherwise.} \end{cases}$$

Let $k \in \mathcal{L}$. Then, $k+1 \in \mathcal{L}$ and $W_{\vdash}(\{k, k+1\}) = \mathcal{L}$. Thus, **gECQ** holds in $\mathcal{S}$. Since **gECQ** holds in $\mathcal{S}$, and $\mathcal{L}$ has at least three distinct elements, by [Theorem 5.3.1](#), **sECQ** also holds in $\mathcal{S}$.

Now, to see that **pfECQ** does not hold in $\mathcal{S}$, let $\Gamma \subsetneq \mathcal{L}$ such that $|\Gamma| \geq 3$. Then, for any $\Delta \subseteq \mathcal{L}$ such that $\Gamma \subseteq \Delta$, $W_{\vdash}(\Delta) = \Delta$. So, for any $\Delta \supseteq \Gamma$, $W_{\vdash}(\Delta) = \mathcal{L}$ iff $\Delta = \mathcal{L}$. Thus, there does not exist any $\Delta \subsetneq \mathcal{L}$ such that $\Gamma \subseteq \Delta$ and $W_{\vdash}(\Delta) = \mathcal{L}$. Hence, **pfECQ** fails in $\mathcal{S}$.

By similar arguments as above, we can say that for any $\Gamma \subsetneq \mathcal{L}$ with $|\Gamma| \geq 3$, there does not exist any $\alpha \in \mathcal{L}$ such that $\Gamma \cup \{\alpha\} \subsetneq \mathcal{L}$ and $W_{\vdash}(\Gamma \cup \{\alpha\}) = \mathcal{L}$. Thus, **spECQ** fails in $\mathcal{S}$.

Remark 5.3.16. The fact that **gECQ** does not imply **spECQ** can also be derived from the fact that **gECQ** does not imply **pfECQ** (shown in the above example) and [Theorem 5.3.12](#), which states that **spECQ** implies **pfECQ**.

Theorem 5.3.17. Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure. If **pfECQ3** holds in $\mathcal{S}$, then **spECQ** holds in it. The converse also holds if $\mathcal{L}$ has at least two distinct elements. Thus, if $\mathcal{L}$ has at least two distinct elements, **pfECQ3** is equivalent to **spECQ**.

Proof. Suppose **pfECQ3** holds in $\mathcal{S}$. Let $\Gamma \subsetneq \mathcal{L}$. Then, by **pfECQ3**, there exists $\emptyset \neq \Delta \subsetneq \mathcal{L}$ such that $\Gamma \cup \Delta \subsetneq \mathcal{L}$, and for every $\emptyset \neq \Delta' \subseteq \Delta$, $W_{\vdash}(\Gamma \cup \Delta') = \mathcal{L}$. Since $\Delta \neq \emptyset$, there exists $\alpha \in \Delta$. Moreover, $\Gamma \cup \{\alpha\} \subseteq \Gamma \cup \Delta \subsetneq \mathcal{L}$. Finally, since $\emptyset \neq \{\alpha\} \subseteq \Delta$, $W_{\vdash}(\Gamma \cup \{\alpha\}) = \mathcal{L}$. Thus, for every $\Gamma \subsetneq \mathcal{L}$, there exists $\alpha \in \mathcal{L}$ such that $\Gamma \cup \{\alpha\} \subsetneq \mathcal{L}$, and $W_{\vdash}(\Gamma \cup \{\alpha\}) = \mathcal{L}$, i.e., **spECQ** holds in $\mathcal{S}$.

Conversely, suppose $\mathcal{L}$ has at least two distinct elements, and **spECQ** holds in $\mathcal{S}$. Let $\Gamma \subsetneq \mathcal{L}$. Then, by **spECQ**, there exists $\alpha \in \mathcal{L}$ such that $\Gamma \cup \{\alpha\} \subsetneq \mathcal{L}$ and $W_{\perp}(\Gamma \cup \{\alpha\}) = \mathcal{L}$. Now, let $\Delta = \{\alpha\}$. Since $\mathcal{L}$ has at least two distinct elements, $\emptyset \neq \Delta \subsetneq \mathcal{L}$. Moreover, $\Gamma \cup \Delta = \Gamma \cup \{\alpha\} \subsetneq \mathcal{L}$. Finally, since Δ is the only non-empty subset of itself and $W_{\perp}(\Gamma \cup \Delta) = W_{\perp}(\Gamma \cup \{\alpha\}) = \mathcal{L}$, we have that for every $\emptyset \neq \Delta' \subseteq \Delta$, $W_{\perp}(\Gamma \cup \Delta') = \mathcal{L}$. Thus, **pfECQ3** holds in $\mathcal{S}$. $\square$

Corollary 5.3.18. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure. If **pfECQ3** holds in $\mathcal{S}$, then **pfECQ** holds in $\mathcal{S}$. If $\mathcal{L}$ has at least two distinct elements, then **sECQ** also holds in $\mathcal{S}$.*

Proof. Suppose **pfECQ3** holds in $\mathcal{S}$. Then, by the above theorem, **spECQ** holds in $\mathcal{S}$. So, by [Theorem 5.3.12](#), **pfECQ** holds in $\mathcal{S}$.

Moreover, if $\mathcal{L}$ has at least three distinct elements, then, by [Theorem 5.3.8](#), since **spECQ** holds in $\mathcal{S}$, **sECQ** also holds in it. $\square$

Corollary 5.3.19. *We can now draw the following conclusions from the results obtained so far.*

- (i) **sECQ** does not imply **pfECQ3**.
- (ii) **pfECQ** does not imply **pfECQ3**.

Proof. (i) Suppose the contrary, i.e., **sECQ** implies **pfECQ3**. Now, by [Theorem 5.3.17](#), **pfECQ3** implies **spECQ**. So, **sECQ** implies **spECQ**. However, by [Example 5.3.9](#), we know that that is not the case. Hence, **sECQ** cannot imply **pfECQ3**.

- (ii) Suppose the contrary, i.e., **pfECQ** implies **pfECQ3**. Now, by [Theorem 5.3.17](#), **pfECQ3** implies **spECQ**. So, **pfECQ** implies **spECQ**. However, by [Example 5.3.14](#), we know that that is not the case. Hence, **pfECQ** does not imply **pfECQ3**. $\square$

We have already seen, via [Example 5.2.14](#), [Example 5.3.3](#), and [Remark 5.3.4](#), and [Remark 5.3.5](#), that neither **gECQ** nor **sECQ** implies **ECQ**. The following example shows that none of the other explosion principles, **pfECQ**, **pfECQ3** or **spECQ** implies **ECQ**.

Example 5.3.20. Let $\mathcal{S} = (\mathcal{L}, \vdash)$ be the logic PWK (as described in [\[27\]](#)). We know that PWK is paraconsistent, and so **ECQ** fails in it. As discussed in [Example 5.2.14](#), PWK is not $\perp$ -paraconsistent as there is a falsity constant $\perp \in \mathcal{L}$. We claim that **spECQ** holds in PWK. Let $\Gamma \subsetneq \mathcal{L}$. If $W_{\perp}(\Gamma) = \mathcal{L}$, then $\Gamma \neq \emptyset$ since not every formula is a theorem of PWK. So, for any $\alpha \in \Gamma$, $\Gamma \cup \{\alpha\} = \Gamma \subsetneq \mathcal{L}$ and $W_{\perp}(\Gamma \cup \{\alpha\}) = W_{\perp}(\Gamma) = \mathcal{L}$. On the other hand, if $W_{\perp}(\Gamma) \neq \mathcal{L}$, then $W_{\perp}(\Gamma \cup \{\perp\}) = \mathcal{L}$. Now, suppose $\Gamma \cup \{\perp\} = \mathcal{L}$, then $\Gamma = \mathcal{L} \setminus \{\perp\}$, which implies that $\perp \wedge \perp \in \Gamma$. However, as $W_{\perp}(\perp \wedge \perp) = \mathcal{L}$, by monotonicity, $W_{\perp}(\Gamma) = \mathcal{L}$, a contradiction to our previous assumption. Thus, $\Gamma \cup \{\perp\} \subsetneq \mathcal{L}$. Hence, we see that for any $\Gamma \subsetneq \mathcal{L}$, there exists $\alpha \in \mathcal{L}$ such that $\Gamma \cup \{\alpha\} \subsetneq \mathcal{L}$ and $W_{\perp}(\Gamma \cup \{\alpha\}) = \mathcal{L}$, i.e., **spECQ** holds in PWK.

Since **spECQ** holds in PWK, **pfECQ** holds in it, by [Theorem 5.3.12](#). Moreover, since $|\mathcal{L}| \geq 2$, **pfECQ3** also holds in it, by [Theorem 5.3.17](#).

Example 5.3.21. The logic P^1 , discussed in [Example 5.3.3](#), is also an example of a logic where **spECQ** holds. This can be shown as follows. Let $P^1 = (\mathcal{L}, \vdash)$, as before, and $\Gamma \subsetneq \mathcal{L}$. Now, as argued in [Example 5.3.3](#), $W_{\vdash}(\neg(\alpha \rightarrow \alpha)) = \mathcal{L}$. Since P^1 is monotonic, $W_{\vdash}(\Gamma \cup \{\neg(\alpha \rightarrow \alpha)\}) = \mathcal{L}$. Now, if $\Gamma \cup \{\neg(\alpha \rightarrow \alpha)\} \subsetneq \mathcal{L}$, then we are done. If not, then $\Gamma = \mathcal{L} \setminus \{\neg(\alpha \rightarrow \alpha)\}$, and hence, $\neg((\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha)) \in \Gamma$. Now, $W_{\vdash}(\neg((\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha))) = \mathcal{L}$ as well, and so, by monotonicity, $W_{\vdash}(\Gamma \cup \{\neg((\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha))\}) = \mathcal{L}$ and $\Gamma \cup \{\neg((\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha))\} = \Gamma \subsetneq \mathcal{L}$. Thus, **spECQ** holds in P^1 .

Therefore, by [Theorem 5.3.12](#), **pfECQ** also holds in P^1 . Hence the logic P^1 is another example of a logic, where **spECQ** and **pfECQ** hold, but **ECQ** fails.

Remark 5.3.22. It can be shown using the same arguments, as in the above example, that **spECQ**, and hence, **pfECQ**, hold in all the gently paraconsistent logics from [\[42\]](#) (discussed in [Remark 5.3.4](#)) and the paraconsistent logics P^m and P^{*m} , $m \geq 1$, from [\[43\]](#) (mentioned in [Remark 5.3.5](#)). This is because all of these logics are monotonic and contain the positive fragment of classical propositional logic. So, for any α , as $\alpha \rightarrow \alpha$ is a theorem in each of them, $W_{\vdash}(\{\neg(\alpha \rightarrow \alpha)\}) = \mathcal{L}$.

It has been established now that **ECQ** is not implied by any of the other principles of explosion. However, **ECQ** implies **sECQ** and **gECQ**. We argue below that **ECQ** does not imply the two remaining explosion principles, viz., **spECQ** and **pfECQ**.

Theorem 5.3.23. Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure. If **spECQ** holds in $\mathcal{S}$, then $\mathcal{L}$ has at least two distinct elements, and there exists $\alpha \in \mathcal{L}$ such that $W_{\vdash}(\{\alpha\}) = \mathcal{L}$.

Proof. Suppose **spECQ** holds in $\mathcal{S}$. Now, $\emptyset \subsetneq \mathcal{L}$. Then, by **spECQ**, there exists $\alpha \in \mathcal{L}$ such that $\emptyset \cup \{\alpha\} = \{\alpha\} \subsetneq \mathcal{L}$, which implies that $\mathcal{L}$ has at least two distinct elements, and $W_{\vdash}(\{\alpha\}) = \mathcal{L}$. $\square$

Remark 5.3.24. We can conclude from the above theorem that if $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logic such that $\neg$ -**ECQ** holds, where $\neg$ is a unary operator in $\mathcal{S}$, i.e., $\mathcal{S}$ is not paraconsistent with respect to $\neg$, but there does not exist $\alpha \in \mathcal{L}$ such that $W_{\vdash}(\{\alpha\}) = \mathcal{L}$, which, in particular, implies that there is no falsity constant, then **spECQ** fails in $\mathcal{S}$. We construct such a logic below.

Example 5.3.25. Let $\mathcal{S} = (\mathcal{L}, \vdash)$ be a logical structure, where $\mathcal{L} = \mathbb{Z}$, and $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ is such that, for any $\Gamma \subseteq \mathcal{L}$,

$$W_{\vdash}(\Gamma) = \begin{cases} \mathcal{L} & \text{if } \Gamma \neq \emptyset \text{ and for each } n \in \Gamma, -n \in \Gamma, \\ \Gamma & \text{otherwise.} \end{cases}$$

Then, with respect to the unary operator $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(n) = -n$ for each $n \in \mathbb{Z}$, we see that **ECQ** holds, i.e., f -**ECQ** holds, since for each $n \in \mathbb{Z}$, $W_{\vdash}(\{n, -n\}) = \mathcal{L}$.

To see that **spECQ** does not hold in $\mathcal{S}$, let $\Gamma = \{n \in \mathbb{Z} \mid n \geq 0\} \subsetneq \mathcal{L}$. Clearly, for any $n \in \mathcal{L}$, $\Gamma \cup \{n\} \subsetneq \mathcal{L}$ but $W_{\vdash}(\Gamma \cup \{n\}) \neq \mathcal{L}$.

In fact, the only subset $\Delta \subseteq \mathcal{L}$ such that $\Gamma \subseteq \Delta$ and $W_{\vdash}(\Delta) = \mathcal{L}$ is $\mathbb{Z}$. Hence, there is no $\Delta \subsetneq \mathcal{L}$ such that $\Gamma \subseteq \Delta$ and $W_{\vdash}(\Delta) = \mathcal{L}$. So, **pfECQ** also fails in $\mathcal{S}$.

Thus, ECQ does not imply spECQ or pfECQ. The following theorem, however, shows that, under some conditions, ECQ does imply spECQ and pfECQ.

Theorem 5.3.26. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a monotonic logic, where $\mathcal{L}$ is the formula algebra over a denumerable set of variables V , using a signature containing a unary (negation) operator $\neg$. Moreover, suppose there exists $\alpha \in \mathcal{L}$ such that $W_{\vdash}(\{\alpha\}) = \mathcal{L}$. If $\neg$ -ECQ holds in $\mathcal{S}$, then spECQ and pfECQ also hold in it.*

Proof. Suppose $\neg$ -ECQ holds in $\mathcal{S}$. Let $\Gamma \subsetneq \mathcal{L}$. To show that spECQ holds, we need to find a $\varphi \in \mathcal{L}$ such that $\Gamma \cup \{\varphi\} \subsetneq \mathcal{L}$ and $W_{\vdash}(\Gamma \cup \{\varphi\}) = \mathcal{L}$. Now, by monotonicity, $W_{\vdash}(\Gamma \cup \{\alpha\}) = \mathcal{L}$. If $\Gamma \cup \{\alpha\} \subsetneq \mathcal{L}$, then we are done. If not, then $\Gamma = \mathcal{L} \setminus \{\alpha\}$. So, $\Gamma \neq \emptyset$ since $\mathcal{L}$ is denumerable. Now, let $p \in V$ such that p does not occur in α . Then $p, \neg p \neq \alpha$, and hence, $p, \neg p \in \Gamma$. Now, since $\neg$ -ECQ holds in $\mathcal{S}$, $W_{\vdash}(\{p, \neg p\}) = \mathcal{L}$. Thus, $\Gamma \cup \{p\} = \Gamma \subsetneq \mathcal{L}$, and, by monotonicity, $W_{\vdash}(\Gamma \cup \{p\}) = W_{\vdash}(\Gamma) = \mathcal{L}$. So, spECQ holds in $\mathcal{S}$. Then, by Theorem 5.3.12, pfECQ also holds in $\mathcal{S}$. $\square$

The connections between the different explosion principles discussed so far can now be summarised in the following figure. The arrows in the figure indicate implications, while the absence of an arrow from one to another principle of explosion indicates a non-implication. Some implications hold under mild conditions, such as that $\mathcal{L}$ must have at least two or three elements. We have not shown these conditions in the picture to preserve clarity.

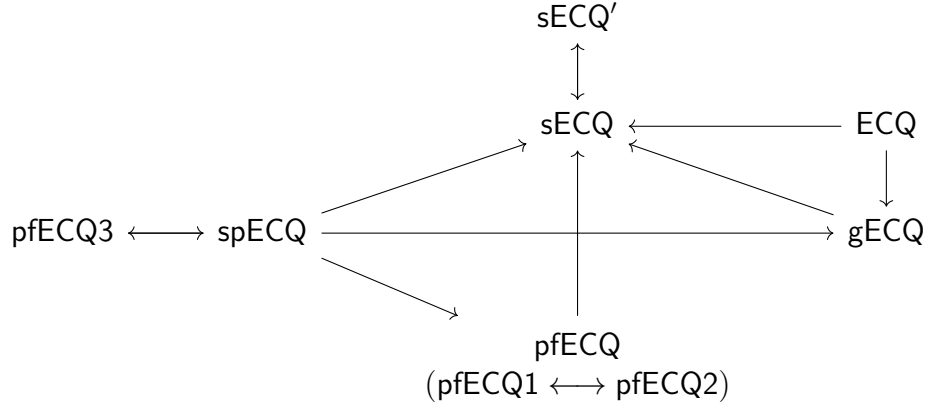


Figure 5.1: Principles of Explosion

We next explore the connections between the various principles of explosion discussed in this section with the property of *finite trivialisability*.

Definition 5.3.27 (FINITELY TRIVIALISABLE LOGICAL STRUCTURE). Let $\mathcal{S} = (\mathcal{L}, \vdash)$ be a logical structure. $\mathcal{S}$ is said to be *finitely trivialisable* if there exists a finite $\Gamma \subseteq \mathcal{L}$ such that $W_{\vdash}(\Gamma) = \mathcal{L}$.

Remark 5.3.28. It is clear that if $\mathcal{L}$ is finite, then finite trivialisability is equivalent to trivialisability. Thus, it makes sense to investigate this property in the case of logics, where

$\mathcal{L}$ is infinite. Hence, $\mathcal{L}$ will be assumed to be infinite in all the following results concerning finite trivialisability.

It is fairly easy to see that finite trivialisability does not necessarily imply any of the principles of explosion: ECQ, gECQ, sECQ, spECQ, pfECQ. The following results expound the conditions under which finite trivialisability does imply gECQ, sECQ, spECQ, and pfECQ.

Theorem 5.3.29. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a monotonic logical structure. If $\mathcal{S}$ is finitely trivialisable, then sECQ holds in $\mathcal{S}$.*

Proof. Suppose $\mathcal{S}$ is finitely trivialisable. Then, there exists a finite $\Gamma \subseteq \mathcal{L}$ such that $W_{\vdash}(\Gamma) = \mathcal{L}$. Let $\alpha \in \mathcal{L}$. Then $\Gamma \cup \{\alpha\}$ is a finite subset of $\mathcal{L}$, and hence $\Gamma \cup \{\alpha\} \subsetneq \mathcal{L}$ since $\mathcal{L}$ is infinite. Clearly, $\alpha \in \Gamma \cup \{\alpha\}$, and, by monotonicity, $W_{\vdash}(\Gamma \cup \{\alpha\}) = \mathcal{L}$. Thus, sECQ holds in $\mathcal{S}$. $\square$

Theorem 5.3.30. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a transitive logic, where $\mathcal{L}$ is the formula algebra over a set of variables $V \neq \emptyset$, using a finite signature containing a conjunction operator $\wedge$. Moreover, suppose the following elimination rule for $\wedge$ holds in $\mathcal{S}$.*

$$\{\alpha \wedge \beta\} \vdash \alpha \quad \text{and} \quad \{\alpha \wedge \beta\} \vdash \beta \quad (\wedge E).$$

Then, $\mathcal{S}$ is finitely trivialisable iff there exists $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$.

Proof. Suppose $\mathcal{S}$ is finitely trivialisable. Let Γ be a finite subset of $\mathcal{L}$ such that $W_{\vdash}(\Gamma) = \mathcal{L}$, and $\varphi = \bigwedge_{\beta \in \Gamma} \beta$ (the conjunction of all the elements in the finite set Γ). Now, by $\wedge E$, $\varphi \vdash \beta$ for all $\beta \in \Gamma$. Since $W_{\vdash}(\Gamma) = \mathcal{L}$, by transitivity, $W_{\vdash}(\{\varphi\}) = \mathcal{L}$. The converse clearly holds since $\{\varphi\}$ is a finite set. $\square$

Theorem 5.3.31. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a monotonic and transitive logic, where $\mathcal{L}$ is as in the above result. Moreover, suppose the elimination rule $(\wedge E)$, as described in the above result, holds in $\mathcal{S}$. If $\mathcal{S}$ is finitely trivialisable, then spECQ, pfECQ, and gECQ hold in $\mathcal{S}$.*

Proof. Suppose $\mathcal{S}$ is finitely trivialisable. So, by Theorem 5.3.30, there exists $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$. Let $\Gamma \subsetneq \mathcal{L}$.

Case 5.3.31.1. $\Gamma \cup \{\varphi\} \subsetneq \mathcal{L}$. By monotonicity, $W_{\vdash}(\Gamma \cup \{\varphi\}) = \mathcal{L}$.

Case 5.3.31.2. $\Gamma \cup \{\varphi\} = \mathcal{L}$. So, $\Gamma = \mathcal{L} \setminus \{\varphi\}$, which implies that $\varphi \wedge \varphi \in \Gamma$ since $\varphi \neq \varphi \wedge \varphi$. Now, by $(\wedge E)$, $\varphi \wedge \varphi \vdash \varphi$. So, by monotonicity, $\Gamma \cup \{\varphi \wedge \varphi\} \vdash \varphi$ (i.e., $\Gamma \vdash \varphi$). Then, as $W_{\vdash}(\{\varphi\}) = \mathcal{L}$, by transitivity, $W_{\vdash}(\Gamma \cup \{\varphi \wedge \varphi\}) = W_{\vdash}(\Gamma) = \mathcal{L}$.

Thus, spECQ holds in $\mathcal{S}$. So, by Theorem 5.3.12, pfECQ also holds in $\mathcal{S}$. Since $\mathcal{L}$ has at least two distinct elements (in fact, $\mathcal{L}$ is denumerable as $\mathcal{L}$ generated inductively over a non-empty set), by Theorem 5.3.8, gECQ also holds in $\mathcal{S}$. $\square$

Remark 5.3.32. Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ such that $\mathcal{L}$ is infinite. It is easy to see that if ECQ or gECQ holds in $\mathcal{S}$, then it is finitely trivialisable. Moreover, by [Theorem 5.3.23](#), if spECQ holds in $\mathcal{S}$, then there exists $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$, which implies that $\mathcal{S}$ is finitely trivialisable. In other words, if $\mathcal{S}$ is not finitely trivialisable, then none of the explosion principles ECQ, gECQ, or spECQ holds in it. The 3-valued logic *LP* (*Logic of Paradox*), introduced in [5] and investigated in [91], is one such logic. An interesting feature of *LP* is that it does not have a detachable implication. The logic *Pac*, obtained by adding a detachable implication to *LP*, is also not finitely trivialisable. This is also a 3-valued paraconsistent logic that first appeared with this name in [7], as $RM_3^{\supset}$ in [6], and as PI^s in [12]. The logic *PI*, introduced in [12], is also not finitely trivialisable. (See also the discussion on *Pac* and *PI* in [37].)

It may, however, be noted that sECQ and pfECQ may or may not hold in logics that are not finitely trivialisable. Clearly, if there does not exist any $\Gamma \subsetneq \mathcal{L}$ such that $W_{\vdash}(\Gamma) = \mathcal{L}$, then sECQ and pfECQ also fail. The following example shows that, under some conditions, pfECQ and sECQ might hold in logics that are not finitely trivialisable, provided there is an infinite proper subset of $\mathcal{L}$ that explodes.

