Distributed Adaptive Parameter Estimation and Control with Relaxed Excitation Conditions

A THESIS

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THESIS CERTIFICATE

This is to certify that the thesis titled **Distributed Adaptive Parameter Estimation and Control with Relaxed Excitation Conditions**, submitted by **Tushar Garg**, to the Indraprastha Institute of Information Technology (IIIT), Delhi, for the award of the degree of **Doctor of Philosophy**, is a bonafide record of the research work done by him under our supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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ABSTRACT

Over the years, distributed adaptive parameter estimation/control for multi-agent systems (MASs) has gained a lot of attention in the form of dynamical systems, where the concept of cooperative persistence of excitation (C-PE) is proposed for accurate estimation of unknown parameters. The C-PE condition relaxes the traditional persistence of excitation (PE) condition in the sense that it can be satisfied by incorporating multiple system signals with each of them not necessarily being PE. However, the C-PE condition is still restrictive due to its persistent nature, which is difficult to satisfy in many practical control applications. The main objective of this dissertation is to relax the stringent C-PE condition requirement while still designing efficient distributed adaptive systems by utilizing different network topologies. The research is structured into three key sub-problems - $1 >$ Developing a relaxed excitation condition based distributed adaptive parameter estimation (DAPE) algorithm considering undirected connected graph network topology along with control application, $2 >$ Extending the framework to strongly connected directed graph network while also analyzing the effect of communication delay, $3 >$ Proposing a generalized relaxed excitation condition for DAPE over weakly connected digraph topology along with extremum-seeking control application. We have conceptualized a new condition, called cooperative initial excitation (C-IE), which is milder than the classical C-PE condition. We have proved that the C-IE condition is sufficient to ensure convergence for the proposed distributed adaptive algorithms using a rigorous Lyapunov analysis. Simulation results validate the efficacy of the proposed algorithms.

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ABBREVIATIONS

C-IE: Cooperative/Collective Intital Excitation

C-PE: Cooperative/Collective Persistent of Excitation

PE: Persistent of Excitation

IE: Intital Excitation

gC-IE: Generalized Cooperative/Collective Initial Excitation

gC-PE: Generalized Cooperative/Collective Persistent of Excitation

DAPE : Distributed Adaptive Parameter Estimation

EL: Euler-Lagrange

GES: Global Exponential Stability

ISS: Input-to-State Stability

MAS: Multi-Agent System

LRE: Linear Regression Equation

UGES: Uniform Global Exponential Stability

UGS: Uniform Global Stability

MRAC: Model Reference Adaptive Control

NOTATIONS

$\mathbb{R}$	Real Space
$\mathbb{R}^n$	Real Vector Space of dimension n
$\mathbb{R}^{n\times n}$	Real Matrix Space of dimension $n \times n$
I	Identity Matrix with appropriate dimension
$A > 0(< 0)$	Positive (Negative) Definite Matrix
$\lambda_{min}(A)$	Minimum eigen-value of the Matrix A
$\lambda_{max}(A)$	Maximum eigen-value of the Matrix A
$i = 1(1)n$	Denotes $i = 1, 1+1, 1+2, \ldots, n$
L	Laplacian Matrix with chosen network graph topologies
$\otimes$	Kronecker product

Chapter 1

INTRODUCTION

1.1 Single-Agent Adaptive Systems

The tasks of control, identification or prediction of the dynamics of an unknown nonlinear system is usually accomplished assuming that there exists an approximation to the true dynamics with a fixed vector of parameters that globally fits the dynamics. A typical scenario is to assume that the dynamics is described by an ordinary differential (or difference) equation with unknown parameters, which are then estimated by designing an on-line parameter estimator. In its simplest formulation, it is assumed that these parameters enter linearly in the dynamic model leading to a relationship of the form;

$$y(t) = \phi^T(x(t))\theta, \quad \forall t \geq t_0 \tag{1.1}$$

where $y(t) \in \mathbb{R}$ represent the measurable output, $x(t) \in \mathbb{R}^n$ represent the measurable input, and $\phi(x(t)) : \mathbb{R}^n \to \mathbb{R}^p$ is known basis function/feature vector (also known as "regressor" in adaptive control literature). Moreover, $\theta \in \mathbb{R}^p$ is an unknown constant parameters vector. We call the equation (1.1) in the sequel as linear regression equation (LRE). LREs associated with many control problems, including system identification (Ljung, 1987; Papusha *et al.*, 2014), adaptive control (Ioannou and Sun, 1996; Chowdhary *et al.*, 2013), (Narendra and Annaswamy, 2012), filtering and prediction (Goodwin and Sin, 2014), reinforcement learning (Lewis *et al.*, 2012) and sparse regression analysis (Brunton *et al.*, 2016), have been reported in literature.

Since the focus of the current discussion is primarily on an online parameter estimation/tuning, hence in the subsequent formulation, we will write $\phi(t)$ instead of $\phi(x(t))$.

1.1.1 Objective

Here, the objective is to develop an online differential parameter update law for $\hat{\theta}(t) \in \mathbb{R}^p$ using the real-time data of $\{\phi(t), y(t)\}$, such that

$$\|\hat{\theta}(t) - \theta\| \to 0 \quad as \quad t \to \infty, \quad \forall t \geq t_0 \tag{1.2}$$

1.1.2 Classical Parameter Tuner/Estimator

This section introduces the fundamentals of classical first-order parameter tuner, which builds on an online gradient-descent algorithm. Consider the following cost function

$$J(\hat{\theta}(t), t) = \frac{1}{2}\epsilon^2(t),\ \epsilon(t) \triangleq \big(y(t) - \underbrace{\phi^T(t)\hat{\theta}(t)}_{\hat{y}(t)}\big), \quad \forall t \geq t_0 \tag{1.3}$$

which penalizes a quadratic function of prediction error $\epsilon(t) \in \mathbb{R}$, capturing the mismatch between the actual output $y(t)$ and the predicted output $\hat{y}(t)$. The parameter tuner is designed using an online gradient-descent algorithm based on the above cost function:

$$\dot{\hat{\theta}}(t) = -\Gamma \nabla_{\hat{\theta}(t)} J(\hat{\theta}(t), t), \quad \forall t \geq t_0 \tag{1.4}$$

where $\Gamma \in \mathbb{R}^{p \times p}$ is a positive-definite (PD) gain matrix, i.e., $\Gamma > 0$. Define parameter estimation error as

$$\Delta\theta(t) \triangleq \hat{\theta}(t) - \theta. \tag{1.5}$$

Substituting the analytical expression of the gradient, the parameter estimation error dynamics become

$$\Delta\dot{\theta}(t) = -\Gamma\phi(t)\phi^T(t)\Delta\theta(t), \quad \forall t \geq t_0. \tag{1.6}$$

The above error dynamics ensure global exponential stability (GES) (Ioannou and Sun, 1996) of the zero equilibrium, iff the regressor signal $\phi(t)$ holds persistence of excitation (PE) condition (Narendra and Annaswamy, 2012). The concept of PE is explained in the next section.

1.1.3 Persistence of Excitation

The persistence of excitation (PE) condition of a bounded vector-valued function $\phi(t)$ is defined as follows (Narendra and Annaswamy, 2012; Tao, 2003):

Definition 1. *A bounded vector-valued function $\phi(t) \in \mathbb{R}^p$, where $t \in [t_0, \infty)$, $t_0 \geq 0$, is PE, if $\exists\, T > 0$ and $\Upsilon > 0$, such that the following inequality holds*

$$\int_{t}^{t+T} \phi(r)\phi^T(r)dr \geq \Upsilon I_p, \quad \forall t \geq t_0. \tag{1.7}$$

$\square$

For example, $\phi(t) = [1 \ sin(t)]^T$ is persistently exciting (PE) (Tao, 2003) but $\phi(t) = [1 \ e^{-t}]^T$ is not PE. The PE condition is restrictive and difficult to verify online as it relies on the future behaviour of the dynamical systems.

1.1.4 Motivation and Relevant Literature

Classical schemes in system identification and adaptive control often rely on stringent persistence of excitation to guarantee parameter convergence (Narendra and Annaswamy, 2012; Ioannou and Sun, 1996), which may be difficult to achieve with a single-agent and a single-input. The PE condition is related to the richness of information content regarding the unknown parameters. Conventional online estimation techniques need the excitation/richness to persist for the entire time-span to ensure accurate estimation at steady-state. The classical PE condition is stringent in nature since it cannot be verified online due to its reliance on the future behaviour of the signal of interest. Since a PE signal contains infinite energy, imposing PE through exogenous input may lead to wastage of energy in many applications (Chowdhary *et al.*, 2014).

In contrast to PE based results, some recent works (Parikh *et al.*, 2019; Adetola and Guay, 2008; Cho *et al.*, 2017; Roy *et al.*, 2017*b*,*a*; Ortega *et al.*, 2021; Roy *et al.*, 2016) have proposed a relaxed condition; called initial/finite excitation (IE/FE), which is shown to be sufficient for parameter convergence in the developed composite adaptive control/identification for single-agent systems. Unlike the PE, the IE/FE condition is online-verifiable and IE is milder than PE condition since it requires the excitation/richness of the signal only in the initial time-window of finite length. Moreover, as compared to PE, it does not require infinite energy exogenous inputs. The concept of IE is discussed in the subsequent section.

1.1.5 Intital Excitation

Definition 2. *A bounded vector-valued function $\phi(t) \in \mathbb{R}^p$, where $t \in [t_0, \infty)$, $t_0 \geq 0$, is IE, if $\exists\, T > 0$ and $\Upsilon > 0$, such that the following inequality holds*

$$\int_{t_0}^{t_0+T} \phi(r)\phi^T(r)dr \geq \Upsilon I_p. \tag{1.8}$$

□

From IE definition in (1.8), it can be inferred that the excitation/richness is entailing only in the initial time window, unlike PE definition (1.7), where the excitation/richness entails throughout the entire time span. In (Basu Roy *et al.*, 2018; Roy *et al.*, 2016; Jha *et al.*, 2019; Roy *et al.*, 2017*b*; Roy and Bhasin, 2019), it is established that the IE condition is less restrictive than PE, since it is online verifiable and does not require infinite energy exogenous inputs, unlike PE.

1.2 Multi-Agent Adaptive Systems

In parallel to these single-agent results, distributed cooperative adaptive parameter estimation/control (Olfati-Saber *et al.*, 2007; Papusha *et al.*, 2014; Chen *et al.*, 2013; Stegagno and Yuan, 2019; Wensing and Slotine, 2018; Yuan *et al.*, 2021; Chung *et al.*, 2019; Phan *et al.*, 2020; Deng *et al.*, 2020; Li *et al.*, 2019; Jun-Min *et al.*, 2013) has gained a lot of attention, where it is shown that distributed identification algorithms outperform conventional identification algorithms in terms of transient response and robustness. Inspired from consensus theory, the fundamental idea of distributed cooperative adaptive identification is that multiple uncertain agents simultaneously perform estimation of a common parameter vector while sharing information with each other in a distributed fashion through a communication graph topology.

1.2.1 Multi-Agent Adaptive System (MAS) Architecture

The communication topology for a MAS architecture, comprising n number of agents, is represented by the graph G, where each vertex is treated as an agent and edge $E = (i, j) \in V$[1] , denote the communication link between i^{th}agent and j^{th}agent. The output of each agent $y_i(t) \in \mathbb{R}$ is defined subsequently:

$$y_i(t) = \phi_i^T(t)\theta, \quad \forall t \geq t_0, \quad \forall i = 1(1)n \tag{1.9}$$

where $\theta \in \mathbb{R}^p$ is a vector of unknown constant parameters and $\phi_i(t) \in \mathbb{R}^p$ is a known, continuous, and uniformly bounded basis function/feature vectors (also known as "regressor" in adaptive control literature).

1.2.2 Objective

Here, the objective is to develop a distributed adaptive parameter estimation algorithm, using the online measurements of input $\big(\phi_i(t)\big)$, output $\big(y_i(t)\big)$ of the model (1.9) while collaborating (sharing instantaneous information) with the neighboring agents, such that[2]

$$\|\hat{\theta}_i(t) - \theta\| \to 0 \quad as \quad t \to \infty, \quad \forall i = 1(1)n \tag{1.10}$$

1.2.3 Classical Distributed Parameter Tuner/Estimator

This section introduces the fundamentals of classical distributed first-order parameter tuner[3], which builds on an online gradient-descent algorithm. Consider the following

[1]Here, V denotes the finite set of n vertices in the graph G which can be defined as $V = \{v_1, \ldots, v_n\}$.

[2]Here, the each agent i in the network has access the following local data: $\phi_i(t)$, $y_i(t)$, $\hat{\theta}_i(t)$ at time t. The data exchanged between the agents is current parameter estimates i.e., $\hat{\theta}_j(t)$ (Each agent j shares its current parameter estimate with its neighbors to facilitate consensus or agreement on the estimated parameter).

[3]For more clarity, we are highlighting the fact that the classical distributed first-order parameter tuner is already known in the existing literature Chen *et al.* (2013); Papusha *et al.* (2014).

cost function for i^{th} agent:

$$J(\hat{\theta}_i(t),t) = \frac{1}{2}\big(\hat{y}_i(t) - y_i(t)\big)^2 + \sum_{j\in N_i}\frac{1}{2}a_{ij}\|\hat{\theta}_j(t) - \hat{\theta}_i(t)\|^2 \tag{1.11}$$

where $\hat{y}_i(t) \triangleq \phi_i^T(t)\hat{\theta}_i(t)$ and N_i is defined as the set of neighbors for the i^{th} agent which is defined as; $N_i = \big\{j \mid (i,j) \in E\big\}$.

The distributed parameter tuner for i^{th} agent is designed using an online gradient-descent algorithm based on the above cost function (1.11):

$$\begin{aligned}\dot{\hat{\theta}}_i(t) &= -\,\Gamma\nabla J_{\hat{\theta}_i(t)}, \quad \forall t \geq t_0 \\ &= -\,\underbrace{\Gamma\phi_i(t)\big(\hat{y}_i(t) - y_i(t)\big)}_{\textit{Local Information}} + \underbrace{\Gamma\sum_{j\in N_i} a_{ij}\big(\hat{\theta}_j(t) - \hat{\theta}_i(t)\big)}_{\textit{Neighboring Information}}.\end{aligned} \tag{1.12}$$

1.2.4 Compact representation of Parameter Tuner/Estimator

Consider the compact representation of distributed parameter tuner (1.12) for all n number of agents:

$$\underbrace{\Delta\dot{\theta}(t)}_{\mathbb{R}^{np\times 1}} = -\Gamma_\theta\big(L\otimes I_p + \Phi_P(t)\big)\Delta\theta(t), \quad \forall t \geq t_0 \tag{1.13}$$

where $L \in \mathbb{R}^{n\times n}$ denotes the Laplacian matrix corresponding to the assumed network communication topology, $\Gamma_\theta = \Gamma \otimes I_n \in \mathbb{R}^{np\times np}$, and

$$\Phi_P(t) = \text{diag}\big\{\phi_1(t)\phi_1^T(t),\ldots,\phi_n(t)\phi_n^T(t)\big\} \in \mathbb{R}^{np\times np}.$$

The above parameter estimation error dynamics (1.13) ensure global exponential stability (GES) of the zero equilibrium, if the set of regressor signals $\phi_i(t)s$ holds stringent cooperative/collective persistence of excitation (C-PE) condition (Papusha *et al.*, 2014; Wensing and Slotine, 2018; Stegagno and Yuan, 2019; Chen *et al.*, 2013). Whereas, the concept of C-PE is explained in the next section.

1.2.5 Cooperative/Collective Persistence of Excitation

The extended version of classical PE condition in multi-agent system (MAS) architecture named as cooperative persistent of excitation (C-PE) is defined as follows (Papusha *et al.*, 2014; Chen *et al.*, 2013; Wensing and Slotine, 2018; Stegagno and Yuan, 2019).

Definition 3. *A group of bounded vector-valued functions $\phi_i(t) \in \mathbb{R}^p$, where $t \in [t_0, \infty)$, $t_0 \geq 0$, $\forall i = 1(1)n$, are C-PE, if $\exists\, T > 0$ and $\Upsilon > 0$, such that the following inequality holds*

$$\int_{t}^{t+T} \sum_{i=1}^{n} \phi_i(r)\phi_i^T(r)dr \geq \Upsilon I_p, \quad \forall t \geq t_0. \tag{1.14}$$

□

C-PE condition implies that each regressor signals $\phi_i(t)$ need not be individually PE, rather the set of signals can collectively satisfy the PE condition. Since in (1.14) the excitation/richness is needed for the entire time-span, hence the C-PE condition is restrictive in nature.

1.3 Objective of the Thesis

The main objective of this thesis is to address the problem of the stringent C-PE condition requirement in distributed cooperative adaptive parameter estimation/control for MAS problems. This thesis focuses on developing relaxed excitation conditions for distributed adaptive parameter estimation/control problems while utilizing different communication topologies.

1.4 Organization of the Thesis

The thesis is organized into seven chapters. Each chapter is composed in a self-consistent manner without requiring heavy cross-referencing between the chapters for the convenience of readability. The summary of the work presented in each chapter is briefly outlined as follows.

Chapter 1: This is an introductory chapter that elaborates on the motivation of this research, problem orientation, the pertaining gaps in the literature, and an outline of the dissertation.

Chapter 2 (Part-1): In this part of the chapter, we propose a consensus based novel PI-like parameter estimator for collaborative system identification. Conventional online parameter estimation algorithms, which are used for system identification, require a restrictive condition of PE for the estimates to converge to the true parameters. Some recent works have shown that collaborative system identification using multiple agents can relax the PE condition to a milder condition of C-PE for parameter convergence. The C-PE condition implies that the PE condition is cooperatively satisfied by all the agents through sharing information between neighbors using a connected graph architecture, where each agent is not required to satisfy the PE condition separately. The proposed work designs a novel collaborative parameter estimator dynamics, which with the help of integral-like components ensure parameter convergence under a further slackened condition; coined as cooperative/collective initial excitation (C-IE). The C-IE condition is an extension of the concept of IE, which was recently proposed in the context of parameter estimation in adaptive control. It has been already established that IE condition is significantly less restrictive than PE. The current work generalizes the concept of IE in multi-agent settings, where information sharing through connected graph guarantees consensus parameter convergence under the C-IE condition. It can be argued that the C-IE condition is milder than all of the other above mentioned conditions of PE, C-PE, and IE.

Chapter 2 (Part-2): In this part of the chapter, the proposed work designs a combined cooperative adaptive cruise control (CACC) architecture for an uncertain homogeneous vehicle platoon. The term "combined" is borrowed from combined model reference adaptive control (MRAC) literature, which is a combination of direct and indirect MRAC (Narendra and Annaswamy, 2012). The combined CACC architecture is composed of a distributed parameter estimator of the uncertain vehicle dynamics parameters and a MRAC control law with a differential control parameter update routine. The control parameter estimator uses information from the vehicle dynamics parameter estimator making the design analogous to combined MRAC. The distributed parameter estimator of the vehicle dynamics is designed based on a two-layer filtering mechanism (Jha *et al.*, 2019) and a consensus-based component using information from immediate preceding and following vehicles' instantaneous estimation. This distributed estimator can ensure exponentially fast parameter convergence using the newly defined condition of C-IE and thereby relaxes the need for excitation (information content regarding the unknown parameters) to persist for all time. The C-IE condition implies that the IE condition is satisfied by all the agents cooperatively instead of individually. So the information content is distributed among all the vehicles' regressors in the initial time-window, which is strategically captured in the distributed estimator dynamics leading to parameter convergence. Further, the designed MRAC law along with the distributed estimator ensures asymptotic convergence of the vehicle platoon to a string stable reference platoon, thus maintaining smooth and safe operation.

Chapter 3 (Part-1): In this part of the chapter, the proposed work extends the concept of IE/FE in the context of a multi-agent setting while invoking the notion of C-IE for distributed estimation. The work builds on the formulation of the papers (Papusha *et al.*, 2014; Chen *et al.*, 2013; Stegagno and Yuan, 2019; Wensing and Slotine, 2018) which consider a network of multi-agent architecture for online parameter estimation. The proposed work develops a distributed consensus-based switched parameter estimator where strategic multiple switching is incorporated to reflect the effect of C-IE condition ensuring parameter convergence. It is analytically proved that the estimation error dynamics shows global exponential stability (GES) under the C-IE condition, which is in contrast to (Papusha *et al.*, 2014; Chen *et al.*, 2013; Stegagno and Yuan, 2019; Wensing and Slotine, 2018), where stringent C-PE condition is required to obtain a similar stability result. The concept of cooperative initial/finite excitation is introduced in a few recent works (Yuan *et al.*, 2021, 2018; Rezaei and Stefanovic, 2020; Poveda *et al.*, 2019; Garg and Roy, 2020*a*,*c*), however, all of them have been considered a bidirectional communication among neighboring agents, i.e., an undirected graph topology is utilized. Unlike these results, the proposed work allows unidirectional communication among agents as long as the directed graph is balanced in nature. Further, the formulation is molded as an online optimization problem. A novel convex cost function is conceptualized in such a way that the proposed distributed estimator acts as a distributed gradient-descent of the cost. Moreover, the cost function is proved to be strongly convex under the C-IE condition thereby making the estimator dynamics a unique global minimizer of the cost. Simulation results reflect the effectiveness of the proposed algorithm in contrast to traditional C-PE based estimators.

Chapter 3 (Part-2): In this part of the chapter, we propose a distributed composite adaptive synchronization algorithm for multiple uncertain Euler-Lagrange (EL) systems, where parameter convergence is achieved under a relaxed mathematical condition as compared to the state-of-the-art. Classical adaptive controllers require an analytical condition, called PE, to ensure parameter convergence, which results in better transient performance and robustness to disturbance. The PE condition is extended to the C-PE condition for distributed adaptive controllers with cooperative estimation strategies. The PE and C-PE conditions are restrictive in nature since these conditions are not satisfied in most practical applications. Recent literature in adaptive control has relaxed the PE condition to IE, which is shown to be sufficient for parameter convergence. The IE condition is argued to be significantly milder than PE and can be satisfied in many practical settings. The proposed result further extends the IE condition to the C-IE condition in distributed adaptive control architecture in the context of synchronizing multiple EL systems. It is established that the C-IE condition is milder than PE, IE, and C-PE conditions. A two-tier filter-based estimation algorithm with strategic switching ensures parameter convergence under the C-IE condition and thereby provides exponential convergence of tracking and parameter estimation error to zero. Simulation results validate the efficacy of the proposed algorithm as compared to conventional distributed adaptive controllers in terms of superior tracking and estimation performance.

Chapter 4: In this chapter, a novel distributed adaptive parameter estimator is proposed for a MAS architecture having a strongly connected digraph based communication topology in the presence of inter-agent communication delay. The proposed algorithm exhibits the following properties; (1) asymptotic consensus of parameter estimates is ensured without any restriction on the regressor or feature vectors and (2) parameter convergence is achieved under the uniform C-IE condition. Here, the notion of uniform C-IE is defined for the regressor concerning the agent dynamics, where each agent is modeled as a single-integrator. Unlike previous results on C-IE, a novel weighting function based integrator is introduced here. The designed integrator dynamics does not have internal instability as well as online rank-checking based multiple switching requirements, which were the major concerns in open-loop and closed-loop filter architectures of C-IE based designs (Garg and Roy, 2019*a*, 2020*c*,*b*; Goel *et al.*, 2022; Garg and Basu Roy).

Moreover, the proposed algorithm utilizes a more generalized graph topology of strongly connected digraph, unlike the previous C-IE based frameworks using undirected graph (Garg and Roy, 2019*a*) and strongly connected and balanced digraph (without communication delay) (Garg and Roy, 2020*c*). Under the condition of uniform C-IE, rigorous stability analysis with a suitable choice of Lyapunov-Krasovskii (LK) functional candidate is performed, which ensures uniform global stability (UGS) and asymptotic convergence in the presence of communication delay and uniform global exponential stability (UGES) in case of a delay-free setting. Simulation results further validate the efficacy of the proposed algorithm. As far as the authors are aware, this is the first work on a relaxed excitation-based distributed adaptive estimator over a strongly connected digraph, providing stability guarantees in the presence of communication delay.

Chapter 5: In this chapter, a novel distributed adaptive parameter estimation (DAPE) algorithm is proposed for multi-agent system (MAS) over weakly connected digraph network, where parameter convergence is ensured under a newly coined relaxed excitation condition, called generalized cooperative initial excitation (gC-IE). This is in contrast to the past literature, where such DAPE algorithms demand C-PE and generalized cooperative persistent of excitation (gC-PE) for strongly connected digraph, and weakly connected digraph networks, respectively, for parameter convergence. The gC-PE and C-PE conditions are restrictive in the sense that they require the richness/excitation of information over the entire time-span of the signal/data, unlike the gC-IE condition where excitation is needed only in the initial time-span. The newly coined gC-IE condition is an extension of the C-IE condition. While the C-IE condition is applies to a strongly connected digraph, the newly proposed gC-IE condition extends the concept to a weakly connected digraph. The proposed algorithm utilizes a novel set of weighted integrator dynamics, which omits the requirement of computationally involved multiples switching mechanisms in past literature, while still ensuring parameter convergence. The proposed algorithm provides global exponential stability (GES) of the origin of the parameter estimation error dynamics under the gC-IE condition. Furthermore, robustness to unmodeled disturbance is also established in the form of input-to-state stability (ISS).

Chapter 6: In this chapter, a novel distributed adaptive extremum seeking control (DAdESC) algorithm for a MAS architecture over a weakly connected digraph network is proposed, where parameter convergence is ensured under a newly coined relaxed excitation condition, called gC-IE. Here, a zeroth-order optimization framework is used, where each agent can only query the numerical value of its cost function at the current coordinate, and it is assumed that only the parameter estimates information is shared among the agents, not the individual cost. Parameter estimation plays a decisive role in the stability and convergence properties of the DAdESC algorithm and it is also well established in the existing literature that to ensure parameter convergence in such context a restrictive gC-PE condition is required. Here, we eliminate the need for a restrictive gC-PE condition by utilizing a novel set of weighted integral filter dynamics, while ensuring sufficient richness using a milder condition, called gC-IE. A detailed Lyapunov function based analysis is performed to establish the closed-loop stability and convergence in the form of exponential stability. Moreover, to validate the robustness guarantees in the presence of unmodeled bounded disturbance, another Lyapunov analysis is performed to establish the closed-loop stability and convergence in the form of ISS.

Chapter 7: This chapter concludes the thesis with a concise summary of primary contributions and a brief discussion about future research work.

Chapter 2

Collaborative System Identification via Consensus based novel PI-like Parameter Estimator along with Control Application

Part I

Collaborative System Identification via Consensus based novel PI-like Parameter Estimator

2.1 Introduction

In this chapter, we propose a consensus based novel PI-like parameter estimator for collaborative system identification. Conventional online parameter estimation algorithms, which are used for system identification, require a restrictive PE condition for the estimates to converge to the true parameters. Some recent works have shown that collaborative system identification using multiple agents can relax the PE condition to a milder condition of C-PE for parameter convergence. The C-PE condition implies that the PE condition is cooperatively satisfied by all the agents through sharing information between neighbors using a connected graph architecture, where each agent is not required to satisfy the PE condition separately. The proposed work designs a novel collaborative parameter estimator dynamics, which with the help of integral-like components ensure parameter convergence under a further slackened condition; coined as cooperative initial excitation (C-IE). The C-IE condition is an extension of the concept of IE, which was recently proposed in the context of parameter estimation in adaptive control. It has been already established that IE condition is significantly less restrictive than PE. The current work generalizes the concept of IE in multi-agent settings, where information sharing through connected graphs guarantees consensus parameter convergence under the C-IE condition. It can be argued that the C-IE condition is milder than all of the other above mentioned conditions of PE, C-PE, and IE. Simulation results further validate the efficacy of the proposed estimation algorithm.

2.2 Preliminaries

2.2.1 Preliminaries on Algebraic Graph Theory

The graph theoretic notations and terminologies, which are typically referred from (Codsil and Royle, 2001), are briefly described subsequently.

An undirected graph $G = (V, E)$ is a finite set of n vertices $V = \{v_1, \ldots, v_n\}$ together with a set of m edges $E = \{e_1, \ldots, e_m\}$. An edge e_p is defined by unordered pair of vertices $\{v_i, v_j\}$ belongs to the set of vertices V. The adjacency matrix A of the undirected graph G is defined as:

$$A = a_{ij} = \begin{cases} +1, & \text{if } \{v_i, v_j\} \in E, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

It can be inferred that the adjacency matrix A is symmetric for undirected graph. The degree of the vertex v_i, denoted as, deg(v_i), is the number of neighbors $|N_i|$ available to that vertex. where N_i is defined as the neighborhood of the vertex v_i defined as: $N_i = \Big\{ j \mid \{v_i, v_j\} \in E \Big\}$.

The degree matrix is a diagonal matrix $D \in \mathbb{R}^{n \times n}$, which is defined as the following

$$D = \begin{bmatrix} deg(v_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & deg(v_n) \end{bmatrix}.$$

The Laplacian matrix $L = L^T \in \mathbb{R}^{n \times n}$, which is defined as

$$L \triangleq D - A \tag{2.2}$$

is a positive semi-definite matrix for the connected graph (a graph is called connected if given any two vertices v_i, v_j, there is a path from v_i to v_j). For connected graph the Laplacian matrix L has the property that all its eigenvalues are positive except for the smallest one, which is zero. For the zero eigenvalue, corresponding eigenvector is $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$. In particular, $L\mathbf{1} = 0$ and $\mathbf{1}^T L = 0$. Since in this part of chapter, we are considering the communication topology for n number of agents as undirected graph. Hence the discussion around equations (2.1)-(2.2) is valid only for undirected graph .

2.2.2 Definitions on Signal Excitation conditions

This sub-section introduces signal excitation definition, which are used for commenting on the convergence properties of the subsequent proposed estimator dynamics.

In contrast to the C-PE condition, a new slackened condition called cooperative/collective initial excitation (C-IE) is introduced in this work and characterized as follows .