Example 5.3.33. Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure, where $\mathcal{L}$ is infinite, and $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ is such that, for any $\Gamma \subseteq \mathcal{L}$,

$$W_{\vdash}(\Gamma) = \begin{cases} \Gamma & \text{if } \Gamma \text{ is finite,} \\ \mathcal{L} & \text{otherwise.} \end{cases}$$

Clearly, $\mathcal{S}$ is not finitely trivialisable. Let $\Gamma \subsetneq \mathcal{L}$. If Γ is infinite, then $W_{\vdash}(\Gamma) = \mathcal{L}$. On the other hand, if Γ is finite, let $\alpha \in \mathcal{L} \setminus \Gamma$ (such an α exists since Γ is finite, while $\mathcal{L}$ is infinite). Now, let $\Delta = \mathcal{L} \setminus \{\alpha\}$. Then, $\Gamma \subseteq \Delta \subsetneq \mathcal{L}$, and since Δ is infinite, $W_{\vdash}(\Delta) = \mathcal{L}$. Thus, pfECQ holds in $\mathcal{S}$, and so, by [Theorem 5.3.13](#), sECQ also holds in $\mathcal{S}$.

We end the section with a class of logics that might serve as important examples to motivate one to consider the above principles of explosion. These are the *Logics of Formal Inconsistency (LFIs)* [36, 37]. For these logics, we assume that $\mathcal{L}$ is the formula algebra over a denumerable set of variables V , using a finite signature containing a unary (negation) operator $\neg$. These logics are also assumed to be of Tarski-type, structural, and finitary, i.e., for any $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}$, if $\Gamma \vdash \alpha$, then there exists a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \alpha$. A Tarski-type, structural, finitary logic is sometimes referred to as a *standard* logic. An LFI is a paraconsistent logic, i.e., ECQ fails in it, where a local or controlled explosion is allowed. It is important to note that not all paraconsistent logics are LFIs, and examples of such logics can be found in the works cited at the beginning of this paragraph. The following definition of an LFI is adapted from [36].

Definition 5.3.34 (LFI). Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a standard logic. Let $p \in V$ and $\bigcirc(p)$ be a non-empty set of formulas depending exactly on p . Moreover, for any $\varphi \in \mathcal{L}$, let $\bigcirc(\varphi) = \{\psi(\varphi) \mid \psi(p) \in \bigcirc(p)\}$. $\mathcal{S}$ is a *Logic of Formal Inconsistency (LFI)* (with respect to $\neg$ and $\bigcirc(p)$) if the following conditions hold.

- (i) There exists $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi, \neg\varphi\}) \neq \mathcal{L}$, i.e., $\neg$ -ECQ fails in $\mathcal{S}$, or equivalently, $\mathcal{S}$ is paraconsistent with respect to $\neg$.

- (ii) There exists $\alpha, \beta \in \mathcal{L}$ such that $\beta \notin W_{\vdash}(\bigcirc(\alpha) \cup \{\alpha\}) \cup W_{\vdash}(\bigcirc(\alpha) \cup \{\neg\alpha\})$, which implies that $W_{\vdash}(\bigcirc(\alpha) \cup \{\alpha\}) \neq \mathcal{L}$ and $W_{\vdash}(\bigcirc(\alpha) \cup \{\neg\alpha\}) \neq \mathcal{L}$.
- (iii) $W_{\vdash}(\bigcirc(\varphi) \cup \{\varphi, \neg\varphi\}) = \mathcal{L}$ for all $\varphi \in \mathcal{L}$.

There are two other ways of introducing LFIs. These are called *weak* and *strong* LFIs. It follows easily from the definitions of LFIs, weak LFIs, and strong LFIs that every strong LFI is an LFI, and every LFI is a weak LFI. The converses are not true. A detailed discussion on these can be found in [36, 37].

Given an $\alpha \in \mathcal{L}$, the set $\bigcirc(\alpha)$ is intended to express, in a sense, the consistency of α relative to the logic $\mathcal{S}$. When this set is a singleton, its element is denoted by $\circ\alpha$, and in this case, $\circ$ defines a *consistency connective* or *consistency operator*. The consistency connective may or may not be a primitive connective of the logic.

Clause (iii) in the above definition is called the *gentle principle of explosion* (not to be confused with the principle of gentle explosion discussed in Remark 5.3.4), and a logic satisfying this clause is called *gently explosive* (with respect to the respective $\bigcirc(p)$). If a logic is gently explosive with respect to some finite set $\bigcirc(p)$, it is called *finitely gently explosive*.

Theorem 5.3.35. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is an LFI with respect to $\neg$ and $\bigcirc(p)$. If $\mathcal{S}$ is finitely gently explosive, i.e., $\bigcirc(p)$ is finite, then it is finitely trivialisable. Moreover, suppose the signature of $\mathcal{S}$ has a conjunction operator $\wedge$ for which the elimination rule $(\wedge E)$, as described in Theorem 5.3.30, holds⁴. Then there exists $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$ ⁵.*

Proof. Suppose $\mathcal{S}$ is finitely gently explosive. Then, for any $\alpha \in \mathcal{L}$, $\bigcirc(\alpha)$ is finite, and hence, $\bigcirc(\alpha) \cup \{\alpha, \neg\alpha\}$ is a finite subset of $\mathcal{L}$. Since $W_{\vdash}(\bigcirc(\alpha) \cup \{\alpha, \neg\alpha\}) = \mathcal{L}$, $\mathcal{S}$ is finitely trivialisable.

Now, in case the signature of $\mathcal{S}$ has a conjunction operator $\wedge$ such that $(\wedge E)$ holds, then, by Theorem 5.3.30, there exists $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$. $\square$

Remark 5.3.36. It cannot, however, be concluded that a finitely trivialisable LFI is always finitely gently explosive since the definition of an LFI does not rule out the presence of some $\alpha \in \mathcal{L}$ such that $W_{\vdash}(\{\alpha, \neg\alpha\}) = \mathcal{L}$ even though $\neg$ -ECQ does not hold.

We already know that LFIs are paraconsistent, so $\neg$ -ECQ fails in all of them. The following theorem lists the status of the other explosion principles in LFIs.

Theorem 5.3.37. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is an LFI with respect to $\neg$ and $\bigcirc(p)$.*

- (i) *pfECQ and sECQ hold in $\mathcal{S}$*
- (ii) *If $\mathcal{S}$ is finitely gently explosive, and the signature of $\mathcal{S}$ has a conjunction operator $\wedge$ for which the elimination rule $(\wedge E)$, as described in Theorem 5.3.30, holds, then spECQ and gECQ hold in it.*

⁴A logic with such a conjunction is called *left-adjunctive* in [37].

⁵Such an element φ is called a *bottom particle* in [37], and a *bottom/bottom formula* in [36].

Proof. (i) Let $\Gamma \subsetneq \mathcal{L}$. If $W_{\vdash}(\Gamma) = \mathcal{L}$, then we are done. So, suppose $W_{\vdash}(\Gamma) \neq \mathcal{L}$. Let $\alpha \in \mathcal{L}$, and $\Delta = \Gamma \cup \bigcirc(\alpha) \cup \{\alpha, \neg\alpha\}$. Clearly, $\Gamma \subseteq \Delta$. Now, by the gentle principle of explosion, $W_{\vdash}(\bigcirc(\alpha) \cup \{\alpha, \neg\alpha\}) = \mathcal{L}$. So, by monotonicity, $W_{\vdash}(\Delta) = \mathcal{L}$.

We claim that $\Delta \subsetneq \mathcal{L}$. Suppose the contrary, i.e., $\Delta = \Gamma \cup \bigcirc(\alpha) \cup \{\alpha, \neg\alpha\} = \mathcal{L}$. Now, $\bigcirc(\alpha)$ is the result of substituting α for the propositional variable p in each element of $\bigcirc(p)$, which is a non-empty set of formulas depending exactly on p . So, $\text{var}(\bigcirc(\alpha)) = \text{var}(\alpha)$, where $\text{var}(\cdot)$ is as described in [Definition 5.2.11](#). Thus, $\text{var}(\bigcirc(\alpha) \cup \{\alpha, \neg\alpha\}) = \text{var}(\alpha)$, a finite set. Since the set of propositional variables V is denumerable, there exists $q \in V \setminus \text{var}(\bigcirc(\alpha) \cup \{\alpha, \neg\alpha\})$. Clearly, $(\bigcirc(q) \cup \{q, \neg q\}) \cap (\bigcirc(\alpha) \cup \{\alpha, \neg\alpha\}) = \emptyset$. Then, as $\Gamma \cup \bigcirc(\alpha) \cup \{\alpha, \neg\alpha\} = \mathcal{L}$, $\bigcirc(q) \cup \{q, \neg q\} \subseteq \Gamma$. Again, by the gentle principle of explosion, $W_{\vdash}(\bigcirc(q) \cup \{q, \neg q\}) = \mathcal{L}$, and hence, by monotonicity, $W_{\vdash}(\Gamma) = \mathcal{L}$. This is contrary to our assumption. So, $\Delta \subsetneq \mathcal{L}$.

Hence, for every $\Gamma \subsetneq \mathcal{L}$, there exists $\Delta \subsetneq \mathcal{L}$ such that $\Gamma \subseteq \Delta$ and $W_{\vdash}(\Delta) = \mathcal{L}$. In other words, **pfECQ** holds in $\mathcal{S}$. Since $\mathcal{L}$ is denumerable, and hence, $|\mathcal{L}| \geq 2$, by [Theorem 5.3.13](#), **sECQ** also holds in $\mathcal{S}$.

(ii) Suppose $\mathcal{S}$ is finitely gently explosive. Then, by [Theorem 5.3.35](#), $\mathcal{S}$ is finitely trivialisable, and moreover, there exists $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$. Since $\mathcal{S}$ is an LFI, it is monotonic and transitive. Hence, by [Theorem 5.3.31](#), **spECQ** and **gECQ** hold in $\mathcal{S}$. □

Remark 5.3.38. We see, from the above theorem, that if $\mathcal{S} = (\mathcal{L}, \vdash)$ is a finitely gently explosive LFI with a conjunction operator $\wedge$ for which $(\wedge E)$ holds, then all the explosion principles discussed in this section, barring, of course, **ECQ**, hold in $\mathcal{S}$.

The logic P^1 was discussed in [Example 5.3.3](#), and it was pointed out that while **ECQ** fails, **gECQ** and **sECQ** hold in it. Subsequently, it was also presented as a logic, where **spECQ** and **pfECQ** hold, in [Example 5.3.21](#). P^1 is, in fact, a finitely gently explosive LFI, as demonstrated in [37]. Thus, the fact that all the principles of explosion, except for **ECQ**, hold in P^1 can be obtained as a corollary of [Theorem 5.3.37](#) as well.

If $\mathcal{S}$ is not finitely trivialisable, and thus, not finitely gently explosive either, as discussed in [Remark 5.3.32](#), none of **ECQ**, **gECQ**, and **spECQ** holds in $\mathcal{S}$. However, from the above theorem, we see that even in this case, **sECQ** and **pfECQ**, the only two principles of explosion that can hold in logics that are not finitely trivialisable, hold in $\mathcal{S}$.

5.4 Principles of Partial Explosion

In this section, we investigate some generalisations of **gECQ**, which differ from those considered in the previous section. The main idea is to try to move away from the universality of explosion. In other words, instead of having each element α of the underlying set $\mathcal{L}$ of a logical structure $\mathcal{S} = (\mathcal{L}, \vdash)$ explode with some $\beta \in \mathcal{L}$, the principles of explosion considered below limit the explosion to the members of some $\Gamma \subseteq \mathcal{L}$ and that too with some or all elements of some $\Delta \subseteq \mathcal{L}$. Hence, we call these *principles of partial explosion*. Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure. Then, these principles can be formulated as follows.

- There exists $\Gamma, \Delta \subseteq \mathcal{L}$ such that $\Gamma \neq \emptyset$, and for every $\alpha \in \Gamma$, there exists $\beta \in \Delta$ such that $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$. (parECQ1)
- There exists $\Gamma, \Delta \subseteq \mathcal{L}$ such that $\Gamma \times \Delta \neq \emptyset$, and for every $(\alpha, \beta) \in \Gamma \times \Delta$, $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$. (parECQ2)

The string ‘par’ in the names of the above principles denote partial.

Theorem 5.4.1. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ be a logical structure. pfECQ1 holds in $\mathcal{S}$ iff pfECQ2 holds in it.*

Proof. It is easy to see that pfECQ2 implies pfECQ1. To prove the converse, suppose pfECQ1 holds in $\mathcal{S}$. Then, there exists $\Gamma, \Delta \subseteq \mathcal{L}$ such that $\Gamma \neq \emptyset$, and for every $\alpha \in \Gamma$, there exists $\beta \in \Delta$ such that $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$. Now, let $\Gamma' = \{\alpha\} \subseteq \Gamma$ and $\Delta' = \{\beta \in \Delta \mid W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}\}$. Clearly, $\Gamma' \neq \emptyset$ and, by pfECQ1, $\Delta' \neq \emptyset$. Hence, $\Gamma' \times \Delta' \neq \emptyset$. Now, by the construction of Γ', Δ' , for every $(\alpha, \beta) \in \Gamma' \times \Delta'$, $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$. Thus, pfECQ2 holds. $\square$

Remark 5.4.2. In light of the above theorem, we will use pfECQ to refer to both principles pfECQ1 and pfECQ2.

Theorem 5.4.3. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure. If gECQ holds in $\mathcal{S}$, then pfECQ also holds in it.*

Proof. Suppose gECQ holds in $\mathcal{S}$. Let $\alpha \in \mathcal{L}$. Then, by gECQ, there exists $\beta \in \mathcal{L}$ such that $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$. Let $\Gamma = \{\alpha\}$ and $\Delta = \{\beta\}$. This choice of Γ, Δ clearly shows that pfECQ holds in $\mathcal{S}$. $\square$

Corollary 5.4.4. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logic with a unary operator $\neg$. If $\neg$ -ECQ holds in $\mathcal{S}$, then pfECQ also holds in it.*

Proof. Suppose $\neg$ -ECQ holds in $\mathcal{S}$. Then, by Theorem 5.2.2, gECQ holds in $\mathcal{S}$. Thus, by Theorem 5.4.3, pfECQ holds in $\mathcal{S}$. $\square$

Corollary 5.4.5. *Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure such that $\mathcal{L}$ has at least two distinct elements. If spECQ holds in $\mathcal{S}$, then pfECQ also holds in it.*

Proof. Suppose spECQ holds in $\mathcal{S}$. Since $\mathcal{L}$ has at least two distinct elements, by Theorem 5.3.8, gECQ holds in $\mathcal{S}$. Then, the conclusion follows from Theorem 5.4.3. $\square$

The converse of Theorem 5.4.3 does not hold as shown by the next example.

Example 5.4.6. Let $\mathcal{S} = (\mathcal{L}, \vdash)$ be a logical structure, where $\mathcal{L} = \mathbb{N}$ and $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ is such that, for any $\Gamma \subseteq \mathcal{L}$,

$$W_{\vdash}(\Gamma) = \begin{cases} \mathcal{L} & \text{if } \Gamma = \{0\}, \\ \Gamma & \text{otherwise.} \end{cases}$$

Taking $\Gamma = \{0\} = \Delta$, we see that pfECQ holds in $\mathcal{S}$. However, for any $\alpha \in \mathcal{L}$ such that $\alpha \neq 0$, $W_{\vdash}(\{\alpha, \beta\}) \neq \mathcal{L}$ for all $\beta \in \mathcal{L}$. Thus, gECQ fails in $\mathcal{S}$.

Remark 5.4.7. We can construct similar examples, as above, to see that none of the other principles of explosion, viz., **sECQ** or **pfECQ**, implies or is implied by **pfECQ**. However, that is unnecessary as it is not hard to see that **pfECQ** is incomparable to **sECQ** and **pfECQ**. This is because **sECQ** and **pfECQ** yield an explosive set, of unknown size, for each element/subset of $\mathcal{L}$, while **pfECQ** claims the existence of two subsets Γ, Δ of $\mathcal{L}$ such that for each $(\alpha, \beta) \in \Gamma \times \Delta$, the set $\{\alpha, \beta\}$ is explosive.

[Example 5.4.6](#) already establishes that **pfECQ** does not imply **gECQ**. [Theorem 5.4.13](#) below further refines this non-implication by giving a condition under which **pfECQ** implies the failure of **gECQ**. We first make the following definition for the ease of the subsequent presentation.

Definition 5.4.8 (COMPLEMENTARY/WEAKLY COMPLEMENTARY PAIRS IN A LOGICAL STRUCTURE). Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure. A pair of sets $(\Gamma, \Delta) \in \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ is said to be

- (a) *complementary* in $\mathcal{S}$ if for every $(\alpha, \beta) \in \Gamma \times \Delta$, $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$.
- (b) *weakly complementary* in $\mathcal{S}$ if for every $\alpha \in \Gamma$, there exists $\beta \in \Delta$ such that $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$.

Remark 5.4.9. It is easy to see that, given a logical structure $\mathcal{S} = (\mathcal{L}, \vdash)$, **pfECQ** (using the **pfECQ1** formulation) holds in $\mathcal{S}$ iff there is a weakly complementary pair of sets $(\Gamma, \Delta) \in \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ in $\mathcal{S}$ with $\Gamma \neq \emptyset$. Also, **pfECQ** (using the **pfECQ2** formulation) holds in $\mathcal{S}$ iff there is a complementary pair of sets $(\Gamma, \Delta) \in \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ in $\mathcal{S}$ with $\Gamma \times \Delta \neq \emptyset$. Thus, there exists a complementary pair in $\mathcal{S}$ iff there is a weakly complementary pair in $\mathcal{S}$. If a pair of sets is complementary, then clearly, it is also weakly complementary. However, not every weakly complementary pair is complementary.

Theorem 5.4.10. Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure. There does not exist $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$ iff for all complementary (Γ, Δ) in $\mathcal{S}$, with $\Gamma \times \Delta \neq \emptyset$, $\Gamma \cap \Delta = \emptyset$.

Proof. Suppose $(\Gamma, \Delta) \in \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ is complementary in $\mathcal{S}$ such that $\Gamma \times \Delta \neq \emptyset$. Then, $\Gamma, \Delta \neq \emptyset$. Suppose further that $\Gamma \cap \Delta \neq \emptyset$, and let $\varphi \in \Gamma \cap \Delta$. Then, $(\varphi, \varphi) \in \Gamma \times \Delta$, and hence, by definition of a complementary pair of sets, $W_{\vdash}(\{\varphi\}) = \mathcal{L}$. Thus, if there is no $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$, then, for any complementary (Γ, Δ) in $\mathcal{S}$, with $\Gamma \times \Delta \neq \emptyset$, $\Gamma \cap \Delta = \emptyset$.

Conversely, suppose that for any complementary (Γ, Δ) in $\mathcal{S}$, with $\Gamma \times \Delta \neq \emptyset$, $\Gamma \cap \Delta = \emptyset$. Let $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$, and $\Gamma = \Delta = \{\varphi\}$. Then, clearly, (Γ, Δ) is complementary in $\mathcal{S}$ such that $\Gamma \times \Delta \neq \emptyset$ but $\Gamma \cap \Delta \neq \emptyset$. This contradicts our assumption regarding the complementary pairs in $\mathcal{S}$. Hence, there cannot be any $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$. $\square$

Remark 5.4.11. It follows from the above result that, given a logical structure $\mathcal{S} = (\mathcal{L}, \vdash)$, if there exists $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$, then there exists a complementary pair (Γ, Δ) in $\mathcal{S}$ with $\Gamma \times \Delta \neq \emptyset$. This implies that **pfECQ** holds in $\mathcal{S}$, per the argument in [Remark 5.4.9](#).

Theorem 5.4.12. Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure. Let

$$\mathfrak{C} = \{(\Gamma, \Delta) \in \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L}) \mid (\Gamma, \Delta) \text{ is weakly complementary, } \Gamma \neq \emptyset, \text{ and } \Gamma \cap \Delta = \emptyset\},$$

and $\preceq \subseteq \mathfrak{C} \times \mathfrak{C}$ be a relation defined as follows.

$$(\Gamma, \Delta) \preceq (\Gamma', \Delta') \quad \text{iff} \quad \Gamma \subseteq \Gamma' \text{ and } \Delta \subseteq \Delta'.$$

Then, $(\mathfrak{C}, \preceq)$ is a partially ordered set.

Proof. Straightforward. □

Theorem 5.4.13. Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure such that there does not exist any $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$, and **pfECQ** holds in it. Let $(\mathfrak{C}, \preceq)$ be the partially ordered set described in the above result. If **gECQ** holds in $\mathcal{S}$, then $(\mathfrak{C}, \preceq)$ has a maximal element (Π, Λ) such that $\Pi \cup \Lambda = \mathcal{L}$.

Proof. We first note that since **pfECQ** holds in $\mathcal{S}$, by [Remark 5.4.9](#), there exists complementary, and hence, also weakly complementary (Γ, Δ) in $\mathcal{S}$ such that $\Gamma \times \Delta \neq \emptyset$. Then, since there is no $\varphi \in \mathcal{L}$ with $W_{\vdash}(\{\varphi\}) = \mathcal{L}$, by [Theorem 5.4.10](#), every such complementary (Γ, Δ) in $\mathcal{S}$ is such that $\Gamma \cap \Delta = \emptyset$. Hence, $\mathfrak{C} \neq \emptyset$. Now, suppose **gECQ** holds in $\mathcal{S}$.

Let $\mathcal{C}$ be a chain in $(\mathfrak{C}, \preceq)$. We claim that $\left(\bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Gamma, \bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Delta \right)$ is an upper bound of $\mathcal{C}$ in $\mathfrak{C}$. Since for every $(\Gamma, \Delta) \in \mathcal{C} \subseteq \mathfrak{C}$, $\Gamma \neq \emptyset$, $\bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Gamma \neq \emptyset$. Let $\alpha \in \bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Gamma$. This implies that there exists $(\Gamma, \Delta) \in \mathcal{C}$ such that $\alpha \in \Gamma$. Since (Γ, Δ) is weakly complementary, there exists $\beta \in \Delta \subseteq \bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Delta$ such that $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$. Thus, $\left(\bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Gamma, \bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Delta \right)$ is weakly complementary in $\mathcal{S}$. Next, to show that $\left(\bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Gamma \right) \cap \left(\bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Delta \right) = \emptyset$, suppose the contrary. Let $\alpha \in \left(\bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Gamma \right) \cap \left(\bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Delta \right)$. So, $\alpha \in \bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Gamma$, which implies that there exists $(\Gamma, \Delta) \in \mathcal{C}$ such that $\alpha \in \Gamma$. Similarly, since $\alpha \in \bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Delta$, there exists $(\Gamma', \Delta') \in \mathcal{C}$ such that $\alpha \in \Delta'$. Now, since $\mathcal{C}$ is a chain, either $(\Gamma, \Delta) \preceq (\Gamma', \Delta')$ or $(\Gamma', \Delta') \preceq (\Gamma, \Delta)$. We can assume, without loss of generality, $(\Gamma, \Delta) \preceq (\Gamma', \Delta')$. Hence, $\Gamma \subseteq \Gamma'$, which implies that $\alpha \in \Gamma'$. Thus, $\alpha \in \Gamma' \cap \Delta'$. However, since $(\Gamma', \Delta') \in \mathcal{C} \subseteq \mathfrak{C}$, $\Gamma' \cap \Delta' = \emptyset$. This is a contradiction. Hence, $\left(\bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Gamma \right) \cap \left(\bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Delta \right) = \emptyset$. So, $\left(\bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Gamma, \bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Delta \right) \in \mathfrak{C}$. Clearly, $(\Gamma, \Delta) \preceq \left(\bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Gamma, \bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Delta \right)$ for each $(\Gamma, \Delta) \in \mathcal{C}$. Thus, $\left(\bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Gamma, \bigcup_{(\Gamma, \Delta) \in \mathcal{C}} \Delta \right)$ is an upper bound of $\mathcal{C}$ in $\mathfrak{C}$. Since $\mathcal{C}$ was an arbitrary chain in $(\mathfrak{C}, \preceq)$, this proves that every

chain in $(\mathfrak{C}, \preceq)$ has an upper bound. Hence, by Zorn's result, we conclude that $(\mathfrak{C}, \preceq)$ has a maximal element.

Let $(\Pi, \Lambda) \in \mathfrak{C}$ be such a maximal element. We claim that $\Pi \cup \Lambda = \mathcal{L}$. Suppose the contrary. Then, there exists $\gamma \in \mathcal{L} \setminus (\Pi \cup \Lambda)$. Since **gECQ** holds in $\mathcal{S}$, there exists $\delta \in \mathcal{L}$ such that $W_{\vdash}(\{\gamma, \delta\}) = \mathcal{L}$. Moreover, since there does not exist any $\varphi \in \mathcal{L}$ such that $W_{\vdash}(\{\varphi\}) = \mathcal{L}$, $\delta \neq \gamma$. We fix such a δ . There are now three possible cases.

Case 5.4.13.1. $\delta \in \Lambda$. (Then, $\delta \notin \Pi$ since $\Pi \cap \Lambda = \emptyset$.) Let $\Pi' = \Pi \cup \{\gamma\}$. Clearly, $\Pi' \neq \emptyset$. Since (Π, Λ) is weakly complementary and $W_{\vdash}(\{\gamma, \delta\}) = \mathcal{L}$, for every $\alpha \in \Pi'$, there exists $\beta \in \Lambda$ such that $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$. So, (Π', Λ) is weakly complementary. Finally, since $\Pi \cap \Lambda = \emptyset$ and $\gamma \notin \Pi \cup \Lambda$, $\Pi' \cap \Lambda = \emptyset$. Thus, $(\Pi', \Lambda) \in \mathfrak{C}$. Now, since $\Pi \subsetneq \Pi'$, $(\Pi, \Lambda) \preceq (\Pi', \Lambda)$ but $(\Pi, \Lambda) \neq (\Pi', \Lambda)$. This contradicts the maximality of (Π, Λ) in $(\mathfrak{C}, \preceq)$.

Case 5.4.13.2. $\delta \in \Pi$. (Then, $\delta \notin \Lambda$, since $\Pi \cap \Lambda = \emptyset$.) We know that $\Pi \neq \emptyset$. Let $\Lambda' = \Lambda \cup \{\gamma\}$. By the same arguments as in Case 1, (Π, Λ') is weakly complementary, and $\Pi \cap \Lambda' = \emptyset$. Thus, $(\Pi, \Lambda') \in \mathfrak{C}$. Again, since $\Lambda \subsetneq \Lambda'$, $(\Pi, \Lambda) \preceq (\Pi, \Lambda')$ but $(\Pi, \Lambda) \neq (\Pi, \Lambda')$. This contradicts the maximality of (Π, Λ) in $(\mathfrak{C}, \preceq)$.

Case 5.4.13.3. $\delta \in \mathcal{L} \setminus (\Pi \cup \Lambda)$. Let $\Pi' = \Pi \cup \{\gamma\}$ and $\Lambda' = \Lambda \cup \{\delta\}$. Clearly, $\Pi' \neq \emptyset$. By the same arguments as in the previous two cases, (Π', Λ') is weakly complementary. Since $\Pi \cap \Lambda = \emptyset$, $\gamma, \delta \notin \Pi \cup \Lambda$, and $\delta \neq \gamma$, $\Pi' \cap \Lambda' = \emptyset$. Then, as $\Pi \subsetneq \Pi'$ and $\Lambda \subsetneq \Lambda'$, $(\Pi, \Lambda) \preceq (\Pi', \Lambda')$ but $(\Pi, \Lambda) \neq (\Pi', \Lambda')$. This contradicts the maximality of (Π, Λ) in $(\mathfrak{C}, \preceq)$.

Since we arrive at a contradiction in each of the above three possible cases, we conclude that $\Pi \cup \Lambda = \mathcal{L}$. $\square$

Remark 5.4.14. We note, from the above theorem, that, given a logical structure $\mathcal{S} = (\mathcal{L}, | -)$ that satisfies the conditions of the theorem, if **gECQ** holds, then $\mathcal{L}$ can be partitioned into two disjoint pieces, viz., the sets Π, Λ of a maximal element $(\Pi, \Lambda) \in \mathfrak{C}$ such that each member of Π has a corresponding member in Λ , with which it explodes. Moreover, the maximality of (Π, Λ) , together with the fact that $\Pi \cap \Lambda = \emptyset$, implies that there does not exist $\alpha, \beta \in \Pi$ such that $W_{\vdash}(\{\alpha, \beta\}) = \mathcal{L}$. Hence, we can think of $\mathcal{S}$ as being 'locally consistent,' as witnessed by Π .

We can also conclude, from the above theorem, that if the partially ordered set $(\mathfrak{C}, \preceq)$ has no such maximal element (Π, Λ) such that $\Pi \cup \Lambda = \mathcal{L}$, then **gECQ** fails in $\mathcal{S}$.

We end this section with a concept of paraconsistency arising from the failure of a special case of **pfECQ** in which $\Gamma = \mathcal{L}$. This is defined below.

Definition 5.4.15 (*K-PARA-CONSISTENCY*). Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure, and $K \subseteq \mathcal{L}$. $\mathcal{S}$ is said to be *K-paraconsistent* if there exists $\alpha \in \mathcal{L}$ such that $W_{\vdash}(\{\alpha, \beta\}) \neq \mathcal{L}$ for all $\beta \in K$.

The following are some easy observations that follow from the above definition.

Theorem 5.4.16. Suppose $\mathcal{S} = (\mathcal{L}, \vdash)$ is a logical structure.

(i) $\mathcal{S}$ is $\emptyset$ -paraconsistent.

- (ii) $\mathcal{S}$ is $\mathcal{L}$ -paraconsistent iff it is NF -paraconsistent.
- (iii) $\mathcal{S}$ is K -paraconsistent for some $K \subseteq \mathcal{L}$ implies that it is K' -paraconsistent for every $K' \subseteq K$.

Proof. These statements follow straightforwardly from the definition of K -paraconsistency. $\square$

Remark 5.4.17. The set $K \subseteq \mathcal{L}$, in [Definition 5.4.15](#), can be varied to obtain different ‘degrees of paraconsistency,’ somewhat like the degrees of unsolvability in Computability theory. More precisely, we can say that a logical structure $\mathcal{S}_1 = (\mathcal{L}, \vdash_1)$ is more paraconsistent than a logical structure $\mathcal{S}_2 = (\mathcal{L}, \vdash_2)$ if there exists a $K \subseteq \mathcal{L}$ such that $\mathcal{S}_1$ is K -paraconsistent, but $\mathcal{S}_2$ is not. We plan to expand on this in future articles.

5.5 Future Works

- We introduced and discussed the notion of quasi-negation. This should be studied further, in general, and for specific logics. In the latter case, this can lead to characterisations of the sets of quasi-negations of a formula in these logics. In the case of logics with algebraic or topological interpretations, such as classical and intuitionistic propositional logics, quasi-negation can also be studied algebraically or topologically.
- It is now imperative that we introduce a notion of paraconsistency via the failure of each of the distinct principles of explosion presented here. However, we have left this for future work. Further investigation is also required to find more examples of logics that satisfy, or fail, one or more of these new principles of explosion. In this connection, LFIs that are not finitely trivialisable or not finitely gently explosive might be particularly interesting to study. As mentioned in the introductory section, we strongly believe that there are other principles of explosion possible. This needs to be investigated further.
- We have introduced a couple of principles of partial explosion, which turned out to be equivalent. The goal with these is to localise consistency in a logic. Further work is needed, not only to find examples but also to see if other such principles are possible. The idea of K -paraconsistency was formulated as the failure of a particular case of the principles of partial explosion. As indicated at the end of the previous section, this needs to be rigorously studied, as well, to see if it can be used to compare paraconsistent logics.
- Another interesting future project would be to connect the present study to Abstract Algebraic Logic (AAL). In this regard, we hope to find similarities between the present work with the research on inconsistency results and antitheorems in AAL (discussed in [\[92\]](#)).

Chapter 6

Abstract Sequent Structures and Rule-elimination Theorems¹

6.1 Introduction

CUT-elimination theorems are one of the most important classes of theorems of proof theory. Since Gentzen's proof of the CUT-elimination theorem for the system **LK**, introduced in [60], several other proofs of the theorem have been proposed (see [66] for several of these proofs, including the original one by Gentzen). These proofs are essentially of a very particular nature. Even though the techniques of these proofs can be modified to sequent systems other than **LK**, each describes an algorithm to transform a given proof into a CUT-free proof. While these algorithms are useful and often can be modified suitably to other sequent systems, they are not general enough. The following paragraph from [40] nicely summarises the issue.

Checking whether a sequent calculus admits cut-elimination is often a rather tedious task. Indeed, the proof is based on case distinctions and has to be checked for all the possible combinations of the rules. This is usually done using heavy syntactic arguments based on case distinctions, written without filling in the details (note that even Gentzen did not formalize all the cases). This renders the cut-elimination process rather opaque. It is then natural to search for general criteria that a sequent calculus has to satisfy in order to admit cut-elimination. Moreover, such criteria, if given on a suitable level of abstraction, would also provide a deeper understanding of the nature of cut-elimination.

Furthermore, despite its important consequences, what is essentially done in the CUT-elimination theorems is nothing but the *elimination* of a certain rule. One may ask the same question for any other rule. For example, one may ask whether WEAKENING is eliminable from **LK** (the case of CONTRACTION has been dealt with in [68]). One may even ask whether there exists some necessary and sufficient condition(s) for eliminating some particular rule

¹A version of this chapter forms part of [95].

from a particular sequent system. For questions like this, the usual algorithmic proofs of ‘elimination theorems’ hardly shed any light. So, as we have quoted above, it is natural to search for a suitable abstract set-up in which questions like these can be posed and (hopefully) answered. And, while one is at it, why focus on the elimination (or non-elimination) of *particular* rules in *particular* sequent systems? This is the starting point of the genesis of this paper. More specifically, the questions in which we are interested are the following:

- What makes the elimination of CUT possible in **LK**?
- Is it possible to characterise sequent systems for which CUT-elimination holds?
- Is it possible to give necessary and sufficient conditions of eliminating any rule from a given sequent system? What does a ‘rule’ mean? What are we supposed to understand by a ‘sequent system’?

The motivation for these questions comes from *universal logic* in the sense of [14]. One of the most important goals of universal logic, according to [21] is “to clarify the fundamental concepts of logic and to construct general proofs.” In this paper, we aim to clarify the essence of the so-called “elimination theorems” and construct general proofs of the same. We will do the former by providing proofs of the CUT-elimination theorem for the propositional fragment of **LK** and the latter by proving what we call RULE-elimination theorems respectively for *normal sequent structures* and *abstract sequent structures* (which are nothing but certain special types of logical structures; cf. Definition 6.3.3 and Definition 6.4.1, respectively).

As we have mentioned earlier, there already are works of a similar spirit. For example, in [8], the authors use semantic methods to provide a necessary and sufficient condition for CUT-elimination, although for multiple-conclusion calculi. In [40], the authors provide necessary and sufficient conditions (both syntactic and semantic) for CUT-elimination for a certain class of “sequent calculus”. Without going into a detailed comparison between these landmark works on the subject, we would like to point out that the work presented here differs from these two (and several others of the same spirit) in the following aspects. First, our main objective is to prove ‘RULE-elimination theorems’. Second, in introducing normal sequent structures and abstract sequent structures, we have not assumed any particular properties of them like the subformula property (as, e.g., was required in [40]). Third, one of the main motivations for introducing the notion of abstract sequent structures was to find the minimal set of properties that is required to admit CUT-elimination. The notion of proof in this context is more general than the usual one.

The rest of the paper is organised as follows. In Section 6.2, we will give two new non-algorithmic proofs of the CUT-elimination theorem for the propositional fragment of **LK**. En route, we also give a new proof of the consistency of the same (see Theorem 6.2.12). From these two proofs, we abstract the essential features of the argument and define normal sequent structures. Two RULE-elimination theorems for these particular classes of sequent structures, and a converse of one of them, will then be proved in Section 6.3. In the next section, we define abstract sequent structures and point out the essential features that made the proof of the RULE-elimination theorems for normal sequent structures work. This will pave the way towards formulating the most general version of the RULE-elimination theorems for certain

types of abstract sequent structures. We will show that the RULE-elimination theorem also has a converse for these structures. Finally, in the concluding section, we will point out some directions for further research.

6.2 CUT-elimination Theorem for the Propositional Fragment of LK

In this section, we provide two non-algorithmic proofs of the CUT-elimination theorem for the propositional fragment of **LK**, which we will denote by **PLK**. The motivation for the first proof comes from Buss's proof of CUT-elimination theorem for a variant of the system **PLK** (see pp. 13-15 of [33]). That proof is also non-algorithmic. However, there are at least three differences between that proof and ours. These are listed below.

- (1) Buss's proof is 'semantic'; ours is 'syntactic'.
- (2) Buss's proof (especially the COMPLETENESS THEOREM) relies on the INVERSION THEOREM, neither of which we require.
- (3) Although Buss's proof works for the system he mentions in [33], it doesn't seem to work for the original system of Gentzen (see the first paragraph on page 14 of [33]). Our proof works for both.

6.2.1 Basics of PLK

The presentation of the basics of **PLK** in this section closely follows [66].

In what follows, we will use a rather standard form of a propositional language for classical logic. Let $\mathcal{L}_{\text{CPL}}$ denote an algebra of formulae: $(\mathcal{L}, \neg, \wedge, \vee, \longrightarrow)$ with a countably infinite set of propositional symbols, which we denote as $\mathcal{P}$. Operations of this algebra correspond to the well-known connectives of *negation*, *conjunction*, *disjunction* and *implication* respectively. We will sometimes write $\alpha \in \mathcal{L}_{\text{CPL}}$ to mean that $\alpha \in \mathcal{L}$.

The *complexity* of a formula is a map $\text{Comp} : \mathcal{L} \longrightarrow \mathbb{N}$ as follows:

$$\begin{aligned} \text{Comp}(p) &:= 0 && \text{if } p \text{ is a propositional variable} \\ \text{Comp}(\neg A) &:= \text{Comp}(A) + 1 \\ \text{Comp}(A \circ B) &:= \text{Comp}(A) + \text{Comp}(B) \end{aligned}$$

where $\circ \in \{\wedge, \vee, \longrightarrow\}$.

The **PLK**-*sequents* (or, when the context is clear, simply *sequents*) are finite ordered lists (including the empty list) of the form $\Gamma \Rightarrow \Delta$ where the elements of both Γ and Δ are elements of $\mathcal{L}$. We denote the empty list as Λ .

The *sequent complexity* of a sequent is a map $\text{SeqComp} : \mathcal{L} \longrightarrow \mathbb{N}$ as follows:

$$\text{SeqComp}(\Gamma \Rightarrow \Delta) := \sum_{\alpha \in \Gamma} \text{Comp}(\alpha) + \sum_{\alpha \in \Delta} \text{Comp}(\alpha)$$

Notice that here Γ and Δ both are finite ordered lists and not sets.

The sequent system **PLK** consists of the following rules (here p denotes a propositional variable and all other letters denote arbitrary formulas):

AXIOMS

$$p \Rightarrow p$$

STRUCTURAL RULES

WL	$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}$	WR
CL	$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}$	CR
EL	$\frac{\Gamma, A, B, \Pi \Rightarrow \Delta}{\Gamma, B, A, \Pi \Rightarrow \Delta}$	ER

LOGICAL RULES

$\neg$L	$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}$	$\neg$R
$\wedge$L	$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$	$\wedge$L
$\wedge$R	$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$	$\vee$L
$\vee$R	$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}$	$\vee$R
$\rightarrow$L	$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}$	$\rightarrow$R

CUT RULE

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

We use the familiar convention of denoting the upper sequents as the *premise(s)* and the lower sequent as the *conclusion* of the rule under consideration.

A formula introduced by the application of a logical rule is the *principal formula* of this rule application. Formulae used for the derivation of the principal formula are *side formulae* of this rule application. The principal and side formulae are the *active formulae* of this rule application. By convention, we take the formula which occurs on both sides of any axiomatic sequent as an active formula.

Proofs in **PLK** are built as trees of sequents (i.e. with nodes labelled with sequents) with axioms as leaves and the proven sequent as a root. Formally, this notion can be defined as follows.

1. An axiomatic sequent S is a proof of a sequent S .
2. If $\mathcal{D}$ is a proof of a sequent S , then $\frac{\mathcal{D}}{S'}$ is a proof of a sequent S' , provided that S is an instance of the premise and S' an instance of the conclusion of some one-premise rule.
3. If $\mathcal{D}$ is a proof of a sequent S and $\mathcal{D}'$ is a proof of a sequent S' , then $\frac{\mathcal{D} \quad \mathcal{D}'}{S''}$ is a proof of a sequent S'' , provided that S and S' are instances of the premises and S'' an instance of the conclusion of some two-premise rule.

Admitting any sequents (not only axiomatic ones) as leaves of a tree, we say that there is a *derivation* of S from $S_1, \dots, S_n$ if there is a tree where some leaves are decorated with sequents that are not axioms but belong to the list $S_1, \dots, S_n$. If $n = 0$, we have a proof of S as a special case of a derivation.

Remark 6.2.1. One may ask the reason for taking the ‘additive’ version of the CUT-rule (and of $\multimap$) (see the discussion on p. 65 of [66]) instead of the ‘multiplicative’ one (which was the original version considered by Gentzen), i.e.,

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

The reason is mainly due to certain technical conveniences (a detailed account is given in Remark 6.2.21) and also because the multiplicative version of CUT in **LK** is eliminable iff the additive version of CUT in **LK** is eliminable. For the sake of completeness, we give an outline of the proof of this claim.

For this, let us denote the additive version of CUT, ACUT and the multiplicative one as MCUT. Let us also denote the system of **PLK** with ACUT as its “CUT” rule as **PLK^A** and the one with MCUT as **PLK^M**.

So, suppose ACUT is eliminable in **PLK^A**, but MCUT is not. This implies that there exists a proof which can’t be done without using MCUT. This implies, in particular, that there exists a proof whose last two lines look like the following:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

We transform the proof in the following way.

$$\text{STRUCTURAL RULES} \quad \frac{\frac{\Gamma \Rightarrow \Delta, A}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', A} \quad \frac{A, \Gamma' \Rightarrow \Delta'}{A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \quad \text{STRUCTURAL RULES}$$

Since ACUT is eliminable in $\mathbf{PLK}^A$, it follows that there exists an ACUT-free $\mathbf{PLK}^A$ -proof of $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. This means that there exists a $\mathbf{PLK}^A$ -proof of $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ using some or all of WEAKENING, EXCHANGE, CONTRACTION and the logical rules. Consequently, it follows that there exists a MCUT-free $\mathbf{PLK}^A$ -proof of $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ as well, a contradiction. This contradiction establishes our claim.

Throughout this section, we will assume that we are working in $\mathbf{PLK}^A$, which we will continue to denote as $\mathbf{PLK}$.

6.2.2 First Proof of the CUT-elimination Theorem for PLK

Henceforth, $\emptyset \vdash_{\mathbf{PLK}} \Gamma \Rightarrow \Delta$ will be used to mean that $\Gamma \Rightarrow \Delta$ has a $\mathbf{PLK}$ -proof. Our first proof of the CUT-elimination theorem for $\mathbf{PLK}$ needs the use of the following theorems. The proofs of these results, being straightforward, are omitted.

Theorem 6.2.2. *Let $\Gamma \Rightarrow \Delta$ be the end sequent of a $\mathbf{PLK}$ -proof. Suppose also that it is the conclusion of an instance of some rule, say R . If $\Gamma \Rightarrow \Delta$ has a $\mathbf{PLK}$ -proof, then each premise of R also has a $\mathbf{PLK}$ -proof.*

Theorem 6.2.3. *Let $\Gamma \Rightarrow \Delta$ be the end sequent of a $\mathbf{PLK}$ -proof. Suppose also that it is the conclusion of an instance of a rule, say R . If this proof of $\Gamma \Rightarrow \Delta$ is CONTRACTION+CUT-free, then each premise of R also has a CONTRACTION+CUT-free $\mathbf{PLK}$ -proof.*

Theorem 6.2.4. *Every $\mathbf{PLK}$ -sequent of the form $A \Rightarrow A$ has a CONTRACTION+CUT-free proof.*

Proof. We use structural induction to prove this result. The base case is done by noting that $p \Rightarrow p$ is provable without using any instance of CONTRACTION or CUT because the one-step proof

$$p \Rightarrow p$$

suffices.

Let us now assume that for $A, B \in \mathcal{L}_{\mathbf{CPL}}$, both $A \Rightarrow A$ and $B \Rightarrow B$ are provable. We now show that each of the sequents $\neg A \Rightarrow \neg A$, $A \wedge B \Rightarrow A \wedge B$, $A \vee B \Rightarrow A \vee B$ and $A \multimap B \Rightarrow A \multimap B$ is provable.

For $\neg A \Rightarrow \neg A$, after the proof of $A \Rightarrow A$, we do the following steps.

$$\frac{\frac{\frac{A \Rightarrow A}{\neg A, A \Rightarrow A} \quad \neg L}{A, \neg A \Rightarrow A} \quad EL}{\neg A \Rightarrow \neg A} \quad \neg R$$

For $A \wedge B \Rightarrow A \wedge B$, after the proofs of $A \Rightarrow A$ and $B \Rightarrow B$, we do the following steps.

$$\wedge L \quad \frac{\frac{A \Rightarrow A}{A \wedge B \Rightarrow A} \quad \frac{B \Rightarrow B}{A \wedge B \Rightarrow B}}{A \wedge B \Rightarrow A \wedge B} \quad \wedge R$$

For $A \vee B \Rightarrow A \vee B$, after the proofs of $A \Rightarrow A$ and $B \Rightarrow B$, we do the following steps.

$$\text{VR} \quad \frac{\frac{A \Rightarrow A}{A \Rightarrow A \vee B} \quad \frac{B \Rightarrow B}{B \Rightarrow A \vee B}}{A \vee B \Rightarrow A \vee B} \quad \text{VL}$$

For $A \longrightarrow B \Rightarrow A \longrightarrow B$, after the proofs of $A \Rightarrow A$ and $B \Rightarrow B$, we do the following steps.

$$\begin{array}{c} \text{WR} \quad \frac{\frac{A \Rightarrow A}{A \Rightarrow A, B} \quad \frac{B \Rightarrow B}{A, B \Rightarrow B}}{A \Rightarrow B, A \quad B, A \Rightarrow B} \quad \text{WL} \\ \text{ER} \quad \frac{\frac{A \Rightarrow B, A \quad B, A \Rightarrow B}{A \longrightarrow B, A \Rightarrow B} \quad \frac{A, A \longrightarrow B \Rightarrow B}{A \longrightarrow B \Rightarrow A \longrightarrow B}}{A \longrightarrow B \Rightarrow A \longrightarrow B} \quad \text{EL} \end{array} \quad \begin{array}{c} \longrightarrow \text{L} \\ \longrightarrow \text{R} \end{array}$$

Each of these proofs is CONTRACTION+ CUT -free. Hence, by the principle of structural induction, we are done. $\square$

Theorem 6.2.5 (CONTRACTION-ELIMINATION THEOREM FOR **PLK**). *Let $\Gamma \Rightarrow \Delta$ has a **PLK**-proof. Then, it has a CONTRACTION-free proof as well.*

Proof. Consider the highest leftmost application of CONTRACTION. Let the sequent obtained by this application be $\Gamma' \Rightarrow \Delta'$. Two cases arise.