Definition 4. *A group of bounded vector-valued functions $\phi_i(t) \in \mathbb{R}^p$, where $t \in [t_0, \infty)$, $t_0 \geq 0$, $\forall i = 1(1)n$, are **C-IE**, if $\exists\, T > 0$ and $\Upsilon > 0$ such that the following inequality holds*

$$\int_{t_0}^{t_0+T} \sum_{i=1}^{n} \phi_i(r)\phi_i^T(r)dr \geq \Upsilon I_p. \tag{2.3}$$

□

Remark 1. *Unlike the C-PE condition* (1.14) *where excitation has to persist for all time, the C-IE condition* (2.3) *demands excitation only in the transient period (initial time-window of finite-length), making it a practically viable condition. Hence, from the above discussion, it can be argued that the C-IE condition is milder than all the PE, C-PE, and IE conditions .*

For illustration - it can be shown that neither $\phi_1(t) = [sin(t), 0]^T$ nor $\phi_2(t) = [0, cos(t)]^T$ satisfy the PE condition. However the matrix $[\phi_1, \phi_2]$ is PE i.e., $\phi_1(t)$, $\phi_2(t)$ cooperatively satisfy the PE condition (C-PE). Further it can be shown that neither $\phi_1(t) = [exp(-t), 0]^T$ nor $\phi_2(t) = [0, exp(-t)cos(t)]^T$ satisfy the IE condition. However the matrix $[\phi_1, \phi_2]$ is IE i.e., $\phi_1(t)$, $\phi_2(t)$ cooperatively satisfy the IE condition (C-IE). □

2.3 Problem Formulation

2.3.1 Model Description

The interaction topology of network for n number of agents is represented by the graph $G = (V, E)$ where, each vertex v_i is treated as a single agent and edge $e_p = \{v_i, v_j\} \in V$ is an available bidirectional communication link between agent i and j, which is based on connected graph architecture as depicted in Figure 2.1 (below). The output of each agent $y_i(t) \in \mathbb{R}$, $\forall i = 1(1)n$ is defined as a linear combination of a common set of unknown parameters as follows:

$$y_i(t) = \theta^T \phi_i(t), \quad \forall t \geq t_0, \quad \forall i = 1(1)n, \tag{2.4}$$

where $\theta \in \mathbb{R}^p$ is a constant unknown parameter vector and, $\phi_i(t) \in \mathbb{R}^p$ is a known regressor signal.

Assumption 1. *Graph G, which represents the communication topology between all the agents, is a connected graph.* □

Remark 2. *The above posed problem with model* (2.4) *or equivalent models are used in adaptive control, composite adaptive control, distributed adaptive control literature (see (Schwager* et al.*, 2009; Abdul Razak* et al.*, 2018; Papusha* et al.*, 2014; Chen* et al.*, 2013; Wensing and Slotine, 2018; Stegagno and Yuan, 2019; Yuan* et al.*, 2021; Basu Roy* et al.*, 2018; Chowdhary* et al.*, 2013) for further details). While this part of the chapter is entirely focused on parameter estimation, its application in distributed control problems, such as distributed model reference adaptive control (MRAC) (Peng* et al.*, 2013; Yuan* et al.*, 2019, 2021), distributed adaptive coverage control (Schwager* et al.*, 2008*a*; Li* et al.*, 2012), distributed adaptive extremum seeking control (Poveda and Quijano, 2013; Poveda* et al.*, 2019; Guay* et al.*, 2015, 2018), etc, are kept as promising future directions.* □

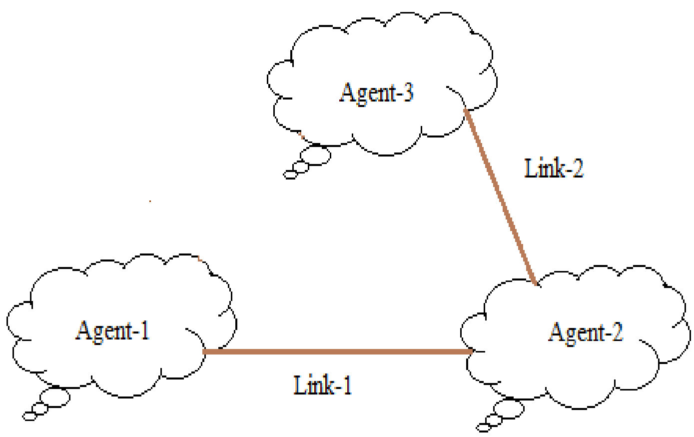

Figure 2.1: Connected graph based communication topology for $n = 3$ agents and $m = 2$ links.

2.3.2 Estimation Objective

Here, the objective is to design a consensus based PI-like parameter estimation algorithm, using the online measurements of input $\big(\phi_i(t)\big)$, output $\big(y_i(t)\big)$ of the model (2.4) while collaborating (sharing instantaneous information) with the neighbouring agents, such that

$$\|\hat{\theta}_i(t) - \theta\| \rightarrow 0 \quad as \quad t \rightarrow \infty, \quad \forall i = 1(1)n, \tag{2.5}$$

without requiring the restrictive C-PE condition.

Here, $\hat{\theta}_i(t) \in \mathbb{R}^p$ denotes the estimate of unknown parameter θ by i^{th} agent. While designing the estimator, local information flow i.e., communication from agent j to agent i is allowed only if $j \in N_i$, which is based on Assumption 1.

For further subsequent formulation and analysis following assumption is considered.

Assumption 2. $\phi_i(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\dot{\phi}_i(t) \in \mathcal{L}_\infty$, $\forall i = 1(1)n$. □

2.4 PI-like Collaborative Estimator Design

2.4.1 Integrator Dynamics

The work in (Papusha *et al.*, 2014) has designed a collaborative estimator using a similar model like (2.4). However, the designs require a C-PE condition for parameter convergence. In contrast, the proposed work has introduced a strategically designed set of integrator dynamics to relax the restrictive C-PE condition for parameter convergence:

$$\dot{Y}_I^i(t) = \phi_i(t)\phi_i^T(t), \quad Y_I^i(t_0) = 0, \quad \forall i = 1(1)n \tag{2.6}$$

$$\dot{y}_I^i(t) = \phi_i(t)y_i(t), \quad y_I^i(t_0) = 0, \quad \forall i = 1(1)n \tag{2.7}$$

where ${Y_I}^i(t) \in \mathbb{R}^{p\times p}$ denotes the integrated regressor and $y_I^i(t) \in \mathbb{R}^p$ is known as the integrated output.

Taking integration of (2.6), (2.7) and using (2.4), it can be concluded that

$$Y_I^i(t)\theta = y_I^i(t), \quad \forall t \geq t_0, \quad \forall i = 1(1)n. \tag{2.8}$$

The integrated regressor matrix $Y_I^i(t)$ has the following two properties, which can be directly infer by integrating the right hand side of (2.6).

Property 1. *$Y_I^i(t)$ is positive semi-definite function of time i.e. $Y_I^i(t) \geq 0$, $\forall t \geq t_0$, $\forall i = 1(1)n$.* $\square$

Property 2. *$Y_I^i(t)$ is a non-decreasing function of time i.e. the matrix inequality $Y_I^i(t_2) \geq Y_I^i(t_1)$, for $t_2 > t_1$, $\forall i = 1(1)n$, holds true.* $\square$

2.4.2 Proposed Parameter Estimator

The consensus based novel PI-like parameter estimator, $\forall i = 1(1)n$, is subsequently defined as follows:

$$\dot{\hat{\theta}}_i(t) = \underbrace{-\gamma\phi_i(t)(\hat{y}_i - y_i)}_{T_P} - \underbrace{k\,\overbrace{(Y_I^i\hat{\theta}_i - y_I^i)}^{\eta_I^i}}_{T_I} + \underbrace{\sum_{j\in N_i} a_{ij}(\hat{\theta}_j - \hat{\theta}_i)}_{T_C} \tag{2.9}$$

Here, the first term T_P of the (2.9) is a proportional-like component, where $\gamma > 0$ is a positive real constant to control the local information fusion rate. The component T_I is an integral-like term and the last term T_C is a term based on the neighbor's current estimates. Together T_I and T_C circumvent the C-PE restriction and lead to parameter convergence under the C-IE condition as revealed subsequently. Moreover, $k, \gamma \in \mathbb{R}_{>0}$ are tuning parameters.

2.5 Stability/Convergence Analysis

The parameter estimation error dynamics for all n number of agents can be compactly represented as

$$\Delta\dot{\theta}(t) = -\gamma\Phi(t)\Delta\theta(t) - k\Phi_I(t)\Delta\theta(t) - (L\otimes I_p)\Delta\theta(t) \tag{2.10}$$

where $\otimes$ denotes the kronecker product and $I_p \in \mathbb{R}^{p\times p}$ is the identity matrix, column vectors $\hat{\theta}(t) = [\hat{\theta}_1^T(t), \ldots, \hat{\theta}_n^T(t)]^T \in \mathbb{R}^{np}$ and $\Delta\theta = [\Delta\theta_1^T(t), \ldots, \Delta\theta_n^T(t)]^T \in \mathbb{R}^{np}$ by stacking the components $\hat{\theta}_i(t) \in \mathbb{R}^p$ and $\Delta\theta_i(t) = \hat{\theta}_i(t) - \theta \in \mathbb{R}^p$, $\forall i = 1(1)n$. And

$\Phi(t), \Phi_I(t) \in \mathbb{R}^{np \times np}$ are the block diagonal matrices, which are defined as

$$\Phi(t) = \begin{bmatrix} \phi_1(t)\phi_1^T(t) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \phi_n(t)\phi_n^T(t) \end{bmatrix}$$

and

$$\Phi_I(t) = \begin{bmatrix} \int\limits_{t_0}^{t} \phi_1(r)\phi_1^T(r)dr & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \int\limits_{t_0}^{t} \phi_n(r)\phi_n^T(r)dr \end{bmatrix}.$$

The following Theorem characterizes several properties of the proposed collaborative parameter estimator dynamics.

Theorem 1. *Provided Assumptions 1-2 hold, the parameter estimation error dynamics* (2.9) *or (2.10) exhibits the following:*

(1) The origin of the parameter estimation error dynamics is Lyapunov stable and all the auxiliary signals remain bounded for all time.

(2) Asymptotic behavior of prediction errors:

$$\Delta y_i(t) = \hat{y}_i(t) - y_i(t) \to 0 \; as \; t \to \infty, \; \forall i = 1(1)n.$$
$$\eta_I^i(t) = \left(Y_I^i(t)\hat{\theta}_i(t) - y_I^i(t)\right) \to 0 \; as \; t \to \infty, \; \forall i = 1(1)n.$$

(3) Asymptotic behavior of parameter consensus:

$$\hat{\theta}_j(t) - \hat{\theta}_i(t) \to 0 \; as \; t \to \infty, \; \forall i, j = 1(1)n.$$

*(4) Parameter Convergence under **C-IE**: In addition if the definition (4) is true for the set of regressor signals $\phi_i(t)'s$, then the parameter estimates will exponentially converge to the true parameter vector i.e. $\Delta\theta_i(t) = \hat{\theta}_i(t) - \theta \to 0$ (exponentially fast) as $t \to \infty$, $\forall i = 1(1)n$.*

Proof. Consider the following Lyapunov candidate

$$V(\Delta\theta) = \frac{1}{2}\Delta\theta^T\Delta\theta = \frac{1}{2}\sum_{i=1}^{n}\Delta\theta_i^T\Delta\theta_i \tag{2.11}$$

Taking time derivative of (2.11) and substituting the parameter estimation error dynamics

(2.10), yields

$$\dot{V}(\Delta\theta) = -\Delta\theta^T\Big((L \otimes I_P) + \gamma\Phi(t) + k\Phi_I(t)\Big)\Delta\theta \leq 0 \tag{2.12}$$

where the inequality holds from the fact that the $(L \otimes I_p)$, $\Phi(t)$ and $\Phi_I(t)$ are positive semi-definite matrix, which implies that the origin of the parameter estimation error dynamics is Lyapunov stable. Based on the Assumption 2, it can be conclude that $\big(\Phi(t), \Phi_I(t), \dot{\Phi}(t), \dot{\Phi}_I \in \mathcal{L}_\infty\big)$. Using these above arguments, it can be concluded that $V(\Delta\theta(t))$ is uniformly bounded above by its initial value, i.e. $V(\Delta\theta(t)) \leq V(\Delta\theta(0))$, so $\Delta\theta(t) \in \mathcal{L}_\infty$, which implies $\hat{\theta}_i(t) \in \mathcal{L}_\infty$ i.e., the local prediction $\hat{y}_i(t)$, $Y_I^i(t)\hat{\theta}_i(t) \in \mathcal{L}_\infty$, $\forall i = 1(1)n$. This completes the proof of part 1.

For the next part of the proof, differentiating (2.12)

$$\begin{aligned}\ddot{V}(\Delta\theta) = &-2\Delta\theta^T(L \otimes I_p)\Delta\dot{\theta} - \gamma\Delta\theta^T\dot{\Phi}(t)\Delta\theta - 2\gamma\Delta\theta^T\Phi(t)\Delta\dot{\theta} - k\Delta\theta^T\dot{\Phi}_I(t)\Delta\theta \\ &- 2k\Delta\theta^T\Phi_I(t)\Delta\dot{\theta}\end{aligned} \tag{2.13}$$

Using the boundedness related arguments in the last part of the proof, it can be concluded that $\ddot{V}(\Delta\theta) \in \mathcal{L}_\infty$ or $\dot{V}(\Delta\theta)$ is uniformly continuous, $(V(\Delta\theta(t) \geq 0)$, $\dot{V}(\Delta\theta(t))$ is decreasing, and it has a finite limit as $t \to \infty$. Then by invoking Barbalat's lemma (Lemma 8.2 at page 323 (Khalil and Grizzle, 2002)), it can be concluded that; $\dot{V}(\Delta\theta(t)) \to 0$ as $t \to \infty$. Hence from (2.12), it can be concluded that $\Delta\theta^T\Phi(t)\Delta\theta$, $\Delta\theta^T\Phi_I(t)\Delta\theta$, $\Delta\theta^T(L \otimes I_p)\Delta\theta \to 0$ as $t \to \infty$.

The first term $\Delta\theta^T\Phi(t)\Delta\theta \to 0$ as $t \to \infty$. In particular this means $\Phi(t)\Delta\theta \to 0$ which ensure that the local prediction error $\Delta y_i(t) \to 0$ as $t \to \infty$, $\forall i = 1(1)n$.

Second term $\Delta\theta^T\Phi_I(t)\Delta\theta \to 0$ as $t \to \infty$. In particular this means $\Phi_I(t)\Delta\theta \to 0$ which ensure that the integrated regressor based prediction error $\eta_I^i(t) \to 0$ as $t \to \infty$, $\forall i = 1(1)n$. This complete the proof of part 2, i.e. the asymptotic behavior of prediction errors are achieved.

Third term $\Delta\theta^T(L \otimes I_p)\Delta\theta \to 0$ as $t \to \infty$, which implies that $(L \otimes I_p)\Delta\theta = (L \otimes I_p)\hat{\theta} \to 0$. For a connected graph G, null(L) = null($L \otimes I_p$)= span$\{\mathbf{1}_p\}$ where null denotes the null space of a matrix, then $\hat{\theta}_j(t) - \hat{\theta}_i(t) \to 0$ as $t \to \infty$, $\forall i, j = 1(1)n$.

This completes the proof of part 3, i.e. the asymptotic behavior of parameter consensus is achieved.

For proof of part 4, since $L\mathbf{1} = 0$ that implies

$$(L \otimes I_p)\Big(\frac{1}{\sqrt{n}}\mathbf{1} \otimes f_j\Big) = 0$$

Moreover, the eigen-decomposition form of the matrix $(L \otimes I_p)$ is

$$(L \otimes I_p)(x_i \otimes f_j) = \lambda_i(x_i \otimes f_j)$$

whereas the right hand side of both the above equations can be verified on the basis of the mixed product property $PQ \otimes RS = (P \otimes R)(Q \otimes S)$ for appropriately sized matrices $P, Q, R,$ and S. And $\lambda_i > 0$, x_i, $\forall i = 2(1)n$ and $f_j \in \mathbb{R}^p$ is the jth unit vector, $\forall j = 1(1)p$.

Lets define the unit vector $z \in \mathbb{R}^{np}$ in this basis as

$$z = \sum_{j=1}^{p} a1_j \frac{1}{\sqrt{n}}\mathbf{1} \otimes f_j + \sum_{i=2}^{n}\sum_{j=1}^{p} \delta_{ij} x_i \otimes f_j \tag{2.14}$$

with $(a1, \delta) \in \mathbb{R}^p \times \mathbb{R}^{(n-1)p}$ has unit norm. Define the new quantity $\Phi_{bar}(t)$, which is the time average of quantity $\Phi(t)$ over the time interval $[t_0, t_0 + T]$

$$\Phi_{bar}(t) \triangleq \frac{1}{T}\int_{t_0}^{t_0+T} \Phi(r)dr \tag{2.15}$$

Then the proof of part 4 for the Theorem 1 can be done in two stages.

First stage is to show the following integral defined below is uniformly strictly positive-definite over the time window $[t_0, t_0 + T]$

$$\frac{1}{T}\int_{t_0}^{t_0+T} z^T\Big((L\otimes I_p)+\gamma\Phi(r)\Big)zdr = z^T(L\otimes I_p)z+\gamma z^T\Phi_{bar}z \geq max\{z^T(L\otimes I_p)z, \gamma z^T\Phi_{bar}z\} > 0. \tag{2.16}$$

By substituting the (2.14) into (2.16) and using the $(L\otimes I_p)\Big(\frac{1}{\sqrt{n}}\mathbf{1}\otimes f_j\Big) = 0$, $(L\otimes I_p)(x_i\otimes f_j) = \lambda_i(x_i \otimes f_j)$, the first term $z^T(L \otimes I_p)z$ from $max\{z^T(L \otimes I_p)z, \gamma z^T\Phi_{bar}z\}$ can be bounded as

$$z^T(L \otimes I_p)z = \sum_{i=2}^{n}\sum_{j=1}^{p} \lambda_i \delta_{ij}^2 \geq \lambda_2(1 - ||a1||_2^2)$$

where $||a1||_2^2 + ||\delta||_2^2 = 1$.

The second term $z^T \Phi_{bar} z$ is

$$\begin{aligned} z^T \Phi_{bar} z =& \frac{1}{n}\sum_{i=1}^{p}\sum_{j=1}^{p} a1_i a1_j (\mathbf{1} \otimes f_i)\Phi_{bar}(\mathbf{1} \otimes f_j) + \frac{2}{\sqrt{n}}\sum_{i=2}^{n}\sum_{j=1}^{p}\sum_{k=1}^{p} a1_k \delta_{ij}(\mathbf{1} \otimes f_k)^T \Phi_{bar}(x_i \otimes f_j) \\ & + \underbrace{\sum_{i=2}^{n}\sum_{j=1}^{p}\sum_{k=2}^{n}\sum_{l=1}^{p} \delta_{ij}\delta_{kl}(x_i \otimes f_j)\Phi_{bar}(x_k \otimes f_l)}_{\geq 0} \end{aligned}$$

using the fact $(\mathbf{1} \otimes f_i)^T \Phi_{bar}(\mathbf{1} \otimes f_j) = \overline{(\phi_1 \phi_1^T)}_{ij} + ... + \overline{(\phi_n \phi_n^T)}_{ij}$,

$z^T \Phi_{bar} z$ can be bound as

$$z^T \Phi_{bar} z \geq \frac{1}{n} a1^T \Big(\sum_{i=1}^{n} \overline{\phi_i \phi_i^T}\Big) a1 - \frac{2\zeta}{\sqrt{n}}\sum_{i=2}^{n}\sum_{j=1}^{p}\sum_{k=1}^{p} |a1_k \delta_{ij}|$$

after more simplification

$$z^T \Phi_{bar} z \geq \frac{||a1||_2^2}{n}\beta - 2\zeta n\sqrt{||a1||_2^2(1 - ||a1||_2^2)}$$

where ζ is the possible upperbound based on the C-IE definition. thus by clubbing these two terms together, required lower bound, which is strictly greater than zero for given time window $[t_0, t_0 + T]$ is achieved i.e.

$$max\{z^T(L \otimes I_p)z, \gamma z^T \Phi_{bar} z\} \geq k_1 > 0$$

where the k_1 is

$$k_1 = \inf_{||a1||_2 \leq 1} max\left\{\lambda_2\Big(1 - ||a1||_2^2\Big), \gamma\frac{||a1||_2^2}{n}\beta - 2\gamma\zeta n\sqrt{||a1||_2^2\Big(1 - ||a1||_2^2\Big)}\right\}. \quad (2.17)$$

By using continuity argument, infimum in (2.17) is attained and is strictly greater than zero (if k_1 is zero and the first term is zero, then the second term should also be zero, $\gamma\frac{\beta}{n}$ is zero, which is a contradiction from C-IE condition requirement for parameter convergence).

In second stage for proof of part 4, define

$$M(t) = (L \otimes I_p) + \Phi_I(t) = (L \otimes I_p) + \int_{t_0}^{t} \Phi(r)dr \tag{2.18}$$

from the above stage, it can be deduced that $M(t_0 + T) > 0, \forall t \geq t_0 + T$. Furthermore, from (2.12), it can be concluded that

$$\dot{V}(\Delta\theta) \leq -\Delta\theta^T M(t)\Delta\theta \leq -\lambda_{min}(M(t))||\Delta\theta||^2, \ \forall t \geq t_0 + T \tag{2.19}$$

using the same argument as in Property 2, $M(t) \geq M(t_0 + T) > 0, \forall t \geq t_0 + T$, which implies $\lambda_{min}(M(t)) \geq c > 0$ where the λ_{min} is the minimum eigenvalue of matrix $M(t)$ and c is positive real constant. Then from (2.19), it becomes

$$\dot{V}(\Delta\theta) \leq -2cV(\Delta\theta), \ \forall t \geq t_0 + T \tag{2.20}$$

after further manipulation, (2.20) becomes

$$V(\Delta\theta(t)) \leq V(\Delta\theta(t_0 + T))e^{-2c(t-t_0-T)}, \ \forall t \geq t_0 + T \tag{2.21}$$

(2.21) implies that the parameter estimates will converge exponentially. This completes the proof of part 4. $\square$

2.6 Simulation Results

Consider the communication network example in Figure 2.1 where $n = 3$ agents, $m = 2$ links, $p = 2$. Three agents are tasked to estimate a true parameter vector $\theta = (\theta_1, \theta_2) = (3, -3) \in \mathbb{R}^p$ using the model (2.4), where

$$\phi_1(t) = \begin{bmatrix} 0 \\ 2e^{-t} \end{bmatrix}, \phi_2(t) = \begin{bmatrix} 7e^{-t}sin(3t) \\ 0 \end{bmatrix}, \phi_3(t) = \begin{bmatrix} 0 \\ cos(t) \end{bmatrix}.$$

The dynamics (2.9) is formulated by considering $\gamma = 1$, $k = 1$, and each $a_{ij} = 1$

i.e., consensus is available.

Considering all the above assumptions, a comparison based simulation analysis is carried-out where both, conventional parameter estimator based on (Papusha *et al.*, 2014; Chen *et al.*, 2013) and a proposed novel PI-lke parameter estimator is taken. Figures 2.2, 2.3, and 2.4 are based on the conventional parameter estimator based on (Papusha *et al.*, 2014; Chen *et al.*, 2013), where Figure 2.2 shows the estimate of θ_2 and Figure 2.3 shows the estimate of θ_1, which signify that, the parameter convergence is not achieved since C-PE is not true here. Figure 2.4 norm of parameter estimation error $\|\Delta\theta(t)\|$ confirms the above argument related to the parameter convergence. Figures 2.5, 2.6, and 2.7 are based on the proposed novel PI-like parameter estimator, where figure 2.5, Figure 2.6 signify that the parameter convergence is achieved since the C-IE condition (which is milder than C-PE) holds for the chosen set of regressors. Figure 2.7; norm of parameter estimation error $\|\Delta\theta(t)\|$ confirms the efficacy of proposed PI-like parameter estimator.

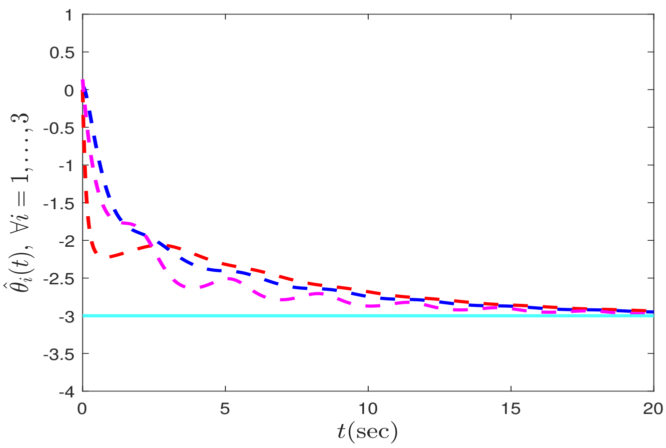

Figure 2.2: Estimation of ($\theta_2 = -3$) by parameter estimator based on (Papusha *et al.*, 2014; Chen *et al.*, 2013).

2.7 Conclusion

This chapter proposes a consensus-based novel PI-like parameter estimator for collaborative system identification. The integral component in the estimator dynamics is instrumental in ensuring parameter convergence under a newly proposed condition called C-IE. The C-IE condition is significantly milder than the classical PE condition or its modification like the C-PE condition usually required for parameter convergence. The

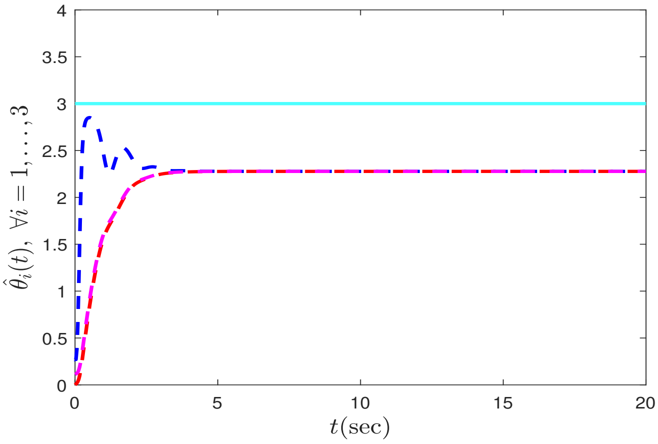

Figure 2.3: Estimation of ($\theta_1 = 3$) by parameter estimator based on (Papusha *et al.*, 2014; Chen *et al.*, 2013).

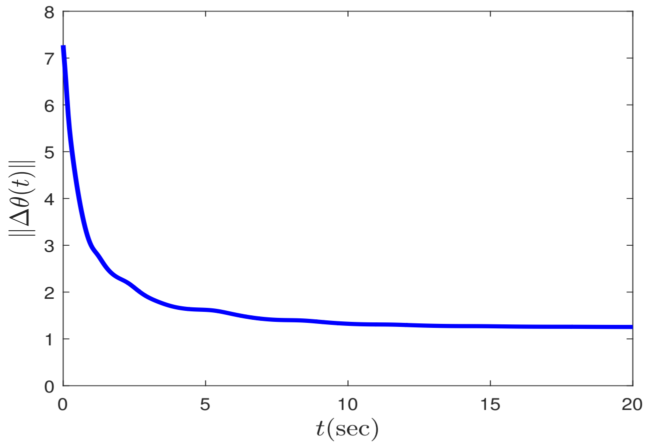

Figure 2.4: Norm of parameter estimation error based on (Papusha *et al.*, 2014; Chen *et al.*, 2013).

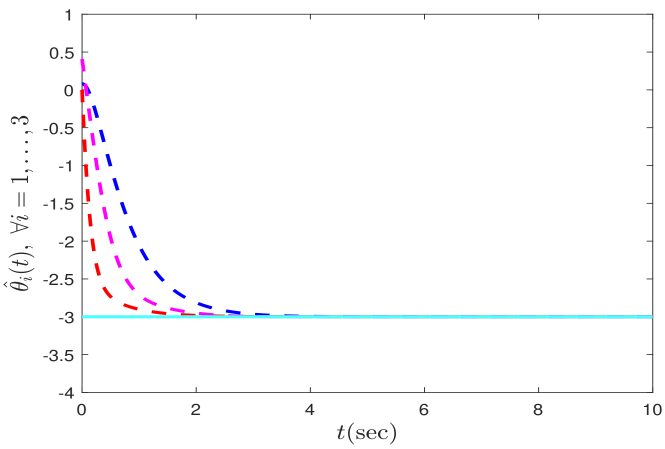

Figure 2.5: Estimation of ($\theta_2 = -3$) by proposed novel PI-like parameter estimator.

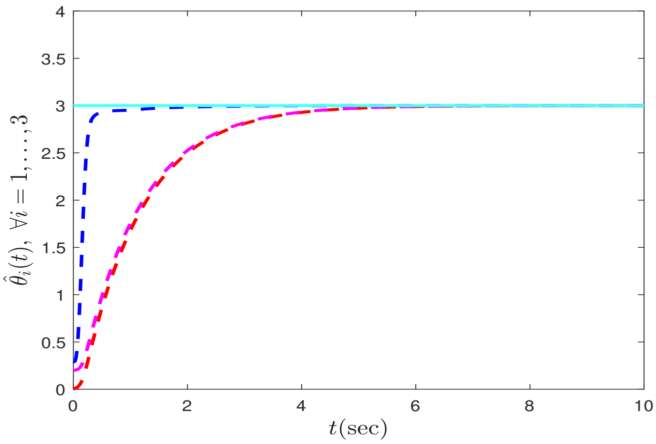

Figure 2.6: Estimation of ($\theta_1 = 3$) by proposed novel PI-like parameter estimator.

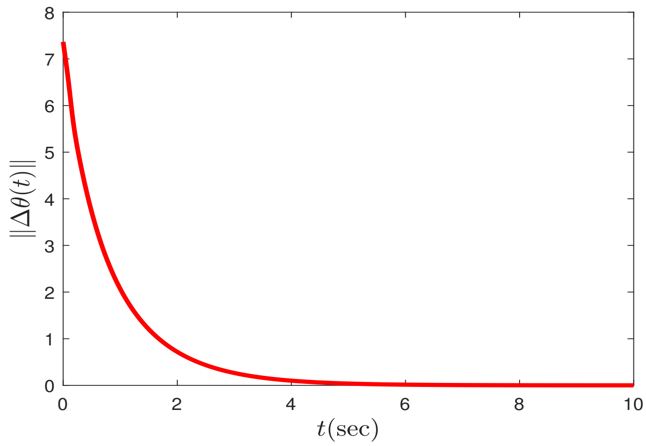

Figure 2.7: Norm of parameter estimation error based on proposed novel PI-like parameter estimator.

C-IE condition is an extension of the concept of IE, which was recently proposed in the context parameter estimation in adaptive control as a milder condition than the PE condition. This paper generalizes the concept of IE in multi-agent settings, where information sharing through connected graph guarantees consensus parameter convergence under the C-IE condition.