Case 6.2.5.1. If $\Gamma' \Rightarrow \Delta'$ is the result of an application of **CL** then we have,

$$\frac{A, A, \Gamma'' \Rightarrow \Delta'}{A, \Gamma'' \Rightarrow \Delta'} \quad \text{CL}$$

where $A, \Gamma'' \Rightarrow \Delta'$ is $\Gamma' \Rightarrow \Delta'$. Replace this with the following application of **CUT**,

$$\frac{A, \Gamma'' \Rightarrow \Delta', A \quad A, A, \Gamma'' \Rightarrow \Delta'}{A, \Gamma'' \Rightarrow \Delta'} \quad \text{CUT}$$

Case 6.2.5.2. If $\Gamma' \Rightarrow \Delta'$ is the result of an application of **CR** then we have,

$$\frac{\Gamma' \Rightarrow \Delta'', A, A}{\Gamma' \Rightarrow \Delta'', A} \quad \text{CR}$$

where $\Gamma' \Rightarrow \Delta'', A$ is $\Gamma' \Rightarrow \Delta'', A$. Replace this by the following application of **CUT**,

$$\frac{\Gamma' \Rightarrow \Delta'', A, A \quad A, \Gamma' \Rightarrow \Delta'', A}{\Gamma' \Rightarrow \Delta'', A} \quad \text{CUT}$$

The upper leftmost sequent (in the first case) and the upper rightmost sequent (in the second case) can be obtained by application(s) of WEAKENING and/or EXCHANGE from $A \Rightarrow A$. Since the proof of $A \Rightarrow A$ can be done without using CONTRACTION+CUT (by [Theorem 6.2.4](#)). Repeat this process until all the instances of CONTRACTION have been removed. $\square$

For the following theorem, we will need the notion of height of a proof.

Definition 6.2.6 (HEIGHT OF A **PLK**-PROOF). The *height* of a **PLK**-proof $\mathcal{D}$, denoted as $h(\mathcal{D})$ is defined recursively as follows:

- If $\mathcal{D}$ is the single-step proof of an axiom, then $h(\mathcal{D}) = 0$.
- If $\mathcal{D}$ is the proof of a sequent S , deduced from S' by an application of a single-premise rule, with $\mathcal{D}'$ being a proof of S' , then $h(\mathcal{D}) = h(\mathcal{D}') + 1$.
- If $\mathcal{D}$ is the proof of a sequent S , deduced from S' and S'' by an application of a two-premise rule, with $\mathcal{D}'$ and $\mathcal{D}''$ being their proofs respectively, then $h(\mathcal{D}) = \max(h(\mathcal{D}'), h(\mathcal{D}'')) + 1$.

We will also need the notion of *degree* of a proof.

Definition 6.2.7 (DEGREE OF A **PLK**-PROOF). Given a **PLK** proof, the *degree* of the proof is the maximum complexity (i.e., the number of connectives appearing in a formula) of the CUT-formula appearing in the proof.

A given sequent $\Gamma \Rightarrow \Delta$ is said to have a *proof in $\mathbf{PLK}_n$* , more simply **PLK_n-proof**, if there is a **PLK**-proof of $\Gamma \Rightarrow \Delta$, whose degree is at most n .

In [Theorem 6.2.12](#), we will show that certain sequents are not **PLK**-provable. In particular, we will show that $\Lambda \Rightarrow \Lambda$ is not **PLK**-provable. Usually, this result is proved as a corollary of the CUT-elimination theorem of **PLK** or via soundness theorem. In fact, both [Theorem 6.2.8](#) and [Theorem 6.2.12](#) admit easier proofs if one uses the soundness theorem for **PLK**. However, due to our reliance on purely syntactic methods, we can't apply the soundness theorem. Also, since we are trying to prove the CUT-elimination theorem for **PLK**, we can't use it to prove either [Theorem 6.2.8](#) or [Theorem 6.2.12](#).

Theorem 6.2.8. *There is no $\mathbf{PLK}_0$ -proof of sequents of the following forms:*

- (1) $\Lambda \Rightarrow \Lambda$
- (2) $p_1, \dots, p_k \Rightarrow \Lambda$
- (3) $\Lambda \Rightarrow q_1, \dots, q_l$
- (4) $p_1, \dots, p_k \Rightarrow q_1, \dots, q_l$

where p_i 's and q_j 's are propositional variables and for (4) $\{p_1, \dots, p_k\} \cap \{q_1, \dots, q_l\} = \emptyset$.

Proof. Notice that, due to [Theorem 6.2.5](#), it is enough to consider only CONTRACTION-free $\mathbf{PLK}_0$ -proofs. We apply strong induction on the height of these proofs, i.e., on $h(\mathcal{D})$ where $\mathcal{D}$ is a CONTRACTION-free $\mathbf{PLK}_0$ -proof.

BASE CASE: In this case $h(\mathcal{D}) = 0$, and in this case the statement is true since the only proofs of height 0 are of axiomatic sequents, which are of the form $r \Rightarrow r$, where r is a propositional variable, and this is not among the forms listed in the theorem.

INDUCTION HYPOTHESIS: There is no CONTRACTION-free $\mathbf{PLK}_0$ -proof of any sequent of the forms given in the theorem, of height $< n$.

INDUCTION STEP: Consider any $\mathbf{PLK}_0$ -proof of any one of the sequent of the form $\Lambda \Rightarrow \Lambda, p_1, \dots, p_k \Rightarrow \Lambda, \Lambda \Rightarrow q_1, \dots, q_l$ and $p_1, \dots, p_k \Rightarrow q_1, \dots, q_l$, where p_i 's and q_j 's are propositional variables, of height n (and for the last case $\{p_1, \dots, p_k\} \cap \{q_1, \dots, q_l\} = \emptyset$). Four cases will arise, one for each sequent in the list. In each case, we will get a contradiction. We discuss them one by one.

Case 6.2.8.1. Suppose that the sequent under consideration is $\Lambda \Rightarrow \Lambda$ and that it has a $\mathbf{PLK}$ -proof of height n .

Then, note that since the proof is CONTRACTION-free and since no logical connective is introduced in $\Lambda \Rightarrow \Lambda$, it can't be the conclusion of any instance of a logical rule. Furthermore, since no formula is introduced and no formulas are interchanged, it can't be the conclusion of any instance of WEAKENING and EXCHANGE-rules. Hence $\Lambda \Rightarrow \Lambda$ can only be the conclusion of an instance of CUT. Hence, the last couple of lines of the proof will look something like the following:

$$\frac{\Lambda \Rightarrow p \quad p \Rightarrow \Lambda}{\Lambda \Rightarrow \Lambda} \quad \text{CUT}$$

where p is a propositional variable (here we use the fact that this proof is a $\mathbf{PLK}_0$ -proof).

But, by [Theorem 6.2.2](#), this would imply that there exists a $\mathbf{PLK}_0$ -proof of $\Lambda \Rightarrow p$ of height less than n , contrary to the induction hypothesis. Consequently, in this case, our conclusion follows.

Case 6.2.8.2. Suppose that the sequent under consideration is $p_1, \dots, p_k \Rightarrow \Lambda$ and that it has a $\mathbf{PLK}$ -proof of height n .

Then, note that since the proof is CONTRACTION-free and since no logical connective is introduced in $p_1, \dots, p_k \Rightarrow \Lambda$, it can only be the conclusion of an instance of CUT or WEAKENING or EXCHANGE.

Now, starting from the bottom, we move upward. Proceeding in this manner, at the first occurrence of a sequent which is not the conclusion of an instance of EXCHANGE. Let that sequent be $p_{\sigma(1)}, \dots, p_{\sigma(k)} \Rightarrow \Lambda$ where $\sigma : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ is a bijection. Then, this sequent should be the conclusion of an instance of CUT or WEAKENING.

Subcase 6.2.8.2.1. If $k > 1$, then the following cases arise.

Subsubcase 6.2.8.2.1.1. Suppose that $p_{\sigma(1)}, \dots, p_{\sigma(k)} \Rightarrow \Lambda$ is the conclusion of an instance of WEAKENING. Then, it must be the conclusion of an instance of **WL**. Hence, the last two lines of the proof will look something like the following:

$$\frac{p_{\sigma(2)}, \dots, p_{\sigma(k)} \Rightarrow \Lambda}{p_{\sigma(1)}, p_{\sigma(2)}, \dots, p_{\sigma(k)} \Rightarrow \Lambda} \quad \text{WL}$$

But then, this would mean that $p_{\sigma(2)}, \dots, p_{\sigma(k)} \Rightarrow \Lambda$ has a **PLK₀**-proof of height $< n$, a contradiction to the induction hypothesis. Consequently, in this case also, our conclusion follows.

Subsubcase 6.2.8.2.1.2. If, on the other hand, it is the conclusion of an instance of CUT, then the last two lines of the proof will look something like the following:

$$\frac{p_{\sigma(1)}, p_{\sigma(2)}, \dots, p_{\sigma(k)} \Rightarrow r \quad r, p_{\sigma(1)}, p_{\sigma(2)}, \dots, p_{\sigma(k)} \Rightarrow \Lambda}{p_{\sigma(1)}, p_{\sigma(2)}, \dots, p_{\sigma(k)} \Rightarrow \Lambda} \quad \text{CUT}$$

For some propositional variable r . (Here again, we use the fact that this proof is a **PLK₀**-proof).

If $r \in \{p_{\sigma(1)}, p_{\sigma(2)}, \dots, p_{\sigma(k)}\}$ then it would imply that the sequent $r, p_{\sigma(1)}, \dots, p_{\sigma(k)} \Rightarrow \Lambda$ has a **PLK₀**-proof of height $< n$, contrary to the induction hypothesis. If $r \notin \{p_{\sigma(1)}, p_{\sigma(2)}, \dots, p_{\sigma(k)}\}$ then by considering the sequent $p_{\sigma(1)}, \dots, p_{\sigma(k)} \Rightarrow r$ we will obtain a similar contradiction. Thus, we are done in this case as well.

Subcase 6.2.8.2.2. If $k = 1$, one can also obtain a contradiction by slightly modifying the above argument. Hence, in this case also, we are done.

Case 6.2.8.3. Suppose that the sequent under consideration is $\Lambda \Rightarrow q_1, \dots, q_l$ and that it has a **PLK**-proof of height n . Then, this case can be dealt with analogously to the previous one.

Case 6.2.8.4. Suppose that the sequent under consideration is $p_1, \dots, p_k \Rightarrow q_1, \dots, q_l$ such that $\{p_1, \dots, p_k\} \cap \{q_1, \dots, q_l\} = \emptyset$ and that it has a **PLK**-proof of height n . Then, in this case both $k, l \geq 1$.

Without loss of generality, we may assume that $p_1, \dots, p_k \Rightarrow q_1, \dots, q_l$ is not the conclusion of any instance of an EXCHANGE-rule (if it were, then we will use the argument as in [Case 6.2.8.2](#) to obtain a sequent which is not).

So, since $p_1, \dots, p_k \Rightarrow q_1, \dots, q_l$ is not the conclusion of any instance of an EXCHANGE-rule it must be the conclusion of some instance of WEAKENING or CUT. In the former case, the possibilities for the last two lines of the proof look something like the following:

$$\frac{p_1, \dots, p_k \Rightarrow q_1, \dots, q_{l-1}}{p_1, \dots, p_k \Rightarrow q_1, \dots, q_{l-1}, q_l} \quad \text{WR}$$

if $l > 1$, or,

$$\frac{p_1, \dots, p_k \Rightarrow \Lambda}{p_1, \dots, p_k \Rightarrow q_1} \quad \text{WR}$$

if $l = 1$, or,

$$\frac{p_1, \dots, p_k \Rightarrow q_1, \dots, q_l}{p_1, p_2, \dots, p_k \Rightarrow q_1, \dots, q_l} \quad \text{WL}$$

if $k > 1$, or,

$$\frac{\Lambda \Rightarrow q_1, \dots, q_l}{p_1 \Rightarrow q_1, \dots, q_l} \quad \text{WL}$$

if $k = 1$, each of which will imply that there exists a proof of height less than n of a sequent of one of the types listed in (1), (2) and (3), a contradiction. This shows that $p_1, \dots, p_k \Rightarrow q_1, \dots, q_l$ can't be the conclusion of any instance of a WEAKENING-rule. If it is the conclusion of an instance of CUT, then the last two lines of the proof will look something like the following.

$$\frac{p_1, \dots, p_k \Rightarrow q_1, \dots, q_l, r \quad r, p_1, \dots, p_k \Rightarrow q_1, \dots, q_l}{p_1, \dots, p_k \Rightarrow q_1, \dots, q_l} \quad \text{CUT}$$

If $r \in \{p_1, \dots, p_k\}$ then consider the sequent $r, p_1, \dots, p_k \Rightarrow q_1, \dots, q_l$ and note that it is among the forms listed, $\{r, p_1, \dots, p_k\} \cap \{q_1, \dots, q_l\} = \emptyset$ but have a proof of height $< n$, a contradiction. If, on the other hand $r \notin \{p_1, \dots, p_k\}$, then consider the $p_1, \dots, p_k \Rightarrow q_1, \dots, q_l, r$ to obtain a contradiction.

Since we have obtained a contradiction in each of the cases, this completes the induction step, and our conclusion follows. $\square$

We now state the following theorem without proof. For a proof of this theorem see [66, Theorem 2.2].

Theorem 6.2.9 (SYNTACTIC INVERTIBILITY OF **PLK**). *In **PLK** the following results hold.*

- (1) If $\emptyset \vdash_{\mathbf{PLK}_n} \Gamma \Rightarrow \Delta, \neg A$ then $\emptyset \vdash_{\mathbf{PLK}_n} A, \Gamma \Rightarrow \Delta$.
- (2) If $\emptyset \vdash_{\mathbf{PLK}_n} \neg A, \Gamma \Rightarrow \Delta$ then $\emptyset \vdash_{\mathbf{PLK}_n} \Gamma \Rightarrow \Delta, A$.
- (3) If $\emptyset \vdash_{\mathbf{PLK}_n} A \wedge B, \Gamma \Rightarrow \Delta$ then $\emptyset \vdash_{\mathbf{PLK}_n} A, B, \Gamma \Rightarrow \Delta$.
- (4) If $\emptyset \vdash_{\mathbf{PLK}_n} \Gamma \Rightarrow \Delta, A \wedge B$ then $\emptyset \vdash_{\mathbf{PLK}_n} \Gamma \Rightarrow \Delta, A$ as well as $\emptyset \vdash_{\mathbf{PLK}_n} \Gamma \Rightarrow \Delta, B$.
- (5) If $\emptyset \vdash_{\mathbf{PLK}_n} A \vee B, \Gamma \Rightarrow \Delta$ then $\emptyset \vdash_{\mathbf{PLK}_n} A, \Gamma \Rightarrow \Delta$ as well as $\emptyset \vdash_{\mathbf{PLK}_n} B, \Gamma \Rightarrow \Delta$.
- (6) If $\emptyset \vdash_{\mathbf{PLK}_n} \Gamma \Rightarrow \Delta, A \vee B$ then $\emptyset \vdash_{\mathbf{PLK}_n} \Gamma \Rightarrow \Delta, A, B$.
- (7) If $\emptyset \vdash_{\mathbf{PLK}_n} A \longrightarrow B, \Gamma \Rightarrow \Delta$ then $\emptyset \vdash_{\mathbf{PLK}_n} \Gamma \Rightarrow \Delta, A$ as well as $\emptyset \vdash_{\mathbf{PLK}_n} B, \Gamma \Rightarrow \Delta$.
- (8) If $\emptyset \vdash_{\mathbf{PLK}_n} \Gamma \Rightarrow \Delta, A \longrightarrow B$ then $\emptyset \vdash_{\mathbf{PLK}_n} A, \Gamma \Rightarrow \Delta, B$.

Remark 6.2.10. This is not the original statement of [66, Theorem 2.2]. However, it follows immediately from the argument given there.

Remark 6.2.11. It is important to mention at this point that even though there are different ways to prove this particular theorem, we are interested only in the proofs using *global* transformations (see, e.g. [45] and [87]). These types of proofs have the advantage that if the proof of the former sequent doesn't have any instance of the application of a particular rule, the proof of the latter also will not.

Using these two theorems, we can prove the following theorem, which says that **PLK** is consistent.

Theorem 6.2.12. *There is no **PLK**-proof of the sequents of the following forms:*

- (1) $\Lambda \Rightarrow \Lambda$
- (2) $p_1, \dots, p_k \Rightarrow \Lambda$
- (3) $\Lambda \Rightarrow q_1, \dots, q_l$
- (4) $p_1, \dots, p_k \Rightarrow q_1, \dots, q_l$

where p_i 's and q_j 's are propositional variables and for (4) $\{p_1, \dots, p_k\} \cap \{q_1, \dots, q_l\} = \emptyset$.

Proof. We show this only for $\Lambda \Rightarrow \Lambda$. The proofs for the other cases are similar.

We show that if there is a **PLK**_n-proof of $\Lambda \Rightarrow \Lambda$, there is a **PLK**₀-proof of the same.

Start with the highest leftmost application of CUT. Let $\Gamma \Rightarrow \Delta$ be the sequent obtained by this application of CUT. Then, the last two lines of the this part of the proof, ending in $\Gamma \Rightarrow \Delta$ will look something like the following.

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{ CUT}$$

Then, it follows that there is a **PLK**_{Comp(A)}-proof of $\Gamma \Rightarrow \Delta$. Furthermore, by our choice of $\Gamma \Rightarrow \Delta$, both $\Gamma \Rightarrow \Delta, A$ and $A, \Gamma \Rightarrow \Delta$ has **PLK**₀-proofs. If $\text{Comp}(A) = 0$, then we do nothing. Otherwise, we do the following.

Case 6.2.12.1. If $A = \neg B$ for some formula B . Then, by [Theorem 6.2.9](#), we replace the proofs of $\Gamma \Rightarrow \Delta, \neg B$ and $\neg B, \Gamma \Rightarrow \Delta$ by the proofs of $B, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, B$ respectively. Then, we get the following proof of $\Gamma \Rightarrow \Delta$.

$$\frac{\Gamma \Rightarrow \Delta, B \quad B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{ CUT}$$

Case 6.2.12.2. Let $A = B \wedge C$ for some formulas B and C . Then, by [Theorem 6.2.9](#), we replace the proofs of $\Gamma \Rightarrow \Delta, B \wedge C$ and $B \wedge C, \Gamma \Rightarrow \Delta$ by the proofs of $\Gamma \Rightarrow \Delta, B, \Gamma \Rightarrow \Delta, C$ and $B, C, \Gamma \Rightarrow \Delta$ respectively. Then, we get the following proof of $\Gamma \Rightarrow \Delta$.

$$\begin{array}{c}
\text{WL} \quad \frac{\Gamma \Rightarrow \Delta, B}{C, \Gamma \Rightarrow \Delta, B} \quad \frac{B, C, \Gamma \Rightarrow \Delta}{C, \Gamma \Rightarrow \Delta} \quad \text{CUT} \\
\frac{\Gamma \Rightarrow \Delta, C}{\Gamma \Rightarrow \Delta} \quad \text{CUT}
\end{array}$$

Case 6.2.12.3. Let $A = B \vee C$ for some formulas B and C . Then, by [Theorem 6.2.9](#), we replace the proofs of $\Gamma \Rightarrow \Delta, B \vee C$ and $B \vee C, \Gamma \Rightarrow \Delta$ by the proofs of $B, \Gamma \Rightarrow \Delta, C, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, B, C$ respectively. Then, we get the following proof of $\Gamma \Rightarrow \Delta$.

$$\begin{array}{c}
\text{WR} \quad \frac{C, \Gamma \Rightarrow \Delta}{C, \Gamma \Rightarrow \Delta, B} \\
\text{CUT} \quad \frac{\Gamma \Rightarrow \Delta, B, C}{\Gamma \Rightarrow \Delta, B} \quad \frac{C, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, B} \quad \text{CUT} \quad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta} \quad B, \Gamma \Rightarrow \Delta
\end{array}$$

Case 6.2.12.4. Let $A = B \longrightarrow C$ for some formulas B and C . Then, by [Theorem 6.2.9](#), we replace the proofs of $\Gamma \Rightarrow \Delta, B \longrightarrow C$ and $B \longrightarrow C, \Gamma \Rightarrow \Delta$ by the proofs of $\Gamma \Rightarrow \Delta, B, C, \Gamma \Rightarrow \Delta$ and $B, \Gamma \Rightarrow \Delta, C$ respectively. Then, we get the following proof of $\Lambda \Rightarrow \Lambda$.

$$\begin{array}{c}
\text{WL} \quad \frac{C, \Gamma \Rightarrow \Delta}{B, C, \Gamma \Rightarrow \Delta} \\
\text{EL} \quad \frac{B, C, \Gamma \Rightarrow \Delta}{C, B, \Gamma \Rightarrow \Delta} \\
\text{CUT} \quad \frac{B, \Gamma \Rightarrow \Delta, C}{B, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta} \quad \text{CUT}
\end{array}$$

In each case, the complexity of the CUT-formula is $< \text{Comp}(A)$. Since $\text{Comp}(A) \leq n$, we thus obtain a proof of $\Gamma \Rightarrow \Delta$ of a lower degree than n . Continuing this process, till there is a CUT-formula of nonzero degree, we will get a **PLK**₀ proof of $\Lambda \Rightarrow \Lambda$, contradicting [Theorem 6.2.8](#). This contradiction ensures that our assumption was wrong, and hence there is no **PLK** _{n} -proof of $\Lambda \Rightarrow \Lambda$. Since n was chosen arbitrarily, we have thus shown that there is no **PLK**-proof of $\Lambda \Rightarrow \Lambda$. $\square$

Theorem 6.2.13 (CUT-ELIMINATION THEOREM FOR **PLK**). *Let $\Gamma \Rightarrow \Delta$ be a **PLK**-provable sequent. Then there is a CUT-free **PLK**-proof of $\Gamma \Rightarrow \Delta$.*

Proof. The proof is by induction on the value of $\text{SeqComp}(\Gamma \Rightarrow \Delta)$, which denotes the total number of connectives in the wffs of Γ and Δ , say n .

BASE CASE: In the base case, $\text{SeqComp}(\Gamma \Rightarrow \Delta) = 0$, the sequent $\Gamma \Rightarrow \Delta$ contains no logical connectives and thus every formula in the sequent is a propositional variable. Since the sequent is provable, by [Theorem 6.2.12](#), there must be some variable p which occurs both in Γ and in Δ . Thus $\Gamma \Rightarrow \Delta$ can be proved without using CUT, although we may need WEAKENING and EXCHANGE in this case.

INDUCTION HYPOTHESIS: For every **PLK**-sequent $\Gamma \Rightarrow \Delta$ if $\emptyset \vdash_{\text{PLK}} \Gamma \Rightarrow \Delta$ and $\text{SeqComp}(\Gamma \Rightarrow \Delta) < m$, then there is a CUT-free **PLK**-proof of $\Gamma \Rightarrow \Delta$.

INDUCTION STEP: Let $\Gamma \Rightarrow \Delta$ be a **PLK**-provable sequent for which $\text{SeqComp}(\Gamma \Rightarrow \Delta) =$

$m(> 0)$. Without loss of generality (since we have EXCHANGE rule), we may assume that at least one of the extreme most formulas is not a propositional variable. Depending on the outermost connective of it, we have the following cases.

Case 6.2.13.1. Let $\Gamma = \neg A, \Gamma'$ for some wff A . Then $\emptyset \vdash_{\text{PLK}} \neg A, \Gamma' \Rightarrow \Delta$. Hence, by Theorem 6.2.9 $\emptyset \vdash_{\text{PLK}} \Gamma' \Rightarrow \Delta, A$. Furthermore, since $\text{SeqComp}(\Gamma' \Rightarrow \Delta, A) < m$ by the induction hypothesis, it follows that $\Gamma' \Rightarrow \Delta, A$ has a CUT-free proof. Consequently, so does $\Gamma \Rightarrow \Delta$ (just apply $\neg\text{L}$ to $\Gamma' \Rightarrow \Delta, A$).

Case 6.2.13.2. $\Delta = \Delta', \neg A$ for some wff A . This can be dealt in an analogous manner as the above.

Case 6.2.13.3. Let $\Gamma = A \vee B, \Gamma'$ for some wffs A, B . Then $\emptyset \vdash_{\text{PLK}} A \vee B, \Gamma' \Rightarrow \Delta$. Hence, by Theorem 6.2.9 $\emptyset \vdash_{\text{PLK}} A, \Gamma' \Rightarrow \Delta$ as well as $\emptyset \vdash_{\text{PLK}} B, \Gamma' \Rightarrow \Delta$. Furthermore, since $\text{SeqComp}(A, \Gamma' \Rightarrow \Delta) < m$ as well as $\text{SeqComp}(B, \Gamma' \Rightarrow \Delta) < m$ by the induction hypothesis, it follows that $A, \Gamma' \Rightarrow \Delta$ has a CUT-free proof as well as $B, \Gamma' \Rightarrow \Delta$ has a CUT-free proof. Consequently, so does $\Gamma \Rightarrow \Delta$ (just apply $\vee\text{R}$ to $A, \Gamma' \Rightarrow \Delta$).

Case 6.2.13.4. Let $\Delta = \Delta', A \wedge B$ for some wffs A, B . This can be dealt in an analogous manner as the above.

Case 6.2.13.5. Let $\Gamma = A \longrightarrow B, \Gamma'$ for some wffs A, B . This can be dealt in an analogous manner as the above.

Case 6.2.13.6. Let $\Delta = \Delta', A \longrightarrow B$ for some wffs A, B . This can be dealt in an analogous manner as the above.

Case 6.2.13.7. Let $\Delta = \Delta', A \vee B$ for some wffs A, B . Then $\emptyset \vdash_{\text{PLK}} \Gamma \Rightarrow \Delta', A \vee B$. By Theorem 6.2.9 $\emptyset \vdash_{\text{PLK}} \Gamma \Rightarrow \Delta', A, B$ as well. Furthermore, since $\text{SeqComp}(\Gamma \Rightarrow \Delta', A, B) < m$ by the induction hypothesis, it follows that $\Gamma \Rightarrow \Delta', A, B$ has a CUT-free proof. Consequently, $\Gamma \Rightarrow \Delta$ has a CUT-free proof as follows:

$$\begin{array}{c}
\frac{\Gamma \Rightarrow \Delta', A, B}{\Gamma \Rightarrow \Delta', A, A \vee B} \quad \vee\text{R} \\
\frac{\Gamma \Rightarrow \Delta', A, A \vee B}{\Gamma \Rightarrow \Delta', A \vee B, A} \quad \text{ER} \\
\frac{\Gamma \Rightarrow \Delta', A \vee B, A}{\Gamma \Rightarrow \Delta', A \vee B, A \vee B} \quad \vee\text{R} \\
\frac{\Gamma \Rightarrow \Delta', A \vee B, A \vee B}{\Gamma \Rightarrow \Delta', A \vee B} \quad \text{CR}
\end{array}$$

Consequently, it follows that $\Gamma \Rightarrow \Delta$ has a CUT-free proof.

Case 6.2.13.8. Let $\Gamma = A \wedge B, \Gamma'$ for some wffs A, B . Then $\emptyset \vdash_{\text{PLK}} A \wedge B, \Gamma' \Rightarrow \Delta$. By Theorem 6.2.9 $\emptyset \vdash_{\text{PLK}} A, B, \Gamma' \Rightarrow \Delta$ as well. Furthermore, $\text{SeqComp}(A, B, \Gamma' \Rightarrow \Delta) < m$ by the induction hypothesis, it follows that $A, B, \Gamma' \Rightarrow \Delta$ has a CUT-free proof. $\Gamma \Rightarrow \Delta$ has a CUT-free proof as follows:

$$\begin{array}{c}
\frac{A, B, \Gamma' \Rightarrow \Delta}{A \wedge B, B, \Gamma' \Rightarrow \Delta} \quad \wedge\text{L} \\
\frac{A \wedge B, B, \Gamma' \Rightarrow \Delta}{B, A \wedge B, \Gamma' \Rightarrow \Delta} \quad \text{EL} \\
\frac{B, A \wedge B, \Gamma' \Rightarrow \Delta}{A \wedge B, A \wedge B, \Gamma' \Rightarrow \Delta} \quad \wedge\text{L} \\
\frac{A \wedge B, A \wedge B, \Gamma' \Rightarrow \Delta}{A \wedge B, \Gamma' \Rightarrow \Delta} \quad \text{CL}
\end{array}$$

Consequently, it follows that $\Gamma \Rightarrow \Delta$ has a CUT-free proof.

□

6.2.3 Second Proof of the CUT-elimination Theorem for PLK

In this section, we provide a different argument to prove a slightly stronger result. We show that if a sequent has a **PLK**-proof, then it has a CONTRACTION+CUT-free **PLK**-proof as well.

Theorem 6.2.14. *Let $\Gamma \Rightarrow \Delta$ be a sequent and q a propositional variable. If $q \longrightarrow q, \Gamma \Rightarrow \Delta$ has a CONTRACTION-free proof in **PLK**, then there is a CONTRACTION+CUT-free **PLK**-proof of $q \longrightarrow q, \Gamma \Rightarrow \Delta$.*

Proof. We apply induction on the sequent complexity of $q \longrightarrow q, \Gamma \Rightarrow \Delta$, $\text{SeqComp}(q \longrightarrow q, \Gamma \Rightarrow \Delta)$.

BASE CASE: In the base case, $\text{SeqComp}(q \longrightarrow q, \Gamma \Rightarrow \Delta) = 1$. Since the sequent $q \longrightarrow q, \Gamma \Rightarrow \Delta$ contains exactly one logical connective, and since $\longrightarrow$ is a connective already occurring, it must be that one. Furthermore, every other formula in the sequent must be a propositional variable.

Now, note that since $q \longrightarrow q, \Gamma \Rightarrow \Delta$ is provable without using CONTRACTION, so are both the sequents $q, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, q$ (by [Theorem 6.2.9](#) and [Remark 6.2.11](#)). We claim that there exists a propositional variable r occurring in both Γ and Δ .

To show this, observe that since $q, \Gamma \Rightarrow \Delta$ is provable, by [Theorem 6.2.12](#), it follows that there exists a propositional variable r_1 belonging to both the lists q, Γ and Δ . Similarly, since $\Gamma \Rightarrow \Delta, q$ is provable, by [Theorem 6.2.12](#), it follows that there exists a propositional variable r_2 belonging to both the lists Γ and Δ, q .

If $r_1 \neq q$ or $r_2 \neq q$, then by applying WEAKENING and/or EXCHANGE as required, we obtain CONTRACTION+CUT-free proofs of $q, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, q$. If, on the other hand, $r_1 = r_2 = q$, then since Δ contains r_1 , it follows that Δ contains q ; similarly, since Γ contains r_2 , it follows that Γ contains q . Hence, in this case also, by applying WEAKENING and/or EXCHANGE as required, we can obtain CONTRACTION+CUT-free proofs of $q, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, q$.

The sequent $q \longrightarrow q, \Gamma \Rightarrow \Delta$ can now be obtained by an application of $\longrightarrow$ L. We have thus obtained a CONTRACTION+CUT-free proof of $q \longrightarrow q, \Gamma \Rightarrow \Delta$. This completes the proof of the base case.

INDUCTION HYPOTHESIS: Suppose that for every sequent $\Gamma \Rightarrow \Delta$ with $\text{SeqComp}(q \longrightarrow q, \Gamma \Rightarrow \Delta) < n$ if there is a CONTRACTION-free **PLK**-proof of $q \longrightarrow q, \Gamma \Rightarrow \Delta$, then there is a CONTRACTION+CUT-free proof of $q \longrightarrow q, \Gamma \Rightarrow \Delta$ as well.

INDUCTION STEP: Consider a sequent $\Gamma \Rightarrow \Delta$ with $\text{SeqComp}(q \longrightarrow q, \Gamma \Rightarrow \Delta) = n$ such

that $q \longrightarrow q, \Gamma \Rightarrow \Delta$ is **PLK**-provable without using **CONTRACTION**. Consequently, by [Theorem 6.2.9](#) and [Remark 6.2.11](#), it follows that both the sequents $q, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, q$ are **PLK**-provable without using **CONTRACTION**.

Depending upon the outermost connective occurring in the sequent $\Gamma \Rightarrow \Delta$, there will be eight cases – four for the right and four for the left. Below, we discuss only the case when $\Gamma = B \vee C, \Gamma'$ for some formulas B and C . Proofs for the other cases are similar.

Since $\Gamma = B \vee C, \Gamma'$, and since both $q, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, q$ are **PLK**-provable without using **CONTRACTION**, it follows that both $q, B \vee C, \Gamma' \Rightarrow \Delta$ and $B \vee C, \Gamma' \Rightarrow \Delta, q$ are **PLK**-provable without using **CONTRACTION**. Consider $q, B \vee C, \Gamma' \Rightarrow \Delta$ and note that since it is provable, so is $B \vee C, q, \Gamma' \Rightarrow \Delta$ (apply **EL**). Then, by [Theorem 6.2.9](#) and [Remark 6.2.11](#), it follows that there are **CONTRACTION**-free proofs of $B, q, \Gamma' \Rightarrow \Delta$ and $C, q, \Gamma' \Rightarrow \Delta$. Similarly, from the **CONTRACTION**-free proof of $B \vee C, \Gamma' \Rightarrow \Delta, q$, we will be able to construct **CONTRACTION**-free proofs of $B, \Gamma' \Rightarrow \Delta, q$ and $C, \Gamma' \Rightarrow \Delta, q$. Now, choose any such proof for each sequent. We extend each of these proofs as follows.

$$\frac{B, \Gamma' \Rightarrow \Delta, q \quad \frac{B, q, \Gamma' \Rightarrow \Delta}{q, B, \Gamma' \Rightarrow \Delta} \text{EL}}{q \longrightarrow q, B, \Gamma' \Rightarrow \Delta} \longrightarrow \text{L}$$

and,

$$\frac{C, \Gamma' \Rightarrow \Delta, q \quad \frac{C, q, \Gamma' \Rightarrow \Delta}{q, C, \Gamma' \Rightarrow \Delta} \text{EL}}{q \longrightarrow q, C, \Gamma' \Rightarrow \Delta} \longrightarrow \text{L}$$

Now, since $\text{SeqComp}(q \longrightarrow q, B, \Gamma' \Rightarrow \Delta) < n$ as well as $\text{SeqComp}(q \longrightarrow q, C, \Gamma' \Rightarrow \Delta) < n$ and since both are provable in **PLK** without using **CONTRACTION**, by the induction hypothesis it follows that each of these sequents has a **CONTRACTION**+**CUT**-free proof in **PLK**. Choose any such for each of them. To these proofs, we add the following lines.

$$\frac{\frac{q \longrightarrow q, B, \Gamma' \Rightarrow \Delta}{B, q \longrightarrow q, \Gamma' \Rightarrow \Delta} \text{EL} \quad \frac{q \longrightarrow q, C, \Gamma' \Rightarrow \Delta}{C, q \longrightarrow q, \Gamma' \Rightarrow \Delta} \text{EL}}{\frac{B \vee C, q \longrightarrow q, \Gamma' \Rightarrow \Delta}{q \longrightarrow q, B \vee C, \Gamma' \Rightarrow \Delta} \text{EL}} \vee \text{L}$$

Which gives a **CONTRACTION**+**CUT**-free proof of $q \longrightarrow q, \Gamma \Rightarrow \Delta$. This completes the proof in this case. $\square$

Theorem 6.2.15 (**CONTRACTION**+**CUT**-ELIMINATION THEOREM FOR **PLK** (PART I)).
Let $\Gamma \Rightarrow \Delta$ be a sequent and A be a formula. If $A \longrightarrow A, \Gamma \Rightarrow \Delta$ has a **CONTRACTION**-free proof in **PLK**, then there is a **CONTRACTION**+**CUT**-free **PLK**-proof of $A \longrightarrow A, \Gamma \Rightarrow \Delta$.

Proof. The proof is done by induction on the complexity of A , $\text{Comp}(A)$. The base case, i.e. when $\text{Comp}(A) = 0$, is proved in the previous theorem. The proof of the induction step follows an argument similar to the induction step of the last theorem. $\square$

A *multiset* may be formally defined as an ordered pair (A, m) where A is a set, called the *underlying set* of the multiset and $m : A \rightarrow \mathbb{N}$, called the *multiplicity function*. For any $a \in A$, $m(a)$ = the multiplicity of a , is the number of occurrences of a in the multiset A (e.g., if $a \notin A$, we have $m(a) = 0$).

Given two multisets (A, m) and (B, n) , we will say that (A, m) is *included in* (B, n) if $A \subseteq B$ and for all $a \in A$ $m(a) \leq n(a)$. In this case, we will write $(A, m) \sqsubseteq (B, n)$. We will write $(A, m) \sqsubset (B, n)$ if $(A, m) \sqsubseteq (B, n)$ and there exists $a \in A$ such that $m(a) < n(a)$.

To each ordered list, one can associate a multiset by simply ignoring the order. Given a sequent $\Gamma \Rightarrow \Delta$, we will denote the multisets corresponding to Γ and Δ as $\text{MultiSet}(\Gamma)$ and $\text{MultiSet}(\Delta)$ respectively.

Definition 6.2.16 (SUBSEQUENT). Let $\Gamma \Rightarrow \Delta$ be a sequent. A sequent $\Gamma' \Rightarrow \Delta'$ is said to be a *subsequent* of $\Gamma \Rightarrow \Delta$ if $\text{MultiSet}(\Gamma') \sqsubseteq \text{MultiSet}(\Gamma)$ and $\text{MultiSet}(\Delta') \sqsubseteq \text{MultiSet}(\Delta)$. In this case, we will write $\Gamma' \Rightarrow \Delta' \preceq \Gamma \Rightarrow \Delta$. We will write $\Gamma' \Rightarrow \Delta' \prec \Gamma \Rightarrow \Delta$ if either $\text{MultiSet}(\Gamma') \sqsubset \text{MultiSet}(\Gamma)$ or $\text{MultiSet}(\Delta') \sqsubset \text{MultiSet}(\Delta)$.

Theorem 6.2.17. *Let $\Gamma \Rightarrow \Delta$ be a sequent. If it is deducible from a sequent $\Gamma' \Rightarrow \Delta'$ by application(s) of either WEAKENING or EXCHANGE, then $\Gamma' \Rightarrow \Delta'$ is a subsequent of $\Gamma \Rightarrow \Delta$.*

Proof. Suppose that $\Gamma' \Rightarrow \Delta'$ is a sequent $\Gamma \Rightarrow \Delta$ is deducible from $\Gamma' \Rightarrow \Delta'$ by an appropriate number of applications of WEAKENING and/or EXCHANGE. Let the part of such a proof from $\Gamma' \Rightarrow \Delta'$ to $\Gamma \Rightarrow \Delta$ look something like the following.

$$\begin{array}{c} \frac{\Gamma_0 \Rightarrow \Delta_0}{\Gamma_1 \Rightarrow \Delta_1} \\ \vdots \\ \overline{\Gamma_k \Rightarrow \Delta_k} \end{array}$$

Where $\Gamma_0 = \Gamma'$, $\Gamma_k = \Gamma$, $\Delta_0 = \Delta'$ and $\Delta_k = \Delta$ and for each $i \in \{0, \dots, k\}$ $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is obtained from $\Gamma_i \Rightarrow \Delta_i$ either by WEAKENING or EXCHANGE.

Then note that for all $i \in \{0, \dots, k\}$ if $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is obtained via an application of WEAKENING from $\Gamma_i \Rightarrow \Delta_i$, we have $\Gamma_i \Rightarrow \Delta_i \prec \Gamma_{i+1} \Rightarrow \Delta_{i+1}$. Else, if it is obtained via an application of EXCHANGE, we have $\Gamma_i \Rightarrow \Delta_i = \Gamma_{i+1} \Rightarrow \Delta_{i+1}$. In either case, at each step, we have $\Gamma_i \Rightarrow \Delta_i \preceq \Gamma_{i+1} \Rightarrow \Delta_{i+1}$, which proves our claim. $\square$

Theorem 6.2.18. *Let $\Gamma \Rightarrow \Delta$ and $\Gamma' \Rightarrow \Delta'$ be two **PLK**-sequents. If there is a derivation of $A \rightarrow A, \Gamma \Rightarrow \Delta$ from $A \rightarrow A, \Gamma' \Rightarrow \Delta'$ using only WEAKENING and/or EXCHANGE, then there is a derivation of $\Gamma \Rightarrow \Delta, A$ from $\Gamma' \Rightarrow \Delta', A$ using only WEAKENING and/or EXCHANGE.*

Proof. Immediate by applying induction on the total number of wffs in Γ and Δ . $\square$

Theorem 6.2.19 (CONTRACTION+CUT-ELIMINATION THEOREM FOR **PLK** (PART II)).
*Let $\Gamma \Rightarrow \Delta$ be a sequent. If $A \longrightarrow A, \Gamma \Rightarrow \Delta$ has a CONTRACTION+CUT-free **PLK**-proof, then so does $\Gamma \Rightarrow \Delta$.*

Proof. The proof is by induction on the values of $\text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta)$.

BASE CASE: In the base case, $\text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta) = 1$, the sequent $A \longrightarrow A, \Gamma \Rightarrow \Delta$ contains exactly one logical connective and since $\longrightarrow$ is a connective already occurring, it must be that one. Hence, $A \longrightarrow A$ is actually of the form $q \longrightarrow q$, where q is a propositional variable. Also, every other formula in the sequent must be a propositional variable.

Furthermore, observe that in this proof, there is no application of CUT as well as CONTRACTION. Hence, the only potentially applicable rules are the logical rules, WEAKENING and EXCHANGE. However, the application of any logical rule other than $\longrightarrow$ L is prohibited because that will increase sequent complexity, and EXCHANGE doesn't introduce any new formula. So, if we consider the place where $q \longrightarrow q$ appeared first in the proof, we are forced to conclude that this must be due to the application of either an instance of WEAKENING or of $\longrightarrow$ L.

In the former case, the sequent just above $A \longrightarrow A, \Gamma \Rightarrow \Delta$ is $\Gamma \Rightarrow \Delta$. Since $A \longrightarrow A, \Gamma \Rightarrow \Delta$ is **PLK**-provable, so is $\Gamma \Rightarrow \Delta$. Hence, there exists some propositional variable r common to both Γ and Δ (by [Theorem 6.2.12](#)). So, by applying WEAKENING and/or EXCHANGE, we can prove $\Gamma \Rightarrow \Delta$.

In the latter case, following an argument similar to the base case of [Theorem 6.2.14](#), we can prove that there is a propositional variable common to both Γ and Δ . Then, applying WEAKENING and/or EXCHANGE, we can prove $\Gamma \Rightarrow \Delta$ in this case as well. Consequently, the base case is proved.

INDUCTION HYPOTHESIS: For every sequent $\Gamma \Rightarrow \Delta$ for which $\text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta) < m$, if there is a CONTRACTION+CUT-free **PLK**-proof of $A \longrightarrow A, \Gamma \Rightarrow \Delta$, then there is a CONTRACTION+CUT-free **PLK**-proof of $\Gamma \Rightarrow \Delta$.

INDUCTION STEP: Let $\Gamma \Rightarrow \Delta$ be a sequent for which $\text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta) = m$. Suppose also that $A \longrightarrow A, \Gamma \Rightarrow \Delta$ has a CONTRACTION+CUT-free **PLK**-proof.

We do the following to transform this particular proof of $A \longrightarrow A, \Gamma \Rightarrow \Delta$ to obtain a CONTRACTION+CUT-free proof of $\Gamma \Rightarrow \Delta$. Starting from the bottom, we go upwards and find the sequent where a logical rule is used to obtain the conclusion sequent, say $\Gamma' \Rightarrow \Delta'$. This means, in particular, that every sequent below $\Gamma' \Rightarrow \Delta'$, in the proof of $A \longrightarrow A, \Gamma \Rightarrow \Delta$, is obtained either by WEAKENING or EXCHANGE. So, by [Theorem 6.2.17](#), it follows that $\Gamma' \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta$.

We divide the argument into the following cases.

1. The first appearance of $A \longrightarrow A$ in $A \longrightarrow A, \Gamma \Rightarrow \Delta$ is after $\Gamma' \Rightarrow \Delta'$. ([Case 6.2.19.1](#))
2. The first appearance of $A \longrightarrow A$ in $A \longrightarrow A, \Gamma \Rightarrow \Delta$ is in or before $\Gamma' \Rightarrow \Delta'$. ([Case 6.2.19.2](#))

2.1. $A \longrightarrow A$ occurs in $\Gamma' \Rightarrow \Delta'$. In this case, let Γ^* be such that $\Gamma' := A \longrightarrow A, \Gamma^*$. (Subcase 6.2.19.2.1)

2.1.1. $\text{SeqComp}(A \longrightarrow A, \Gamma^* \Rightarrow \Delta') = \text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta)$. (Subsubcase 6.2.19.2.1.1)

2.1.2. $\text{SeqComp}(A \longrightarrow A, \Gamma^* \Rightarrow \Delta') < \text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta)$. (Subsubcase 6.2.19.2.1.2)

2.2. $A \longrightarrow A$ occurs before $\Gamma' \Rightarrow \Delta'$. (Subcase 6.2.19.2.2)

2.2.1. $A \longrightarrow A$ occurs in $\Gamma' \Rightarrow \Delta'$. (Subsubcase 6.2.19.2.2.1)

2.2.2. $A \longrightarrow A$ does not occur in $\Gamma' \Rightarrow \Delta'$. (Subsubcase 6.2.19.2.2.2)

Case 6.2.19.1. If the first appearance of $A \longrightarrow A$ in $A \longrightarrow A, \Gamma \Rightarrow \Delta$ is after $\Gamma' \Rightarrow \Delta'$, then it must be the result of an application of a WEAKENING-rule (since EXCHANGE doesn't introduce new formulas). Then, that part of the proof looks something like the following.

$$\frac{\Gamma'' \Rightarrow \Delta''}{A \longrightarrow A, \Gamma'' \Rightarrow \Delta''} \quad \text{WL}$$

For some sequent $\Gamma'' \Rightarrow \Delta''$ occurring in the proof of $A \longrightarrow A, \Gamma \Rightarrow \Delta$. Consequently, we have a CONTRACTION+CUT-free **PLK**-proof of $\Gamma'' \Rightarrow \Delta''$.

Finally, we apply WEAKENING and EXCHANGE rules, as required, to obtain a CONTRACTION+CUT-free **PLK**-proof of $\Gamma \Rightarrow \Delta$.

Case 6.2.19.2. So suppose that the appearance of $A \longrightarrow A$ in $A \longrightarrow A, \Gamma \Rightarrow \Delta$ is in or before $\Gamma' \Rightarrow \Delta'$.

Subcase 6.2.19.2.1. If $A \longrightarrow A$ occurs in $\Gamma' \Rightarrow \Delta'$, then this is possible only if $\Gamma' \Rightarrow \Delta'$ has been obtained by an application of $\longrightarrow L$. Then $\Gamma' = A \longrightarrow A, \Gamma^*$. In this case, the line where it is applied will look something like the following.

$$\frac{\Gamma^* \Rightarrow \Delta', A \quad A, \Gamma^* \Rightarrow \Delta'}{A \longrightarrow A, \Gamma^* \Rightarrow \Delta'} \quad \longrightarrow L$$

For some sequent $\Gamma^* \Rightarrow \Delta'$ occurring in the proof of $A \longrightarrow A, \Gamma \Rightarrow \Delta$. Then, two cases arise.

Subsubcase 6.2.19.2.1.1. $\text{SeqComp}(A \longrightarrow A, \Gamma^* \Rightarrow \Delta') = \text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta)$. The difference between the sequents $A \longrightarrow A, \Gamma^* \Rightarrow \Delta'$ and $A \longrightarrow A, \Gamma \Rightarrow \Delta$ lies in the number of propositional variables only. Hence, $\Gamma^* \Rightarrow \Delta', A \preceq \Gamma \Rightarrow \Delta, A$. Consequently, by Theorem 6.2.18, we can easily obtain a proof of $\Gamma \Rightarrow \Delta, A$ by appropriate applications of WEAKENING and/or EXCHANGE.