Part II

Combined Cooperative Adaptive Cruise Control (CACC) using Collective Initial Excitation based Distributed Parameter Estimator

2.8 Introduction

In this chapter, we design a combined CACC architecture for an uncertain homogeneous vehicle platoon. The term "combined" is borrowed from combined model reference adaptive control (MRAC) literature, which is a combination of direct and indirect MRAC (Narendra and Annaswamy, 2012). The combined CACC architecture is composed of a distributed parameter estimator of the uncertain vehicle dynamics parameters and a MRAC control law with a differential control parameter update routine. The control parameter estimator uses information from the vehicle dynamics parameter estimator making the design analogous to combined MRAC. The distributed parameter estimator of the vehicle dynamics is designed based on a two-layer filtering mechanism (Jha *et al.*, 2019) and a consensus-based component using information from immediate preceding and following vehicles' instantaneous estimation. This distributed estimator can ensure exponentially fast parameter convergence using the newly defined condition of C-IE and thereby relaxes the need for excitation (information content regarding the unknown parameters) to persist for all time. The C-IE condition implies that the IE condition is satisfied by all the agents cooperatively instead of individually. So the information content is distributed among all the vehicles' regressors in the initial time-window, which is strategically captured in the distributed estimator dynamics leading to parameter convergence. Further, the designed MRAC law along with the distributed estimator ensures asymptotic convergence of the vehicle platoon to a string stable reference platoon, thus maintaining smooth and safe operation.

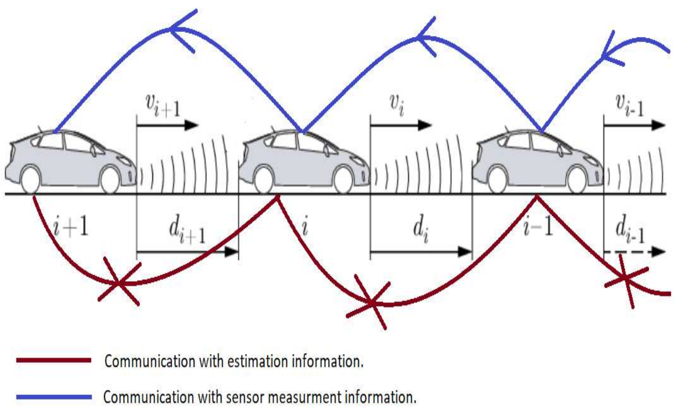

Figure 2.8: CACC based homogeneous vehicle platoon.

2.9 Model Description

2.9.1 Model Description for Vehicle Platoon Architecture

Consider a homogeneous platoon with n number of vehicles. Figure 2.8, shows the platoon where $v_i(t) \in \mathbb{R}$ denotes the velocity (m/s) of vehicle i, and $d_i(t) \in \mathbb{R}$ is the distance (m) between vehicle i and its preceding vehicle $i-1$. This distance is measured using a radar or lidar mounted on the front bumper of each vehicle. Furthermore, each vehicle in the platoon string can communicate with its preceding vehicle via wireless communication. The main task of every vehicle in the platoon, except the Leader (virtual leader in the proposed model), is to maintain some desired inter-vehicle distance $d_{r,i}(t) \in \mathbb{R}$ between itself and its preceding vehicle. Define the set $S_n = \{i \in \mathbb{N} | 1 \leq i \leq n\}$ with the index $i = O$ is fixed for the virtual leader. To regulate the inter-vehicle distance, a constant time headway (CTH) spacing policy is chosen, which is based on (Rajamani and Zhu, 2002). The CTH is formulated by defining the $d_{r,i}(t)$ as

$$d_{r,i}(t) = r_i + hv_i(t), \ \forall i \in S_n \tag{2.22}$$

where $r_i \in \mathbb{R}$ is the standstill distance (meters) and $h > 0$ is the time headway (seconds). Hence the spacing error (meters) of the i^{th} vehicle is defined as

$$\begin{aligned} e_i(t) =& d_i(t) - d_{r,i}(t) & (2.23) \\ =& (q_{i-1}(t) - q_i(t) - L_i) - (r_i + hv_i(t)), \ \forall i \in S_n & (2.24) \end{aligned}$$

where $q_i(t) \in \mathbb{R}$ and $L_i \in \mathbb{R}$ representing the rear-bumper position (m) and length (m) of vehicle i, respectively. The desired behavior of the string of vehicle platoon is coined in terms of string stability, which captures the notion of attenuation of disturbances like emergency braking (Ploeg *et al.*, 2013). An established definition of string stability is as follows.

Definition 3 (String Stability (Ploeg *et al.*, 2013))

Consider the acceleration of vehicle i is denoted by $a_i(t) \in \mathbb{R}$. Then, a platoon can be considered as string stable if

$$\sup_w |X_i(jw)| = \sup_w \left| \frac{a_i(jw)}{a_{i-1}(jw)} \right| \leq 1, \ \forall i \in S_n \tag{2.25}$$

where $a_i(jw)$ is the Laplace transform of the acceleration of vehicle i. $\square$

The dynamics of the i^{th} vehicle is represented by the following model.

$$\begin{pmatrix} \dot{e}_i \\ \dot{v}_i \\ \dot{a}_i \end{pmatrix} = \begin{pmatrix} 0 & -1 & -h \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{\tau} \end{pmatrix} \begin{pmatrix} e_i \\ v_i \\ a_i \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} v_{i-1} + \begin{pmatrix} 0 \\ 0 \\ \frac{\Omega}{\tau} \end{pmatrix} u_i, \ \forall i \in S_n \tag{2.26}$$

where $u_i(t) \in \mathbb{R}$ is the control input (m/s^2) of vehicle i and τ denotes each vehicle's unknown driveline time constant (seconds) and Ω denotes the engine's performance. Engine's performance is effected by the different type of disturbances such as wind gust, slope of road, etc. Based on model (2.26) proposed in (Ploeg *et al.*, 2013; Harfouch *et al.*, 2017) and considering the ideal engine's performance, the virtual leader vehicle model is defined as

$$\begin{pmatrix} \dot{e}_O \\ \dot{v}_O \\ \dot{a}_O \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{\tau_O} \end{pmatrix} \begin{pmatrix} e_O \\ v_O \\ a_O \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{\Omega_O}{\tau_O} \end{pmatrix} u_O \tag{2.27}$$

2.9.2 Baseline Controller and CACC Reference Model

By considering various baseline conditions such as ideal engine performance, persistent communication availability between consecutive vehicles, the authors in (Ploeg *et al.*, 2013) derived a controller and a spacing policy, which ensures string stability of the platoon. The CACC baseline controller is defined as

$$\dot{u}_{bl,i} = \frac{1}{h}\Big(- u_{bl,i} + K_p e_i + K_d \dot{e}_i + u_{bl,i-1}\Big), \ u_{bl,i}(t_0) = 0 \tag{2.28}$$

where K_p and K_d are the tuning parameters for controller. The term, $u_{bl,i-1}$ introduces information from the precedent vehicle $(i-1)$, which makes CACC a powerful scheme in contrast to ACC. Further the control input of the virtual leader is designed as

$$\dot{u}_O = \frac{1}{h}(-u_O + u_r) \tag{2.29}$$

where $u_r(t) \in \mathbb{R}$ is an external input acting as the desired acceleration (m/s^2) of the virtual leader.

To design the adaptive component of the controller based on MRAC approach, the CACC reference model is defined subsequently as provided in (Harfouch *et al.*, 2017).

$$\underbrace{\begin{pmatrix} \dot{e}_{i,r} \\ \dot{v}_{i,r} \\ \dot{a}_{i,r} \\ \dot{u}_{i,r} \end{pmatrix}}_{\dot{x}_{i,r}} = \underbrace{\begin{pmatrix} 0 & -1 & -h & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{\tau_O} & \frac{\Omega_O}{\tau_O} \\ \frac{K_p}{h} & -\frac{K_d}{h} & -K_d & -\frac{1}{h} \end{pmatrix}}_{A_r} \underbrace{\begin{pmatrix} e_{i,r} \\ v_{i,r} \\ a_{i,r} \\ u_{i,r} \end{pmatrix}}_{x_{i,r}} + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{K_d}{h} & \frac{1}{h} \end{pmatrix}}_{B_{w,r}} \underbrace{\begin{pmatrix} v_{i-1} \\ u_{bl,i-1} \end{pmatrix}}_{w_i}, \ \forall i \in S_n \tag{2.30}$$

where $x_{i,r}(t) \in \mathbb{R}^4$ and $w_i(t) \in \mathbb{R}^2$ are i^{th} vehicle's reference state vector and input vector, respectively; and $A_r \in \mathbb{R}^{4\times4}$, $B_{w,r} \in \mathbb{R}^{4\times2}$ are the system matrix and input matrix, respectively. Further, combining (2.27) with (2.29), the virtual leader dynamics become

$$\underbrace{\begin{pmatrix} \dot{e}_O \\ \dot{v}_O \\ \dot{a}_O \\ \dot{u}_O \end{pmatrix}}_{\dot{x}_O} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{\tau_O} & \frac{\Omega_O}{\tau_O} \\ 0 & 0 & 0 & -\frac{1}{h} \end{pmatrix}}_{A_{lr}} \underbrace{\begin{pmatrix} e_O \\ v_O \\ a_O \\ u_O \end{pmatrix}}_{x_O} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{h} \end{pmatrix}}_{B_{lr}} u_r \tag{2.31}$$

It has been proved in (Ploeg *et al.*, 2013) that, the reference model (2.30) is asymptotically stable around the equilibrium point

$$x_{i,r,eq} = (0 \ v_O \ 0 \ 0)^T, \text{for } x_O = x_{i,r,eq} \text{ and } u_r = 0 \tag{2.32}$$

where v_O is a constant velocity, provided that the following Routh–Hurwitz conditions are satisfied:

$$h > 0,\ K_p, K_d > 0,\ K_d > \tau_O K_p. \tag{2.33}$$

To invoke string stability of the CACC reference platoon dynamics (2.30), the following transfer function model is considered,

$$X_i(s) = \frac{1}{hs+1},\ \forall i \in S_n \tag{2.34}$$

which satisfies the string stability condition (2.25) based of Definition 3 for any $h > 0$.

2.9.3 MRAC in conjunction with Baseline Controller

In this section, CACC reference model (2.30) will be used to design the control input $u_i(t),\ \forall i \in S_n$, such that the uncertain platoon's dynamics described by (2.25) and (2.26) converge to the string stable nominal dynamics. To achieve this, the baseline controller is augmented with an adaptive controller as

$$u_i(t) = u_{bl,i}(t) + u_{ad,i}(t),\ \forall i \in S_n \tag{2.35}$$

where $u_{ad,i}(t) \in \mathbb{R}$ is the adaptive controller to be constructed subsequently. Now substituting (2.35) in (2.26) and exploiting (2.28), yields, $\forall i \in S_n$

$$\underbrace{\begin{pmatrix} \dot{e}_i \\ \dot{v}_i \\ \dot{a}_i \\ \dot{u}_{bl,i} \end{pmatrix}}_{\dot{x}_i} = \underbrace{\begin{pmatrix} 0 & -1 & -h & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{\tau} & \frac{\Omega}{\tau} \\ \frac{K_p}{h} & -\frac{K_d}{h} & -K_d & -\frac{1}{h} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} e_i \\ v_i \\ a_i \\ u_{bl,i} \end{pmatrix}}_{x_i} + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{K_d}{h} & \frac{1}{h} \end{pmatrix}}_{B_w} \underbrace{\begin{pmatrix} v_{i-1} \\ u_{bl,i-1} \end{pmatrix}}_{w_i} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \frac{\Omega}{\tau} \\ 0 \end{pmatrix}}_{B_u} u_{ad,i} \tag{2.36}$$

where $x_i \in \mathbb{R}^4$, $A \in \mathbb{R}^{4\times 4}$, $B_u \in \mathbb{R}^4$, and $u_{ad,i}(t)$ is defined as

$$u_{ad,i} = \hat{K}^T x_i,\ \forall i \in S_n \tag{2.37}$$

where $\hat{K}(t) \in \mathbb{R}^4$.

As compared to conventional CACC architectures (Harfouch *et al.*, 2017), this work modifies the reference model (2.30) by incorporating actual state information in the reference model dynamics as

$$\dot{x}_{i,c} = A_r x_{i,c} + B_w w_i + l\Big(x_i(t) - x_{i,c}(t)\Big),\ \forall i \in S_n \tag{2.38}$$

where $l > 0$ is the free design parameter. In (Gibson *et al.*, 2013), this type of reference model modification is denoted as closed-loop reference model. The parameter l plays a crucial role in the proposed CACC architecture as revealed in the subsequent stability analysis. Moreover, note that if asymptotic convergence of $x_i(t)$ to $x_{i,c}(t)$ is satisfied, $x_{i,c}(t)$ will also tend to $x_{i,r}(t)$, which implies that the fundamental objective of following the open-loop string stable model (2.30) is not hampered in the proposed closed-loop modification.

To facilitate the design objective of making system (2.36) respond as the chosen reference model (2.38), the following matching condition is introduced (Gibson *et al.*, 2013).

Assumption 3. *There exist constant matrix $K^* \in \mathbb{R}^4$ such that*

$$A_r = A + B_u K^{*T} \tag{2.39}$$

The tracking-error (between actual and closed-loop reference model) is defined as

$$\zeta_i(t) \triangleq x_i(t) - x_{i,c}(t),\ \forall i \in S_n \tag{2.40}$$

using (2.40), (2.39), (2.38), (2.37), and (2.36), the tracking-error dynamics $\dot{\zeta}_i(t)$ can be expressed as

$$\dot{\zeta}_i(t) = A_r \zeta_i - l\zeta_i + B_u \tilde{K}^T x_i,\ \forall i \in S_n \tag{2.41}$$

where $\zeta_i(t) \in \mathbb{R}^4$ and $\tilde{K}(t) \triangleq \hat{K}(t) - K^*$. The standard direct projection based adaptive update law for $\hat{K}$ inspired from (Gibson *et al.*, 2013) is, $\dot{\hat{K}}(t) = Proj_{\Omega}(-\Gamma \zeta_i^T x_i B_u^T P, K)$. Since in present context B_u is unknown, the control parameter update law is designed as

$$\dot{\hat{K}}(t) = Proj_{\Omega}(-\Gamma \zeta_i^T x_i \hat{B}_u^T P, K),\ \forall i \in S_n \tag{2.42}$$

where $\Gamma > 0$ and $P = P^T \in \mathbb{R}^{4\times4}$ is positive definite solution of following algebraic Lyapunov equation

$$A_r{}^T P + PA_r = -Q \tag{2.43}$$

where $Q \in \mathbb{R}^{4\times4}$ is a chosen positive definite matrix.

The quantity $\hat{B}_u(t)$ is an online estimate of the unknown input matrix. The following subsection develops the distributed platoon parameter estimator, which supplies $\hat{B}_u(t)$ for the control parameter update.

2.9.4 Online Identification for Unknown Platoon Parameters

From the structure of (2.36), it can be conclude that the dynamics

$$\dot{a}_i = -\frac{1}{\tau}a_i + \frac{\Omega}{\tau}u_{bl,i} + \frac{\Omega}{\tau}u_{ad,i},\ \forall i \in S_n \tag{2.44}$$

only makes (2.36) uncertain, otherwise, all other parameters are known. Hence taking advantage of that structure, the linear parameterization of (2.44) is obtained as

$$\dot{a}_i = \begin{bmatrix} -a_i \ u_i \end{bmatrix} \begin{bmatrix} \frac{1}{\tau} \\ \frac{\Omega}{\tau} \end{bmatrix} = y_i^T(a_i, u_i)\theta,\ \forall i \in S_n \tag{2.45}$$

where $y_i(a_i, u_i) \in \mathbb{R}^2$ is the known regressor and $\theta \in \mathbb{R}^2$ is the unknown platoon parameter needs to be estimated.

Assumption 4. $||\theta|| < \delta_1$, *for some known constant* $\delta_1 > 0$. □

To handle the unavailability of the acceleration measurement $\dot{a}_i(t)$, the following filter equations are designed as

$$\dot{z}_i(t) = -kz_i(t) + y_i(t), \quad z_i(t_0) = 0, \quad \forall i \in S_n \tag{2.46}$$

$$\dot{g}_i(t) = -kg_i(t) + \dot{a}_i(t), \quad g_i(t_0) = 0, \quad \forall i \in S_n \tag{2.47}$$

where $z_i(t) \in \mathbb{R}^2$ denotes the filtered regressor matrix and $g_i(t) \in \mathbb{R}$ denotes the filtered version of $\dot{a}_i(t)$ and k is a positive scalar introduced to stabilize the above filter equations.

Analytically solving (2.46) and (2.47) and utilizing (2.45), the following relation can be deduced.

$$g_i(t) = z_i^T(t)\theta,\ \forall i \in S_n \tag{2.48}$$

From (2.47), $g_i(t)$ cannot be explicitly computed since $\dot{a}_i(t)$ is unknown. However, after analytically solving (2.47) and applying the by-parts rule of integration, it can be shown that

$$g_i(t) = a_i(t) - e^{-kt}a_i(t_0) - kh_i(t),\ a_i(t_0) = 0,\ \forall i \in S_n \tag{2.49}$$

where $h_i(t) \in \mathbb{R}$ is the output of the subsequently designed filter dynamics.

$$\dot{h}_i(t) = -kh_i(t) + a_i(t),\ h_i(t_0) = 0,\ \forall i \in S_n \tag{2.50}$$

Since $a_i(t)$ is measurable, (2.48) and (2.49) can be utilized to obtain $g_i(t)$ online. Hence, it can be argued that the above filter equations (2.46) and (2.47) converts the differential equation in (2.45) to an algebraic one in (2.48), leading to the omission of $\dot{a}_i(t)$ information. A gradient-based law using (2.48) can be designed to estimate the system parameter $\theta,\ \forall i \in S_n$. However, this type of law requires the stringent PE condition on $z_i(t)$ for parameter convergence (Narendra and Annaswamy, 2012). To overcome this restriction, another pair of projection-based integral law is introduced, inspired by (Basu Roy *et al.*, 2018).

$$\dot{M}_i(t) = proj\big(z_i(t)z_i^T(t)\big), \quad M_i(t_0) = 0, \quad \forall i \in S_n \tag{2.51}$$

$$\dot{w}_i(t) = proj\big(z_i(t)g_i(t)\big), \quad w_i(t_0) = 0, \quad \forall i \in S_n \tag{2.52}$$

where the square matrix $M_i(t) \in \mathbb{R}^{2\times 2}$ denotes the integrated filtered regressor and $w_i(t) \in \mathbb{R}^2$ can be thought of as integrated filtered version of $\dot{a}_i(t)$ (although dimensionally $w_i(t)$ is different from $\dot{a}_i(t)$). Unlike (Adetola and Guay, 2008; Roy *et al.*, 2016), the use of $proj(.)$ (Basu Roy *et al.*, 2018), which denotes projection operator, restrict the variables $M_i(t)$ and $w_i(t)$ within a compact set.

Proposition 1. *Integrating (2.51) and (2.52) and using (2.48), it can be shown that*

$$w_i(t) = M_i(t)\theta,\ \forall t \geq t_0,\ \forall i \in S_n \tag{2.53}$$

Proof. For proof refer the (Basu Roy *et al.*, 2018). □

The matrices $M_i(t)$'s have the following properties:

Property 3. *$M_i(t)$ is a positive semi-definite function of time i.e. $M_i(t) \geq 0, \forall t \geq t_0$.* □

Property 4. *$M_i(t)$ is a non-decreasing function of time in the sense of matrix inequality i.e., $M_i(t_2) \geq M_i(t_1), \forall t_2 \geq t_1 \geq t_0$.* □

By exploiting these two properties, above filtering, and $proj(.)$ based arguments, a novel distributed consensus-based parameter estimation law is proposed as follows (Garg and Roy, 2019*b*).

$$\dot{\hat{\theta}}_i(t) = \underbrace{k_\theta z_i(g_i - z_i^T \hat{\theta}_i)}_{P} + \underbrace{\Gamma_\theta(w_i - M_i\hat{\theta}_i)}_{I} + \underbrace{\sum_{j \in \mathbb{N}_i}(\hat{\theta}_j - \hat{\theta}_i)}_{C} \tag{2.54}$$

where $\hat{\theta}_i(t) \in \mathbb{R}^2$ is an online estimate of the unknown platoon parameter vector $\theta,\ \forall i \in S_n$, and $k_\theta > 0$ and $\Gamma_\theta > 0$ are two positive scalar gains used to tune the rate of convergence and the neighbors sets are defined as $\mathbb{N}_1 = \{2\}, \mathbb{N}_i = \{(i-1),(i+1)\}, \forall i = 2, 3, .., n-1, \mathbb{N}_n = \{n-1\}$. Here, the first term P of (2.54) is a proportional-like component, the component I is an integral-like term and the last term C is a term based on the neighbor's current estimates according to chosen network topology. Together I and C circumvents the C-PE restriction and leads to parameter convergence under the C-IE condition as revealed subsequently.

Assumption 5. *The set of filtered regressors $z_i(t)$, $\forall i \in S_n$, satisfy the C-IE condition as per the Definition 4.* □

2.9.5 Compact Representation for Parameter Estimation Error Dynamics

The parameter estimation error dynamics for all the n number of unknown follower vehicles in the platoon can be compactly represented as

$$\Delta\dot{\theta}(t) = -k_\theta \Phi(t)\Delta\theta(t) - \Gamma_\theta \Phi_I(t)\Delta\theta(t) - (L \otimes I_2)\Delta\theta(t) \tag{2.55}$$

where $L \in \mathbb{R}^{n\times n}$ denotes the laplacian matrix, which is used to represent the estimation information sharing phenomena over the given network topology as in (Garg and Roy, 2019*a*), $\otimes$ denotes the kronecker product and $I_2 \in \mathbb{R}^{2\times 2}$ is the identity matrix, column vectors $\hat{\theta}(t) = [\hat{\theta}_1(t), \ldots, \hat{\theta}_n(t)]^T \in \mathbb{R}^{n2}$ and $\Delta\theta(t) = [\Delta\theta_1(t), \ldots, \Delta\theta_n(t)]^T \in \mathbb{R}^{n2}$ by stacking the components $\hat{\theta}_i(t) \in \mathbb{R}^2$ and $\Delta\theta_i(t) = \hat{\theta}_i(t) - \theta \in \mathbb{R}^2$, $\forall i \in S_n$. And $\Phi(t), \Phi_I(t) \in \mathbb{R}^{n2\times n2}$ are block diagonal matrices, which are defined as

$$\Phi(t) = \begin{bmatrix} z_1(t)z_1^T(t) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & z_n(t)z_n^T(t) \end{bmatrix}$$

and

$$\Phi_I(t) = \begin{bmatrix} proj(z_1(r)z_1^T(r)) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & proj(z_n(r)z_n^T(r)) \end{bmatrix}.$$

Theorem 2. *The origin of the parameter estimation error dynamics (2.55) is Lyapunov stable, in addition if the Assumption 5 is satisfied, then $||\Delta\theta(t)||$ exponentially converges to zero, $\forall t \geq t_0 + T$ i.e.,*

$$||\Delta\theta(t)|| \leq \gamma_1 e^{-\gamma_2 t},\ \forall t \geq t_0 + T \tag{2.56}$$

for some positive scalars γ_1 and γ_2.

Proof. Consider the following Lyapunov candidate

$$V_1(\Delta\theta) = \frac{1}{2}\Delta\theta^T\Delta\theta \tag{2.57}$$

Taking the time derivative of (2.57) along the dynamics (2.55) yields

$$\dot{V}_1(\Delta\theta) = -\Delta\theta^T\Big((L\otimes I_2) + k_\theta\Phi(t) + \Gamma_\theta\Phi_I(t)\Big)\Delta\theta \leq 0 \tag{2.58}$$

which implies $V_1(\Delta\theta) \in \mathcal{L}_\infty$ and it is non-increasing in time $\forall t \geq t_0$, i.e., the origin of the dynamics of $\Delta\theta(t)$ is Lyapunov stable.

Let's assume the matrix $J(t)$, which is defined as

$$J(t) \triangleq (L \otimes I_2) + \Phi_I(t) \triangleq (L \otimes I_2) + proj(\Phi(t)) \tag{2.59}$$

then by referring the proof of Theorem 1 from (Garg and Roy, 2019*a*), it can be concluded that $J(t_0 + T) > 0,\ \forall t \geq t_0 + T$.

Hence, (2.58) can be upper bounded as

$$\dot{V}_1(\Delta\theta) \leq -\Delta\theta^T J(t)\Delta\theta \leq -\lambda_{min}(J(t))||\Delta\theta||^2 \tag{2.60}$$

using the same argument as in Property 4, $J(t) \geq J(t_0 + T) > 0$, $\forall t \geq t_0 + T$, which implies that $\lambda_{min}(J(t)) \geq c > 0$, where the λ_{min} is the minimum eigenvalue of matrix $J(t)$ and c is positive real constant.

Using (2.57), (2.60) can be expressed as

$$\dot{V}_1(\Delta\theta) \leq -2cV_1(\Delta\theta),\ \forall t \geq t_0 + T \tag{2.61}$$

This differential inequality leads to the following exponentially convergent bound on $V(\Delta\theta)$

$$V_1(\Delta\theta(t)) \leq V_1(\Delta\theta(t_0 + T))e^{-2c(t-t_0-T)}, \quad \forall t \geq t_0 + T \tag{2.62}$$

From (2.57), $||\Delta\theta(t)|| = \sqrt{2V_1(\Delta\theta(t))}$, which implies that $||\Delta\theta(t)||$ is exponentially convergent to zero for $t \geq t_0 + T$, i.e., (2.56) holds true. Since $V_1(\Delta\theta(t))$ in (2.57) is radially unbounded, the mentioned result is globally valid. □

Remark 3. *From classical control literature, it is well established that an integral action in conjunction with a proportional control improves the steady-state accuracy. Motivated by the power of integral action, the integral-like component I is introduced in the update law, which reduces the steady state parameter estimation error. The integral term in conjunction with the cooperative term C circumvents the restrictive C-PE condition for parameter convergence while requiring a milder condition of C-IE.* □

2.10 Tracking-Error Stability/Convergence Analysis

The tracking-error dynamics $\zeta(t)$ for all n number of vehicles in the platoon are compactly represented as

$$\dot{\zeta}(t) = \Big(I_n \otimes (A_r - lI_n)\Big)\zeta(t) + (\mathbf{1}_n^T \otimes B_u)\tilde{K}^T(t)x(t) \tag{2.63}$$

where $\zeta(t) \in \mathbb{R}^{n4}$, $x(t) \in \mathbb{R}^{n4}$, $\tilde{K}(t) \in \mathbb{R}^{n4}$ and $\mathbf{1}_n = \{1, \dots, 1\} \in \mathbb{R}^{1\times n}$.

The compact representation for $\hat{K}(t)$ dynamics from (2.42) is given as

$$\dot{\hat{K}}(t) = Proj_{\Omega}(-\Gamma(\mathbf{1}_n \otimes \hat{B}_u^T)(I_n \otimes P)\zeta x^T, K) \tag{2.64}$$

where the Proj operator in (2.64) ensures that $\hat{K}(t)$ remains within a compact set for all time (Lavretsky and Wise, 2013).

Theorem 3. *For the system (2.36) along with control input (2.37), the parameter update laws (2.55) and (2.64), and the overall error dynamics $\eta(t) = [\zeta^T(t), \Delta\theta^T(t), \tilde{K}^T(t)]^T$ is Lyapunov stable $\forall t \geq t_0 + T$, provided Assumption 5 holds. In addition, the tracking-error $\zeta(t)$ tends to zero asymptotically with asymptotic string stability i.e.,* $\lim_{t\to\infty}[x_i(t) - x_{i,r}(t)] = 0,\ \forall i \in S_n$.

Proof. Consider the following Lyapunov candidate.

$$V_2 = \frac{1}{2}\zeta^T(I_n \otimes P)\zeta + \frac{1}{2}Tr(\tilde{K}^T\Gamma^{-1}\tilde{K}) + \frac{1}{2}\Delta\theta^T\Delta\theta \tag{2.65}$$

Taking the time derivative of (2.65) along the system trajectories, yields

$$\dot{V}_2 = \frac{1}{2}\zeta^T(I_n \otimes P)\dot{\zeta} + \frac{1}{2}\dot{\zeta}^T(I_n \otimes P)\zeta + Tr(\tilde{K}^T\Gamma^{-1}\dot{\tilde{K}}) + \Delta\theta^T\Delta\dot{\theta} \tag{2.66}$$

after putting (2.63), (2.64) and using the compact representation of Lyapunov equation (2.43) and resulting argument from proof of Theorem 2, the relation in (2.66) can be modified as

$$\dot{V}_2 \leq -\frac{1}{2}\zeta^T(I_n \otimes Q)\zeta - l\zeta^T(I_n \otimes P)\zeta - k_1||\Delta\theta||^2 + \zeta^T(I_n \otimes P)(\mathbf{1}_n^T \otimes \tilde{B}_u)\tilde{K}^T x,\ \forall t \geq t_0 + T \tag{2.67}$$

where k_1 is the $\lambda_{min}(J(t_0+T))$ and the inequality is due to (2.64). Further (2.67) can be upper-bound as

$$\dot{V}_2 \leq -\frac{1}{2}\lambda_{min}(I_n \otimes Q)||\zeta||_2^2 - l\lambda_{min}(I_n \otimes P)||\zeta||_2^2 - k_1||\Delta\theta||_2^2 + \zeta^T(I_n \otimes P)(\mathbf{1}_n^T \otimes \tilde{B}_u)\tilde{K}^T x \tag{2.68}$$

where $\lambda_{min}(\cdot)$ is the minimum eigenvalue of the specified matrix. From (2.40), (2.64) and by considering the proof of Theorem 2, (2.68) can be further modified as

$$\begin{aligned}\dot{V}_2 \leq & -\frac{1}{2}\lambda_{min}(I_n \otimes Q)||\zeta||_2^2 - l\lambda_{min}(I_n \otimes P)||\zeta||_2^2 - k_1||\Delta\theta||_2^2 \\ & + ||(I_n \otimes P)||_F||\Delta\theta||_2||\tilde{K}^T||_2||\zeta||_2^2 + ||(I_n \otimes P)||_F||\Delta\theta||_2||\tilde{K}^T||_2||\zeta||_2||x_{i,c}||_2\end{aligned} \tag{2.69}$$

where $||\cdot||_F$ denotes the Forbenious norm of a matrix and $||\cdot||_2$ denotes the 2-norm, which is used for vectors.