Subsubcase 6.2.19.2.1.2. $\text{SeqComp}(A \longrightarrow A, \Gamma^* \Rightarrow \Delta') < \text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta)$. Then, since $A \longrightarrow A, \Gamma^* \Rightarrow \Delta'$ has a CONTRACTION+CUT-free proof, by the induction hypothesis, we can conclude that there exists a CONTRACTION+CUT-free proof of $\Gamma^* \Rightarrow \Delta'$ as well. Now, observe that,

$$A \longrightarrow A, \Gamma^* \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \implies \Gamma^* \Rightarrow \Delta' \preceq \Gamma \Rightarrow \Delta$$

Consequently, since $\Gamma^* \Rightarrow \Delta'$ has a CONTRACTION+CUT-free proof and since by Theorem 6.2.18 $\Gamma \Rightarrow \Delta$ can be obtained from $\Gamma^* \Rightarrow \Delta'$ by applications of WEAKENING and/or EXCHANGE; it follows that $\Gamma \Rightarrow \Delta$ has a CONTRACTION+CUT-free proof as well.

Subcase 6.2.19.2.2. If $A \longrightarrow A$ occurs before $\Gamma' \Rightarrow \Delta'$, then two cases arise.

Subsubcase 6.2.19.2.2.1. Suppose that $A \longrightarrow A$ occurs in $\Gamma' \Rightarrow \Delta'$ as well. Suppose further that it is not due to any application of $\longrightarrow \mathbf{L}$ (because otherwise by an argument similar to the one in [Subcase 6.2.19.2.1](#), we are done). Then some other rule must have been applied in the previous step. Since it must be a logical rule, the following seven cases arise, one for each rule .

Subsubsubcase 6.2.19.2.2.1.1. $\Gamma' \Rightarrow \Delta'$ is the result of an application of $\wedge \mathbf{L}$. Then that part of the proof looks something like the following.

$$\frac{B, \Gamma^* \Rightarrow \Delta'}{B \wedge C, \Gamma^* \Rightarrow \Delta'} \quad \wedge \mathbf{L}$$

Where $\Gamma' = B \wedge C, \Gamma^*$. Also note that since $A \longrightarrow A$ was introduced earlier in the proof, it must be in the list Γ^* (otherwise, refer to [Subsubcase 6.2.19.2.2.2](#)). Let $\Gamma^* := \Gamma_1, A \longrightarrow A, \Gamma_2$. Hence $\Gamma' = B \wedge C, \Gamma^* = B \wedge C, \Gamma_1, A \longrightarrow A, \Gamma_2$. Now,

$$\begin{aligned} & \Gamma' \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & B \wedge C, \Gamma^* \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & B \wedge C, \Gamma_1, A \longrightarrow A, \Gamma_2 \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & A \longrightarrow A, B \wedge C, \Gamma_1, \Gamma_2 \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & B \wedge C, \Gamma_1, \Gamma_2 \Rightarrow \Delta' \preceq \Gamma \Rightarrow \Delta \end{aligned}$$

In other words, $B \wedge C, \Gamma_1, \Gamma_2 \Rightarrow \Delta'$ is a subsequence of $\Gamma \Rightarrow \Delta$.

Now note that since $B \wedge C, \Gamma^* \Rightarrow \Delta'$ has a CONTRACTION+CUT-free proof, so does $B, \Gamma^* \Rightarrow \Delta'$ (by [Theorem 6.2.9](#)). Now we extend this proof of $B, \Gamma^* \Rightarrow \Delta'$ in the following way.

$$\frac{\frac{B, \Gamma^* \Rightarrow \Delta'}{B, \Gamma_1, A \longrightarrow A, \Gamma_2 \Rightarrow \Delta'}}{A \longrightarrow A, B, \Gamma_1, \Gamma_2 \Rightarrow \Delta'} \quad \mathbf{EL}$$

This, now constitutes a CONTRACTION+CUT-free proof of $A \longrightarrow A, B, \Gamma_1, \Gamma_2 \Rightarrow \Delta'$. Furthermore, since we have, $\text{SeqComp}(A \longrightarrow A, B, \Gamma_1, \Gamma_2 \Rightarrow \Delta') < \text{SeqComp}(B \wedge C, \Gamma_1, A \longrightarrow A, \Gamma_2 \Rightarrow \Delta') \leq \text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta)$, by the induction hypothesis, there exists a CONTRACTION+CUT-free proof of $B, \Gamma_1, \Gamma_2 \Rightarrow \Delta'$. From this proof, we obtain a CONTRACTION+CUT-free proof of $B \wedge C, \Gamma_1, \Gamma_2 \Rightarrow \Delta'$ as follows.

$$\frac{B, \Gamma_1, \Gamma_2 \Rightarrow \Delta'}{B \wedge C, \Gamma_1, \Gamma_2 \Rightarrow \Delta'} \quad \wedge \mathbf{L}$$

Now note that, since $B \wedge C, \Gamma_1, \Gamma_2 \Rightarrow \Delta' \preceq \Gamma \Rightarrow \Delta$, by [Theorem 6.2.18](#) it follows that $\Gamma \Rightarrow \Delta$ is obtainable from $B \wedge C, \Gamma_1, \Gamma_2 \Rightarrow \Delta'$ by suitable application(s) of WEAKENING and/or EXCHANGE-rule(s).

Subsubsubcase 6.2.19.2.2.1.2. $\Gamma' \Rightarrow \Delta'$ is the result of an application of $\vee\text{L}$. Then that part of the proof looks something like the following.

$$\frac{B, \Gamma^* \Rightarrow \Delta' \quad C, \Gamma^* \Rightarrow \Delta'}{B \vee C, \Gamma^* \Rightarrow \Delta'} \quad \vee\text{L}$$

Where $\Gamma' = B \vee C, \Gamma^*$. Now note that since $B \vee C, \Gamma^* \Rightarrow \Delta'$ has a $\text{CONTRACTION}+\text{CUT}$ -free proof, so does $B, \Gamma^* \Rightarrow \Delta'$ as well as $C, \Gamma^* \Rightarrow \Delta'$ (by [Theorem 6.2.9](#)). Since $A \longrightarrow A$ was introduced earlier in the proof, it must be in the list Γ^* (otherwise, refer to [Subsubcase 6.2.19.2.2.2](#)). Let $\Gamma^* := \Gamma_1, A \longrightarrow A, \Gamma_2$. Then $\Gamma' = B \vee C, \Gamma^* = B \vee C, \Gamma_1, A \longrightarrow A, \Gamma_2$. Now,

$$\begin{aligned} & \Gamma' \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & B \vee C, \Gamma^* \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & B \vee C, \Gamma_1, A \longrightarrow A, \Gamma_2 \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & A \longrightarrow A, B \vee C, \Gamma_1, \Gamma_2 \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & B \vee C, \Gamma_1, \Gamma_2 \Rightarrow \Delta' \preceq \Gamma \Rightarrow \Delta \end{aligned}$$

In other words, $B \vee C, \Gamma_1, \Gamma_2 \Rightarrow \Delta'$ is a subsequent of $\Gamma \Rightarrow \Delta$.

Now, after the proofs of $B, \Gamma^* \Rightarrow \Delta'$ and $C, \Gamma^* \Rightarrow \Delta'$ we add the following lines, respectively.

$$\frac{\frac{B, \Gamma^* \Rightarrow \Delta'}{B, \Gamma_1, A \longrightarrow A, \Gamma_2 \Rightarrow \Delta'}}{A \longrightarrow A, B, \Gamma_1, \Gamma_2 \Rightarrow \Delta'} \quad \text{EL}$$

$$\frac{\frac{C, \Gamma^* \Rightarrow \Delta'}{C, \Gamma_1, A \longrightarrow A, \Gamma_2 \Rightarrow \Delta'}}{A \longrightarrow A, C, \Gamma_1, \Gamma_2 \Rightarrow \Delta'} \quad \text{EL}$$

These constitute $\text{CONTRACTION}+\text{CUT}$ -free proofs of $A \longrightarrow A, B, \Gamma_1, \Gamma_2 \Rightarrow \Delta'$ and $A \longrightarrow A, C, \Gamma_1, \Gamma_2 \Rightarrow \Delta'$ respectively. Since $\text{SeqComp}(A \longrightarrow A, B, \Gamma_1, \Gamma_2 \Rightarrow \Delta') < \text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta)$ and $\text{SeqComp}(A \longrightarrow A, C, \Gamma_1, \Gamma_2 \Rightarrow \Delta') < \text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta)$, by the induction hypothesis we can conclude that there exist $\text{CONTRACTION}+\text{CUT}$ -free proofs of $B, \Gamma_1, \Gamma_2 \Rightarrow \Delta'$ and $C, \Gamma_1, \Gamma_2 \Rightarrow \Delta'$. From these proofs, we obtain a $\text{CONTRACTION}+\text{CUT}$ -free proof of $B \vee C, \Gamma_1, \Gamma_2 \Rightarrow \Delta'$ as follows.

$$\frac{B, \Gamma_1, \Gamma_2 \Rightarrow \Delta' \quad C, \Gamma_1, \Gamma_2 \Rightarrow \Delta'}{B \vee C, \Gamma_1, \Gamma_2 \Rightarrow \Delta'} \quad \vee\text{L}$$

Now note that since $B \vee C, \Gamma_1, \Gamma_2 \Rightarrow \Delta'$ is a subsequent of $\Gamma \Rightarrow \Delta$, by [Theorem 6.2.18](#) we can obtain a $\text{CONTRACTION}+\text{CUT}$ -free proof of $\Gamma \Rightarrow \Delta$ as well.

Subsubsubcase 6.2.19.2.2.1.3. $\Gamma' \Rightarrow \Delta'$ is the result of an application of $\neg\text{L}$. Then that part of the proof looks something like the following.

$$\frac{\Gamma^* \Rightarrow \Delta', B}{\neg B, \Gamma^* \Rightarrow \Delta'} \quad \neg\text{L}$$

Where $\Gamma' = \neg B, \Gamma^*$. Now note that since $\neg B, \Gamma^* \Rightarrow \Delta'$ has a CONTRACTION+CUT-free proof, so does $\Gamma^* \Rightarrow \Delta', B$ (by [Theorem 6.2.9](#)). Since $A \longrightarrow A$ was introduced earlier in the proof, it must be in the list Γ^* (otherwise, refer to [Subsubcase 6.2.19.2.2.2](#)). Without loss of generality, let $\Gamma^* := \overline{\Gamma}_1, A \longrightarrow A, \overline{\Gamma}_2$. Then $\Gamma' = \neg B, \overline{\Gamma}_1, A \longrightarrow A, \overline{\Gamma}_2$. Now,

$$\begin{aligned} & \Gamma' \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & \neg B, \overline{\Gamma}_1, A \longrightarrow A, \overline{\Gamma}_2 \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & A \longrightarrow A, \neg B, \overline{\Gamma}_1, \overline{\Gamma}_2 \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & \neg B, \overline{\Gamma}_1, \overline{\Gamma}_2 \Rightarrow \Delta' \preceq \Gamma \Rightarrow \Delta \end{aligned}$$

In other words, $\neg B, \overline{\Gamma}_1, \overline{\Gamma}_2 \Rightarrow \Delta'$ is a subsequence of $\Gamma \Rightarrow \Delta$.

Now, we extend the proof of $\Gamma^* \Rightarrow \Delta', B$ as follows.

$$\frac{\frac{\Gamma^* \Rightarrow \Delta', B}{\overline{\Gamma}_1, A \longrightarrow A, \overline{\Gamma}_2 \Rightarrow \Delta', B}}{A \longrightarrow A, \overline{\Gamma}_1, \overline{\Gamma}_2 \Rightarrow \Delta', B} \quad \text{EL}$$

This constitutes a CONTRACTION+CUT-free proof of $A \longrightarrow A, \overline{\Gamma}_1, \overline{\Gamma}_2 \Rightarrow \Delta', B$. Since $\text{SeqComp}(A \longrightarrow A, \overline{\Gamma}_1, \overline{\Gamma}_2 \Rightarrow \Delta', B) < \text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta)$ by the induction hypothesis we can conclude that there exist a CONTRACTION+CUT-free proof of $\overline{\Gamma}_1, \overline{\Gamma}_2 \Rightarrow \Delta', B$. From this proof, we obtain a CONTRACTION+CUT-free proof of $\neg B, \overline{\Gamma}_1, \overline{\Gamma}_2 \Rightarrow \Delta'$ as follows.

$$\frac{\overline{\Gamma}_1, \overline{\Gamma}_2 \Rightarrow \Delta', B}{\neg B, \overline{\Gamma}_1, \overline{\Gamma}_2 \Rightarrow \Delta'} \quad \neg\text{L}$$

Since $\neg B, \overline{\Gamma}_1, \overline{\Gamma}_2 \Rightarrow \Delta'$ is a subsequence of $\Gamma \Rightarrow \Delta$, applying WEAKENING and EXCHANGE rules suitably, we can obtain $\Gamma \Rightarrow \Delta$, by [Theorem 6.2.18](#).

Subsubsubcase 6.2.19.2.2.1.4. $\Gamma' \Rightarrow \Delta'$ is the result of an application of $\neg\text{R}$. Then that part of the proof looks something like the following.

$$\frac{B, \Gamma' \Rightarrow \Delta^*}{\Gamma' \Rightarrow \Delta^*, \neg B} \quad \neg\text{R}$$

Where $\Delta' = \Delta^*, \neg B$.

Now note that since $\Gamma' \Rightarrow \Delta^*, \neg B$ has a CONTRACTION+CUT-free proof, so does $B, \Gamma' \Rightarrow \Delta^*$ (by [Theorem 6.2.9](#)). Since $A \longrightarrow A$ was introduced earlier in the proof, it must be in the

list Γ' (otherwise, refer to [Subsubcase 6.2.19.2.2.2](#)). Let $\Gamma' := \Gamma_1, A \longrightarrow A, \Gamma_2$. Now,

$$\begin{aligned}
& \Gamma' \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\
\Rightarrow & \Gamma' \Rightarrow \Delta^*, \neg B \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\
\Rightarrow & \Gamma_1, A \longrightarrow A, \Gamma_2 \Rightarrow \Delta^*, \neg B \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\
\Rightarrow & A \longrightarrow A, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, \neg B \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\
\Rightarrow & \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, \neg B \preceq \Gamma \Rightarrow \Delta
\end{aligned}$$

In other words, $\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, \neg B$ is a subsequence of $\Gamma \Rightarrow \Delta$.

Now, after the proof of $B, \Gamma' \Rightarrow \Delta^*$, we add the following steps.

$$\frac{\frac{B, \Gamma' \Rightarrow \Delta^*}{B, \Gamma_1, A \longrightarrow A, \Gamma_2 \Rightarrow \Delta^*}}{A \longrightarrow A, B, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*} \text{ EL}$$

This constitutes a CONTRACTION+CUT-free proof of $A \longrightarrow A, B, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*$. Since $\text{SeqComp}(A \longrightarrow A, B, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*) < \text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta)$, by the induction hypothesis, we can conclude that there exists a CONTRACTION+CUT-free proof of $B, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*$. From this proof, we obtain a CONTRACTION+CUT-free proof of $\Gamma_1, \Gamma_2 \Rightarrow \Delta', \neg B$ as follows.

$$\frac{B, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*}{\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, \neg B} \text{ } \neg\text{R}$$

Since $\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, \neg B$ is a subsequence of $\Gamma \Rightarrow \Delta$, by [Theorem 6.2.18](#) we can obtain $\Gamma \Rightarrow \Delta$.

Subsubsubcase 6.2.19.2.2.1.5. $\Gamma \Rightarrow \Delta$ is the result of an application of $\wedge\text{R}$. Then that part of the proof looks something like the following.

$$\frac{\Gamma' \Rightarrow \Delta^*, B \quad \Gamma' \Rightarrow \Delta^*, C}{\Gamma' \Rightarrow \Delta^*, B \wedge C} \text{ } \wedge\text{R}$$

Where $\Delta' = \Delta^*, B \wedge C$. Also, note that since $A \longrightarrow A$ was introduced earlier in the proof, it must be in the list Γ' . Let $\Gamma' := \Gamma_1, A \longrightarrow A, \Gamma_2$ (otherwise, refer to [Subsubcase 6.2.19.2.2.2](#)). Now,

$$\begin{aligned}
& \Gamma' \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\
\Rightarrow & \Gamma' \Rightarrow \Delta^*, B \wedge C \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\
\Rightarrow & \Gamma_1, A \longrightarrow A, \Gamma_2 \Rightarrow \Delta^*, B \wedge C \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\
\Rightarrow & A \longrightarrow A, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \wedge C \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\
\Rightarrow & \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \wedge C \preceq \Gamma \Rightarrow \Delta
\end{aligned}$$

In other words, $\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \wedge C$ is a subsequence of $\Gamma \Rightarrow \Delta$.

Since $\Gamma' \Rightarrow \Delta^*, B \wedge C$ has a CONTRACTION+CUT-free proof, so does $\Gamma' \Rightarrow \Delta^*, B$ and $\Gamma' \Rightarrow \Delta^*, C$ (by [Theorem 6.2.9](#)). We extend both these proofs respectively as follows.

$$\frac{\frac{\Gamma' \Rightarrow \Delta^*, B}{\Gamma_1, A \longrightarrow A, \Gamma_2 \Rightarrow \Delta^*, B}}{A \longrightarrow A, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B} \text{ EL}$$

$$\frac{\frac{\Gamma' \Rightarrow \Delta^*, C}{\Gamma_1, A \longrightarrow A, \Gamma_2 \Rightarrow \Delta^*, C}}{A \longrightarrow A, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, C} \text{ EL}$$

These now constitute CONTRACTION+CUT-free proofs of $A \longrightarrow A, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B$ and $A \longrightarrow A, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, C$ respectively. Since $\text{SeqComp}(A \longrightarrow A, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B) < \text{SeqComp}(\neg A, \Gamma \Rightarrow \Delta)$ and $\text{SeqComp}(A \longrightarrow A, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, C) < \text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta)$, by the induction hypothesis, there exist CONTRACTION+CUT-free proofs of $\Gamma_1, \Gamma_2 \Rightarrow \Delta', B$ and $\Gamma_1, \Gamma_2 \Rightarrow \Delta', C$. From these proofs, we obtain a CONTRACTION+CUT-free proof of $\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \wedge C$ as follows.

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \quad \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, C}{\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \wedge C} \text{ } \wedge R$$

Since $\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \wedge C \preceq \Gamma \Rightarrow \Delta$ and since there is a CONTRACTION+CUT-free proof of $\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \wedge C$, by [Theorem 6.2.18](#) it follows that $\Gamma \Rightarrow \Delta$ is obtainable from $\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \wedge C$ by suitable application(s) of WEAKENING and/or EXCHANGE-rule(s).

Subsubsubcase 6.2.19.2.2.1.6. $\Gamma' \Rightarrow \Delta'$ is the result of an application of $\vee R$. Then that part of the proof looks something like the following.

$$\frac{\Gamma' \Rightarrow \Delta^*, B}{\Gamma' \Rightarrow \Delta^*, B \vee C} \text{ } \vee R$$

Where $\Delta' = \Delta^*, B \vee C$.

Since $\Gamma' \Rightarrow \Delta^*, B \vee C$ has a CONTRACTION+CUT-free proof, so does $\Gamma' \Rightarrow \Delta^*, B$ (by [Theorem 6.2.9](#)). Since $A \longrightarrow A$ was introduced earlier in the proof, it must be in the list Γ' (otherwise, refer to [Subsubcase 6.2.19.2.2.2](#)). Let $\Gamma' := \Gamma_1, A \longrightarrow A, \Gamma_2$. Now,

$$\begin{aligned} & \Gamma' \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & \Gamma' \Rightarrow \Delta^*, B \vee C \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & \Gamma_1, A \longrightarrow A, \Gamma_2 \Rightarrow \Delta^*, B \vee C \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & A \longrightarrow A, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \vee C \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \vee C \preceq \Gamma \Rightarrow \Delta \end{aligned}$$

In other words, $\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \vee C$ is a subsequence of $\Gamma \Rightarrow \Delta$.

Now, after the proof of $\Gamma' \Rightarrow \Delta^*, B$, we add the following steps.

$$\frac{\frac{\Gamma' \Rightarrow \Delta^*, B}{\Gamma_1, A \longrightarrow A, \Gamma_2 \Rightarrow \Delta^*, B}}{A \longrightarrow A, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B} \text{ EL}$$

This constitutes a CONTRACTION+CUT-free proof of $A \longrightarrow A, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B$. Since we have $\text{SeqComp}(A \longrightarrow A, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B) < \text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta)$, by the induction hypothesis, we can conclude that there exists a CONTRACTION+CUT-free proof of $\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B$. From this proof, we obtain a CONTRACTION+CUT-free proof of $\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \vee C$ as follows.

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B}{\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \vee C} \quad \text{VR}$$

Since $\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \vee C$ is a subsequence of $\Gamma \Rightarrow \Delta$, by [Theorem 6.2.18](#) we can obtain $\Gamma \Rightarrow \Delta$.

Subsubsubcase 6.2.19.2.2.1.7. $\Gamma' \Rightarrow \Delta'$ is the result of an application of $\longrightarrow \text{R}$. Then that part of the proof looks something like the following.

$$\frac{B, \Gamma' \Rightarrow \Delta^*, C}{\Gamma' \Rightarrow \Delta^*, B \longrightarrow C} \quad \longrightarrow \text{R}$$

i.e., in this case $\Delta' = \Delta^*, B \longrightarrow C$.

Since $\Gamma' \Rightarrow \Delta^*, B \longrightarrow C$ has a CONTRACTION+CUT-free proof, so does $B, \Gamma' \Rightarrow \Delta^*, C$ (by [Theorem 6.2.9](#)). Since $A \longrightarrow A$ was introduced earlier in the proof, it must be in the list Γ' (otherwise, refer to [Subsubsubcase 6.2.19.2.2.2](#)). Without loss of generality, let $\Gamma' := \Gamma_1, A \longrightarrow A, \Gamma_2$. Now,

$$\begin{aligned} & \Gamma' \Rightarrow \Delta' \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & \Gamma_1, A \longrightarrow A, \Gamma_2 \Rightarrow \Delta^*, B \longrightarrow C \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & A \longrightarrow A, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \longrightarrow C \preceq A \longrightarrow A, \Gamma \Rightarrow \Delta \\ \implies & \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \longrightarrow C \preceq \Gamma \Rightarrow \Delta \end{aligned}$$

In other words, $\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \longrightarrow C$ is a subsequence of $\Gamma \Rightarrow \Delta$.

Now, after the proof of $B, \Gamma' \Rightarrow \Delta^*, C$, we add the following steps.

$$\frac{\frac{B, \Gamma' \Rightarrow \Delta^*, C}{B, \Gamma_1, A \longrightarrow A, \Gamma_2 \Rightarrow \Delta^*, C}}{A \longrightarrow A, B, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, C} \quad \text{EL}$$

This constitute a CONTRACTION+CUT-free proof of $A \longrightarrow A, B, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, C$. Since we have $\text{SeqComp}(A \longrightarrow A, B, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, C) < \text{SeqComp}(A \longrightarrow A, \Gamma \Rightarrow \Delta)$, by the induction hypothesis, we can conclude that there exists a CONTRACTION+CUT-free proof of $B, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, C$. From this proof, we obtain a CONTRACTION+CUT-free proof of $B, \Gamma_1, \Gamma_2 \Rightarrow \Delta'$ as follows.

$$\frac{B, \Gamma_1, \Gamma_2 \Rightarrow \Delta^*, C}{\Gamma_1, \Gamma_2 \Rightarrow \Delta^*, B \longrightarrow C} \quad \longrightarrow \text{R}$$

Since $\Gamma_1, \Gamma_2 \Rightarrow \Delta', B \longrightarrow C$ is a subsequence of $\Gamma \Rightarrow \Delta$, applying WEAKENING and EXCHANGE rules suitably, we can obtain $\Gamma \Rightarrow \Delta$, by [Theorem 6.2.18](#).

Subsubcase 6.2.19.2.2.2. If $A \longrightarrow A$ does not occur in the sequent $\Gamma' \Rightarrow \Delta'$, we can immediately conclude that $\Gamma' \Rightarrow \Delta'$ is a subsequence of $\Gamma \Rightarrow \Delta$. Then, since $\Gamma' \Rightarrow \Delta'$ has a CONTRACTION+CUT-free proof, by [Theorem 6.2.18](#), $\Gamma \Rightarrow \Delta$ has a CONTRACTION+CUT-free proof as well.