Since $||I_n \otimes P||_F \in \mathcal{L}_\infty$, $||\Delta\theta||_2 \in \mathcal{L}_\infty$ based on proof of Theorem 2 and $||\tilde{K}||_2 \in \mathcal{L}_\infty$, using $proj(.)$ operator, and (2.69) can be restructured as

$$\begin{aligned}\dot{V}_2 \leq & -\frac{1}{2}\lambda_{min}(I_n \otimes Q)||\zeta||_2^2 - l\lambda_{min}(I_n \otimes P)||\zeta||_2^2 - k_1||\Delta\theta||_2^2 + \delta_3||\Delta\theta||_2||\zeta||_2^2 \\ & + \delta_4||\Delta\theta||_2||\zeta||_2\end{aligned} \tag{2.70}$$

where $\delta_3, \delta_4 > 0$. If the tracking-error $\zeta(t)$ fulfills the following condition

$$||\zeta||_2 \leq \frac{m-\delta_4}{\delta_3} \tag{2.71}$$

where $m > \delta_4$, the following inequality can be written.

$$\delta_3||\zeta||_2^2 + \delta_4||\zeta||_2 \leq m||\zeta||_2. \tag{2.72}$$

Hence, based on (2.72), inequality (2.70) can be further simplified as

$$\dot{V}_2 \leq -\frac{1}{2}\lambda_{min}(I_n \otimes Q)||\zeta||_2^2 - l\lambda_{min}(I_n \otimes P)||\zeta||_2^2 + m||\zeta||_2||\Delta\theta||_2 - k_1||\Delta\theta||_2^2 \tag{2.73}$$

It can be deduced that if the subsequent gain condition

$$k_1 > \frac{m^2}{4l\lambda_{min}(I_n \otimes P)} \tag{2.74}$$

is satisfied, then $\dot{V}_2(t)$ is negative semi-definite. Therefore, $V_2 \in \mathcal{L}_\infty$ which imply that the overall error dynamics $\eta(t) \in \mathcal{L}_\infty$. Further, the tracking-error $\zeta(t)$ can be upper-bounded by the following inequality

$$||\zeta(t)|| \leq \sqrt{\frac{2V(t)}{\Lambda_{min}(I_n \otimes P)}} \quad \forall t \geq t_0. \tag{2.75}$$

Since $\dot{V}_2 \leq 0$, $\forall t \geq t_0$, the inequality in (2.75) can be alternatively expressed as

$$||\zeta(t)|| \leq \sqrt{\frac{2V(t_0+T)}{\Lambda_{min}(I_n \otimes P)}} \quad \forall t \geq t_0 + T. \tag{2.76}$$

The Lyapunov function $V_2(t)$ can be further upper-bounded as

$$V_2(t) \leq \frac{1}{2}\Big(\lambda_{max}(I_n \otimes P)||\zeta(t)||_2^2 + \lambda_{min}^{-1}(\Gamma)||\tilde{K}^T(t)||^2 + ||\Delta\theta(t)||_2^2\Big), \quad \forall t \geq t_0. \tag{2.77}$$

Selecting $||\hat{\theta}(0)|| \leq \delta_1$ and considering Assumption 4, it can be claimed that

$$||\Delta\theta(t)|| \leq 2\delta_1 \quad \forall t \geq t_0. \tag{2.78}$$

Therefore, using (2.77), (2.78) and (2.64), an upper bound of $\zeta(t)$ can be calculated analytically from (2.76) as

$$\zeta(t) \leq \underbrace{\sqrt{\frac{\lambda_{max}(I_n \otimes P)||\zeta(t_0+T)||_2^2 + 2\lambda_{min}^{-1}(\Gamma)\delta_2^2 + 2\delta_1^2}{\lambda_{min}(I_n \otimes P)}}}_{\nu}, \; \forall t \geq t_0 + T. \tag{2.79}$$

Where δ_2 is the upper bound $\tilde{K}(t)$, based on $proj$ operator. Since the stability proof requires (2.71) to be satisfied, it implies that the error bound ν should be less than $\frac{m-\delta_4}{\delta_3}$. Thus, m is chosen in such a way that, it should satisfying the following inequality

$$m > \delta_4 + \delta_3\nu \tag{2.80}$$

where a crude estimate of ν is utilized using $\zeta(t_0)$. The choice of m, which satisfies (2.80) is finally used in (2.70) to derive the sufficient gain condition for Lyapunov stability. Further using Barbalat's Lemma (Lemma 8.2 at 323 page (Khalil and Grizzle, 2002)) on (2.73), it can be conclude that the tracking-error $\zeta(t)$ is asymptotically converging to zero. $\square$

Remark 4. *Since the actual system (2.36) is a linear system with the right-hand side to be globally Lipschitz, using Global Existence and Uniqueness Theorem (Theorem 3.2 at page 93 (Khalil and Grizzle, 2002)), it can be claimed that the system dynamics will remain bounded in the initial time-window $[t_0, t_0 + T)$. Moreover, note that there is a crucial difference between the gain condition (2.74) and a similar gain condition obtained in (Roy* et al.*, 2017*a*) for a single agent linear MRAC problem. Unlike (Roy* et al.*, 2017*a*), the gain condition is shared between l and k_1. Hence, due to the introduction of the closed-loop reference model, the burden of the gain condition is divided between the closed-loop gain l and the distributed estimator gain k_1.* $\square$

2.11 Simulation Results

The proposed algorithm is simulated by considering the protocol as in Figure 2.8, where 3 unknown homogeneous vehicles are forming a vehicle platoon using a virtual leader, which have following system parameters.

$\tau_0 = 0.1$, $\Omega_0 = 1$, $\tau = 0.4$, $\Omega = 0.8$, $\forall i \in \{1, 2, 3\}$. The time gap $h = 0.7s$. The baseline controllers' gains are chosen as $K_p = 0.2$ and $K_d = 0.7$ in order to maintain both string stability conditions (2.33) and (2.34). The desired acceleration is selected as $u_r(t) = 80exp(-2t)$; the design parameter l is chosen as $l = 5$, the adaptation gains are chosen as $k_\theta = 5$, $\Gamma_\theta = 5$.

Figure 2.9 shows the convergence of the norm of tracking error $\zeta(t)$ to zero and Figure 2.10 shows the norm of controller parameter estimation error $\tilde{K}(t) \in \mathcal{L}_\infty$. Figure 2.11 shows the norm of error between the CACC reference model (2.30) and the CACC closed-loop reference model (2.38). From Figure 2.11, it can be concluded that

$\lim_{t\to\infty}[x_i(t) - x_{i,r}(t)] = 0,\ \forall i \in \{1, 2, 3\}$, which is the primary design objective. Figure 2.12 represents the comparison of velocity profile, which portrays velocity synchronization of the entire platoon. Figure 2.13 represents the comparison of the norm of parameter estimation-error $\Delta\theta(t)$ for various cases, like P: means parameter estimator consist only proportional error like the term, $P + C$: means only proportional and consensus terms are used in the estimator, $P + I + C$: means all terms including integral term are used. Moreover from Figure 2.13, it can be inferred that integral and consensus terms are significant in achieving the exponential convergence under the C-IE condition. The C-IE condition is satisfied approximately after some finite time as verified in the simulation by checking the determinant of the matrix $J(t)$.

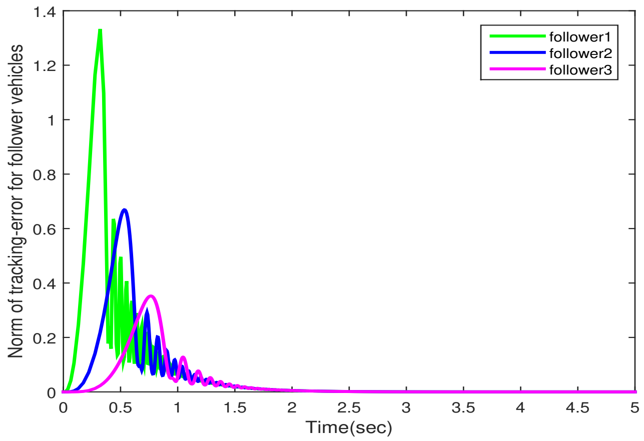

Figure 2.9: Norm of tracking-error $\zeta(t)$ for uncertain vehicles in the platoon.

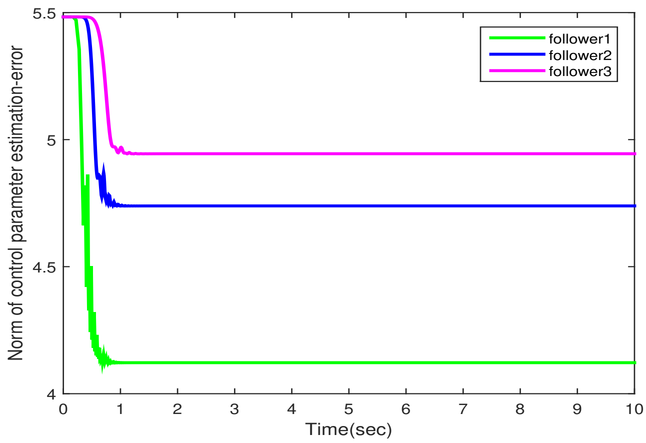

Figure 2.10: Norm of controller parameter estimation-error $\tilde{K}(t)$ for uncertain vehicles in the platoon.

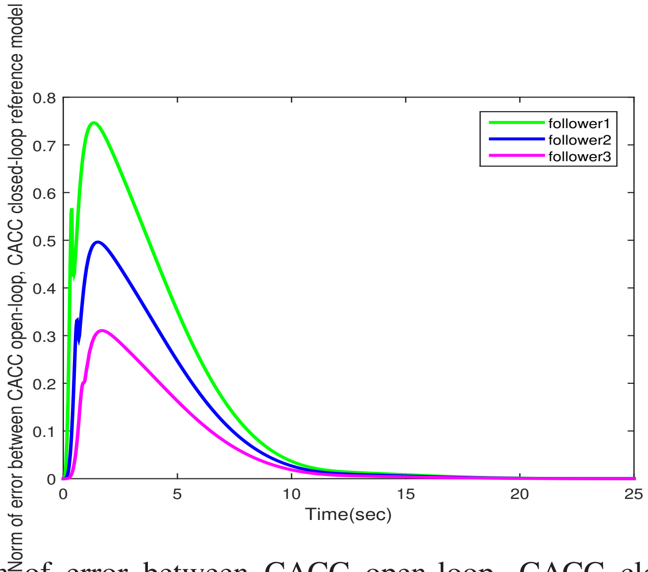

Figure 2.11: Norm of error between CACC open-loop, CACC closed-loop reference model for uncertain vehicles in the platoon.

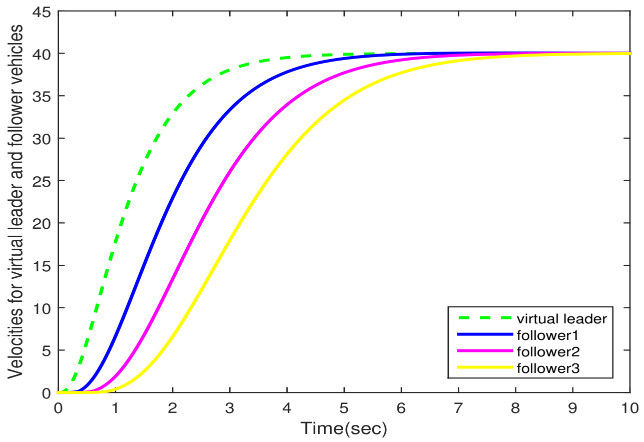

Figure 2.12: Comparison of velocities for virtual leader as well as uncertain vehicles in the platoon.

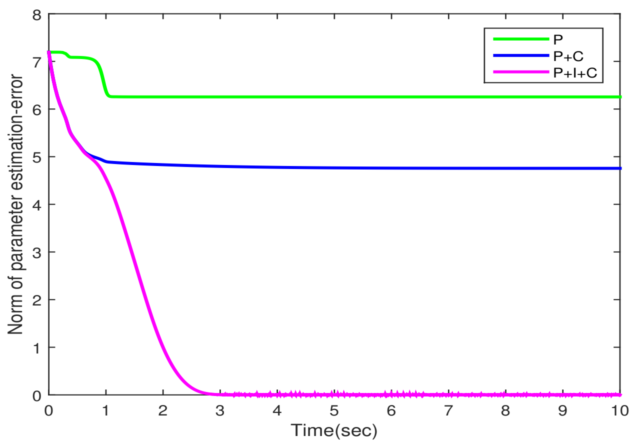

Figure 2.13: Comparison of norm of parameter estimation error $\Delta\theta(t)$.

2.12 Conclusion

This chapter proposes a combined CACC architecture using a closed-loop reference model based MRAC algorithm for a homogeneous platoon, without knowledge of the platoon vehicle dynamics parameters Ω (engine performance) and τ (driveline constant). The method is composed of a novel distributed consensus-based plant parameter estimator in conjunction with a differential adaptive update law for the control parameters. Provided the set of filtered regressors $z_i(t)$, $\forall i \in S_n$, is C-IE, the algorithm guarantees exponential convergence of parameter estimation error $\Delta\theta(t)$ as well as asymptotic convergence of tracking error $\zeta(t)$ to zero. The C-IE condition is milder than all the conditions for parameter convergence available in literature like PE, C-PE, IE, etc. The use of the closed-loop reference model instead of the open-loop reference model in MRAC protocol provides an additional design freedom in the CACC algorithm, which is used to achieve better transient as well as stability guarantee.

Chapter 3

Distributed Adaptive Parameter Estimation without Persistence of Excitation along with Control Application

Part III

Distributed Adaptive Estimation without Persistence of Excitation: an Online Optimization Perspective

3.1 Introduction

In this chapter (part-1), the proposed work extends the concept of IE/FE in the context of a multi-agent setting while invoking the notion of C-IE for distributed estimation. The work builds on the formulation of the papers (Papusha *et al.*, 2014; Chen *et al.*, 2013; Stegagno and Yuan, 2019; Wensing and Slotine, 2018) which consider a network of multi-agent architecture for online parameter estimation. The proposed work develops a distributed consensus-based switched parameter estimator where strategic multiple switching is incorporated to reflect the effect of C-IE condition ensuring parameter convergence. It is analytically proved that the estimation error dynamics shows global exponential stability (GES) under the C-IE condition, which is in contrast to (Papusha *et al.*, 2014; Chen *et al.*, 2013; Stegagno and Yuan, 2019; Wensing and Slotine, 2018), where stringent C-PE condition is required to obtain a similar stability result. The concept of cooperative initial/finite excitation is introduced in a few recent works (Yuan *et al.*, 2021, 2018; Rezaei and Stefanovic, 2020; Poveda *et al.*, 2019; Garg and Roy, 2020*a*,*c*), however, all of them have considered a bidirectional communication among neighboring agents, i.e., an undirected graph topology is utilized. Unlike these results, the proposed work allows unidirectional communication among agents as long as the directed graph is balanced in nature. Further, the formulation is molded as an online optimization problem. A novel convex cost function is conceptualized in such a way that the proposed distributed estimator acts as a distributed gradient-descent of the cost. Moreover, the cost function is proved to be strongly convex under the C-IE condition thereby making the estimator dynamics a unique global minimizer of the cost. Simulation results reflect the effectiveness of the proposed algorithm in contrast to traditional C-PE based estimators.

3.2 Preliminaries

3.2.1 Preliminaries on Algebraic Graph Theory

The communication topology among n number of agents are described by a directed graph (Godsil and Royle, 2001). Let $G \triangleq (V, E)$ be a directed graph, where $V \triangleq \{1, \ldots, n\}$ is a set of n vertices, $E \subseteq V \times V$ is a set of edges. In a directed graph, an edge $(i, j) \in E$

signifies that j^{th} agent can get information from i^{th} agent but converse is not true. In this case, i^{th} agent is called a neighbor of j^{th} agent. Here, N_i is defined as the set of neighbors for the i^{th} agent, defined as: $N_i \triangleq \{j \in V \mid (j,i) \in E\}$. Note that self edges (i,i) are not allowed, thus $(i,i) \notin E$ and $i \notin N_i$. The adjacency matrix of the graph G is defined as $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, where $a_{ij} = 1$ if $(j,i) \in E$ and $a_{ij} = 0$ if $(j,i) \notin E$. A directed path from node i to node j is a sequence of edges in a directed graph. If there is a path between every pair of nodes in a graph, then the graph is connected. A directed graph is strongly connected, if there exists a directed path connecting every pair of nodes. The degree of the vertex i, denoted as, deg(i), is the number of neighbors $|N_i|$ available to that vertex. The Laplacian matrix $L \in \mathbb{R}^{n\times n}$ of the graph G is defined as $L \triangleq D - A$, where $D \triangleq diag\{deg(1), \ldots, deg(n)\} \in \mathbb{R}^{n\times n}$ is the degree matrix. A node is balanced if its in-degree equals its out-degree, i.e., $\sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} a_{ji}$. A directed graph is balanced if all its nodes are balanced.

3.3 Problem Formulation

3.3.1 Multi-Agent System (MAS) Architecture

The communication topology for a MAS architecture, comprising n number of agents, is represented by the graph G, where each vertex is treated as an agent and edge $E = (i,j) \in V$, denote the communication link between i^{th}agent and j^{th}agent. The output of each agent $y_i(t) \in \mathbb{R}$ is defined subsequently.

$$y_i(t) = \phi_i^T(t)\theta,\ \forall t \geq t_0,\ \forall i = 1(1)n \tag{3.1}$$

where $\theta \in \mathbb{R}^p$ is a set of constant unknown parameters and $\phi_i(t) \in \mathbb{R}^p$ is a known, continuous, and uniformly bounded basis functions/feature vectors (also known as “regressor” in adaptive control literature).

Assumption 6. *Graph G, which represents the communication topology among n number of agents, is directed strongly connected and balanced.* □

3.3.2 Objective

The objective of this work is to develop a distributed adaptive parameter estimation algorithm, using the online measurements of input $\big(\phi_i(t)\big)$, output $\big(y_i(t)\big)$ of the model (3.1) while collaborating (sharing instantaneous information) with the neighbouring agents, such that

$$\|\hat{\theta}_i(t) - \theta\| \to 0 \; as \; t \to \infty, \; \forall i = 1(1)n, \tag{3.2}$$

without requiring the restrictive C-PE condition.

Here, $\hat{\theta}_i(t) \in \mathbb{R}^p$ denotes the estimate of unknown parameter θ by i^{th} agent. While designing the estimator, local information flow i.e., communication from agent j to agent i is allowed only if $j \in N_i$, which is based on Assumption 6 (see Figure 3.1 for more illustration).

Lemma 1. *For a strongly connected directed graph G, Laplacian matrix $L \in \mathbb{R}^{n\times n}$ has the following properties.*

(1) It has a simple zero eigenvalue corresponding to the right eigenvector $\mathbf{1}_n$, and all non zero eigenvalues have positive real part.

(2) Let $s = [s_1, s_2, \ldots, s_n]^T$, $s_i \in \mathbb{R}_{>0}$, $i = 1(1)n$, be the left eigenvector of L associated with the zero eigenvalue and $S = diag(s_1, s_2, \ldots, s_n)$. Then, $min_{s^T z=0, z\neq 0}$ $\frac{z^T\tilde{L}z}{z^Tz} > \frac{\lambda_2(\tilde{L})}{n}$, where $\tilde{L} \triangleq SL + L^TS \geq 0$. Moreover, $s = \mathbf{1}_n$ iff G is strongly connected and balanced.

Proof. For proof refer to Lemma 3 (chapter 1) (Li and Duan, 2017). □

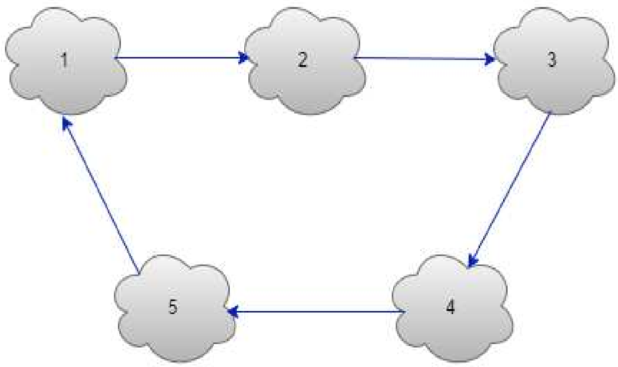

Figure 3.1: Communication topology for $n = 5$ agents based on Assumption 6, where blue arrow represents the communication link between agents.

Further subsequent assumptions are considered to facilitate the distributed adaptive law design.

Assumption 7. $\phi_i(t),\ \dot{\phi}_i(t) \in \mathcal{L}_\infty,\ \forall i = 1(1)n.$ $\square$

Assumption 8. *The group of regressors $\phi_i(t)'s$, $\forall i = 1(1)n$, satisfy the C-IE condition (as per the Definition 4).* $\square$

3.4 Distributed Adaptive Parameter Estimator Design

This section elaborates the implementation procedure for the distributed adaptive parameter estimator design.

3.4.1 Filter Dynamics

Classical distributed adaptive parameter estimation algorithms demand C-PE condition (Papusha *et al.*, 2014; Chen *et al.*, 2013; Wensing and Slotine, 2018; Stegagno and Yuan, 2019) for parameter convergence, which is restrictive in the sense that it requires excitation/richness of the information content over the infinite time window. A layer of filter dynamics are introduced (motivated from (Kreisselmeier, 1977; Cho *et al.*, 2017; Roy *et al.*, 2017*b*)) to overcome this C-PE limitation and instead exploit the benefit of C-IE, $\forall i = 1(1)n$

$$\dot{Y}_{C,i}(t) = -\delta Y_{C,i}(t) + \phi_i(t)\phi_i^T(t),\ Y_{C,i}(t_0) = 0 \tag{3.3}$$

$$\dot{W}_{C,i}(t) = -\delta W_{C,i}(t) + \phi_i(t)y_i(t),\ W_{C,i}(t_0) = 0 \tag{3.4}$$

where $Y_{C,i}(t) \in \mathbb{R}^{p\times p}$ denotes the filtered regressor, $W_{C,i}(t) \in \mathbb{R}^p$ is known as the filtered output and $\delta \in \mathbb{R}_{>0}$ is a positive constant tuning parameter.

Remark 5. *An open-loop counter-part of these closed-loop filter dynamics (3.3)-(3.4) (i.e., with $\delta = 0$) is used in the literature (Adetola and Guay, 2008, 2010; Krause and Khargonekar, 1987) to estimate the unknown parameters, however, the open-loop version may lead to unboundedness of the filter dynamics if the computation is continued* *for infinite time. Also, in the appearance of bounded disturbance in the dynamics, and measurement noise in the sensors, these open-loop integrals can lead to instability of the overall error dynamics. These issues can be overcome by employing a closed-loop stable filter design as in (3.3)-(3.4).* $\square$

Taking integration of (3.3), (3.4) and using (3.1), it can be concluded that

$$Y_{C,i}(t)\theta = W_{C,i}(t),\ \forall t \geq t_0,\ \forall i = 1(1)n. \tag{3.5}$$

From (3.3), the square matrix $Y_{C,i}(t)$, $\forall i = 1(1)n$ can be expressed as

$$Y_{C,i}(t) = \underbrace{exp\{-\delta t\}}_{\geq 0} \int_{t_0}^{t} \underbrace{exp\{\delta r\}}_{\geq 1}\ \underbrace{\phi_i(r)\phi_i^T(r)}_{\geq 0}\, dr \tag{3.6}$$

using (3.6), the following property can be deduced to be utilized subsequently.

Property 5. *$Y_{C,i}(t)$ is positive semi-definite function of time i.e., $Y_{C,i}(t) \geq 0$, $\forall t \geq t_0$, $\forall i = 1(1)n$.* $\square$

The following subsection analyzes several features of the second-layer filter outputs in terms of two lemmas, which acts as a cornerstone of the proposed distributed adaptive estimator design.

Insights to the filtered regressors $Y_{C,i}(t)$s

Lemma 2. *rank$\big(Y_{C,i}(t)\big)$=rank$\big(Y_{O,i}(t)\big)$, $\forall t \in [t_0, \infty)$, where $Y_{O,i}(t)$ is described as*

$$\dot{Y}_{O,i}(t) = \phi_i(t)\phi_i^T(t),\ Y_{O,i}(t_0) = 0,\ \forall i = 1(1)n \tag{3.7}$$

and the operator rank($\cdot$) implies the rank of the argument matrix.

Proof. From (3.7), the regressor $Y_{C,i}(t)$ can be upper bounded as

$$\begin{aligned} Y_{C,i}(t) &= \int_{t_0}^{t} exp\{-\delta(t-r)\}\phi_i(r)\phi_i^T(r)dr \\ &\leq \int_{t_0}^{t} \phi_i(r)\phi_i^T(r)dr = Y_{O,i}(t) \end{aligned} \tag{3.8}$$

further using (3.7), the regressor $Y_{C,i}(t)$ can be lower bounded as

$$\begin{aligned} Y_{C,i}(t) &\geq exp\{-\delta(t-t_0)\} \int_{t_0}^{t} \phi_i(r)\phi_i^T(r)dr \\ &= exp\{-\delta(t-t_0)\}Y_{O,i}(t) \end{aligned} \tag{3.9}$$

clubbing (3.8) and (3.9), $\forall i = 1(1)n$, yields

$$exp\{-\delta(t-t_0)\}Y_{O,i}(t) \leq Y_{C,i}(t) \leq Y_{O,i}(t),\ \forall t \geq t_0 \tag{3.10}$$

from (3.10), Proposition 2 (Appendix A), proof of the Lemma 2 can be inferred. □

Lemma 3. *rank$\big(Y_{C,i}(t_0 + T^i_{m_i})\big)$=$m_i$, where $m_i \in \mathbb{I}^+$, and $T^i_{m_i} \in \mathbb{R}_{\geq 0}$, implies that rank$\big(Y_{C,i}(t)\big) \geq m_i$, $\forall t \in [t_0 + T^i_{m_i}, \infty)$, $\forall i = 1(1)n$.*

Proof. From (3.7), it can be inferred that

$$Y_{O,i}(t_2) \leq Y_{O,i}(t_1),\ for\ t_2 > t_1 > t_0,\ \forall i = 1(1)n \tag{3.11}$$

using Lemma 2, rank$\big(Y_{C,i}(t_0 + T^i_{m_i})\big)$= rank$\big(Y_{O,i}(t_0 + T^i_{m_i})\big)$= m_i. Further using (3.11), it can be deduced that

$$Y_{O,i}(t) \geq Y_{O,i}(t_0 + T^i_{m_i}),\ \forall t \in [t_0 + T^i_{m_i}, \infty) \tag{3.12}$$

$$\implies rank\big(Y_{O,i}(t)\big) \geq m_i$$

$$\implies rank\big(Y_{C,i}(t)\big) \geq m_i,\ \forall t \in [t_0 + T^i_{m_i}, \infty) \tag{3.13}$$

where (3.13) is deduced from Lemma 2. □

3.5 Proposed Distributed Adaptive Parameter Estimator

Consider the following distributed adaptive parameter estimator, $\forall i = 1(1)n$

$$\begin{aligned}\dot{\hat{\theta}}_i(t) = &- \gamma\phi_i(t)\underbrace{\phi_i^T\Delta\theta_i(t)}_{E_P} - \gamma_1\underbrace{\Big(Y_{C,i}(t)\hat{\theta}_i(t) - W_{C,i}(t)\Big)}_{E_{FC}} - \gamma_2\underbrace{\Big(Y_{SW,i}(t)\hat{\theta}_i(t) - W_{SW,i}(t)\Big)}_{E_{SW}} \\ &+ \underbrace{\sum_{j\in N_i} a_{ij}\big(\hat{\theta}_j(t) - \hat{\theta}_i(t)\big)}_{E_C}\end{aligned} \tag{3.14}$$

Constant estimation gains $\gamma,\ \gamma_1,\ \gamma_2 \in \mathbb{R}_{>0}$ controls the local information fusion rate, $Y_{SW,i}(t) \in \mathbb{R}^{p\times p}$, $W_{SW,i}(t) \in \mathbb{R}^p$ are the piece-wise constant switching signals, which are defined as

$$W_{SW,i}(t) \triangleq \begin{cases} W_{C,i}\big(t_0 + T_{k_i}^i\big), & \forall t \in \big[t_0 + T_{k_i}^i,\ t_0 + T_{k_i+1}^i\big) \\ W_{C,i}\big(t_0 + T_{z_i}^i\big), & \forall t \in \big[t_0 + T_{z_i}^i,\ \infty\big) \end{cases} \tag{3.15}$$

$$Y_{SW,i}(t) \triangleq \begin{cases} Y_{C,i}\big(t_0 + T_{k_i}^i\big), & \forall t \in \big[t_0 + T_{k_i}^i,\ t_0 + T_{k_i+1}^i\big) \\ Y_{C,i}\big(t_0 + T_{z_i}^i\big), & \forall t \in \big[t_0 + T_{z_i}^i,\ \infty\big) \end{cases} \tag{3.16}$$

where $i = 1(1)n$ and $k_i = 0(1)z_i - 1$ and $T_0^i \triangleq 0$. Moreover, $\Delta\theta_i(t) \in \mathbb{R}^p$ denotes the parameter estimation error.

$T_{k_i}^i$ is the time instant such that $t \geq t_0 + T_{k_i}^i$ implies that $rank\{Y_{C,i}(t)\} \geq k_i$, $\forall i = 1(1)n$.

$T_{z_i}^i$ is time instant such that $t \geq t_0 + T_{z_i}^i$ implies that $rank\{Y_{C,i}(t)\} = z_i$, $\forall i = 1(1)n$. The variable z_i is used to denote the highest rank that $Y_{C,i}(t)$ attains online.

In (3.14), E_P component comprises proportional type local prediction error, E_{FC} comprises integral type of local prediction error, E_{SW} comprises switching based local prediction error, whereas the other term E_C represents neighbouring interaction among the agents using communication graph topology.

Remark 6. *The main reason for embedding the switching signals (3.15)-(3.16) in the above proposed novel estimator (3.14) is the forgetting factor $\delta > 0$ in stable closed-loop filter dynamics (3.3)-(3.4), which leads to exponential vanishing of the richness of*

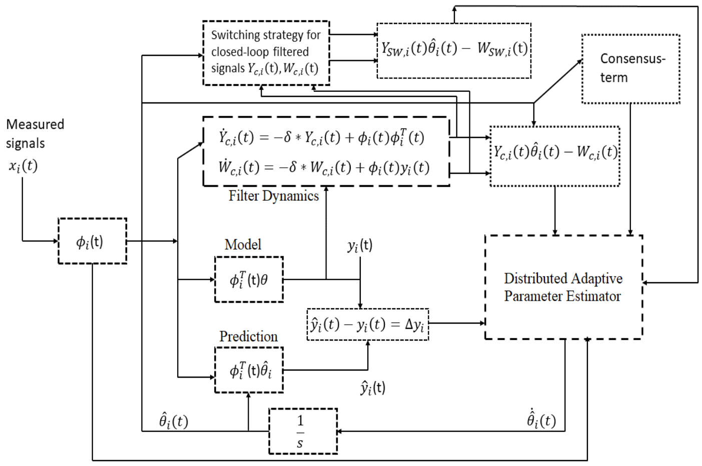

Figure 3.2: Block Diagram of the Proposed Distributed Adaptive Parameter Estimator Architecture.

information to zero as the time approaches infinity; whereas switching signals captures the occurrence of C-IE condition as revealed subsequently in the stability analysis. □

Remark 7. *Since the knowledge of T in Assumption 8 is not known a priori, the online estimator (3.14) switches multiple times at $t = T^i_{k_i}$ for $i = 1(1)n$ and $k_i = 0(1)z_i - 1$ as the information about the time instants $T^i_{k_i}$ is gradually acquired. The proposed switched parameter estimator is implementable in real-time since knowledge of $T^i_{k_i}$ can be gathered online by scrutinizing the rank of $Y_{C,i}(t)$, $\forall i = 1(1)n$. The instant, the rank of $Y_{C,i}(t)$ attains $k_i \leq z_i \leq n$, the above formulated switching concept (3.15), (3.16) comes into picture. However, $T^i_{z_i}$ is the time where $Y_{C,i}(t)$ attain its highest possible rank after the multiple switching, which analytically implies that $T^i_{z_i} \leq T$, $\forall i = 1(1)n$. To make the switching based rank-check operation practically verifiable, rank checking can be performed in a regular interval, say at, $t = w^i\eta$, $\forall w^i \in \mathbb{I}^+$, $\forall i = 1(1)n$, rather than continuously. In that case, $\exists\, h^i \in \mathbb{I}^+$ such that $T^i_{k_i} \in ((h^i - 1)\eta,\ h^i\eta]$, and the switching is performed in the update law at $t = t_0 + h^i\eta$, instead of $t = t_0 + T^i_{k_i}$, $\forall k_i = 1(1)z_i$, $\forall i = 1(1)n$, which will not hamper the rank property as per Lemma 3.* □

For subsequent formulation of stability/convergence analysis, a C-IE assumption based Lemma is needed to formulate as.

Lemma 4. *Provided Assumption 8 holds, the matrix $M(t) \triangleq \big((\tilde{L} \otimes I_p) + \gamma_2\Phi_{SW}(t)\big)$ is uniformly strictly positive-definite over the time window $[t_0 + T, \infty)$ i.e.,*

$$Y^T M(t) Y > 0,\ \forall t \geq t_0 + T,\ \forall Y \in \mathbb{R}^{np} - \{\boldsymbol{0}_{np}\}. \tag{3.17}$$

Proof. Since $\tilde{L}\mathbf{1}_n = 0$ based on Lemma 1, that implies

$$(\tilde{L} \otimes I_p)\Big(\frac{1}{\sqrt{n}}\mathbf{1}_n \otimes f_j\Big) = 0 \tag{3.18}$$

Moreover, the eigen-decomposition form of the matrix $(\tilde{L} \otimes I_p)$ is

$$(\tilde{L} \otimes I_p)(p_i \otimes f_j) = \lambda_i(p_i \otimes f_j) \tag{3.19}$$

whereas the right hand side of both the above equations can be verified on the basis of the mixed product property $PQ \otimes RS = (P \otimes R)(Q \otimes S)$ for appropriately sized matrices P, Q, R, and S. And $\lambda_i > 0$, p_i for all $i = \{2, \ldots, n\}$ and $f_j \in \mathbb{R}^p$ is the jth unit vector for all $j = \{1, \ldots, p\}$. Considering Y^1 as a unit vector and expressing it in this basis as

$$Y = \sum_{j=1}^{p} \varsigma_j \frac{1}{\sqrt{n}}\mathbf{1}_n \otimes f_j + \sum_{i=2}^{n}\sum_{j=1}^{p} \delta_{ij} p_i \otimes f_j \tag{3.20}$$

with $(\varsigma, \delta) \in \mathbb{R}^p \times \mathbb{R}^{(n-1)p}$ has unit norm. Define the quantity $\Phi_{bar}(t)$, which is the time average of quantity $\Phi_P(t)$ over the time interval $[t_0, t_0 + T]$

$$\Phi_{bar}(t) \triangleq \frac{1}{T} \int_{t_0}^{t_0+T} \Phi_P(r) dr \tag{3.21}$$

by substituting relation (3.20) into (3.17), using the first term $Y^T(\tilde{L} \otimes I_p)Y$ of (3.17) can be bounded based on Lemma 1 as

$$Y^T(\tilde{L} \otimes I_p)Y = \sum_{i=2}^{n}\sum_{j=1}^{p} \lambda_i \delta_{ij}^2 \geq \lambda_2(1 - ||\varsigma||_2^2)$$

where $||\varsigma||_2^2 + ||\delta||_2^2 = 1$.

[1] $f_j \in \mathbb{R}^p$ is the j^{th} unit vector, $\forall j = \{1, \ldots, p\}$, i.e., f_j is the j^{th} column of the identity matrix I_p; and p_i is the eigen-vector of $(\tilde{L})$, $\forall i = \{2, \ldots, n\}$.

The second term $\gamma_2 Y^T \Phi_{SW}(t) Y$ is expanded as

$$\begin{aligned}
\gamma_2 Y^T \Phi_{SW} Y =& \frac{\gamma_2}{n} \sum_{i=1}^{p} \sum_{j=1}^{p} \varsigma_i \varsigma_j (\mathbf{1}_n \otimes f_i)^T \Phi_{SW} (\mathbf{1}_n \otimes f_j) \\
&+ \frac{2\gamma_2}{\sqrt{n}} \sum_{i=2}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \varsigma_k \delta_{ij} (\mathbf{1}_n \otimes f_k)^T \Phi_{SW} (p_i \otimes f_j) \\
&+ \gamma_2 \underbrace{\sum_{i=2}^{n} \sum_{j=1}^{p} \sum_{k=2}^{n} \sum_{l=1}^{p} \delta_{ij} \delta_{kl} (p_i \otimes f_j)^T \Phi_{SW} (p_k \otimes f_l)}_{\geq 0}
\end{aligned} \tag{3.22}$$

Using the Propositions 2-3 (see the Appendix A), Lemmas 2-3, and it can be concluded that

$$\Phi_{SW} \geq \beta_2 \Phi_{bar}, \ t \geq t_0 + T_{z_j} \tag{3.23}$$

where $\beta_2 > 0$ is a positive scalar and $T_{z_j} \geq T_{z_i}, \forall i$. Substituting (3.23) into (3.22), yields

$$\begin{aligned}
\gamma_2 Y^T \Phi_{SW} Y \geq& \frac{\beta_2 \gamma_2}{n} \sum_{i=1}^{p} \sum_{j=1}^{p} \varsigma_i \varsigma_j (\mathbf{1}_n \otimes f_i)^T \Phi_{bar} (\mathbf{1}_n \otimes f_j) \\
&+ \frac{2\beta_2 \gamma_2}{\sqrt{n}} \sum_{i=2}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \varsigma_k \delta_{ij} (\mathbf{1}_n \otimes f_k)^T \Phi_{bar} (p_i \otimes f_j) \\
&+ \beta_2 \gamma_2 \underbrace{\sum_{i=2}^{n} \sum_{j=1}^{p} \sum_{k=2}^{n} \sum_{l=1}^{p} \delta_{ij} \delta_{kl} (p_i \otimes f_j)^T \Phi_{bar} (p_k \otimes f_l)}_{\geq 0}
\end{aligned} \tag{3.24}$$

where

$$(\mathbf{1}_n \otimes f_i)^T \Phi_{bar} (\mathbf{1}_n \otimes f_j) = \frac{1}{T} \int_{t_0}^{t_0+T} \sum_{i=1}^{n} \phi_i(r) \phi_i^T(r) dr \tag{3.25}$$

Based on (3.25), $\gamma_2 Y^T \Phi_{SW} Y$ can be lower-bounded as

$$\gamma_2 Y^T \Phi_{SW} Y \geq \frac{\beta_2 \gamma_2}{n} \varsigma^T \left(\frac{1}{T} \int_{t_0}^{t_0+T} \sum_{i=1}^{n} \phi_i(r) \phi_i^T(r) dr \right) \varsigma - \frac{2\beta_2 \gamma_2 n}{\sqrt{n}} \sum_{i=2}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} |\varsigma_k \delta_{ij}| \tag{3.26}$$

after further simplification and provided Assumption 8 (C-IE) holds, (3.26) yields

$$\gamma_2 Y^T \Phi_{SW} Y \geq \frac{||\varsigma||_2^2}{n}\gamma_2\beta_2\Upsilon - 2\Upsilon_1\beta_2\gamma_2 n\sqrt{||\varsigma||_2^2(1-||\varsigma||_2^2)}$$

where $\Upsilon_1 > 0$ is the upperbound based on the C-IE definition. Thus by clubbing these two terms together, required lower bound, which is strictly greater than zero, is achieved i.e.,

$$Y^T M(t) Y > max\{Y^T(\tilde{L}\otimes I_p)Y, \gamma_2 Y^T\Phi_{SW}Y\} \geq \varrho_1 > 0$$

where ϱ_1 is

$$\varrho_1 = \inf_{||\varsigma||_2\leq 1} max\left\{\lambda_2\left(1-||\varsigma||_2^2\right), \gamma_2\frac{||\varsigma||_2^2}{n}\beta_2\Upsilon - 2\Upsilon_1\beta_2\gamma_2 n\sqrt{||\varsigma||_2^2\left(1-||\varsigma||_2^2\right)}\right\} \tag{3.27}$$

by using continuity argument, infimum in (3.27) is attained and is strictly greater than zero (if ϱ_1 is zero and the first term is zero, then the second term should also be zero, $\frac{\gamma_2\beta_2\Upsilon}{n}$ is zero, which is a contradiction from the C-IE condition requirement for parameter convergence). □

3.5.1 Stability/Convergence Analysis

The parameter estimation error dynamics for all n number of agents can be compactly represented as

$$\Delta\dot{\theta}(t) = -\gamma\Phi_P(t)\Delta\theta(t) - \gamma_1\Phi_{FC}(t)\Delta\theta(t) - \gamma_2\Phi_{SW}(t)\Delta\theta(t) - (\tilde{L}\otimes I_p)\Delta\theta(t) \tag{3.28}$$

where the column vectors $\hat{\theta} = [\hat{\theta}_1, \ldots, \hat{\theta}_n]^T \in \mathbb{R}^{np}, \Delta\theta = [\Delta\theta_1, \ldots, \Delta\theta_n]^T \in \mathbb{R}^{np}$ are obtained by stacking the components $\hat{\theta}_i \in \mathbb{R}^p$ and $\Delta\theta_i = \hat{\theta}_i - \theta \in \mathbb{R}^p$, respectively, $\forall i = 1(1)n$. $\Phi_P(t)$, $\Phi_{FC}(t)$, $\Phi_{SW}(t) \in \mathbb{R}^{np\times np}$ are block diagonal matrices, defined as

$$\begin{aligned}
&\Phi_P(t) \triangleq diag\left\{\phi_1(t)\phi_1^T(t), \ldots, \phi_n(t)\phi_n^T(t)\right\}\\
&\Phi_{FC}(t) \triangleq diag\left\{Y_{C,1}(t), \ldots, Y_{C,n}(t)\right\}\\
&\Phi_{SW}(t) \triangleq diag\left\{Y_{C,1}\left(t_0 + T_{z_1}^1\right), \ldots, Y_{C,n}\left(t_0 + T_{z_n}^n\right)\right\}
\end{aligned}$$

The following Theorem characterizes several properties of the proposed novel parameter estimator dynamics.

Theorem 4. *For agents model in (3.1), provided Assumptions 6-7 hold, the dynamics (3.14) or (3.28) exhibits:*

(1) The origin of the parameter estimation error dynamics (3.14) *is Lyapunov stable and all the auxiliary signals remain bounded for all time.*

(2) Asymptotic behavior of prediction errors, $\forall i = 1(1)n$

$$E_p(t) \triangleq \hat{y}_i(t) - y_i(t) \to 0 \; as \; t \to \infty$$
$$E_{FC}(t) \triangleq \Big(Y_{C,i}(t)\hat{\theta}_i(t) - W_{C,i}(t)\Big) \to 0 \; as \; t \to \infty$$
$$E_{SW}(t) \triangleq \Big(Y_{SW,i}(t)\hat{\theta}_i(t) - W_{SW,i}(t)\Big) \to 0 \; as \; t \to \infty$$

(3) Asymptotic behavior of parameter consensus:

$$\hat{\theta}_j(t) - \hat{\theta}_i(t) \to 0 \; as \; t \to \infty, \; \forall i, j = 1(1)n$$

(4) Parameter Convergence under (C-IE): Utilizing the Lemma 4 proof argument which is based on Assumption 8 (C-IE), then the origin of the parameter estimation error dynamics $\Delta\theta(t)$ *is globally exponential stable (GES) in a delayed sense i.e.,*

$$\|\Delta\theta(t)\| \leq \|\Delta\theta(t_0 + T)\| e^{-\alpha_1(t - t_0 - T)}, \; \forall t \geq t_0 + T \tag{3.29}$$

where $\alpha_1 \in \mathbb{R}_{>0}$ *is a constant positive scalar.*

Proof. Consider the following Lyapunov candidate

$$V(\Delta\theta) = \frac{1}{2}\Delta\theta^T \Delta\theta \tag{3.30}$$

Taking time derivative of (3.30) and substituting the parameter estimation error dynamics (3.28), yields

$$\dot{V}(\Delta\theta) = -\Delta\theta^T \Big(\gamma\Phi_P(t) + \gamma_1\Phi_{FC}(t) + \gamma_2\Phi_{SW}(t) + (\tilde{L} \otimes I_p)\Big)\Delta\theta \leq 0 \tag{3.31}$$

which is negative semi-definite; holds from the fact that $(\tilde{L} \otimes I_p)$, $\Phi_P(t)$, $\Phi_{FC}(t)$ and $\Phi_{SW}(t)$ are the positive semi-definite matrices, which implies that the origin of the estimation error dynamics is Lyapunov stable. Based on the Assumption 7 and (3.3), (3.4), (3.15), (3.16), and it can be concluded that $\big(\Phi_P(t), \Phi_{FC}(t), \Phi_{SW}(t), \dot{\Phi}_P(t), \dot{\Phi}_{FC}(t), \dot{\Phi}_{SW}(t)$[2]

[2]Here the boundedness is argued for $t \geq max\{T^i_{z_i}\}$.

$\in \mathcal{L}_\infty$). Using the above arguments, it can also be concluded that $V(\Delta\theta(t))$ is uniformly bounded above by its initial value, which implies $\hat{\theta}_i(t) \in \mathcal{L}_\infty$ i.e., the local prediction $\hat{y}_i(t) \in \mathcal{L}_\infty$, $\forall i = 1(1)n$. This completes the proof of part 1.

For the next part of the proof, differentiating the (3.31) and using the Lemma 1, yields

$$\begin{aligned}\ddot{V}(\Delta\theta) =& -2\Delta\theta^T(\tilde{L}\otimes I_p)\Delta\dot{\theta} - \gamma\Delta\theta^T\dot{\Phi}_p(t)\Delta\theta - 2\gamma\Delta\theta^T\Phi_p(t)\Delta\dot{\theta} - \gamma_1\Delta\theta^T\dot{\Phi}_{FC}(t)\Delta\theta \\ & - 2\gamma_1\Delta\theta^T\Phi_{FC}(t)\Delta\dot{\theta} - \gamma_2\Delta\theta^T\dot{\Phi}_{SW}(t)\Delta\theta - 2\gamma_2\Delta\theta^T\Phi_{SW}(t)\Delta\dot{\theta}\end{aligned}$$

using the boundedness related arguments in the last part of the proof, it can be concluded that $\ddot{V}(\Delta\theta) \in \mathcal{L}_\infty$ or $\dot{V}(\Delta\theta)$ is uniformly continuous. Moreover, $V(\Delta\theta(t))$ is bounded below $(V(\Delta\theta(t)) \geq 0)$ and $\dot{V}(\Delta\theta(t))$ is decreasing, it has a finite limit as $t \to \infty$. Then by invoking Barbalat's Lemma (Lemma 8.2 at page 323 (Khalil and Grizzle, 2002)), $V(\Delta\theta(t))$ is bounded below $(V(\Delta\theta(t)) \geq 0)$ and $\dot{V}(\Delta\theta(t))$ is decreasing, it has a limit as $t \to \infty$, implies that $\dot{V}(\Delta\theta(t)) \to 0$ as $t \to \infty$. Then from (3.31), it can be concluded that $\Delta\theta^T\Phi_P(t)\Delta\theta$, $\Delta\theta^T\Phi_{FC}(t)\Delta\theta$, $\Delta\theta^T\Phi_{SW}(t)\Delta\theta$, $\Delta\theta^T(\tilde{L}\otimes I_p)\Delta\theta \to 0$ as $t \to \infty$, which implies that all the local prediction errors $E_P(t), E_{FC}(t), E_{SW}(t) \to 0$ as $t \to \infty$, $\forall i = 1(1)n$ i.e., the asymptotic behaviour of prediction errors are achieved.

Fourth term $\Delta\theta^T(\tilde{L}\otimes I_p)\Delta\theta \to 0$ as $t \to \infty$, which implies that $(\tilde{L}\otimes I_p)\Delta\theta = (\tilde{L}\otimes I_p)\hat{\theta} \to 0$. For a strongly connected and balanced graph G, Lemma 1, it can be concluded that null($\tilde{L}$) = null($\tilde{L}\otimes I_p$)= span$\{\mathbf{1}_p\}$, where null denotes the null space of a matrix, then $\hat{\theta}_j(t) - \hat{\theta}_i(t) \to 0$ as $t \to \infty$ for all $i, j = 1(1)n$. This completes the proof of part 3 i.e., the asymptotic behavior of parameter consensus is achieved.

For last part of proof, invoking the C-IE assumption based Lemma 4 in (3.31), yields:

$$\dot{V}(\Delta\theta) = -\Delta\theta^T M\Delta\theta < 0,\ \forall t \geq t_0 + T. \tag{3.32}$$

Using (3.30), the above inequality modifies as:

$$\dot{V} \leq -\underbrace{2\lambda_{min}(M)}_{\alpha_1>0}V,\ \forall t \geq t_0 + T. \tag{3.33}$$

Hence using the comparison Lemma (Lemma 3.4 at page 103 (Khalil and Grizzle, 2002)), the differential inequality in (3.33) leads to the subsequent exponentially convergent bound

$$\|\Delta\theta(t)\| \leq \|\Delta\theta(t_0+T)\|e^{-\alpha_1(t-t_0-T)},\ \forall t \geq t_0+T \tag{3.34}$$

Since the Lyapunov function in (3.30) is radially unbounded and the constant α_1 are independent of initial conditions, the algebraic inequality in (3.34) proves GES (in a delayed sense) of the error dynamics $\Delta\theta(t)$, $\forall t \geq t_0+T$. □

3.6 Online Optimization Perspective

In this section, the above C-IE based distributed adaptive estimation problem is molded as a distributed online optimization problem, where a novel objective or cost function is formulated corresponding to the adaptive estimation objective. It is analytically shown that the subsequent online optimization framework agrees with the stability claims of the previous section.

Each agent has its own local cost function, which is assumed to be unaccessible to agents other than i, is defined as, $\forall t \geq t_0$

$$\begin{aligned}\underbrace{J_i(\hat{\theta}_i(t),t)}_{\mathbb{R}^p\times\mathbb{R}_{\geq 0}\to\mathbb{R}} :=\Bigg\{ &\frac{\gamma}{2}\underbrace{\|\phi_i^T(t)\Delta\theta_i(t)\|^2}_{J_i^P}+\underbrace{\frac{\gamma_1}{2}\int_{t_0}^{t}\|\phi_i^T(r)\Delta\theta_i(t)\|^2e^{-\delta(t-r)}dr}_{J_i^{FC}}\\ &+\underbrace{\frac{\gamma_2}{2}\sum_{k_i=1}^{z_i}S_{i,k_i}(t)\int_{t_0}^{t}P_{i,k_i}(t)\|\phi_i^T(r)\Delta\theta_i(t)\|^2e^{-\delta(t-r)}dr}_{J_i^{SW}}\\ &+\underbrace{\frac{a_{ij}}{2}\sum_{j\in\mathcal{N}_i}\|\hat{\theta}_j(t)-\hat{\theta}_i(t)\|^2}_{J_i^C}\Bigg\}\end{aligned} \tag{3.35}$$

where the parameter estimate $\hat{\theta}_i(t) \in \mathbb{R}^p$ is the decision variable. The switching signals

$S_{i,k_i}(t)$, $P_{i,k_i}(t)$, $\forall i = 1(1)n$, are subsequently defined as

$$\underbrace{S_{i,k_i}(t)}_{\forall k_i=1,\ldots,z_i-1} \triangleq \begin{cases} 0 & , \big[t_0 + T_0^i,\ t_0 + T_{k_i}^i\big) \\ 1 & , \big[t_0 + T_{k_i}^i,\ t_0 + T_{k_{i+1}}^i\big) \\ 0 & , \big[t_0 + T_{k_{i+1}}^i,\ \infty\big) \end{cases}$$

$$S_{i,z_i}(t) \triangleq \begin{cases} 0 & , \big[t_0 + T_0^i,\ t_0 + T_{z_i}^i\big) \\ 1 & , \big[t_0 + T_{z_i}^i,\ \infty\big) \end{cases}$$

$$\underbrace{P_{i,k_i}(t)}_{\forall k_i=1,\ldots,z_i} \triangleq \begin{cases} 1 & , \big[t_0 + T_0^i,\ t_0 + T_{k_i}^i\big) \\ 0 & , \big[t_0 + T_{k_i}^i,\ \infty\big) \end{cases}$$

switching signals $S_{i,k_i}(t)$, $P_{i,k_i}(t)$ are strategically formulated according to the switching concept described above as in (3.15), (3.16).

Remark 8. *Based on the above formulation of switching signals in the local objective function (quadratic, time-varying), it can be inferred that the local objective function $J_i\big(\hat{\theta}_i(t), t\big)$, for i^{th} agent is twice continuously differentiable with respect to $\hat{\theta}_i(t)$, and piece-wise continuous with respect to t, respectively.* □

The global or team objective function is defined as

$$\underbrace{J\big(\hat{\theta}(t), t\big)}_{\mathbb{R}^{np}\times\mathbb{R}_{\geq 0}\to\mathbb{R}} := \sum_{i=1}^{n} J_i\big(\hat{\theta}_i(t), t\big) \tag{3.36}$$

The optimization problem is conceptualized as

$$\theta^* = \arg\min_{\hat{\theta}(t)} J\big(\hat{\theta}(t), t\big) \tag{3.37}$$

Remark 9. *Different components in the local cost function* (3.35) *- J_i^P, J_i^{FC}, J_i^{SW} and J_i^C are having a one-to-one correspondence with different components of the estimator* (3.14) *- E_P, E_{FC}, E_{SW} and E_C, respectively.* □

In the subsequent arguments, it is established that the above cost function has a global minima at the true parameter vector (i.e., $\theta^* = \mathbf{1}_n \otimes \theta$) of the model (3.1) under the C-IE condition and the proposed distributed adaptive estimator (3.14) is a distributed optimizer of the cost function (3.36).

From (3.35), it can be inferred that the local objective function $J_i\big(\hat{\theta}_i(t), t\big)$ are uniformly convex in $\hat{\theta}_i(t)$, $\forall t$. However for global minimization, strong convexity is needed, which is established under the C-IE condition in the following Lemma.

Lemma 5. *For the team objective function (3.36), the proposed distributed adaptive estimator (3.28) is a distributed gradient-descent based unique minimizer under C-IE condition.*

Proof. Taking the first order partial derivative or gradient of the global/team objective function (3.36) with respect to $\hat{\theta}(t)$, yields

$$\begin{aligned}\nabla J_{\hat{\theta}(t)}\big(\hat{\theta}(t), t\big) = &\sum_{i=1}^{n} \Bigg\{ \gamma \phi_i(t)\phi_i^T(t)\Delta\theta_i(t) + \gamma_1 \int_{t_0}^{t} \phi_i(r)\phi_i^T(r)e^{-\delta(t-r)}dr\ \Delta\theta_i(t) \\ &+ \gamma_2 \sum_{k_i=1}^{z_i} S_{i,k_i}(t) \int_{t_0}^{t} P_{i,k_i}(t)\phi_i(r)\phi_i^T(r)e^{-\delta(t-r)}dr\ \Delta\theta_i(t) \\ &- \sum_{j\in\mathcal{N}_i} a_{ij}\big(\hat{\theta}_j(t) - \hat{\theta}_i(t)\big)\Bigg\},\ \forall t \geq t_0 \end{aligned} \tag{3.38}$$

From (3.38), it can be concluded that the distributed adaptive parameter estimator (3.28) is exactly same as the negative of the gradient (3.38). In a compact form, the following can be derived.

$$\begin{aligned}\Delta\dot{\theta}(t) &= -\gamma\Phi_P(t)\Delta\theta(t) - \gamma_1\Phi_{FC}(t)\Delta\theta(t) - \gamma_2\Phi_{SW}(t)\Delta\theta(t) - (\tilde{L}\otimes I_p)\Delta\theta(t) \\ &= -\nabla J_{\hat{\theta}(t)}\big(\hat{\theta}(t), t\big),\ \forall t \geq t_0 \end{aligned} \tag{3.39}$$

Hence, it is proved that (3.28) is a distributed gradient descent algorithm w.r.t. (3.36). Further putting $\hat{\theta}(t) = \mathbf{1}_n \otimes \theta$, it can be concluded that

$$\nabla J_{\hat{\theta}(t)}\big(\hat{\theta}(t), t\big)|_{\hat{\theta}(t)=\mathbf{1}_n\otimes\theta} = \mathbf{0},\ \forall t \geq t_0$$

indicating that the true-parameter vector is a minima point of the cost function (3.36).

Now taking the second partial derivative or Hessian w.r.t $\hat{\theta}(t)$ from (3.36), yields

$$\nabla^2 J_{\hat{\theta}(t)}\big(\hat{\theta}(t),t\big) = \gamma\Phi_P(t) + \gamma_1\Phi_{FC}(t) + \gamma_2\Phi_{SW}(t) + (\tilde{L}\otimes I_p),\ \forall t \geq t_0 \tag{3.40}$$

Since $\Phi_P(t)$, $\Phi_{FC}(t)$, $\Phi_{SW}(t)$, $(\tilde{L}\otimes I_p)$, all are positive semi-definite matrices, then it can be concluded that $\nabla^2 J_{\hat{\theta}(t)}\big(\hat{\theta}(t),t\big)$ is at least positive semi-definite, $\forall t \geq t_0$, which proves uniform convexity of the cost function. Furthermore, invoking Lemma 4 in (3.40), yields $\nabla^2 J_{\hat{\theta}(t)}\big(\hat{\theta}(t),t\big)$ is positive definite, $\forall t \geq t_0 + T$, which proves uniform strong convexity of the cost function, $\forall t \geq t_0 + T$.

Hence, $\forall t \geq t_0 + T$, $\hat{\theta}(t) = \mathbf{1}_n \otimes \theta$ is the unique global minima of the cost function (3.36) and the gradient-descent (3.39) will converge to the global minima as $t \to \infty$, which is inline with the stability theorem (Theorem 4).

This completes the proof. □

3.7 Simulation Results

Based on Assumption 6 and Figure 3.1, where a numerical example with 5 agents are tasked to estimate a true parameter vector $\theta = [1, -1, 0.5]^T$ with neighboring sets $N_1 = \{5\}$, $N_2 = \{1\}$, $N_3 = \{2\}$, $N_4 = \{3\}$, $N_5 = \{4\}$ using the proposed distributed consensus-based adaptive parameter estimator (3.14) and each of the agent regressor is chosen such that, its neither IE nor PE [3] i.e.,

$$\phi_1(t) = [1, exp(-t), 0]^T$$

$$\phi_2(t) = [1, 0, 2exp(-6t)]^T$$

$$\phi_3(t) = [0, 0, cos(8t)]^T$$

$$\phi_4(t) = [0, 0, 0.01exp(-3t)cos(4t)]^T$$

$$\phi_5(t) = [0, 0, exp(-7t)]^T.$$

This simulation study considers all the initial conditions in a random nature.

[3] Note that practical real-time control applications mostly encounter situations, where information gradually dies out over time. Hence the choice of the exponentially decaying regressors in this simulation is reasonably realistic.

Based on C-IE based proposed estimation algorithm (3.14), Figures 3.3-3.5 show consensus by all the $n = 5$ agents individually for given true parameters, whereas Figure 3.6 represents the C-IE based combined plot of consensus. Figure 3.7 represents the C-PE based combined consensus plot, where parameter convergence is not achieved. Figure 3.8 (comparison plot) shows that proposed C-IE based parameter estimator outperforms all existing conventional parameter estimation algorithms (Slotine and Li, 1989; Roy *et al.*, 2016; Papusha *et al.*, 2014; Chen *et al.*, 2013; Wensing and Slotine, 2018; Stegagno and Yuan, 2019; Cho *et al.*, 2017), even if the regressors are neither IE nor PE/C-PE. Moreover, the details of the different algorithms in Figure 3.8 are following:

Red solid curve shows our proposed C-IE based algorithm (3.14) or (3.28).

Blue solid curve shows C-PE based distributed adaptive parameter estimation algorithm, defined as

$$\left(\dot{\hat{\theta}}_i(t) = -\gamma\phi_i(t)\underbrace{\phi_i^T\Delta\theta_i(t)}_{E_P} + \underbrace{\sum_{j\in\mathcal{N}_i} a_{ij}(\hat{\theta}_j - \hat{\theta}_i)}_{E_C}\right)$$

as in (Papusha *et al.*, 2014; Chen *et al.*, 2013; Wensing and Slotine, 2018; Stegagno and Yuan, 2019).

Yellow solid curve shows IE based adaptive parameter estimation algorithm, defined as

$$\left(\dot{\hat{\theta}}_i(t) = -\gamma\phi_i(t)\underbrace{\phi_i^T\Delta\theta_i(t)}_{E_P} - \gamma_1\underbrace{\Big(Y_{C,i}(t)\hat{\theta}_i(t) - W_{C,i}(t)\Big)}_{E_{FC}} - \gamma_2\underbrace{\Big(Y_{SW,i}(t)\hat{\theta}_i(t) - W_{SW,i}(t)\Big)}_{E_{SW}}\right)$$

as in (Basu Roy *et al.*, 2018; Roy *et al.*, 2016; Jha *et al.*, 2019; Roy *et al.*, 2017*b*; Roy and Bhasin, 2019).