□

Theorem 6.2.20 (CONTRACTION+CUT-ELIMINATION THEOREM FOR **PLK** (PART III)).
*Let $\Gamma \Rightarrow \Delta$ be a sequent. If $\Gamma \Rightarrow \Delta$ is provable in **PLK**, then there is a CONTRACTION+CUT-free **PLK**-proof of $\Gamma \Rightarrow \Delta$.*

Proof. First, transform the given proof of $\Gamma \Rightarrow \Delta$ into a CONTRACTION-free proof by the procedure mentioned in [Theorem 6.2.5](#). Choose any formula A and obtain a CONTRACTION-free proof $A \longrightarrow A, \Gamma \Rightarrow \Delta$ by an application of **WL**. Thus, we have obtained a CONTRACTION-free proof of $A \longrightarrow A, \Gamma \Rightarrow \Delta$. Consequently, by [Theorem 6.2.15](#), it follows that there exists a CONTRACTION+CUT-free proof of $A \longrightarrow A, \Gamma \Rightarrow \Delta$ as well. Finally, by [Theorem 6.2.19](#), it follows that there exists a CONTRACTION+CUT-free proof of $\Gamma \Rightarrow \Delta$. □

Remark 6.2.21. [Theorem 6.2.5](#) has been one of the crucial ingredients for proving the above theorem. It should be pointed out that if one considered the multiplicative version of CUT instead of the additive version, the result would not hold (see [68] for details). Furthermore, the **SeqComp** function, so crucial in proving [Theorem 6.2.19](#), induces an ordering in the rules; namely, the **SeqComp**-value of every premise of a rule is less than or equal to the value of the conclusion. This order is essentially reversed in the case of (additive) CUT and CONTRACTION-rules. To sum up, in each case, we could conclude something about the ordering of the premises and conclusion in a *uniform* manner. This is not the case if we consider **MCUT**. For example, if $\Gamma, \Delta, \Pi, \Phi$ are all nonempty, having at least one non-atomic formula, and if p is a propositional variable, then considering the following instance of **MCUT**,

$$\frac{\Gamma \Rightarrow \Delta, \neg p \quad \neg p, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \Delta, \Phi} \quad \text{CUT}$$

we note that the **SeqComp**-value of each premise is strictly less than that of the conclusion. If, on the other hand, we consider the following instance of **MCUT** where each element of $\Gamma, \Delta, \Pi, \Phi$ is a propositional variable,

$$\frac{\Gamma \Rightarrow \Delta, p \quad p, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \Delta, \Phi} \quad \text{CUT}$$

we note that the **SeqComp**-value of each premise is the same as that of the conclusion. If, on the other hand, we consider the following instance of **MCUT**:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \Delta, \Phi} \quad \text{CUT}$$

where $\text{Comp}(A) > \text{SeqComp}(\Gamma, \Pi \Rightarrow \Delta, \Phi)$, it follows that the value of SeqComp for each of the premise is strictly greater than that of the conclusion. Consequently, no uniform conclusion about the ordering of rules via the number of connectives can be maintained for the MCUT . Although this doesn't exclude the possibility of finding some *other* function relative to which the rules can be ordered, it is certain that *that* function cannot be SeqComp . These are the main reasons for choosing the additive version of CUT instead of the multiplicative one.

6.2.4 Non-Eliminability of the Other Rules of PLK

So far, we have been successful in eliminating the CONTRACTION -rules and the CUT -rule. Is there any other rule in PLK which can be eliminated? In this section, we delve into this question and show that no other PLK -rule can be eliminated. In fact, we show something stronger. We show that for each rule except CONTRACTION -rules and the CUT -rule there exists a sequent, the proof of which can't be done without using that rule. It is important to note that all of these conclusions follow as corollaries of [Theorem 6.2.20](#).

Theorem 6.2.22 (NON-ELIMINABILITY OF THE STRUCTURAL RULES OF PLK). *Let p and q denote distinct propositional variables. Then, the following statements hold.*

- (1) $q, p \Rightarrow p$ is not provable without using **WL**.
- (2) $p \Rightarrow p, q$ is not provable without using **WR**.
- (3) $p \Rightarrow q, p$ is not provable without using **EL**.
- (4) $p, q \Rightarrow p$ is not provable without using **ER**.

Proof. We prove (1) only. The proofs for the other rules are similar.

Note that $q, p \Rightarrow p$ is a PLK -provable sequent as shown below:

$$\frac{p \Rightarrow p}{q, p \Rightarrow p} \quad \text{WL}$$

Consider any PLK -proof of $q, p \Rightarrow p$. Then, by [Theorem 6.2.20](#), there exists a $\text{CONTRACTION} + \text{CUT}$ -free proof of the same. Consider any such proof. Note that $q, p \Rightarrow p$ can't be the conclusion of any instance of any logical rule since it does not contain any connective. So, it must be the conclusion of an instance of some EXCHANGE -rule or WEAKENING -rule.

But, it can't be the conclusion of any instance of **WR** because that would imply the provability of the sequent $q, p \Rightarrow \Lambda$, an impossibility due to [Theorem 6.2.12](#). Also, it can't be the conclusion of any instance of **ER** because to apply **ER**, one needs at least two wffs on the right of the premise.

Hence, $q, p \Rightarrow p$ can only be the conclusion of some instance of **WL** or **EL**. In case it is the conclusion of an instance of **WL**, it must be the following:

$$\frac{p \Rightarrow p}{q, p \Rightarrow p} \quad \text{WL}$$

In case it is not the conclusion of an instance of **WL**, it must be the conclusion of an instance of **EL**. The last two lines of the proof then look like the following:

$$\frac{p, q \Rightarrow p}{q, p \Rightarrow p} \quad \text{EL}$$

Consider now the sequent $p, q \Rightarrow p$. By an argument similar to the above, we can conclude that it can't be the conclusion of any instance of CUT, CONTRACTION, **WR**, **ER** or logical rules. However, in this case, it can't be the conclusion of an instance **WL** either since that would imply the **PLK**-provability of $q \Rightarrow p$, an impossibility – by [Theorem 6.2.12](#) – since $p \neq q$. Hence, it must be the conclusion of an instance of **EL**. Consequently, the last two lines of the proof will look something like the following:

$$\frac{q, p \Rightarrow p}{p, q \Rightarrow p} \quad \text{EL}$$

and we are back to the same sequent we started with. So, we continue the argument and go upwards in the proof, i.e., we check whether the concerned sequent is the conclusion of some instance of an EXCHANGE-rule. At the first occurrence of a sequent, which is not the conclusion of any instance of an EXCHANGE-rule, we stop (such a sequent must exist since the proof is *finite*). Hence, this sequent must be the conclusion of an instance of **WL**, and we are done. $\square$

Theorem 6.2.23 (NON-ELIMINABILITY OF THE LOGICAL RULES OF **PLK**). *Let p and q denote distinct propositional variables. Then, the following statements hold.*

- (1) $\neg p, p \Rightarrow \Lambda$ is not provable without using $\neg\text{L}$.
- (2) $\Lambda \Rightarrow p, \neg p$ is not provable without using $\neg\text{R}$.
- (3) $p \wedge q \Rightarrow p$ is not provable without using $\wedge\text{L}$.
- (4) $p \Rightarrow p \wedge p$ is not provable without using $\wedge\text{R}$.
- (5) $p \vee p \Rightarrow p$ is not provable without using $\vee\text{L}$.
- (6) $p \Rightarrow p \vee q$ is not provable without using $\vee\text{R}$.
- (7) $p \longrightarrow q, p \Rightarrow q$ is not provable without using $\longrightarrow\text{L}$.
- (8) $\Lambda \Rightarrow p \longrightarrow p$ is not provable without using $\longrightarrow\text{R}$.

Proof. We proof only (7). The proofs for other rules are similar.

Note that $p \longrightarrow q, p \Rightarrow q$ is a **PLK**-provable sequent as shown below.

$$\begin{array}{c} \text{WR} \\ \text{ER} \end{array} \frac{\frac{p \Rightarrow p}{p \Rightarrow p, q} \quad \frac{q \Rightarrow q}{p, q \Rightarrow q}}{\frac{p \Rightarrow q, p}{q, p \Rightarrow q}} \quad \begin{array}{c} \text{WL} \\ \text{EL} \end{array} \frac{}{\longrightarrow\text{L}}$$

Consider any **PLK**-proof of $p \longrightarrow q, p \Rightarrow q$. Then, by [Theorem 6.2.20](#), there exists a **CONTRACTION**+**CUT**-free proof of the same. Consider any such proof. Note that $p \longrightarrow q, p \Rightarrow q$ can't be the conclusion of any instance of any logical rule other than $\longrightarrow L$ since it does not contain any other connective. It can't be the conclusion of any instance of **WL** or **WR** because in the former case we would be able to conclude that $p \Rightarrow q$ is **PLK**-provable, while in the latter we will be able to conclude that $p \longrightarrow q, p \Rightarrow \emptyset$ is **PLK**-provable. The first case is impossible by [Theorem 6.2.12](#) (since p and q are distinct variables); the second case is also impossible since provability of $p \longrightarrow q, p \Rightarrow \emptyset$ will imply (due to [Theorem 6.2.9](#)) that there is a proof of $q, p \Rightarrow \emptyset$, contrary to [Theorem 6.2.12](#).

Furthermore, $p \longrightarrow q, p \Rightarrow q$ can't be the conclusion of any instance of **ER** because to apply **ER** one needs at least two wffs on the right in the premise.

Hence, $p \longrightarrow q, p \Rightarrow q$ can only be the conclusion of some instance of $\longrightarrow L$ or **EL**. Without loss of generality, we may assume that $p \longrightarrow q, p \Rightarrow q$ is the conclusion of an instance of $\longrightarrow L$ because if it is the conclusion of an instance of **EL**, we will go upwards in the proof to find the sequent which is not the conclusion of an instance of **EL**. This sequent must be the conclusion of an instance of $\longrightarrow L$, and in either case, that part of the proof must look something like the following:

$$\frac{p \Rightarrow q, p \quad q, p \Rightarrow q}{p \longrightarrow q, p \Rightarrow q} \quad \longrightarrow L$$

Consider now the sequent $q, p \Rightarrow q$. By an argument similar to the above, we will be able to conclude that it can't be the conclusion of any instance of **CUT**, **CONTRACTION**, **WR**, **ER** or logical rules. However, in this case, it can't be the conclusion of any instance of **WL** either since that would imply the **PLK**-provability of $p \Rightarrow q$, an impossibility (again by [Theorem 6.2.12](#)). Hence, it must be the conclusion of some instance of **EL**. Consequently, the last two lines of the proof of $q, p \Rightarrow q$ in the proof of $p \longrightarrow q, p \Rightarrow q$ will look something like the following:

$$\frac{p, q \Rightarrow q}{q, p \Rightarrow q} \quad EL$$

We continue the argument and go upwards in the proof, i.e., we check whether the corresponding sequent is the conclusion of some instance of **EXCHANGE**-rules. At the first occurrence of a sequent, which is not the conclusion of any instance of **EXCHANGE**-rules, we stop (such a sequent must exist since the proof is *finite*). Hence, this sequent must be the conclusion of an instance of **WL**, and it must be $p, q \Rightarrow q$. By an analogous argument we can show that the case for the sequent $p \Rightarrow q, p$ will be similar. To sum up, the proof of $p \longrightarrow q, p \Rightarrow q$ will always look something like the following:

$$\begin{array}{c}
\text{WR} \quad \frac{p \Rightarrow p}{p \Rightarrow p, q} \quad \frac{q \Rightarrow q}{p, q \Rightarrow q} \quad \text{WL} \\
\text{ER} \quad \frac{p \Rightarrow p, q}{p \Rightarrow q, p} \quad \frac{p, q \Rightarrow q}{p \Rightarrow q, p} \quad \text{EL} \\
\text{ER} \quad \frac{\vdots}{p \Rightarrow q, p} \quad \frac{\vdots}{q, p \Rightarrow q} \quad \text{EL} \\
\frac{p \Rightarrow q, p \quad q, p \Rightarrow q}{p \multimap q, p \Rightarrow q} \quad \text{EL} \rightarrow L \\
\vdots \\
\frac{p \multimap q, p \Rightarrow q}{p \multimap q, p \Rightarrow q}
\end{array}$$

and we are done. □

6.2.5 Observations

We now make a couple of observations.

- (a) The proofs of the CUT-elimination theorem, detailed in the previous sections, cannot be applied in their present form to prove CUT-elimination theorem for any system in which either one or both of the WEAKENING-rules and/or the EXCHANGE-rules are restricted. This, however, does not restrict the possibility of eliminating some restricted form of CUT from the sequent system by applying the same technique.
- (b) As remarked earlier, in each case of the proof of [Theorem 6.2.13](#), the only thing which makes the induction work, apart from [Theorem 6.2.2](#), is the fact that the complexity of each of the premise sequent of a logical rule is strictly less than that of the conclusion sequent. In particular, it should be noted that in the argument, the number of premises is also not important.
- (c) We also note that since the notion of sequent complexity is independent of particular connectives, the proof can be easily adapted to other sequent systems as well. However, analysing carefully, we find that even the notion of complexity is not necessary, and we can generalise the argument even more.
- (d) The only structural rules needed were WEAKENING and EXCHANGE.
- (e) The fact that the sequent $\Lambda \Rightarrow \Lambda$ has no proof in **PLK** (proven in [Theorem 6.2.12](#)) is crucial for the proof of [Theorem 6.2.19](#).

These observations motivate us to generalise the argument to a broader class of sequent systems. In the next section, we will give a sufficient condition for a sequent system to have the property of CUT-elimination.

6.3 Normal Sequent Structures

In this section, we generalise the arguments of the previous section to a broader class of sequent systems, taking note of the observations just mentioned. We introduce the notion

of *normal sequent structures* and prove two RULE-elimination theorems for the same.

We begin with the following definitions.

Definition 6.3.1 (SEQUENT). Let $\mathcal{L}$ be a set and let $\Rightarrow \subseteq \mathbf{FO}(\mathcal{L}) \times \mathbf{FO}(\mathcal{L})$, where $\mathbf{FO}(\mathcal{L})$ denotes the set of all finite ordered lists of elements of $\mathcal{L}$. Then $(\Gamma, \Delta) \in \Rightarrow$ will be said to be a $\mathcal{L}$ -*sequent with respect to* $\Rightarrow$. In this case, instead of writing $(\Gamma, \Delta) \in \Rightarrow$, we will write $\Gamma \Rightarrow \Delta$. The set of all $\mathcal{L}$ -sequents with respect to $\Rightarrow$, will henceforth be denoted by $\text{Seq}(\mathcal{L}, \Rightarrow)$.

Definition 6.3.2 (RULE). Let $\text{Seq}(\mathcal{L}, \Rightarrow)$ be the set of all $\mathcal{L}$ -sequents with respect to $\Rightarrow$. A set R is said to be a $\text{Seq}(\mathcal{L}, \Rightarrow)$ -rule if,

$$\emptyset \subsetneq R \subseteq \text{Seq}(\mathcal{L}, \Rightarrow)^n \times \text{Seq}(\mathcal{L}, \Rightarrow)$$

for some $n \in \mathbb{N}$ (where $\mathbb{N} := \{0, 1, \dots\}$). It is said to be a $\text{Seq}(\mathcal{L}, \Rightarrow)$ -*axiomatic rule* if $\emptyset \subsetneq R \subseteq \{\emptyset\} \times \text{Seq}(\mathcal{L}, \Rightarrow)$, else, a *non-axiomatic rule*. An *instance* of a rule R is an element of R . Given $((\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n), \Gamma \Rightarrow \Delta) \in R$, $\Gamma_i \Rightarrow \Delta_i$ is to be a *premise* of this instance of R for each $i \in \{1, \dots, n\}$ and $\Gamma \Rightarrow \Delta$ is said to be a *conclusion* of this instance of R . In this case, we will also sometimes say $\Gamma \Rightarrow \Delta$ to be *obtained by an application of* R .

Recall that in [Section 6.2](#), the language for **PLK** is defined to be an algebra, constructed recursively over a countably infinite set of propositional variables using a finite set of connectives. In this section, we generalise this and consider our language to be a formula algebra. To be more precise, we assume that $\mathcal{L}$ is the set of *formulas* defined inductively in the usual way over a nonempty set of variables V using a finite set of connectives/operators called the *signature/type*. The formula algebra has the universal mapping property for the class of all algebras of the same type as $\mathcal{L}$ over V , i.e., any function $f : \mathcal{P} \rightarrow A$, where A is the universe of an algebra $\mathbf{A}$ of the same type as $\mathcal{L}$, can be uniquely extended to a homomorphism from $\mathcal{L}$ to $\mathbf{A}$ (see [\[52\]](#) for more details).

Definition 6.3.3 (NORMAL SEQUENT STRUCTURE). Let $\mathcal{L}$ be a formula algebra of some type over a set of variables V and $\text{Seq}(\mathcal{L}, \Rightarrow)$ be the set of all $\mathcal{L}$ -sequents with respect to $\Rightarrow$ and $\vdash \subseteq \mathcal{P}(\text{Seq}(\mathcal{L}, \Rightarrow)) \times \text{Seq}(\mathcal{L}, \Rightarrow)$. The pair $(\text{Seq}(\mathcal{L}, \Rightarrow), \vdash)$ is then said to be a *normal sequent structure* if the following conditions hold.

- (i) $p \Rightarrow p$ for all $p \in V$. Furthermore, $\emptyset \vdash p \Rightarrow p$ for all $p \in V$.
- (ii) Both the WEAKENING and EXCHANGE rules are included in the set of rules.
- (iii) $\emptyset \vdash \Gamma \Rightarrow \Delta$ means that there is a proof of $\Gamma \Rightarrow \Delta$; or, alternatively, that $\Gamma \Rightarrow \Delta$ is *provable*. Proofs are, as in [Section 6.2](#), built as trees of sequents (i.e. with nodes labelled with sequents), with leaves being conclusions of an instance of some axiomatic rule and the proven sequent as a root. Formally this notion may be defined inductively as follows:

- If S_0 is the conclusion of an instance of an axiomatic rule, then it is a proof of a sequent S_0 .

- If $\mathcal{D}_1$ is a proof of a sequent S_0 , then $\frac{\mathcal{D}_1}{S_1}$ is a proof of a sequent S_1 , provided that S_0 is an instance of the premise and S_1 an instance of the conclusion of some one-premise rule.
- If $\mathcal{D}_1$ is a proof of a sequent S_0 and $\mathcal{D}_2$ is a proof of a sequent S_1 , then $\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{S_2}$ is a proof of a sequent S_2 , provided that S_0 and S_1 are instances of the premises and S_2 an instance of the conclusion of some two-premise rule.
- If $\mathcal{D}_1$ is a proof of a sequent S_0 ; $\mathcal{D}_2$ is a proof of a sequent S_1 and $\mathcal{D}_3$ is a proof of a sequent S_2 , then $\frac{\mathcal{D}_1 \quad \mathcal{D}_2 \quad \mathcal{D}_3}{S_3}$ is a proof of a sequent S_3 , provided that S_0, S_1 and S_2 are instances of the premises, and S_3 an instance of the conclusion of some three-premise rule.

... and so on, for each $n \in \mathbb{N} \setminus \{0\}$ and for each n -premise rule.

- (iv) If $\Gamma \Rightarrow \Delta$ is provable in $\text{Seq}(\mathcal{L}, \Rightarrow)$ and if Γ and Δ both contains only variables, then there exists $p \in V$ that is common to both the lists Γ and Δ .
- (v) If $\Gamma \Rightarrow \Delta$ is provable in $\text{Seq}(\mathcal{L}, \Rightarrow)$, then either Γ is not the empty list or Δ is not the empty list.

In the case of **PLK**, **SeqComp** satisfies the following interesting property: for each rule, if the side formulae are non-atomic, then (except CUT and CONTRACTION-rules) the **SeqComp**-value of each of the premise is less than or equal to that of the conclusion. This is not true for CUT and CONTRACTION-rules, and they were eliminable. The motivation for the following definition was the (conjectural) belief that this is not a coincidence. To be more precise, we think that both the CUT and CONTRACTION-rules were eliminable *precisely because* they did not harmonise with other rules. **SeqComp** merely ‘witnessed’ this.

Definition 6.3.4 (WITNESS FUNCTION). Let $(\text{Seq}(\mathcal{L}, \Rightarrow), \vdash)$ be a normal sequent structure and $v : \text{Seq}(\mathcal{L}, \Rightarrow) \rightarrow \mathbb{N}$. Then v is said to be a *witness function for $(\text{Seq}(\mathcal{L}, \Rightarrow), \vdash)$ relative to the rule T* if the following conditions hold.

- (i) There exists $n_0 \in \mathbb{N}$ such that if for some sequent $\Gamma \Rightarrow \Delta$, $v(\Gamma \Rightarrow \Delta) = n_0$ then both Γ and Δ contain at most variables. Furthermore, for all $\Gamma \Rightarrow \Delta \in \text{Seq}(\mathcal{L}, \Rightarrow)$, $v(\Gamma \Rightarrow \Delta) \geq n_0$.
- (ii) Suppose R is a $\text{Seq}(\mathcal{L}, \Rightarrow)$ -non-axiomatic rule. We say that v *respects* (respectively, *strictly respects*) an instance $((\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_m \Rightarrow \Delta_m), \Gamma \Rightarrow \Delta)$ of R if $v(\Gamma_i \Rightarrow \Delta_i) \leq v(\Gamma \Rightarrow \Delta)$ (respectively, if $v(\Gamma_i \Rightarrow \Delta_i) < v(\Gamma \Rightarrow \Delta)$), for all $i \in \{1, \dots, m\}$.
- (iii) v does not respect any instance of T .

Theorem 6.3.5 (RULE-ELIMINATION THEOREM FOR NSSs (I)). *Let $(\text{Seq}(\mathcal{L}, \Rightarrow), \vdash)$ be a normal sequent structure having a witness function relative to a rule T (which is disjoint from WEAKENING and EXCHANGE rules). Suppose that for every $\text{Seq}(\mathcal{L}, \Rightarrow)$ -provable sequent $\Gamma \Rightarrow \Delta$, the following statements hold.*

- (1) If $n_0 < v(\Gamma \Rightarrow \Delta)$ then there exists a provable $\Gamma' \Rightarrow \Delta'$ such that $v(\Gamma' \Rightarrow \Delta') < v(\Gamma \Rightarrow \Delta)$.
- (2) If $\Gamma' \Rightarrow \Delta'$ has a T -free proof, then so does $\Gamma \Rightarrow \Delta$.

Then every provable sequent has a T -free proof.

Proof. Let $v : \text{Seq}(\mathcal{L}, \Rightarrow) \rightarrow \mathbb{N}$ be the witness function relative to T and $\Gamma \Rightarrow \Delta$ be a provable sequent. We will show this by induction on the values of v .

BASE CASE: In the base case, $n = n_0$; thus every formula in the sequent $\Gamma \Rightarrow \Delta$ is a propositional variable (by Definition 6.3.4(i)). Since the sequent is provable, there must be some variable p , which occurs both in Γ and Δ (by Definition 6.3.3(iv)). Since $p \Rightarrow p$ is provable, $\Gamma \Rightarrow \Delta$ can be proved without using T by application (or applications, as needed) of the WEAKENING and/or EXCHANGE rules.²

INDUCTION HYPOTHESIS: For every sequent $\Gamma \Rightarrow \Delta$ such that $n_0 \leq v(\Gamma \Rightarrow \Delta) < n$, if there is a $\text{Seq}(\mathcal{L}, \Rightarrow)$ -proof of $\Gamma \Rightarrow \Delta$, then there is a T -free $\text{Seq}(\mathcal{L}, \Rightarrow)$ -proof of $\Gamma \Rightarrow \Delta$.