Magenta solid curve shows PE based adaptive parameter estimation algorithm, defined as

$$\left(\dot{\hat{\theta}}_i(t) = -\gamma\phi_i(t)\underbrace{\phi_i^T\Delta\theta_i(t)}_{E_P}\right)$$

as in (Tao, 2003; Narendra and Annaswamy, 2012).

Remark 10. *The proposed distributed adaptive estimation algorithm* (3.14) *or* (3.28) *is*

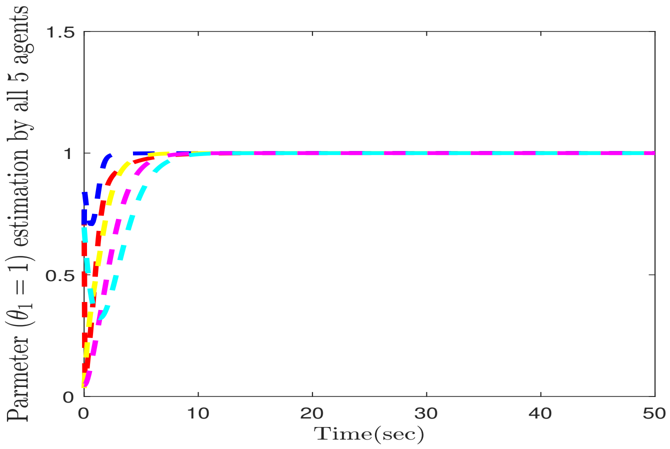

Figure 3.3: Parameter estimation ($\theta_1 = 1$) by all 5 agents using C-IE based proposed distributed parameter estimator (3.14).

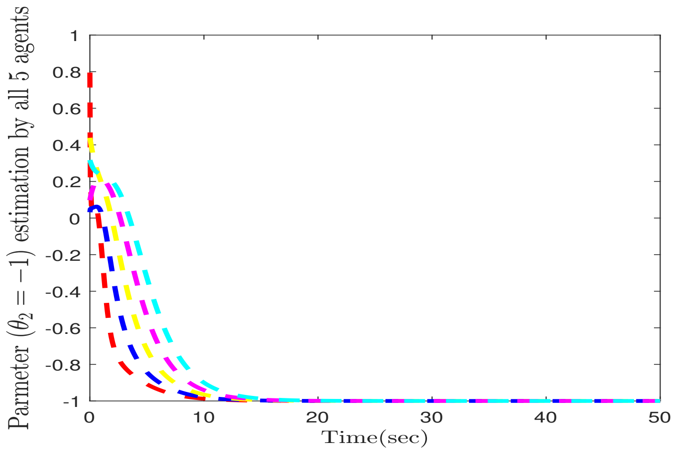

Figure 3.4: Parameter estimation ($\theta_2 = -1$) by all 5 agents using C-IE based proposed distributed parameter estimator (3.14).

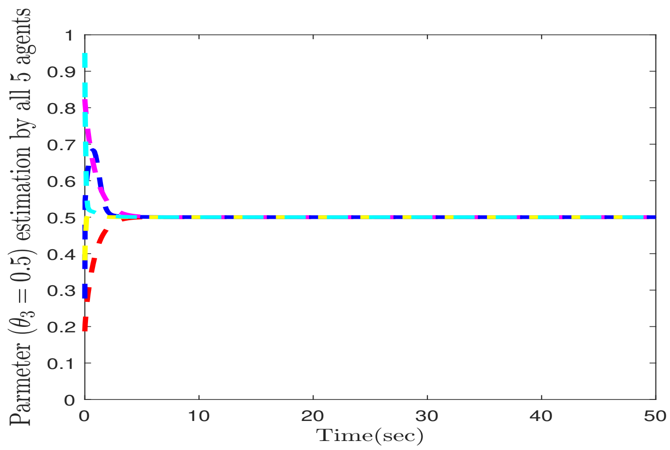

Figure 3.5: Parameter estimation ($\theta_3 = 0.5$) by all 5 agents using C-IE based proposed distributed parameter estimator (3.14).

"adaptive" in sense that it is estimating the unknown constant parameter θ which is inline with the literature of adaptive control. However, with small modification in the proposed

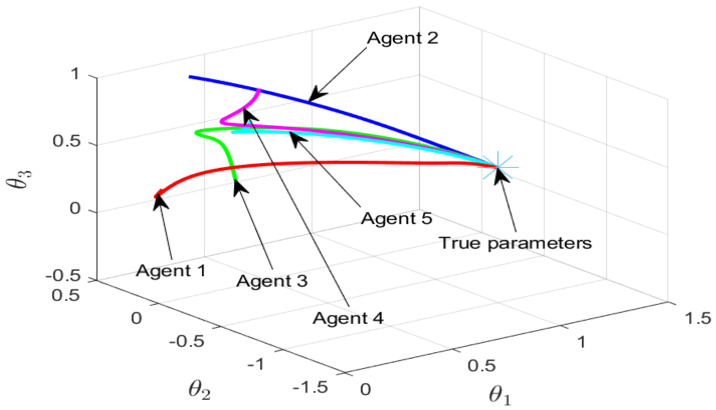

Figure 3.6: Parameter estimation ($\theta_1 = 1$, $\theta_2 = -1$, $\theta_3 = 0.5$) by all 5 agents using C-IE based proposed distributed parameter estimator (3.14).

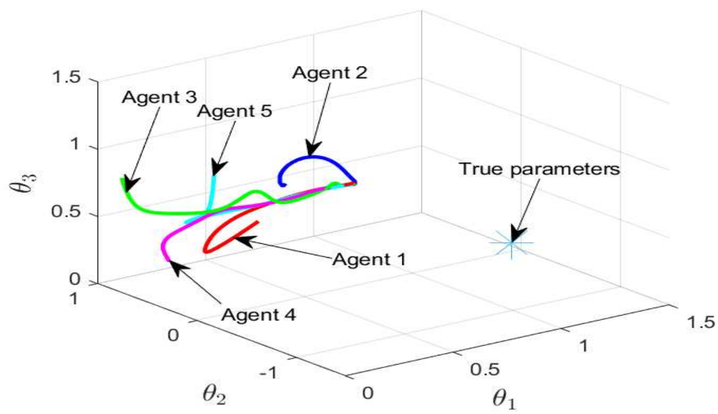

Figure 3.7: Parameter estimation ($\theta_1 = 1$, $\theta_2 = -1$, $\theta_3 = 0.5$) by all 5 agents using C-PE based distributed parameter estimator ((Papusha *et al.*, 2014; Chen *et al.*, 2013; Wensing and Slotine, 2018; Stegagno and Yuan, 2019)).

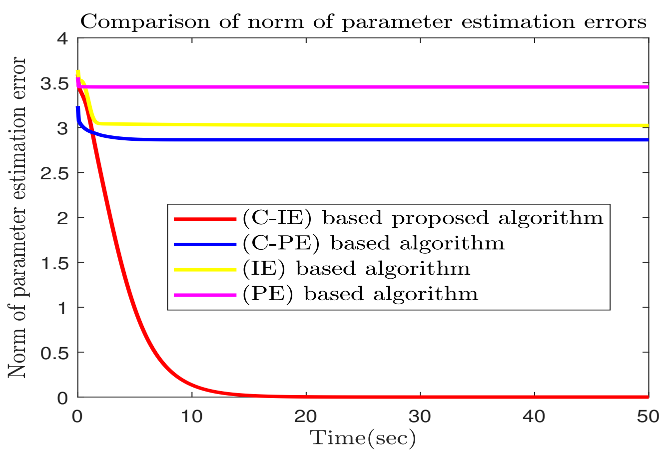

Figure 3.8: Comparison of norm of parameter estimation error for different algorithms.

algorithm could be capable of following time-varying θ as well (for more detail please see the (Goel and Roy, 2022)). □

3.8 Conclusion

This chapter designs a novel distributed adaptive estimation algorithm for a network of MAS architecture. Unlike classical parameter estimation algorithms, which demand to satisfy restrictive PE or C-PE conditions for parameter convergence, the proposed algorithm ensures parameter convergence by a milder condition, called C-IE. The work designs a distributed estimator based on strategic multiple switching such that exponential parameter convergence is obtained under C-IE condition and, thereby, improves transient performance significantly as compared to the status quo. The formulation is further augmented by providing an online optimization perspective. A novel convex cost function is proposed in such a way that a formulated distributed adaptive estimator acts as a distributed gradient-descent of the cost function. The cost function is proved to be uniformly strongly convex under the C-IE condition thereby making the estimator dynamics unique global minimizer of the cost function.

Part IV

Distributed Composite Adaptive Synchronization of Multiple Uncertain Euler-Lagrange Systems using Cooperative Initial Excitation

3.9 Introduction

This chapter (part-2) proposes a distributed composite adaptive synchronization algorithm for multiple uncertain Euler-Lagrange (EL) systems, where parameter convergence is achieved under a relaxed mathematical condition as compared to the state-of-the-art. Classical adaptive controllers require an analytical condition, called PE, to ensure parameter convergence, which results in better transient performance and robustness to disturbance. The PE condition is extended to the C-PE condition for distributed adaptive controllers with cooperative estimation strategies. The PE and C-PE conditions are restrictive in nature since these conditions are not satisfied in most practical applications. Recent literature in adaptive control has relaxed the PE condition to IE, which is shown to be sufficient for parameter convergence. The IE condition is argued to be significantly milder than PE and can be satisfied in many practical settings. The proposed result further extends the IE condition to the C-IE condition in distributed adaptive control architecture in the context of synchronizing multiple EL systems. It is established that the C-IE condition is milder than PE, IE, and C-PE conditions. A two-tier filter based estimation algorithm with strategic switching ensures parameter convergence under the C-IE condition and thereby provides exponential convergence of tracking and parameter estimation error to zero. Simulation results validate the efficacy of the proposed algorithm as compared to conventional distributed adaptive controllers in terms of superior tracking and estimation performance.

3.10 Preliminaries

3.10.1 Preliminaries on Algebraic Graph Theory

The network topology among s number of follower Euler-Lagrange systems is described by an undirected graph $G \triangleq (V, E)$, which captures their local information flow. The graph is modeled as having s vertices $V \triangleq \{1, \ldots, s\}$ together with the edge set $E \subseteq V \times V$. The adjacency matrix of the graph is defined as $A = [a_{ij}]_{s\times s}$, where $a_{ij} = 1$ if $(i, j) \in E$ and $a_{ij} = 0$ otherwise. The leader's state is shared to a subset of followers, which is captured using a directed graph. The degree of the vertex i, denoted as, deg(i), is the number of neighbors $|N_i|$ available to that vertex. where N_i is defined as the neighborhood of the vertex i defined as, $N_i \triangleq \{j \mid (i,j) \in E\}$. The Laplacian matrix $L \in \mathbb{R}^{s\times s}$ of the graph G is defined as $L \triangleq D - A$ where $D \triangleq \text{diag}\{\text{deg}(1), \ldots, \text{deg}(s)\} \in \mathbb{R}^{s\times s}$ is the degree matrix. For connected graph the Laplacian matrix L has the property that, its all eigenvalues are positive except for the smallest one, which is zero. For the zero eigenvalue, corresponding eigenvector is $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^s$. In particular, $L\mathbf{1} = 0$.

Lemma 6. *Let $G \triangleq (V, E)$ be the undirected graph that dictates the interaction among the s agents with the associated graph Laplacian matrix L, $G_0 \triangleq (V_0, E_0)$ be the directed graph characterizing the interaction among the leader and the s agents corresponding to graph G, and $a_{i0} = 1$ if the leader is a neighbor of follower i and $a_{i0} = 0$ otherwise. If in G_0 the leader has directed paths to all followers, then the matrix $B \triangleq L + diag(a_{10}, \ldots, a_{s0})$ is symmetric positive-definite.*

Proof. For proof refer to the (Mei *et al.*, 2011). □

3.11 Problem Formulation

3.11.1 Model Description

Suppose that there exist s number of followers, labeled as agents or followers 1 to s and a leader labeled as agent 0, in a team. The followers are represented by Euler–Lagrange equations of the following form (Spong and Vidyasagar, 2008), $\forall i \in \{1, \ldots, s\}$,

$$M_i(q_i)\ddot{q}_i + V_i(q_i, \dot{q}_i)\dot{q}_i + F_i(\dot{q}_i) + G_i(q_i) = \tau_i \tag{3.41}$$

where $q_i(t) \in \mathbb{R}^n$, $\dot{q}_i(t) \in \mathbb{R}^n$ and $\ddot{q}_i(t) \in \mathbb{R}^n$ represent generalized position, velocity and acceleration, respectively; $M_i(q_i) \in \mathbb{R}^{n\times n}$ represents Inertia Matrix, $V_i(q_i, \dot{q}_i) \in \mathbb{R}^{n\times n}$ is Coriolis and Centrifugal Matrix, $G(q_i) \in \mathbb{R}^n$ represents gravity, $F(\dot{q}_i) \in \mathbb{R}^n$ is frictional term, $\tau_i(t) \in \mathbb{R}^n$ denotes generalized torque or control input for i^{th} agent. The leader has the generalized vector coordinates of position and velocity, denoted as $q_0(t) \in \mathbb{R}^n$ and $\dot{q}_0 \in \mathbb{R}^n$, [4] respectively. Some fundamental properties of the above mentioned dynamics,

[4]velocity of the leader is treated as constant, while time-varying velocity using a similar EL model will be considered in future works.

used in the subsequent formulation, are stated as follows (Spong and Vidyasagar, 2008).

Property 6. *$\forall i \in \{1, \dots, s\}$ there exist positive constants μ_1, μ_2, μ_3, and μ_4 such that $\mu_1 I_n \leq M_i(q_i) \leq \mu_2 I_n$, $\|V_i(q_i, \dot{q}_i)\| \leq \mu_3 \|\dot{q}_i\|$, $\|G_i(q_i)\| \leq \mu_4$.* □

Property 7. *The Matrix $\dot{M}_i(q_i) - 2V_i(q_i, \dot{q}_i)$ is a skew-symmetric i.e.,*

$$\xi_i^T \Big(\dot{M}_i(q_i) - 2V_i(q_i, \dot{q}_i)\Big)\xi_i = 0,\ \forall \xi_i \in \mathbb{R}^n,\ \forall i \in \{1, \dots, s\}.$$

□

Property 8. *The L.H.S of (3.41) has the linearity in parameters property i.e., it can be alternatively expressed as*

$$N_i(q_i, \dot{q}_i, \ddot{q}_i)\theta \triangleq \tau_i(t),\ \forall i \in \{1, \dots, s\} \tag{3.42}$$

where $N_i(q_i, \dot{q}_i, \ddot{q}_i) \in \mathbb{R}^{n \times p}$ is the regressor matrix and $\theta \in \mathbb{R}^p$ is a vector for unknown parameters common to all the agents (i.e., the follower manipulators are homogeneous). □

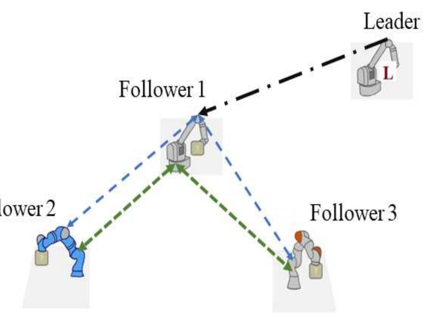

Figure 3.9: Local interaction scenario for leader-follower robot model.

3.12 Distributed Adaptive Controller Design for Synchronization

The objective is to design a distributive adaptive synchronization algorithm based on a local interaction protocol between the robot manipulators and the leader, like Figure 3.9, using (3.41) such that $\|q_i(t) - q_0(t)\| \to 0$, $\dot{q}_i(t) \to \dot{q}_0$ as $t \to \infty$, $\forall i \in \{1, \ldots, s\}$.

The following set of auxiliary signals are introduced to facilitate the design (Mei *et al.*, 2011), $\forall i \in \{1, \ldots, s\}$,

$$\dot{q}_{mi}(t) \triangleq \hat{z}_i(t) - \gamma \sum_{j=0}^{s} a_{ij}(q_i - q_j) \tag{3.43}$$

$$r_i(t) \triangleq \dot{q}_i(t) - \dot{q}_{mi}(t) = \dot{q}_i(t) - \hat{z}_i(t) + \gamma \sum_{j=0}^{s} a_{ij}(q_i - q_j) \tag{3.44}$$

where $\gamma > 0$, $\hat{z}_i(t) \in \mathbb{R}^n$ is the i^{th} follower's estimate of the leader's velocity. Using (3.41), (3.43), and (3.44), the open-loop error dynamics for each agent is obtained as below.

$$M_i(q_i)\dot{r}_i = \tau_i - F_i(\dot{q}_i) - V_i(q_i, \dot{q}_i)\dot{q}_i - M_i(q_i)\ddot{q}_{mi} - G_i(q_i) \tag{3.45}$$

Using Property 8, the above error dynamics can be expressed as follows.

$$M_i(q_i)\dot{r}_i = \tau_i + Z_i(q_i, \dot{q}_i, \ddot{q}_{mi}, \dot{q}_{mi})\theta - V_i(q_i, \dot{q}_i)r_i \tag{3.46}$$

where $Z_i(q_i, \dot{q}_i, \ddot{q}_{mi}, \dot{q}_{mi})\theta$ is defined as

$$Z_i(q_i, \dot{q}_i, \ddot{q}_{mi}, \dot{q}_{mi})\theta = -M_i(q_i)\ddot{q}_{mi} - V_i(q_i, \dot{q}_i)\dot{q}_{mi} - F_i(\dot{q}_i) - G_i(q_i) \tag{3.47}$$

and $Z_i(q_i, \dot{q}_i, \ddot{q}_{mi}, \dot{q}_{mi}) \in \mathbb{R}^{n \times p}$ is a known regressor matrix (Roy *et al.*, 2016). To facilitate the design of a composite adaptive law based on the information related to prediction error and parameter estimation error (which are defined subsequently), the following set of low-pass filters are designed (Slotine and Li, 1989).

$$\dot{N}_{F,i}(t) = -\alpha N_{F,i}(t) + N_i(t), \ N_{F,i}(t_0) = 0, \ \forall i \in \{1, \ldots, s\} \tag{3.48}$$

$$\dot{\tau}_{F,i}(t) = -\alpha \tau_{F,i}(t) + \tau_i(t), \ \tau_{F,i}(t_0) = 0, \ \forall i \in \{1, \ldots, s\} \tag{3.49}$$

where $\alpha > 0$ is the scalar filter gain and $N_{F,i}(t) \in \mathbb{R}^{n\times p}$, $\tau_{F,i}(t) \in \mathbb{R}^n$ are the filtered regressor, filtered torque, respectively. By analytically solving (3.48) and (3.49) and utilising (3.42), the following relation can be derived, $\forall t \in [t_0, \infty)$.

$$N_{F,i}(t, q_i, \dot{q}_i)\theta = \tau_{F,i}(t),\ \forall i \in \{1, \dots, s\}. \tag{3.50}$$

Although $\tau_{F,i}(t)$ can be obtained directly by solving (3.49), $N_{F,i}(t)$ cannot be solved from (3.48) since N_i is unknown due to its dependence on $\ddot{q}$. However, $N_i(q_i, \dot{q}_i, \ddot{q}_i)$ can be split into measurable and unmeasurable terms as

$$N_i(q_i, \dot{q}_i, \ddot{q}_i) = N_i^1(q_i, \ddot{q}_i) + N_i^2(q_i, \dot{q}_i), \quad \forall i \in \{1, \dots, s\} \tag{3.51}$$

where N_i^1, N_i^2 are related to the dynamics by following relation, $\forall i \in \{1, \dots, s\}$:

$$\begin{aligned} N_i^1(q_i, \ddot{q}_i)\theta &= M_i(q_i)\ddot{q}_i \\ N_i^2(q_i, \dot{q}_i)\theta &= V_i(q_i, \dot{q}_i)\dot{q}_i + F_i(\dot{q}_i) + G_i(q_i) \end{aligned}$$

Since $N_i^2(q_i, \dot{q}_i)$ is known, the subsequent differential law can be solved online.

$$\dot{N}_{F,i}^2 = -\alpha N_{F,i}^2 + N_i^2, \quad N_{F,i}^2(t_0) = 0, \quad \forall i \in \{1, \dots, s\} \tag{3.52}$$

Further, consider the following differential equation.

$$\dot{N}_{F,i}^1 = -\alpha N_{F,i}^1 + N_i^1, \quad N_{F,i}^1(t_0) = 0, \quad \forall i \in \{1, \dots, s\} \tag{3.53}$$

The solution of (3.53) can be expressed as

$$N_{F,i}^1(t) = exp\{-\alpha t\} \int_{t_0}^{t} exp\{\alpha r\}\ N_i^1(r)dr, \quad \forall i \in \{1, \dots, s\} \tag{3.54}$$

where, due to the structure $M_i(q_i)\ddot{q}_i$, the elements of $N_i^1(q_i, \ddot{q}_i)$ are of the form $f_i(q_{i,k})\ddot{q}_{i,l}$ for some $l, k \in \{1, \dots, n\}$. Thus, the elements of $N_{F,i}^1$ can be obtained by using integration by parts, as

$$N^1_{F,i}(t) = exp\{-\alpha t\} \int_{t_0}^{t} exp\{\alpha r\}\ f_i(q_{i,k}(r))\ddot{q}_{i,l}(r)dr$$

$$= f_i(q_{i,k}(t))\dot{q}_{i,l}(t) - exp\{-\alpha t\} f_i(q_{i,k}(0))\dot{q}_{i,l}(0) - h_i(t) \tag{3.55}$$

where $h_i(t)$ is computed from the following differential law, $\forall i = \{1, \ldots, s\}$:

$$\dot{h}_i(t) = -\alpha h_i + \dot{f}_i(q_{i,k}(t))\dot{q}_{i,l}(t) + \alpha f_i(q_{i,k}(t))\dot{q}_{i,l}(t), \quad h_i(t_0) = 0. \tag{3.56}$$

Using (3.53), (3.55) and (3.56), $N^1_{F,i}$ can also be computed online, implying that $N_{F,i} = N^1_{F,i} + N^2_{F,i}$ can be obtained online (for further details, see the (Basu Roy *et al.*, 2018)) . Therefore (3.50) can be used in the design of parameter update law . Further, to overcome the PE/C-PE restriction, another layer of filter equations are introduced, $\forall i \in \{1, \ldots, s\}$, inspired from (Kreisselmeier, 1977; Roy *et al.*, 2016, 2017*b*,*a*).

$$\dot{N}_{FC,i}(t) = -\alpha_1 N_{FC,i}(t) + N^T_{F,i}(t) N_{F,i}(t),\ N_{FC,i}(t_0) = 0 \tag{3.57}$$

$$\dot{W}_{FC,i}(t) = -\alpha_1 W_{FC,i}(t) + N^T_{F,i}(t)\tau_{F,i}(t),\ W_{FC,i}(t_0) = 0 \tag{3.58}$$

where $N_{FC,i}(t) \in \mathbb{R}^{p\times p}$ denotes the double-filtered regressor, $W_{FC,i}(t) \in \mathbb{R}^p$ denotes double-filtered torque and $\alpha_1 > 0$ is a scalar gain. Taking integration of (3.57), (3.58) and using (3.50), the following can be concluded $\forall t \in [t_0, \infty)$.

$$W_{FC,i}(t) = N_{FC,i}(t)\theta,\ \forall i \in \{1, \ldots, s\}. \tag{3.59}$$

From (3.57), the square matrix $N_{FC,i}(t)$ can be expressed as

$$N_{FC,i}(t) = \underbrace{exp\{-\alpha_1 t\}}_{\geq 0} \int_{t_0}^{t} \underbrace{exp\{\alpha_1 r\}}_{\geq 1}\ \underbrace{N^T_{F,i}(r) N_{F,i}(r)}_{\geq 0}\, dr \tag{3.60}$$

while utilizing (3.60), the following property can be deduced.

Property 9. *$N_{FC,i}(t)$ is positive semi-definite function of time i.e., $N_{FC,i}(t) \geq 0$, $\forall t \in [t_0, \infty)$, $\forall i \in \{1, \ldots, s\}$.* □

3.12.1 Insights to Double-Filtered Regressors $N_{FC,i}(t)$'s

Lemma 7. *rank*$\big(N_{FC,i}(t)\big)$= *rank*$\big(N_{FO,i}(t)\big)$ [5], $\forall t \in [t_0, \infty)$, $\forall i \in \{1, \ldots, s\}$, *where* $N_{FO,i}(t)$ *is described as below.*

$$\dot{N}_{FO,i}(t) = N_{F,i}^T(t) N_{F,i}(t), \quad N_{FO,i}(t_0) = 0 \tag{3.61}$$

Proof. For proof, refer to the Lemma 1 of (Basu Roy and Bhasin, 2019). □

Lemma 8. *rank*$\big(N_{FC,i}(t_0 + T^i_{k_i})\big) = k_i$, *where* $k_i \in \mathbb{I}^+$, *and* $T^i_{k_i} \in \mathbb{R}^+$, *implies that* *rank*$\big(N_{FC,i}(t)\big) \geq k_i$, $\forall t \in [t_0 + T^i_{k_i}, \infty)$, $\forall i \in \{1, \ldots, s\}$.

Proof. For proof, refer to the Lemma 2 of (Basu Roy and Bhasin, 2019). □

Assumption 9. *The set of filtered regressors* $N_{F,i}(t)'s$ *are C-IE (as per the Definition 4).* □

3.12.2 Proposed Distributed Switched Parameter Estimator

Consider the following novel parameter estimation law, $\forall i \in \{1, \ldots, s\}$, $\forall t \in [t_0, \infty)$.

$$\begin{aligned}\dot{\hat{\theta}}_i(t) = &\underbrace{Z_i^T r_i}_{C_{TR}} - \gamma_1 \underbrace{N_{F,i}^T(t)\Big(N_{F,i}(t)\hat{\theta}_i(t) - \tau_{F,i}(t)\Big)}_{C_{FP}} - \gamma_2 \underbrace{\Big(N_{FC,i}(t)\hat{\theta}_i(t) - W_{FC,i}(t)\Big)}_{C_{FC}} \\ &- \gamma_3 \underbrace{\Big(N_{SW,i}(t)\hat{\theta}_i(t) - W_{SW,i}(t)\Big)}_{C_{SW}} - \underbrace{\sum_{j \in \mathcal{N}_i} a_{ij}(\hat{\theta}_i - \hat{\theta}_j)}_{C_C}\end{aligned} \tag{3.62}$$

where $\gamma_1, \gamma_2, \gamma_3$ are positive scalars, and the piece-wise constant switching signals $N_{SW,i}(t) \in \mathbb{R}^{p \times p}$, $W_{SW,i}(t) \in \mathbb{R}^p$ are defined as

$$W_{SW,i}(t) \triangleq \begin{cases} W_{FC,i}\big(t_0 + T^i_{m_i}\big), & \forall t \in \big[t_0 + T^i_{m_i},\ t_0 + T^i_{m_i+1}\big) \\ W_{FC,i}\big(t_0 + T^i_{h_i}\big), & \forall t \in \big[t_0 + T^i_{h_i},\ \infty\big) \end{cases} \tag{3.63}$$

$$N_{SW,i}(t) \triangleq \begin{cases} N_{FC,i}\big(t_0 + T^i_{m_i}\big), & \forall t \in \big[t_0 + T^i_{m_i},\ t_0 + T^i_{m_i+1}\big) \\ N_{FC,i}\big(t_0 + T^i_{h_i}\big), & \forall t \in \big[t_0 + T^i_{h_i},\ \infty\big) \end{cases} \tag{3.64}$$

[5] The operator rank$(\cdot)$ implies the rank of the argument matrix.

where $i \in \{1, \ldots, s\}$ and $m_i = 0(1)(h_i - 1)$ [6] and $T_0^i \triangleq 0$.

In (3.62), the term C_{TR} is proportional to the filtered tracking error $r_i(t)$, whereas the other three components C_{FP}, C_{FC}, C_{SW} can be shown to be related to parameter estimation error $\Delta\theta_i(t)$, the consensus term C_C represents the local interaction among the followers using undirected graph topology as in Figure 3.9. For (3.46), the corresponding torque (control-input) is chosen as

$$\tau_i(t) = -\Gamma r_i - \zeta(\dot{q}_i - \hat{z}_i) - Z_i\hat{\theta}_i \tag{3.65}$$

$$\dot{\hat{z}}_i = \beta\left(\sum_{j=1}^{s} c_{ij}(\hat{z}_j - \hat{z}_i) + c_{i0}(\dot{q}_0 - \hat{z}_i)\right), \ \forall i \in \{1, \ldots, s\} \tag{3.66}$$

where $\Gamma, \zeta, \beta > 0$ are scalar gain's.

Using (3.65), the error dynamics (3.46) yields $\forall i \in \{1, \ldots, s\}$

$$M_i(q_i)\dot{r}_i = -Z_i(q_i, \dot{q}_i, \ddot{q}_{mi}, \dot{q}_{mi})\Delta\theta_i - V_i(q_i, \dot{q}_i)r_i - \Gamma r_i - \zeta\Big(r_i - \gamma\sum_{j=0}^{s} a_{ij}(q_i - q_j)\Big) \tag{3.67}$$

where $\hat{\theta}_i(t) - \theta \in \mathbb{R}^p$.

3.13 Stability/Convergence Analysis

The parameter estimation error dynamics (3.62), control input (3.65), (3.66), the closed-loop error dynamics (3.67) for all the agents can be compactly represented as

$$\Delta\dot{\theta}(t) = -\gamma_1\Phi_{FP}(t)\Delta\theta - \gamma_2\Phi_{FC}(t)\Delta\theta - \gamma_3\Phi_{SW}(t)\Delta\theta - (L \otimes I_p)\Delta\theta + \Phi_{TR}(t)r \tag{3.68}$$

[6] The notation $m = 0(1)(h_i - 1)$ implies that $0, 1, 2, h_i - 1, \forall i \in \{1, \ldots, s\}$.

$$\tau(t) = -\Gamma r - \zeta\big(r - \gamma(B \otimes I_n)\Delta q\big) - Z\Delta\theta \tag{3.69}$$

$$\Delta\dot{z}(t) = -\beta(B \otimes I_n)\Delta z(t) \tag{3.70}$$

$$M(q)\dot{r} = -Z\Delta\theta - V(q,\dot{q})r - \Gamma r - \zeta\Big(r - \gamma(B \otimes I_n)\Delta q\Big) \tag{3.71}$$

where $\otimes$ denotes the kronecker product and $I_n \in \mathbb{R}^{n\times n}$, $I_p \in \mathbb{R}^{p\times p}$ are the identity matrices, column vectors $\hat{\theta} = [\hat{\theta}_1^T, \ldots, \hat{\theta}_s^T]^T \in \mathbb{R}^{sp}$ and $\Delta\theta = [\Delta\theta_1^T, \ldots, \Delta\theta_s^T]^T \in \mathbb{R}^{sp}$ by stacking the components $\hat{\theta}_i \in \mathbb{R}^p$ and $\Delta\theta_i = \hat{\theta}_i - \theta \in \mathbb{R}^p$, $\forall i \in \{1, \ldots, s\}$. $\Delta q = [(q_1 - q_0)^T, \ldots, (q_s - q_0)^T]^T \in \mathbb{R}^{sn}$, $\Delta z(t) = [(\hat{z}_1 - \dot{q}_0)^T, \ldots, (\hat{z}_s - \dot{q}_0)^T]^T \in \mathbb{R}^{sn}$ and $r = [r_1^T, \ldots, r_s^T]^T \in \mathbb{R}^{sn}$. The matrices $\Phi_{FP}(t), \Phi_{FC}(t), \Phi_{SW}(t) \in \mathbb{R}^{sp\times sp}$, $\Phi_{TR} \in \mathbb{R}^{sp\times sn}$, $M(q) \in \mathbb{R}^{sn\times sn}$, $V(q,\dot{q}) \in \mathbb{R}^{sn\times sn}$, $Z \in \mathbb{R}^{sn\times sp}$ are the block diagonal matrices, which are defined as $\Phi_{TR}(t) \triangleq diag\big\{Z_1^T, \ldots, Z_s^T\big\}$, $\Phi_{FC}(t) \triangleq diag\big\{N_{FC,1}(t), \ldots, N_{FC,s}(t)\big\}$,
$Z \triangleq diag\big\{Z_1, \ldots, Z_s\big\}$,
$V(q,\dot{q}) \triangleq diag\big\{V_1(q_1,\dot{q}_1), \ldots, V_s(q_s,\dot{q}_s)\big\}$,
$M(q) \triangleq diag\big\{M_1(q_1), \ldots, M_s(q_s)\big\}$,
$\Phi_{SW}(t) \triangleq diag\big\{N_{FC,1}\big(t_0 + T_{m_1}^1\big), \ldots, N_{FC,s}\big(t_0 + T_{m_s}^s\big)\big\}$,
$\Phi_{FP}(t) \triangleq diag\big\{N_{F,1}^T N_{F,1}, \ldots, N_{F,s}^T N_{F,s}\big\}$.

Lemma 9. *Provided Assumption 9 is true, the matrix* $M(t) \triangleq (L \otimes I_p) + \gamma_3\Phi_{SW}(t)$ *is uniformly strictly positive-definite over the time window* $[t_0 + T, \infty)$ *i.e.,*

$$Y^T M(t) Y > 0,\ \forall t \geq t_0 + T,\ \forall Y \in \mathbb{R}^{sp} - \{\boldsymbol{0}_{sp}\}. \tag{3.72}$$

Proof. The detailed proof of this lemma is quite elaborate and involved. Hence, it is omitted here to honor the page limit and will be reported in the future publication of the authors. Some insights of the proof can be obtained from Theorem 1 of (Garg and Roy, 2019*a*). □

Theorem 5. *Considering the closed-loop error dynamics, the following properties can be ensured:*

(1) The origin of the extended state-space error dynamics; $\eta(t) \triangleq [r^T(t), \Delta\theta^T(t), \Delta q^T(t), \Delta z^T(t)]^T$ *is uniformly globally stable (UGS),* $\forall t \in [t_0, \infty)$.

(2) Asymptotic convergence of tracking, prediction errors: $r(t)$, $\Delta\theta(t)$, $\Delta q(t)$, $\Delta z(t) \to 0$ *as* $t \to \infty$.

(3) Asymptotic behavior of parameter consensus: $\hat{\theta}_j(t) - \hat{\theta}_i(t) \to 0$ *as* $t \to \infty, \forall i, j \in \{1, \ldots, s\}$.

(4) In addition if the Assumption 9 hold, the origin of the overall error dynamics $\eta(t)$ *is uniformly globally exponential stable (UGES) in a delayed sense i.e.,*

$$\|\eta(t)\| \leq \delta_1 \|\eta(t_0 + T)\| e^{-\delta_1(t - t_0 - T)}, \forall t \in [t_0 + T, \infty) \tag{3.73}$$

where δ_1 *is a positive scalars, which is independent of initial conditions, provided the following gain conditions are satisfied.*

$$K > K_1/2,\ \beta > \big(K_2 \lambda_{max}^2(B \otimes I_n)/2\lambda_{min}(B \otimes I_n)\big),$$
$$K_3 > \big(K_2/2\lambda_{min}(S) + K\lambda_{max}^2(B \otimes I_n)/\lambda_{min}(S)\big).$$

Proof. Consider the following Lyapunov candidate.

$$V(r, \Delta\theta, \Delta q, \Delta z) = \frac{1}{2} r^T M(q) r + \frac{1}{2} \Delta\theta^T \Delta\theta + \frac{1}{2} \Delta z^T \Delta z + \frac{1}{2} \gamma \zeta \Delta q^T (B \otimes I_n) \Delta q \tag{3.74}$$

Taking time derivative of (3.74) and substituting the dynamics (3.68), (3.70), (3.71), using Property 8 and a few cancellation yields $\forall t \in [t_0, \infty)$

$$\begin{aligned}\dot{V} \leq & -K\|r\|^2 - \beta\lambda_{min}(B \otimes I_n)\|\Delta z\|^2 - \Delta\theta^T \Big(\gamma_1 \Phi_{FP}(t) + \gamma_2 \Phi_{FC}(t) + \gamma_3 \Phi_{SW}(t) + (L \otimes I_p)\Big)\Delta\theta \\ & + K_1 r^T (B \otimes I_n)\Delta q + K_2 \Delta q^T (B \otimes I_n) \Delta z - K_3 \lambda_{min}(S) \|\Delta q\|^2 \end{aligned} \tag{3.75}$$

where $\beta, K, K_1, K_2, K_3 > 0$ are constants related to the design parameters, $\lambda_{min}(.)$ dictates the minimum eigenvalue of augmented matrix $(.)$, the matrix $S = (B \otimes I_n)^2$. Using Cauchy-Schwartz inequality in (3.75), is further upper-bounded $\forall t \in [t_0, \infty)$ as

$$\begin{aligned}\dot{V} \leq & -\Big(K - \frac{K_1}{2}\Big)\|r\|^2 - \Big(\beta\lambda_{min}(B \otimes I_n) - \frac{K_2}{2}\lambda_{max}^2(B \otimes I_n)\Big)\|\Delta z\|^2 \\ & - \Delta\theta^T \Big(\gamma_1 \Phi_{FP}(t) + \gamma_2 \Phi_{FC}(t) + \gamma_3 \Phi_{SW}(t) + (L \otimes I_p)\Big)\Delta\theta \\ & - \Big(K_3 \lambda_{min}(S) - \frac{K_2}{2} - \frac{K_1}{2}\lambda_{max}^2(B \otimes I_n)\Big)\|\Delta q\|^2 \leq 0 \end{aligned} \tag{3.76}$$

which is negative semi-definite; it holds from the fact that $(L \otimes I_p)$, $\Phi_{FP}(t)$, $\Phi_{FC}(t)$ and $\Phi_{SW}(t)$ are positive semi-definite matrices, which implies that the origin of the estimation error dynamics is Lyapunov stable. Hence, part 1 of Theorem is proved. Part 2 and 3 of the theorem can be proved by invoking Barbalat's Lemma (Lemma 8.2 at page 323 (Khalil and Grizzle, 2002)) on the Lyapunov function and using the fact that $L\mathbf{1} = 0$.

For part 4 of the theorem : Considering (3.76), Lemma 9 and the mentioned gain conditions in the theorem statement yields the following.

$$\dot{V} \leq -K\|r\|^2 - \beta\|\Delta z\|^2 - \lambda_{min}(M)\|\Delta\theta\|^2 - K_3\|\Delta q\|^2 \leq 0,\ \forall t \geq t_0 + T \quad (3.77)$$

using (3.74), the above inequality modifies to

$$\dot{V} \leq -\underbrace{min(K, \beta, \lambda_{min}(M), K_3)/\lambda_M}_{\delta_1} V,\ \forall t \geq t_0 + T \quad (3.78)$$

where $\lambda_M \triangleq max\Big(\lambda_{max}(M(q)), 1, \lambda_{max}(B \otimes I_n)\Big)$.

Hence using the comparison Lemma (Lemma 3.4 at page 103 (Khalil and Grizzle, 2002)), the differential inequality in (3.78) leads to the subsequent exponentially convergent bound.

$$\|\eta(t)\| \leq \delta_1\|\eta(t_0 + T)\|e^{-\delta_1(t-t_0-T)}, \forall t \in [t_0 + T, \infty). \quad (3.79)$$

Since, the Lyapunov function in (3.74) is radially unbounded and the constant δ_1 are independent of initial conditions, the algebraic inequality in (3.79) proves UGES (in a delayed sense) of the error dynamic $\eta(t), \forall t \geq t_0 + T$. □

3.14 Simulation Results

In this section, a numerical simulations based on Figure 3.9 are preformed to show the effectiveness of the proposed distributed composite adaptive synchronization algorithm. The structures of the respective matrices for all the three followers are provided here. $M_i(q_i) = \big[w_1 + 2w_3c_{i,2}\ w_2 + w_3c_{i,2}; w_2 + w_3c_{i,2}\ w_2\big]$, $V_i(q_i, \dot{q}_i) = [-w_3s_{i,2}\dot{q}_{i,2} - w_3s_{i,2}(\dot{q}_{i,1} + \dot{q}_{i,2}); w_3s_2\dot{q}_{i,1}\ 0]$, $F_i(\dot{q}_i) = [f_1\ 0; 0\ f_2]$. System parameter are given by $w_1 = 3.473\ kg.m^2$, $w_2 = 0.196\ kg.m^2$, $w_3 = 0.242\ kg.m^2$, $f_1 = 5.3\ Nm.sec$, $f_2 = 1.1\ Nm.sec$. Further $c_{i,2} = cos(q_{i,2})$ and $s_{i,2} = sin(q_{i,2})$. The gains are chosen as $\gamma = 1$, $\alpha = 0.5$, $\alpha_1 = 0.5$, $\zeta = 3.5$, $\Gamma = 3.5$, and $\gamma_1 = 150I_5$, $\gamma_2 = 150I_5$,

$\gamma_3 = 150 I_5$, $\beta = 10$, where I_5, denotes the identity matrix. The leader's velocity is $\dot{q}_0 = [0.4, 0.3]^T rad/sec$. The performance of the proposed adaptive controller is compared with (Mei *et al.*, 2011). Figure 3.10 shows the comparison of norm of concatenated position tracking-error $\|\Delta q(t)\|$. Figure 3.11 shows the comparison of norm of concatenated velocity tracking-error $\|\Delta \dot{q}(t)\|$. Figure 3.12 shows the comparison of norm of concatenated parameter estimation-error $\|\Delta \theta(t)\|$. The comparative plots depict that the proposed adaptive controller outperforms the classical one in terms of transient response of tracking error and transient as well as steady-state response of parameter estimation error.

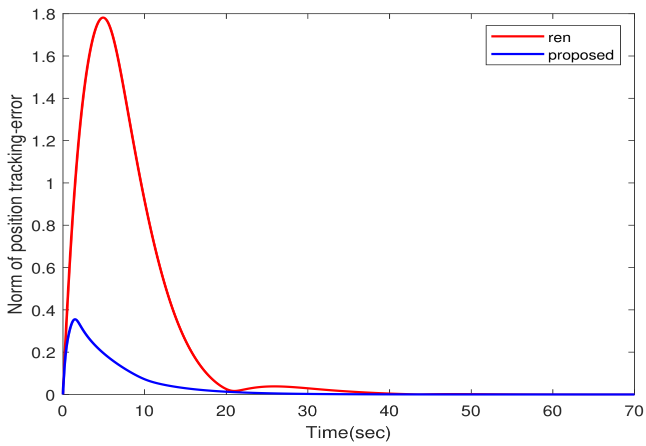

Figure 3.10: Comparison of $\|\Delta q(t)\|$

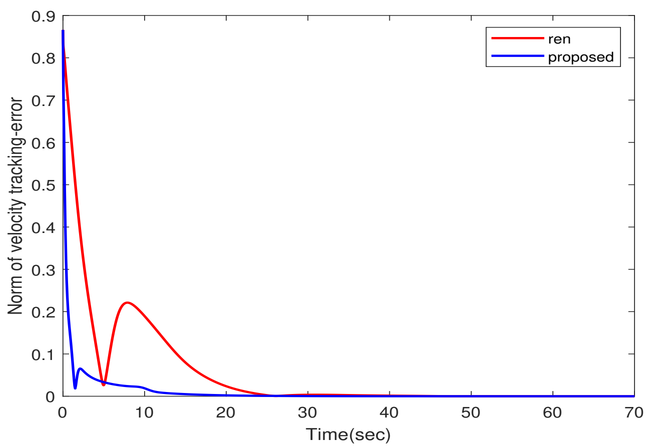

Figure 3.11: Comparison of $\|\Delta \dot{q}(t)\|$

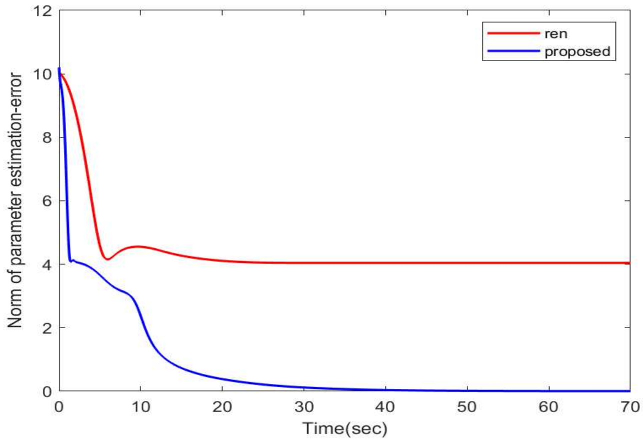

Figure 3.12: Comparison of $\|\Delta\theta(t)\|$

3.15 Conclusion

This chapter designs a novel distributed composite adaptive controller for the synchronization of multiple uncertain follower EL systems to the leader's trajectory. Unlike past literature in adaptive control, where parameter convergence is ensured by restrictive conditions like PE or C-PE, the proposed algorithm ensures parameter convergence by a milder condition, called C-IE. It is an extension of the recently introduced concept of IE condition. The proposed work builds an estimator based on strategic multiple switching such that parameter convergence is obtained under the C-IE condition and, thereby, improves transient synchronization performance significantly as compared to status quo. Simulation results corroborate the claims of the proposed algorithm.

Chapter 4

Relaxed Excitation based Distributed Adaptive Parameter Estimator with Communication Delay

4.1 Introduction

In this chapter, a novel distributed adaptive parameter estimator is proposed for a MAS architecture having a strongly connected digraph based communication topology in the presence of inter-agent communication delay. The proposed algorithm exhibits the following properties; (1) asymptotic consensus of parameter estimates is ensured without any restriction on the regressor or feature vectors and (2) parameter convergence is achieved under the uniform C-IE condition. Here, the notion of uniform C-IE is defined for the regressor concerning the agent dynamics, where each agent is modeled as single-integrator. Unlike previous results on C-IE, a novel weighting function based integrator is introduced here. The designed integrator dynamics does not have internal instability as well as online rank-checking based multiple switching requirements, which were the major concerns in open-loop and closed-loop filter architectures of C-IE based designs (Garg and Roy, 2019*a*, 2020*c*,*b*; Goel *et al.*, 2022; Garg and Basu Roy). Moreover, the proposed algorithm utilizes a more generalized graph topology of strongly connected digraph, unlike the previous C-IE based frameworks using undirected graph (Garg and Roy, 2019*a*) and strongly connected and balanced digraph (without communication delay) (Garg and Roy, 2020*c*). Under the condition of uniform C-IE, rigorous stability analysis with a suitable choice of Lyapunov-Krasovskii (LK) functional candidate is performed, which ensures UGS and asymptotic convergence in the presence of communication delay and UGES in case of a delay-free setting. Simulation results further validate the efficacy of the proposed algorithm. As far as the authors are aware, this is the first work on a relaxed excitation-based distributed adaptive estimator over a strongly connected digraph, providing stability guarantees in the presence of communication delay.

4.2 Definitions on Signal Excitation Conditions

This section introduces some preliminary definitions related to signal excitation, which are used to comment on the convergence properties of proposed distributed adaptive parameter estimators.

The classical definitions of PE and IE can be found in the references (Narendra and Annaswamy, 1987; Tao, 2003; Ioannou and Chien, 1993),(Roy *et al.*, 2016; Basu Roy *et al.*, 2018; Roy *et al.*, 2017*a*), respectively. To capture the initial condition dependence of the regressor, a concept of uniform PE (u-PE) is introduced in (Panteley *et al.*, 2001) for a pair (ϕ, f), where $\phi(t,x)$ is a general function of time t and state $x(t)$ (regressor in the framework of adaptive control) and $f(t,x)$ is the function governing the dynamics of the state $x(t)$. In contrast to u-PE condition, a further slackened condition called uniform IE (u-IE) is introduced in (Basu Roy and Bhasin, 2019; Roy *et al.*, 2017*b*; Roy and Bhasin, 2019).

The extended version of (u-PE) condition in the MAS architecture named uniform C-PE, inspired from (Papusha *et al.*, 2014; Chen *et al.*, 2013; Wensing and Slotine, 2018; Stegagno and Yuan, 2019; Cho *et al.*, 2017; Yuan *et al.*, 2018), is defined as follows.

Definition 5. *A group of vector valued function* $\phi_i(t) \triangleq \phi(t, x_i) : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^p$ *is called uniformly C-PE with respect to the dynamics* $\dot{x}_i = f(t, x_i)$*,* $\forall i = 1(1)q$*, if* $\exists\, \varsigma, T > 0$*, such that* $\forall (t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{nq}$ [1]*, all corresponding solutions satisfy*

$$\int_t^{t+T} \sum_{i=1}^{q} \phi_i(r, x(r)) \phi_i^{\mathrm{T}}(r, x(r)) dr \geq \varsigma I_p, \quad \forall t \geq t_0. \tag{4.1}$$

□

The concept of uniform C-IE, taken from (Garg and Roy, 2019*a*, 2020*b*,*c*; Goel *et al.*, 2022; Garg and Basu Roy), is defined as follows.

Definition 6. *A group of vector valued function* $\phi_i(t) \triangleq \phi(t, x_i) : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^p$ *is called uniformly C-IE with respect to the dynamics* $\dot{x}_i = f(t, x_i)$*,* $\forall i = 1(1)q$*, if* $\exists\, \varsigma, T >$

[1] $x_0 \triangleq [x_1^{\mathrm{T}}(t_0), \cdots, x_q^{\mathrm{T}}(t_0)]^{\mathrm{T}}$.

0, *such that,* $\forall(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{nq}$, *all corresponding solutions satisfy*

$$\int_{t_0}^{t_0+T} \sum_{i=1}^{q} \phi_i(r, x(r))\phi_i^{\mathrm{T}}(r, x(r))dr \geq \varsigma I_p. \tag{4.2}$$

□

Remark 11. *The concept of uniformity indicates that the parameters* ς, T *are independent of the initial conditions* (t_0, x_0)*. The uniform C-PE condition implies a group of signals* $\phi_i(t, x)$*,* $\forall i = 1(1)q$ *being exciting in a cooperative fashion over the entire time-span, i.e., [*$t,\ t+T$*],* $\forall t \geq t_0$*, while each regressor* $\phi_i(t, x)$ *may not be individually uniform PE (Narendra and Annaswamy, 2012; Loria and Panteley, 2002; Panteley* et al.*, 2001). Similarly, uniform C-IE condition implies the satisfaction of excitation requirement in the initial time-window* $[t_0, t_0+T]$ *in a cooperative manner by a group of signals, while every individual signal not necessarily being uniform IE.* □

4.3 Problem Formulation

Consider a mapping problem to learn an unknown vector-valued function over a compact set $M \subset \mathbb{R}^n$ through a team of robots/agents, where the mathematical structure of the unknown function can be represented as

$$y = f(x) = \Theta^{\mathrm{T}}\phi(x),\ \forall x \in M \subset \mathbb{R}^n \tag{4.3}$$

Here, $f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the unknown mapping to be learned; $\Theta \in \mathbb{R}^{p\times m}$ denotes matrix of unknown constant parameters and $\phi(x) : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is a known basis function/feature vector (also known as "regressor" in adaptive control literature). The variable $x \in \mathbb{R}^n$ represents the input and $y \in \mathbb{R}^m$ represents the output of the mapping. The set M is a compact subset of $\mathbb{R}^n$ and the parameterization holds $\forall x \in M$. Here, it is assumed that if $x \in M$, then $\phi(x) \in M_\phi \subset \mathbb{R}^p$, where M_ϕ is a compact subset of $\mathbb{R}^p$. Due to the introduced parameterization in (4.3), the above mapping problem boils down to an estimation problem of the unknown parameter matrix Θ. The problem is solved using a team of robots in a distributed and online setting as described subsequently.

4.3.1 Agents' Communication Topology and Model

The communication topology for comprising q number of agents/robots, is represented by the graph G, where each vertex is treated as an agent and edge $(i, j) \in E$, denotes that j^{th} agent can get information from i^{th} agent. It is assumed that each communication link (i, j) has some unknown constant communication delay T_{ji}, which is depicted in Figure 4.1 (below). Each agent/robot can interact only with its local neighbors through existing communication links. Each robot is modelled as a single-integrator dynamics.

$$\dot{x}_i(t) = u_i(t),\ x_i(t_0) \in M_i \subset M,\ \forall i = 1(1)q \tag{4.4}$$

where $x_i(t) \in \mathbb{R}^n$ is the state/position of robot i and $u_i(t) \in \mathbb{R}^n$ is the velocity/control input of robot i. Each robot is equipped with appropriate sensors to measure it's own position $x_i(t)$ and the functional value of the mapping $y_i(t) \in \mathbb{R}^m$ at the current position, i.e., the input-output pair $\{x_i(t), y_i(t)\}$ satisfies the following relation based on the defined model in (4.3).

$$y_i(t) = \Theta^{\mathrm{T}}\phi(x_i(t)),\ \forall t \geq t_0,\ \forall i = 1(1)q \tag{4.5}$$

In (4.5), M_i's are sub-regions (sub-sets) of the region of interest M, such that the following is satisfied.

$$\bigcup_{i=1}^{q} M_i = M \tag{4.6}$$

$$M_i \cap M_j = \emptyset,\ \forall i, j \in 1(1)q,\ i \neq j \tag{4.7}$$

Note that the first condition (4.6) ensures that the entire area is spanned by the team of robots and the second condition (4.7) obviates the possibility of inter-robot collision.

Assumption 10. *Graph G, which represents the communication topology among q number of agents, is a strongly connected digraph.* □

Assumption 11. $\|\Theta\| < \bar{\Theta}$*, for some constant* $\bar{\Theta}$[2] $\in \mathbb{R}_{>0}$. □

[2]Here, it is assumed that the information of $\bar{\Theta}$ is known for further subsequent formulation.

4.3.2 Objective

The objective of this work is to develop an online distributed adaptive parameter estimation/learning algorithm, i.e., an distributed adaptive parameter update law for $\hat{\Theta}_i(t)$[3], $\forall t \geq t_0$, using the online measurements of $\phi_i(t)$[4], $y_i(t)$ of the model (4.5) while collaborating (sharing delayed information of the parameter estimate) with the neighboring agents, such that

$$\|\hat{\Theta}_i(t) - \Theta\| \rightarrow 0 \; as \; t \rightarrow \infty, \; \forall t \geq t_0, \; \forall i = 1(1)q \tag{4.8}$$

and ensure a bounded and finite energy control input for each robot, i.e., $u_i(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\forall i = 1(1)q$.

An illustration of the multi-robot network, where the communicating protocol is based on Assumption 10 is shown in Figure 4.1.

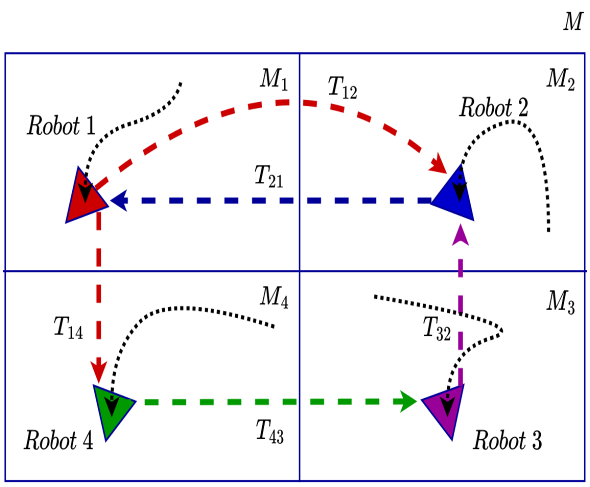

Figure 4.1: Mapping problem formulation with $q = 4$ robots. Here, the dotted curves with the black arrow represent the robots' trajectories; the colored dotted arrows with T_{ji} delay represents the communication link among robots, and colored triangles are used to denote the current locations of the robots/agents.

For subsequent formulation, the following preliminary results of graph theory are used in this paper.

Lemma 10. *For a strongly connected directed graph G, the Laplacian matrix $L \in \mathbb{R}^{q\times q}$ has the following properties.*

[3] $\hat{\Theta}_i(t) \in \mathbb{R}^{p\times m}$ denotes the i^{th} agent estimate of Θ.
[4] Without loss of generality, in this paper $\phi(x_i(t))$ is written as $\phi_i(t)$.

(1) It has a simple zero eigen-value corresponding to the right eigen-vector $\mathbf{1}_q$, and all non-zero eigen-values have the positive real-part.

(2) Let $w = [w_1, w_2, \dots, w_q]^{\mathrm{T}}$, $w_i \in \mathbb{R}_{>0}$, $\forall i = 1(1)q$, be the left eigen-vector of L associated with the zero eigen-value. Then $\tilde{L} = WL + L^{\mathrm{T}}W \geq 0$, where $W =$ diag$\{w_1, w_2, \dots, w_q\} \in \mathbb{R}^{q \times q} > 0$.

Proof. For proof refer to the Lemma 5 and 11 of (Zhang *et al.*, 2011). □

4.3.3 Finite Energy Control Input Design

Consider the following design.

$$u_i(t) = \mathrm{Proj}_{M_i}\big(u_{i,n}(t) + u_{i,ex}(t)\big),\ \forall t \geq t_0,\ \forall i = 1(1)q \tag{4.9}$$

where the Projection operator $\mathrm{Proj}_{M_i}(.)$ is utilized from (Lavretsky and Gibson, 2011) which ensures $x_i(t) \in M_i$, $\forall t \geq t_0$, $\forall i = 1(1)q$. Based on the above facts, it can be ensured that $x_i(t) \in \mathcal{L}_\infty$ which implies $\phi_i(t) \in \mathcal{L}_\infty$ and hence $y_i(t) \in \mathcal{L}_\infty$, $\forall i = 1(1)q$. Here, $u_{i,n}(t) \in \mathbb{R}^n$ is a nominal stabilizing controller, and $u_{i,ex}(t) \in \mathbb{R}^n$ is an exploratory signal. The nominal controller $u_{i,n}(t)$ is typically designed based on the control objective, such as - coverage control (Schwager *et al.*, 2008*b*; Cortes *et al.*, 2004) or extremum-seeking control (Garg *et al.*, 2022), etc. Since there is no specific control objective in this work, the nominal controller can simply be designed to be a goal-reaching controller with a final destination of each robot in their corresponding sub-region while maintaining $u_{i,n}(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$[5], uniform continuity. The exploratory signal $u_{i,ex}(t)$ facilitates to satisfy excitation condition, however, PE or C-PE condition demands an infinite energy (non-square-integrable) persistent exploration. In contrast, C-IE or IE condition can be satisfied by an appropriately designed decaying exploration. Hence, $u_{i,ex}(t)$ for each robot is chosen to be continuously differentiable and have the property - $u_{i,ex}(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, uniform continuity, which implies $u_{i,ex}(t) \to 0$ as $t \to \infty$, $\forall i = 1(1)q$. In fact, $u_{i,ex}(t)$ can be generated using the following dynamics.

$$a_{i,N}\frac{d^N u_{i,ex}(t)}{dt^N} + a_{i,(N-1)}\frac{d^{N-1}u_{i,ex}(t)}{dt^{N-1}} + \cdots + a_{i,1}u_{i,ex}(t) = g_{i,ex}(t) \tag{4.10}$$

[5]Details of $u_{i,n}(t)$ are provided in simulation section.

where $g_{i,ex}(t)$ is an auxiliary input satisfying $g_{i,ex}(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\forall i = 1(1)q$; the constants $a_{i,1}, \dots, a_{i,N}$, and initial conditions and the order N can be suitably chosen for the above properties of $u_{i,ex}(t)$ to hold. For instance, $u_{i,ex}(t)$ can take the forms of e^{-kt}, $e^{-k_1 t} sin(k_2 t)$, $e^{-k_3 t} cos(k_4 t)$ or $e^{-k_5 t}\big(sin(k_6 t) + cos(k_7(t))\big)$, etc. where all the different constants $k_{(.)} \in \mathbb{R}_{>0}$.

Remark 12. *The purpose of embedding such decaying exploratory signal $u_{i,ex}(t)$ is to provide sufficient excitation through transient oscillation which can facilitate to satisfy uniform C-IE condition.* □

4.4 Distributed Adaptive Parameter Estimator Design with Communication Delay

To address the mapping problem of learning an unknown vector-valued function over M, this section elaborates on the implementation procedure of the distributed adaptive parameter estimation algorithm design in the presence of communication delay. However, to omit the requirement of restrictive C-PE condition, a novel set of weighted integrator dynamics is proposed, which is inspired from (Garg *et al.*, 2022) [6].