INDUCTION STEP: Let $\Gamma \Rightarrow \Delta$ be a sequent such that $v(\Gamma \Rightarrow \Delta) = n$. Suppose also that $\Gamma \Rightarrow \Delta$ is $\text{Seq}(\mathcal{L}, \Rightarrow)$ -provable. Then, by (1), it follows that there exists a provable $\Gamma' \Rightarrow \Delta'$ with $v(\Gamma' \Rightarrow \Delta') < v(\Gamma \Rightarrow \Delta)$ such that $\Gamma' \Rightarrow \Delta'$ is provable. Since $v(\Gamma' \Rightarrow \Delta') < v(\Gamma \Rightarrow \Delta)$ by the induction hypothesis it follows that $\Gamma' \Rightarrow \Delta'$ has a T -free proof. Consequently, by (2), it follows that $\Gamma \Rightarrow \Delta$ has a T -free proof as well. $\square$

Remark 6.3.6. Note that in this proof the fact that $\mathcal{L}$ is an algebra is needed only to prove the base case.

Theorem 6.3.7 (RULE-ELIMINATION THEOREM FOR NSSs (II)). *Let $(\text{Seq}(\mathcal{L}, \Rightarrow), \vdash)$ be a normal sequent structure and suppose that for every $\text{Seq}(\mathcal{L}, \Rightarrow)$ -provable sequent $\Gamma \Rightarrow \Delta$, the following statements hold.*

- (1) *If Γ is not the empty list, then there exists a rule R and a sequent $\Gamma^R \Rightarrow \Delta^R$ that is deducible from a set of sequents which includes $\Gamma \Rightarrow \Delta$ as one of the premise(s), by an application of the rule R . If Δ is not the empty list, then there exists a rule L and a sequent $\Gamma^L \Rightarrow \Delta^L$ such that a similar conclusion holds. Furthermore, each of the premises are provable.*
- (2) *If there exists a proof of $\Gamma^R \Rightarrow \Delta^R$, then there exists a T -free $\text{Seq}(\mathcal{L}, \Rightarrow)$ -proof of $\Gamma \Rightarrow \Delta$. Similarly, for $\Gamma^L \Rightarrow \Delta^L$.*

Then every provable sequent has a T -free proof.

²Here we use the fact that T is disjoint from both WEAKENING and EXCHANGE.

Proof. Consider any $\text{Seq}(\mathcal{L}, \Rightarrow)$ -provable $\Gamma \Rightarrow \Delta$. Without loss of generality, let us assume that Γ is nonempty. Then, by (1), it follows that there exists a rule R and a sequent $\Gamma^R \Rightarrow \Delta^R$ such that it is deducible from a set of sequents which includes $\Gamma \Rightarrow \Delta$ is one of the premises and $\Gamma^R \Rightarrow \Delta^R$ the conclusion, of some instance of R . Furthermore, since each of the premises is provable, it follows that $\Gamma^R \Rightarrow \Delta^R$ is provable as well. Consequently, by (2), it follows that $\Gamma \Rightarrow \Delta$ has a T -free proof. $\square$

Theorem 6.3.8. *Let $(\text{Seq}(\mathcal{L}, \Rightarrow), \vdash)$ be a normal sequent structure. Then, the following statements are equivalent.*

- (1) *Every provable sequent has a T -free proof.*
- (2) *The conditions of Theorem 6.3.7 are satisfied.*

Proof. We only show the proof of (1) $\implies$ (2) because the other implication is precisely Theorem 6.3.7.

Note that, since, by hypothesis $\Gamma \Rightarrow \Delta$ is provable, so are $A, \Gamma \Rightarrow \Delta$ (by **WL**) and $\Gamma \Rightarrow \Delta, A$ (by **WR**). We take $R = \text{WR}$ and $L = \text{WL}$. In other words, both $A, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, A$ are deducible from $\Gamma \Rightarrow \Delta$. Since $\Gamma \Rightarrow \Delta$ is provable, it follows that both $A, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, A$ are provable. This proves (1) of Theorem 6.3.7. The proof of (2) of Theorem 6.3.7 is immediate since every provable sequent has a T -free proof. Since $\Gamma \Rightarrow \Delta$ was chosen arbitrarily, we have thus shown (2). $\square$

Remark 6.3.9. The proof of Theorem 6.3.5 is an abstraction of the argument given in Theorem 6.2.13, and the proof of Theorem 6.3.7 is an abstraction of the argument given in Theorem 6.2.20. In the proof of Theorem 6.2.20, the rules R and L were both $\longrightarrow \text{L}$.

Remark 6.3.10. One may take the rule T to be CUT, or some restricted version of CUT as needed. In general, there is no restriction on how T is chosen, apart from the fact that it should be disjoint from both WEAKENING and EXCHANGE.

Remark 6.3.11. If a sequent structure has a witness function, then there may be more than one rule which it does not respect.

6.4 Abstract Sequent Structures

In the previous section, we have seen how to eliminate a rule T , provided we are dealing with normal sequent structures. Note that in the whole discussion we have not specified what the rule T might be; but the sequents were constructed based on some algebraic language. In this section, we remove this restriction.

Note that since we are not considering an algebraic language, the elements of the sequents do not arise from an algebra. Hence, the rules that can potentially be eliminated via the methods discussed in this section must be purely “structural” in nature.

In this section, we propose a generalisation of normal sequent structures, the latter connecting to the theory of logical structures. We prove a **RULE**-elimination theorem for a particular type of these structures and show that the converse of this theorem holds as well.

We start by analysing the proof of [Theorem 6.3.5](#) in the previous section. We observe the following.

- (a) As noted earlier, even though we used $\mathbb{N}$ in the above section, the only property of $\mathbb{N}$ used is that it is a well-ordered set. More specifically, we have relied on the *inductive* structure of $\mathbb{N}$.
- (b) Even though we have assumed the notion of proof/deduction as is standard in the literature on sequent calculus, the properties that we required are limited only to results like [Theorem 6.2.2](#) and [Theorem 6.2.3](#) (or some suitable generalisation of it).
- (c) The base case of [Theorem 6.3.5](#) shows that if the value of v on a given sequent is the least, then it has a T -free proof (where T is disjoint from both **WEAKENING** and **EXCHANGE**).

Definition 6.4.1 (ABSTRACT SEQUENT STRUCTURE). Let $\mathcal{L}$ be a set. An *abstract sequent structure* is a tuple $\mathbf{AbSS} := (\text{Seq}(\mathcal{L}, \Rightarrow), \vdash)$ where,

- (i) $\text{Seq}(\mathcal{L}, \Rightarrow)$ denotes the set of all $\mathcal{L}$ -sequents with respect to $\Rightarrow$ for some set $\mathcal{L}$.
- (ii) $\vdash \subseteq \mathcal{P}(\text{Seq}(\mathcal{L}, \Rightarrow)) \times \text{Seq}(\mathcal{L}, \Rightarrow)$

Definition 6.4.2 (AbSS SEQUENT). Let $\mathcal{L}$ be a set and let $\Rightarrow \subseteq \mathbf{O}(\mathcal{L}) \times \mathbf{O}(\mathcal{L})$, where $\mathbf{O}(\mathcal{L})$ denotes the class of all ordered lists of $\mathcal{L}$.³ Then $(\Gamma, \Delta) \in \Rightarrow$ will be said to be a $\mathcal{L}$ -*sequent with respect to* $\Rightarrow$. In this case, instead of writing $(\Gamma, \Delta) \in \Rightarrow$, we will write $\Gamma \Rightarrow \Delta$. The *set of all $\mathcal{L}$ -sequents with respect to* $\Rightarrow$ will henceforth be denoted by $\text{Seq}(\mathcal{L}, \Rightarrow)$.

Definition 6.4.3 (AbSS RULE). Let $\text{Seq}(\mathcal{L}, \Rightarrow)$ be the set of all $\mathcal{L}$ -sequents with respect to $\Rightarrow$. A set R is said to be a $\text{Seq}(\mathcal{L}, \Rightarrow)$ -*rule* if,

$$\emptyset \subsetneq R \subseteq \text{Seq}(\mathcal{L}, \Rightarrow)^I \times \text{Seq}(\mathcal{L}, \Rightarrow)$$

for some nonempty index set I . It is said to be a $\text{Seq}(\mathcal{L}, \Rightarrow)$ -*axiomatic rule* if $\emptyset \subsetneq R \subseteq \{\emptyset\} \times \text{Seq}(\mathcal{L}, \Rightarrow)$; otherwise it is called a *non-axiomatic rule*. An *instance* of a rule R is an element of R . Given $((\Gamma_j \Rightarrow \Delta_j)_{j \in J}, \Gamma \Rightarrow \Delta) \in R$, $\Gamma_i \Rightarrow \Delta_i$ is to be a *premise* of this instance of R for each $i \in \{1, \dots, n\}$ and $\Gamma \Rightarrow \Delta$ is said to be a *conclusion* of this instance of R . In this case, we will also sometimes say $\Gamma \Rightarrow \Delta$ to be *obtained by an application of* R .

Definition 6.4.4 (RULE-INDUCED AbSS). Let $\mathbf{AbSS} := (\text{Seq}(\mathcal{L}, \Rightarrow), \vdash)$ be an abstract sequent structure and $\mathcal{R} := \{R_i\}_{i \in I}$ be a family of **AbSS**-rules. Then, $\mathbf{AbSS} := (\text{Seq}(\mathcal{L}, \Rightarrow), \vdash)$ is said to be *induced/generated by* $\{R_i\}_{i \in I}$ if the following statements hold.

³More formally, $\mathbf{O}(\mathcal{L}) := \{\mathcal{L}^I : I \text{ is a set}\}$; hence it is a class.

- (i) $\emptyset \vdash \Gamma \Rightarrow \Delta$ if $(\emptyset, \Gamma \Rightarrow \Delta) \in R_i$ whenever R_i is an **AbSS**-axiomatic rule.
- (ii) $\emptyset \vdash \Gamma \Rightarrow \Delta$ implies that there exists R_i such that $(\emptyset, \Gamma \Rightarrow \Delta) \in R_i$ where R_i is a **AbSS**-axiomatic rule or $((\Gamma_j \Rightarrow \Delta_j)_{j \in J}, \Gamma \Rightarrow \Delta) \in R_i$ for some family of sequents $(\Gamma_j \Rightarrow \Delta_j)_{j \in J}$ such that $\emptyset \vdash_{\mathcal{R}} \Gamma_j \Rightarrow \Delta_j$ for all $j \in J$.
- (iii) For all non-axiomatic **AbSS**-rule R_i and $((\Gamma_j \Rightarrow \Delta_j)_{j \in J}, \Gamma \Rightarrow \Delta) \in R_i$, the following statement holds: if $\emptyset \vdash_{\mathcal{R}} \Gamma_j \Rightarrow \Delta_j$ for each $j \in J$ then $\emptyset \vdash_{\mathcal{R}} \Gamma \Rightarrow \Delta$.

In this case, we will write $\vdash = \vdash_{\mathcal{R}}$.

Remark 6.4.5. In informal terms, the first property asserts that in an **AbSS**, axiomatic inferences are provable; the second one asserts that if there is a proof of some sequent then either it is the conclusion of an axiomatic rule or is deduced by the application of some non-axiomatic rule, each of whose premise sequents are provable; the last property says that if each premise of an instance of a rule is provable then so is its conclusion.

Remark 6.4.6. It is important to note that [Definition 6.4.4](#) is motivated by the ‘root-based’ definition of proofs instead of the ‘leaf-based’ one that we have used till [Section 6.3](#) in the context of **PLK**, or for normal sequent structures (for details and differences between these two approaches, see [\[66, §1.6\]](#) and [\[97\]](#)). Nevertheless, there is an important difference: we are not giving an explicit definition of proof or deduction; instead, we are simply specifying certain properties that an abstract sequent structure must satisfy in order to be a rule-induced abstract sequent structure.

Definition 6.4.7 (RULE-INDUCED **AbSS**-ELIMINABLE RULE). Given an abstract sequent structure $\mathbf{AbSS} := (\text{Seq}(\mathcal{L}, \Rightarrow), \vdash)$ generated by a family $\mathcal{R} := \{R_i\}_{i \in I}$ of **AbSS**-rule(s), an **AbSS**-rule R_i is said to be **AbSS-eliminable** if $\emptyset \vdash \Gamma \Rightarrow \Delta$ implies that $\emptyset \vdash_{\mathcal{R} \setminus \{R_i\}} \Gamma \Rightarrow \Delta$, i.e., if $\emptyset \vdash_{\mathcal{R}} \Gamma \Rightarrow \Delta$ implies that $\emptyset \vdash_{\mathcal{R} \setminus \{R_i\}} \Gamma \Rightarrow \Delta$.

Definition 6.4.8. Let $\mathbf{AbSS} := (\text{Seq}(\mathcal{L}, \Rightarrow), \vdash)$ be an abstract sequent structure generated by a family $\mathcal{R} := \{R_i\}_{i \in I}$ of **AbSS**-rule(s). Let $(P, \leq)$ be a poset and $v : \text{Seq}(\mathcal{L}, \Rightarrow) \rightarrow P$. Then v is said to *respect* an instance $((\Gamma_j \Rightarrow \Delta_j)_{j \in J}, \Gamma \Rightarrow \Delta) \in R_i$ of the **AbSS**-rule R_i if $v(\Gamma_j \Rightarrow \Delta_j) \leq v(\Gamma \Rightarrow \Delta)$ for all $j \in J$. It is said to *strictly respect* $((\Gamma_j \Rightarrow \Delta_j)_{j \in J}, \Gamma \Rightarrow \Delta)$ if $v(\Gamma_j \Rightarrow \Delta_j) < v(\Gamma \Rightarrow \Delta)$ for all $j \in J$.

Remark 6.4.9. Even though we have formulated the notion of rule-induced abstract sequent structures using the language of sequents, it is not necessary. Nevertheless, we have chosen to do so mainly due to the continuity of the presentation.

Definition 6.4.10. A *relation space* is a pair $(P, \mathfrak{R})$ where P is a set and $\mathfrak{R} \subseteq P \times P$.

Theorem 6.4.11 (RULE-ELIMINATION THEOREM FOR **AbSS**s). *Let $\mathbf{AbSS} := (\text{Seq}(\mathcal{L}, \Rightarrow), \vdash_{\mathcal{R}})$ be an abstract sequent structure generated by a family $\mathcal{R} := \{R_i\}_{i \in I}$ of **AbSS**-rules and let $(P, \mathfrak{R})$ be a relation space. Let $v : \text{Seq}(\mathcal{L}, \Rightarrow) \rightarrow P$ and R_k be an **AbSS**-non-axiomatic rule. Suppose that the following conditions are satisfied.*

- (1) $R_i \cap R_k = \emptyset$ for all $R_i \in \mathcal{R} \setminus \{R_k\}$
- (2) For every $((\Gamma_t \Rightarrow \Delta_t)_{t \in T}, \Gamma \Rightarrow \Delta) \in R_k$, $(v(\Gamma_t \Rightarrow \Delta_t), v(\Gamma \Rightarrow \Delta)) \in \mathfrak{R}$ for all $t \in T$.
- (3) If $\emptyset \vdash_{\mathcal{R}} \Gamma \Rightarrow \Delta$ then either $(\emptyset, \Gamma \Rightarrow \Delta) \in R_i$ for some **AbSS**-axiomatic rule or there exists some **AbSS**-non-axiomatic rule R_i and $((\Gamma_j \Rightarrow \Delta_j)_{j \in J}, \Gamma \Rightarrow \Delta) \in R_i$ such that $\emptyset \vdash_{\mathcal{R}} \Gamma_j \Rightarrow \Delta_j$ for all $j \in J$ and $(v(\Gamma_j \Rightarrow \Delta_j), v(\Gamma \Rightarrow \Delta)) \notin \mathfrak{R}$ for some $j \in J$. (Note that any R_i satisfying the second condition should be disjoint from R_k because of the previous condition.)
- (4) If $((\Gamma_j \Rightarrow \Delta_j)_{j \in J}, \Gamma \Rightarrow \Delta) \in R_i$ for some **AbSS**-rule R_i such that $R_i \cap R_k = \emptyset$ with $(v(\Gamma_j \Rightarrow \Delta_j), v(\Gamma \Rightarrow \Delta)) \notin \mathfrak{R}$ for some $j \in J$, and $\emptyset \vdash_{\mathcal{R}} \Gamma_j \Rightarrow \Delta_j$ for all $j \in J$, then $\emptyset \vdash_{\mathcal{R} \setminus \{R_k\}} \Gamma_j \Rightarrow \Delta_j$ for all $j \in J$.

Then R_k is **AbSS**-eliminable.

Proof. Let $\emptyset \vdash_{\mathcal{R}} \Gamma \Rightarrow \Delta$. If $(\emptyset, \Gamma \Rightarrow \Delta) \in R_i$ for some **AbSS**-axiomatic rule, it follows that $\emptyset \vdash_{\mathcal{R} \setminus \{R_k\}} \Gamma \Rightarrow \Delta$. Otherwise, by (3), there exists a non-axiomatic R_i with $((\Gamma_j \Rightarrow \Delta_j)_{j \in J}, \Gamma \Rightarrow \Delta) \in R_i$ such that $R_i \cap R_k = \emptyset$, $\emptyset \vdash_{\mathcal{R}} \Gamma_j \Rightarrow \Delta_j$ for every $j \in J$ and $(v(\Gamma_j \Rightarrow \Delta_j), v(\Gamma \Rightarrow \Delta)) \notin \mathfrak{R}$ for some $j \in J$. Then, by (4), it follows that $\emptyset \vdash_{\mathcal{R} \setminus \{R_k\}} \Gamma_j \Rightarrow \Delta_j$ for all $j \in J$. Consequently, since $R_i \notin \mathcal{R} \setminus \{R_k\}$, it follows that, $\emptyset \vdash_{\mathcal{R} \setminus \{R_k\}} \Gamma \Rightarrow \Delta$. So, R_k is **AbSS**-eliminable. $\square$

Theorem 6.4.12. Let $\mathbf{AbSS} := (\text{Seq}(\mathcal{L}, \Rightarrow), \vdash_{\mathcal{R}})$ be an abstract sequent structure generated by a family $\mathcal{R} := \{R_i\}_{i \in I}$ of **AbSS**-rule(s). Let R_k be an **AbSS**-eliminable non-axiomatic rule such that it is disjoint from every rule in $\mathcal{R} \setminus \{R_k\}$. Then there exists a relation space $(P, \mathfrak{R})$ and a function $v : \text{Seq}(\mathcal{L}, \Rightarrow) \longrightarrow P$ such that,

- (1) For every $((\Gamma_t \Rightarrow \Delta_t)_{t \in T}, \Gamma \Rightarrow \Delta) \in R_k$, $(v(\Gamma_j \Rightarrow \Delta_j), v(\Gamma \Rightarrow \Delta)) \in \mathfrak{R}$ for all $t \in T$.
- (2) If $\emptyset \vdash_{\mathcal{R}} \Gamma \Rightarrow \Delta$ then either $(\emptyset, \Gamma \Rightarrow \Delta) \in R_i$ for some **AbSS**-axiomatic rule or $((\Gamma_j \Rightarrow \Delta_j)_{j \in J}, \Gamma \Rightarrow \Delta) \in R_i$ for some **AbSS**-rule R_i such that $\emptyset \vdash_{\mathcal{R}} \Gamma_j \Rightarrow \Delta_j$ for all $j \in J$ and $(v(\Gamma_j \Rightarrow \Delta_j), v(\Gamma \Rightarrow \Delta)) \notin \mathfrak{R}$ for some $j \in J$.
- (3) If $((\Gamma_j \Rightarrow \Delta_j)_{j \in J}, \Gamma \Rightarrow \Delta) \in R_i$ for some **AbSS**-rule R_i with $(v(\Gamma_j \Rightarrow \Delta_j), v(\Gamma \Rightarrow \Delta)) \notin \mathfrak{R}$ and $\emptyset \vdash_{\mathcal{R}} \Gamma_j \Rightarrow \Delta_j$ for all $j \in J$ then $\emptyset \vdash_{\mathcal{R} \setminus \{R_k\}} \Gamma_j \Rightarrow \Delta_j$ for all $j \in J$.

Proof. We define a relation space $(\text{Seq}(\mathcal{L}, \Rightarrow), \mathfrak{R})$ as follows,

$$(\Gamma \Rightarrow \Delta, \Pi \Rightarrow \Lambda) \in \mathfrak{R} \iff \begin{array}{l} \Gamma \Rightarrow \Delta \text{ is a premise of an instance of } R_k \text{ and} \\ \Pi \Rightarrow \Lambda \text{ is the conclusion of the same instance} \end{array}$$

The function v is the identity function.

- (1) is immediately satisfied by observing the relation we defined.

For (2), suppose $\emptyset \vdash_{\mathcal{R}} \Gamma \Rightarrow \Delta$. Since R_k is eliminable, $\emptyset \vdash_{\mathcal{R} \setminus \{R_k\}} \Gamma \Rightarrow \Delta$. Consequently, since **AbSS** is an abstract sequent structure, it follows that either there exists R_i such that $(\emptyset, \Gamma \Rightarrow \Delta) \in R_i$ where R_i is an **AbSS**-axiomatic rule (since R_k is non-axiomatic and hence the sets of axiomatic rules in $\mathcal{R}$ and $\mathcal{R} \setminus \{R_k\}$ are same) or $((\Gamma_j \Rightarrow \Delta_j)_{j \in J}, \Gamma \Rightarrow \Delta) \in R_i \in \mathcal{R} \setminus \{R_k\}$ for some family of sequents $(\Gamma_j \Rightarrow \Delta_j)_{j \in J}$ such that $\emptyset \vdash_{\mathcal{R} \setminus \{R_k\}} \Gamma_j \Rightarrow \Delta_j$ for all $j \in J$. In the former case, the conclusion is immediate. Otherwise, since $R_i \in \mathcal{R} \setminus \{R_k\}$ and it is disjoint from R_k , the latter conclusion follows.

For (3), note that if $(\Gamma_j \Rightarrow \Delta_j, \Gamma \Rightarrow \Delta) \notin \mathfrak{R}$ for some $j \in J$ with $((\Gamma_j \Rightarrow \Delta_j)_{j \in J}, \Gamma \Rightarrow \Delta) \in R_i$ for some **AbSS**-rule R_i , it follows that $R_i \neq R_k$. Hence, by hypothesis, $R_i \cap R_k = \emptyset$. So $R_i \in \mathcal{R} \setminus \{R_k\}$. Now, if $((\Gamma_j \Rightarrow \Delta_j)_{j \in J}, \Gamma \Rightarrow \Delta) \in R_i$ for some **AbSS**-rule $R_i \in \mathcal{R} \setminus \{R_k\}$ and if $\emptyset \vdash_{\mathcal{R}} \Gamma_j \Rightarrow \Delta_j$ for all $j \in J$ then since R_k is eliminable, we immediately have $\emptyset \vdash_{\mathcal{R} \setminus \{R_k\}} \Gamma_j \Rightarrow \Delta_j$ for all $j \in J$. This completes the proof. $\square$

Remark 6.4.13. The condition of mutual disjointness of the family of rules in the above theorems is due to the intuition that instances of two distinct rules should be *formally/schematically* distinguishable.

6.5 Future Works

- We have proved several RULE-elimination theorems. Even though CUT-elimination can be seen as a special case of these theorems, applying these theorems is difficult due mainly to the level of abstraction. Finding some more easily applicable criteria remains a project for the future.
- Even though our discussion centred only around the single-conclusion sequent structures, we think that a similar theory can be worked out for multiple conclusion sequent structures with suitable modifications. One of our long-term aims is to capture in a single theoretical framework most of the generalisations of sequent calculus and prove RULE-elimination theorems for them. And, it seems to us that the problem can be stated even more generally independent of the notion of ‘sequents’. More specifically, as we have hinted in [Remark 6.4.9](#), one can define something like ‘rule-induced logical structure’ and investigate similar questions; thus connecting directly to the theory of logical structures. Satisfactory answers to such questions will not only have important consequences for proof-theoretic studies but also for the model-theoretic investigations as well.
- The idea of using an *arbitrary* poset as the value set of a witness function can be useful, in particular, if one wants to carry out ‘induction-like’ arguments. We would like to work with the ideas introduced in [\[67\]](#) to see if it is possible to give an inductive version of the RULE-elimination theorem for abstract sequent structures where the value set is an arbitrary poset.

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