4.4.1 Weighted Integrator Dynamics

Consider the following weighted integrator dynamics $\forall t \geq t_0$.

$$\dot{Y}_i(t) = \beta(t)\phi_i(t)\phi_i^{\mathrm{T}}(t),\ Y_i(t_0) = 0,\ \forall i = 1(1)q \tag{4.11}$$

$$\dot{Z}_i(t) = \beta(t)y_i(t)\phi_i^{\mathrm{T}}(t),\ Z_i(t_0) = 0,\ \forall i = 1(1)q \tag{4.12}$$

where $Y_i(t) \in \mathbb{R}^{p\times p}$ denotes the weighted integrated regressor, $Z_i(t) \in \mathbb{R}^{m\times p}$ is known as the weighted integrated output, and $\beta(t) \in \mathbb{R}$ is strategically introduced weighting function, which has the following properties

Property 10. $\beta(t) > 0,\ \forall t \in [t_0, \infty)$. □

[6] While the work in (Garg *et al.*, 2022) has introduced the weighted integrator in the context of single-agent parameter estimation, the current work extends the idea for multi-agent setting.

Property 11. $\beta(t) \leq \bar{\beta} < \infty,\ \forall t \in [t_0, \infty) \implies \beta(t) \in \mathcal{L}_\infty$. $\square$

Property 12. $\beta(t) \in \mathcal{L}_1$. $\square$

whereas $\bar{\beta} \in \mathbb{R}_{>0}$ is the upper-bound of $\beta(t)$. Based on the above Properties 10-12, weighting function $\beta(t)$ can be generated by the following dynamics

$$b_N \frac{d^N(\beta(t))}{dt^N} + b_{N-1} \frac{d^{N-1}(\beta(t))}{dt^{N-1}} + \cdots + b_1(\beta(t)) = g_\beta(t) \tag{4.13}$$

where the function $g_\beta(t)$ satisfies; $g_\beta(t) \geq 0$ and $g_\beta(t) \in \mathcal{L}_1 \cap \mathcal{L}_\infty$; the constants $b_1, \ldots, b_N$, initial conditions, N, and function $g_\beta(t)$ are chosen such that the above properties of $\beta(t)$ hold. For instance, $\beta(t)$ can take the shape like $k_0 e^{-k_1 t},\ k_0 t e^{-k_1 t},\ k_0 t^2 e^{-k_1 t}$, etc. with $k_{(\cdot)} > 0$.

Analytically solving the (4.11), (4.12), along with (4.4)-(4.5), it can be deduced that

$$Z_i(t) \triangleq \Theta^{\mathrm{T}} Y_i(t),\ \forall t \geq t_0,\ \forall i = 1(1)q. \tag{4.14}$$

Next, few lemmas are proposed, which quantifies crucial features of the weighted integrated regressor matrix $Y_i(t)$.

Lemma 11. *$Y_i(t)$ is a positive semi-definite function of time i.e., $Y_i(t) \geq 0$, $\forall t \geq t_0$, $\forall i = 1(1)q$.*

Proof. From (4.11), the square matrix $Y_i(t)$ can be represented as

$$Y_i(t) = \int_{t_0}^{t} \underbrace{\beta(r)}_{>0}\ \underbrace{\phi_i(r)\phi_i^{\mathrm{T}}(r)}_{\geq 0}\, dr \tag{4.15}$$

Utilizing Property 10 in (4.15), it can be deduced that $Y_i(t) \geq 0$, $\forall t \geq t_0$, $\forall i = 1(1)q$. $\square$

Lemma 12. *$Y_i(t)$ is a non-decreasing function of time in the sense of matrix inequality i.e., $Y_i(t_2) \geq Y_i(t_1)$, $\forall t_2 \geq t_1 \geq t_0$, $\forall i = 1(1)q$.*

Proof. From (4.11), the square matrix $Y_i(t)$ can also be expressed as

$$Y_i(t_2) = Y_i(t_1) + \underbrace{\int_{t_1}^{t_2} \beta(r)\phi_i(r)\phi_i^{\mathrm{T}}(r) dr}_{\geq 0} \tag{4.16}$$

From (4.16), utilizing the Property 10, it can be concluded that $Y_i(t_2) \geq Y_i(t_1)$, $\forall t_2 \geq t_1 \geq t_0, \forall i = 1(1)q$. $\square$

Lemma 13. *Weighted integrated variables $Y_i(t), Z_i(t) \in \mathcal{L}_\infty$, $\forall t \geq t_0$, $\forall i = 1(1)q$.*

Proof. Based on the (4.9), Properties 10-12 of weighting function $\beta(t)$ in (4.15), $\|Y_i(t)\|$ can be expressed

$$\begin{aligned}\|Y_i(t)\| &= \|\int_{t_0}^{t} \underbrace{\beta(r)}_{>0}\ \underbrace{\phi_i(r)\phi_i^{\mathrm{T}}(r)}_{\geq 0} dr\| \\ &\leq \int_{t_0}^{t} \|\beta(r)\| \|\phi_i(r)\phi_i^{\mathrm{T}}(r)\| dr \\ &\leq \underbrace{\max(\|\phi_i(t)\phi_i^{\mathrm{T}}(t)\|)}_{\nabla^2 \in \mathbb{R}_{>0}} \underbrace{\int_{t_0}^{t} \|\beta(r)\| dr}_{\bar{\beta} \in \mathbb{R}_{>0}} < \infty \end{aligned} \tag{4.17}$$

which implies $Y_i(t) \in \mathcal{L}_\infty$. Utilizing a similar argument and Assumption 11, based on (4.12), it can be ensured that $Z_i(t) \in \mathcal{L}_\infty$, $\forall i = 1(1)q$. $\square$

Remark 13. *Compared to (4.11)-(4.12), the open-loop filter architecture; $Y_i(t) = \int\limits_{t_0}^{t} \phi_i(r)\phi_i^{\mathrm{T}}(r)dr$, $Z_i(t) = \int\limits_{t_0}^{t} y_i(r)\phi_i^{\mathrm{T}}(r)dr$ (used in (Krause and Khargonekar, 1987; Adetola and Guay, 2008; Roy* et al.*, 2017*b*; Yuan* et al.*, 2021; Cho* et al.*, 2017; Garg and Roy, 2020*b*, 2019*a*)) are not BIBO stable and susceptible to disturbances. On the other hand, closed-loop filter architecture; $Y_i(t) = \int\limits_{t_0}^{t} e^{-k(t-r)}\phi_i(r)\phi_i^{\mathrm{T}}(r)dr$, $Z_i(t) = \int\limits_{t_0}^{t} e^{-k(t-r)}y_i(r)\phi_i^{\mathrm{T}}(r)dr$, (used in (Garg and Roy, 2020*c*; Goel* et al.*, 2022)) the above issues are resolved. However, due to the forgetting factor $k > 0$ (i.e., unlearning of information), a computationally expensive switching mechanism is required in the parameter update law for incorporating the C-IE condition, which may not be realistic in many practical settings. Unlike past literature, the use of weighting function $\beta(t)$ in the proposed architecture ensures BIBO stability of the integrator dynamics* (4.11)-(4.12) *and further removes the computationally burdensome switching mechanism as revealed subsequently.* $\square$

4.4.2 Local Objective Function for i^{th} Agent

Consider the following i^{th} agent objective function, $\forall t \geq t_0, \forall i = 1(1)q$:

$$\underbrace{J\big(\hat{\Theta}_i(t), \hat{\Theta}_j(t-T_{ji}), t\big)}_{\mathbb{R}^{p\times m}\times\mathbb{R}^{p\times m}\times\mathbb{R}_{\geq 0}\to\mathbb{R}} = \frac{\gamma}{2}\underbrace{\|\hat{\Theta}_i^{\mathrm{T}}(t)\phi_i(t) - y_i(t)\|^2}_{J_P} + \underbrace{\frac{\gamma_I}{2}\int_{t_0}^{t}\beta(r)\|\hat{\Theta}_i^{\mathrm{T}}(t)\phi_i(r) - y_i(r)\|^2 dr}_{J_I}$$
$$+ \underbrace{\frac{1}{2}\sum_{j\in N_i}\|\hat{\Theta}_j(t-T_{ji}) - \hat{\Theta}_i(t)\|^2}_{J_C}, \tag{4.18}$$

where γ, $\gamma_I \in \mathbb{R}_{>0}$ are positive tuning parameters. Each agent has its local objective or cost function to be minimized.

4.4.3 Proposed Distributed Adaptive Parameter Estimator

The adaptive parameter estimator is designed as gradient-descent of (4.18) with respect to $\hat{\Theta}_i(t), \forall t \geq t_0, \forall i = 1(1)q$:

$$\begin{aligned}\dot{\hat{\Theta}}_i(t) &= -\Gamma_i \nabla J_{\hat{\Theta}_i(t)}\big(\hat{\Theta}_i(t), \hat{\Theta}_j(t-T_{ji}), t\big)\\ &= -\gamma\Gamma_i \underbrace{\phi_i(t)\big(\hat{y}_i(t) - y_i(t)\big)^{\mathrm{T}}}_{T_P} - \gamma_I\Gamma_i \underbrace{\big(\hat{Z}_i(t) - Z_i(t)\big)^{\mathrm{T}}}_{T_I} + \underbrace{\Gamma_i\sum_{j\in N_i}\big(\hat{\Theta}_j(t-T_{ji}) - \hat{\Theta}_i(t)\big)}_{T_C},\end{aligned} \tag{4.19}$$

where $\Gamma_i \in \mathbb{R}^{p\times p}$ is a positive-definite learning gain matrix, $\hat{y}_i(t) \triangleq \hat{\Theta}_i^{\mathrm{T}}(t)\phi_i(t)$, and $\hat{Z}_i(t) \triangleq \hat{\Theta}_i^{\mathrm{T}}(t)Y_i(t)$.

Remark 14. *To reinterpret the dynamics* (4.19) *as an instantaneous minimization of a particular objective/cost function, a local objective function is proposed in* (4.18) *which has the following features. The term J_P in the objective function represents the quadratic local prediction cost which is used to capture the proportional type of prediction error as T_P in* (4.19)*, local quadratic term J_I is used to capture past information of prediction error with appropriate weighting as T_I in* (4.19)*, and J_I denotes the quadratic disagreement function which is used to capture the neighbor's information corresponding to each i^{th} agent as T_C in* (4.19). □

In (4.19), T_P component is a proportional-type local prediction error, T_I[7] component is a novel weighted integral-type local prediction error, whereas the last component T_C is to penalize the disagreement w.r.t to neighbors.

Inspired from (Wensing and Slotine, 2018), let's define an auxiliary variable

$$\tau_{ji} = \Delta\Theta_j(t - T_{ji}) - \Delta\Theta_i(t),\ \forall i = 1(1)q,\ j \in N_i \tag{4.20}$$

where the subscript ji indicates communication from j to i. Utilizing (4.20) in (4.19), yields

$$\Delta\dot{\Theta}_i(t) = -\gamma\Gamma_i \underbrace{\phi_i(t)\phi_i^{\mathrm{T}}(t)\Delta\Theta_i(t)}_{T_P} - \gamma_I\Gamma_i \underbrace{Y_i(t)\Delta\Theta_i(t)}_{T_I} + \Gamma_i \underbrace{\sum_{j\in N_i} \tau_{ji}}_{T_C},\ \forall i = 1(1)q \tag{4.21}$$

Here, $\Delta\Theta_i(t) = \hat{\Theta}_i(t) - \Theta \in \mathbb{R}^{p\times m}$ denotes parameter estimation error.

Utilizing (4.20), the above can be alternatively represent as, $\forall i = 1(1)q$

$$\Delta\dot{\Theta}_i(t) = -\gamma\Gamma_i \underbrace{\phi_i(t)\phi_i^{\mathrm{T}}(t)\Delta\Theta_i(t)}_{T_P} - \gamma_I\Gamma_i \underbrace{Y_i(t)\Delta\Theta_i(t)}_{T_I} + \Gamma_i \underbrace{\sum_{j\in N_i} \big((\hat{\Theta}_j(t) - \hat{\Theta}_i(t)) + \Delta\delta_{ji}(t)\big)}_{T_C} \tag{4.22}$$

where $\Delta\delta_{ji}(t) \in \mathbb{R}^{p\times m}$ denotes j^{th} neighbors delay deferential error for i^{th} agent, which is defined as

$$\Delta\delta_{ji}(t) \triangleq \Delta\Theta_j(t - T_{ji}) - \Delta\Theta_j(t),\ \forall i = 1(1)q,\ j \in N_i \tag{4.23}$$

Utilizing Assumption 10 based graph preliminaries, compact representation of the (4.22) becomes

$$\Delta\dot{\Theta}(t) = -\gamma\Gamma\Phi_P(t)\Delta\Theta(t) - \gamma_I\Gamma\Phi_I(t)\Delta\Theta(t) - \Gamma(\tilde{L}\otimes I_p)\Delta\Theta(t) + \Gamma\Delta\delta(t) \tag{4.24}$$

where the column vectors $\hat{\Theta}(t) = [\hat{\Theta}_1^{\mathrm{T}}(t), \ldots, \hat{\Theta}_q^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{pq\times m}$,

[7]The main motive is to capture the past information through the integral-action since it reduces the steady state error to zero as known from classical control literature.

$\Delta\Theta(t) = [\Delta\Theta_1^{\mathrm{T}}(t), \ldots, \Delta\Theta_q^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{pq\times m}$ are obtained by stacking the components $\hat{\Theta}_i(t)$ and $\Delta\Theta_i(t)$, respectively, $\forall i = 1(1)q$. $\Phi_P(t)$, $\Phi_I(t) \in \mathbb{R}^{pq\times pq}$ are the block diagonal matrices, defined as

$$\Phi_P(t) \triangleq \mathrm{diag}\Big\{\phi_1(t)\phi_1^{\mathrm{T}}(t), \ldots, \phi_q(t)\phi_q^{\mathrm{T}}(t)\Big\}$$
$$\Phi_I(t) \triangleq \mathrm{diag}\Big\{Y_1(t), \ldots, Y_q(t)\Big\}.$$

Further, $\Delta\delta(t) \in \mathbb{R}^{pq\times m}$ denotes the compact representation of $\Delta\delta_{ji}(t)$, which is defined as

$$\Delta\delta(t) \triangleq \left[\sum_{j\in N_1} \Delta\delta_{j1}^{\mathrm{T}}(t), \ldots, \sum_{j\in N_q} \Delta\delta_{jq}^{\mathrm{T}}(t)\right]^{\mathrm{T}}.$$

$\Gamma \in \mathbb{R}^{pq\times pq}$ is a block diagonal matrix which is defined as $\Gamma = \mathrm{diag}\{\Gamma_1, \ldots, \Gamma_q\}$.

4.4.4 Stability/Convergence Analysis

The following Lemma characterizes several properties of the above proposed distributed adaptive parameter estimator dynamics (4.19) or (4.21).

Lemma 14. *For MAS architecture model* (4.4)-(4.5)*, provided Assumptions 10-11 hold, the distributed adaptive parameter estimator dynamics (4.19) or* (4.21) *ensures the following.*

(1) The origin of (4.21) *is Lyapunov stable (UGS) and all the auxiliary signals remain bounded,* $\forall t \geq t_0$, $\forall i = 1(1)q$.

(2) Asymptotic convergence of prediction errors, $\forall i = 1(1)q$

$$\big(\hat{y}_i(t) - y_i(t)\big) \to 0 \; as \; t \to \infty$$
$$\big(\hat{Z}_i(t) - Z_i(t)\big) \to 0 \; as \; t \to \infty$$

(3) Asymptotic consensus of parameter estimates:

$$\hat{\Theta}_j(t) - \hat{\Theta}_i(t) \to 0 \; as \; t \to \infty, \; \forall i = 1(1)q, \; j \in N_i.$$

Proof. Consider the Lyapunov-Krasovskii (LK) functional candidate $V = \sum V_i$ with

$$V_i = \frac{1}{2}\mathrm{tr}\Big(\Delta\Theta_i^{\mathrm{T}}\Gamma_i^{-1}\Delta\Theta_i\Big) + \frac{1}{2}\mathrm{tr}\left(\sum_{j\in N_i} \int_{t-T_{ji}}^{t} \Delta\Theta_j^{\mathrm{T}}\Delta\Theta_j dr\right) \tag{4.25}$$

Taking the time derivative of (4.25) and substituting the parameter estimator dynamics (4.21), yields

$$\begin{aligned}\dot{V}_i =& -\operatorname{tr}\Bigg(\Big(\gamma\Delta\Theta_i^{\mathrm{T}}\phi_i(t)\phi_i^{\mathrm{T}}(t)\Delta\Theta_i + \gamma_I\Delta\Theta_i^{\mathrm{T}}Y_i(t)\Delta\Theta_i\Big)\\ &- \sum_{j\in N_i}\Big(\Delta\Theta_i^{\mathrm{T}}\tau_{ji} + \frac{1}{2}\big(\Delta\Theta_j^{\mathrm{T}}\Delta\Theta_j - \Delta\Theta_j^{\mathrm{T}}(t-T_{ji})\Delta\Theta_j(t-T_{ji})\big)\Big)\Bigg)\end{aligned} \tag{4.26}$$

Noting that each $\Big(\Delta\Theta_j^{\mathrm{T}}(t-T_{ji})\Delta\Theta_j(t-T_{ji})\Big)$ can be expressed via (4.20) as

$$\Delta\Theta_j^{\mathrm{T}}(t-T_{ji})\Delta\Theta_j(t-T_{ji}) = \Delta\Theta_i^{\mathrm{T}}\Delta\Theta_i + \tau_{ji}^{\mathrm{T}}\tau_{ji} + 2\Delta\Theta_i^{\mathrm{T}}\Big(\Delta\Theta_j(t-T_{ji}) - \Delta\Theta_i(t)\Big)$$

which implies

$$\dot{V} = -\operatorname{tr}\Bigg(\sum_{i=1}^{q}\Big(\gamma\Delta\Theta_i^{\mathrm{T}}\phi_i(t)\phi_i^{\mathrm{T}}(t)\Delta\Theta_i + \gamma_I\Delta\Theta_i^{\mathrm{T}}Y_i(t)\Delta\Theta_i\Big) + \frac{1}{2}\sum_{i=1}^{q}\sum_{j\in N_i}\Big(\tau_{ji}^{\mathrm{T}}\tau_{ji}\Big)\Bigg) \leq 0 \tag{4.27}$$

The above inequality implies $\dot{V}(\Delta\Theta)$ is negative semi-definite, $\forall t \geq 0$. Hence, the origin of (4.21) is UGS. Based on (4.9) and Lemma 13, it can be conclude that $\big(\phi_i(t), Y_i(t), \dot{\phi}_i(t), \dot{Y}_i(t) \in \mathcal{L}_\infty\big)$. Using the above arguments, it can also be concluded that $V(\Delta\Theta(t))$ is uniformly bounded above by its initial value, which implies $\hat{\Theta}_i(t) \in \mathcal{L}_\infty$ i.e., the local prediction $\hat{y}_i(t), \hat{Z}_i(t) \in \mathcal{L}_\infty$, $\forall i = 1(1)q$. This completes the proof of part-1.

For part-2 of the proof, differentiating (4.27) and using the boundedness related arguments in the last paragraph, it can be concluded that $\ddot{V}(\Delta\Theta) \in \mathcal{L}_\infty \implies \dot{V}(\Delta\Theta)$ is uniformly continuous. By invoking Barbalat's Lemma (Lemma 8.2 at page 323 (Khalil and Grizzle, 2002)), $\dot{V}(\Delta\Theta(t)) \to 0$ as $t \to \infty$. Then from (4.27), it can be concluded that each $\Delta\Theta_i^{\mathrm{T}}\big(\phi_i(t)\phi_i^{\mathrm{T}}(t)\big)\Delta\Theta_i$, $\Delta\Theta_i^{\mathrm{T}}\big(Y_i(t)\big)\Delta\Theta_i$, $\tau_{ji} \to 0$ as $t \to \infty$, $\forall i = 1(1)q, j \in N_i$, which implies that all the local prediction errors $\big(\hat{y}_i(t) - y_i(t)\big)$, $\big(\hat{Z}_i(t) - Z_i(t)\big) \to 0$ as $t \to \infty$, $\forall i = 1(1)q$, respectively i.e., the asymptotic convergence of prediction errors is achieved.

For part-3, utilizing the above facts, it can be seen that each $\Delta\dot{\Theta}_i(t) \to 0$ asymptotically. Based on Assumption 10, it can be inferred that there exists a cycle as $\{(i,\omega_1)$, (ω_1,ω_2) ,..., $(\omega_o, i)\}$, $\forall i = 1(1)q$, here $\omega_{(.)}$ denotes the intermediate vertices of the graph.

Now, $\tau_{ji} \to 0$ as $t \to \infty$, $\forall i = 1(1)q$ and $j \in N_i$, implies that each $\Delta\Theta_i(t)$ converges to a trajectory with period $(T_{i\omega_1} + T_{\omega_1\omega_2} + \cdots + T_{\omega_o i})$. Since $\Delta\dot{\Theta}_i(t) \to 0$ the only possibility for this periodic trajectory is that $\hat{\Theta}_i(t)$ converges to a constant value $\forall i = 1(1)q$. Further, for $j \in N_i$ $\tau_{ji} \to 0$; which implies $\hat{\Theta}_j(t - T_{ji}) \to \hat{\Theta}_i(t)$ as $t \to \infty$. Thus due to Assumption 10, all the estimates $\hat{\Theta}_i(t)$ must converge to a common constant vector (parameter consensus). As a result, this also implies each $\Delta\delta_{ji}(t) = \hat{\Theta}_j(t - T_{ji}) - \hat{\Theta}_j(t) \to 0$, as $t \to \infty$ $\forall i = 1(1)q$, $j \in N_i$. This completes the proof of part-3 i.e., the parameter estimates asymptotically reach consensus. □

4.4.5 Asymptotic Stability/Convergence with uniform C-IE

Consider the following assumption, which can be facilitated by appropriate choices of exploratory signals $u_{i,ex}(t)$, $\forall i = 1(1)q$ of the robots.

Assumption 12. *The group of regressors $\phi_i(t)$s hold uniform C-IE w.r.t the set of dynamics* (4.4), (4.9)-(4.10) *as per Definition 6.* □

Lemma 15. *Provided Assumption 12 holds, the matrix $M(t) \triangleq \big((\tilde{L} \otimes I_p) + \gamma_I \bar{W}\Phi_I(t)\big)$ is uniformly positive-definite over the time window $[t_0 + T, \infty)$ i.e.,*

$$\Lambda^{\mathrm{T}} M(t)\Lambda \geq \nu\|\Lambda\|^2,\ \forall t \geq t_0 + T,\ \forall \Lambda \in \mathbb{R}^{pq} - \{\boldsymbol{0}_{pq}\} \tag{4.28}$$

where $\nu \in \mathbb{R}_{>0}$ is a constant positive scalar, which will be defined subsequently.

Proof. Since $L\mathbf{1}_q = 0$ (based on Lemma 10) and hence $\tilde{L}\mathbf{1}_q = 0$ (please refer to the Lemma 9 of (Zhang *et al.*, 2011)), and which implies

$$(\tilde{L} \otimes I_p)\Big(\frac{1}{\sqrt{q}}\mathbf{1}_q \otimes f_j\Big) = 0 \tag{4.29}$$

Here, the eigen-decomposition form of the matrix $(\tilde{L} \otimes I_p)$ is written as

$$(\tilde{L} \otimes I_p)(\vartheta_i \otimes f_j) = \lambda_i(\vartheta_i \otimes f_j) \tag{4.30}$$

whereas the right hand side of both the above equations can be verified on the basis of the mixed product property $PQ \otimes RS = (P \otimes R)(Q \otimes S)$ for appropriately sized matrices

P, Q, R, and S. Here, $\lambda_i > 0$, ϑ_i are the eigen-value and eigen-vector of $(\tilde{L})$ $\forall i = 2(1)q$, respectively. $f_j \in \mathbb{R}^p$ is the j^{th} unit vector for all $j = 1(1)(p)$. Considering $\Lambda \in \mathbb{R}^{pq}$ as a unit vector and expressing it in this basis as

$$\Lambda = \sum_{j=1}^{p} \kappa_j \frac{1}{\sqrt{q}} \mathbf{1}_q \otimes f_j + \sum_{i=2}^{q} \sum_{j=1}^{p} \varepsilon_{ij} \vartheta_i \otimes f_j \tag{4.31}$$

with $(\kappa, \varepsilon) \in \mathbb{R}^p \times \mathbb{R}^{(q-1)p}$ has unit norm. Define the new variable $\Phi_{bar}(t)$ of $\Phi_P(t)$ over the time interval $[t_0, t_0 + T]$ is.

$$\Phi_{bar}(t) \triangleq \int_{t_0}^{t_0+T} \Phi_P(r) dr \tag{4.32}$$

by substituting relation (4.31) into (4.28), using the first term $\Lambda^{\mathrm{T}}(\tilde{L} \otimes I_p)\Lambda$ of (4.28), can be bounded based on Lemma 10 as

$$\Lambda^{\mathrm{T}}(\tilde{L} \otimes I_p)\Lambda = \sum_{i=2}^{q} \sum_{j=1}^{p} \lambda_i \varepsilon_{ij}^2 \geq \lambda_2 (1 - ||\kappa||_2^2) \tag{4.33}$$

where $||\kappa||_2^2 + ||\varepsilon||_2^2 = 1$.

The second term $\gamma_I \Lambda^{\mathrm{T}} \bar{W} \Phi_I(t) \Lambda$ is

$$\begin{aligned}\gamma_I \Lambda^{\mathrm{T}} \bar{W} \Phi_I \Lambda =& \frac{\gamma_I}{q} \sum_{i=1}^{p} \sum_{j=1}^{p} \kappa_i \kappa_j (\mathbf{1}_q \otimes f_i)^{\mathrm{T}} \bar{W} \Phi_I (\mathbf{1}_q \otimes f_j) \\ &+ \frac{2\gamma_I}{\sqrt{q}} \sum_{i=2}^{q} \sum_{j=1}^{p} \sum_{k=1}^{p} \kappa_k \varepsilon_{ij} (\mathbf{1}_q \otimes f_k)^{\mathrm{T}} \bar{W} \Phi_I (\vartheta_i \otimes f_j) \\ &+ \gamma_I \underbrace{\sum_{i=2}^{q} \sum_{j=1}^{p} \sum_{k=2}^{q} \sum_{l=1}^{p} \varepsilon_{ij} \varepsilon_{kl} (\vartheta_i \otimes f_j)^{\mathrm{T}} \bar{W} \Phi_I (\vartheta_k \otimes f_l)}_{\geq 0}\end{aligned} \tag{4.34}$$

Utilizing the Properties 10-12 of weighting function $\beta(t)$, (4.11), it can be concluded that

$$\Phi_I(t) \geq \beta_1 \Phi_{bar}(t), \ \forall t \geq t_0 + T \tag{4.35}$$

where $\beta_1 > 0$ is a positive scalar which is depend on lower-bound $\underline{\beta}$ of weighting function

$\beta(t)$. Substituting the (4.35) into (4.34), yields

$$\begin{aligned}\gamma_I \Lambda^{\mathrm{T}} \bar{W} \Phi_I(t) \Lambda \geq & \frac{\beta_1 \gamma_I}{q} \sum_{i=1}^{p} \sum_{j=1}^{p} \kappa_i \kappa_j (\mathbf{1}_q \otimes f_i)^{\mathrm{T}} \bar{W} \Phi_{bar}(t) (\mathbf{1}_q \otimes f_j) \\ & + \frac{2\beta_1 \gamma_I}{\sqrt{q}} \sum_{i=2}^{q} \sum_{j=1}^{p} \sum_{k=1}^{p} \kappa_k \varepsilon_{ij} (\mathbf{1}_q \otimes f_k)^{\mathrm{T}} \bar{W} \Phi_{bar}(t) (\vartheta_i \otimes f_j) \\ & + \beta_1 \gamma_I \underbrace{\sum_{i=2}^{q} \sum_{j=1}^{p} \sum_{k=2}^{q} \sum_{l=1}^{p} \varepsilon_{ij} \varepsilon_{kl} (\vartheta_i \otimes f_j)^{\mathrm{T}} \bar{W} \Phi_{bar}(t) (\vartheta_k \otimes f_l)}_{\geq 0} \end{aligned} \tag{4.36}$$

Moreover utilizing the C-IE definition and Cauchy-Schwarz inequality followed by Holder's inequality on cross-coupled term, $\gamma_I \Lambda^{\mathrm{T}} \bar{W} \Phi_I(t) \Lambda$ can be lower-bounded as

$$\gamma_I \Lambda^{\mathrm{T}} \bar{W} \Phi_I(t) \Lambda \geq \frac{\beta_1 \gamma_I}{q} \kappa^{\mathrm{T}} \left(\int_{t_0}^{t_0+T} \sum_{i=1}^{q} w_i \phi_i(r) \phi_i^{\mathrm{T}}(r) dr \right) \kappa - \frac{2\varsigma_1 \beta_1 \gamma_I q}{\sqrt{q}} \sum_{i=2}^{q} \sum_{j=1}^{p} \sum_{k=1}^{p} |\kappa_k \varepsilon_{ij}| \tag{4.37}$$

After further simplification and provided the Assumption 12 uniform C-IE holds, (4.37) yields

$$\gamma_I \Lambda^{\mathrm{T}} \bar{W} \Phi_I(t) \Lambda \geq \frac{||\kappa||_2^2 \gamma_I \beta_1 \varsigma \underline{w}}{q} - 2\varsigma_1 \beta_1 \gamma_I q \sqrt{||\kappa||_2^2 (1 - ||\kappa||_2^2)} \tag{4.38}$$

where $\varsigma_1 > 0$ is the possible upper-bound based on the uniform C-IE definition and $\underline{w} = \min_i w_i$. Thus by clubbing these two terms (4.33) and (4.38) together, required lower bound, which is strictly greater than zero, is achieved i.e.,

$$\Lambda^{\mathrm{T}} M(t) \Lambda > \max\{\Lambda^{\mathrm{T}} (\tilde{L} \otimes I_p) \Lambda, \gamma_I \Lambda^{\mathrm{T}} \bar{W} \Phi_I(t) \Lambda\} \geq \nu > 0$$

where ν is

$$\nu = \inf_{||\kappa||_2 \leq 1} \max \left\{ \lambda_2 \Big(1 - ||\kappa||_2^2\Big), \frac{||\kappa||_2^2 \gamma_I \beta_1 \varsigma \underline{w}}{q} - 2\varsigma_1 \beta_1 \gamma_I q \sqrt{||\kappa||_2^2 \Big(1 - ||\kappa||_2^2\Big)} \right\} \tag{4.39}$$

Using continuity argument, infimum in (4.39) is attained and is strictly greater than zero (if ν is zero and the first term is zero, then the second term should also be zero, $\frac{\gamma_I \beta_1 \varsigma \underline{w}}{q}$ is zero, which is a contradiction from the uniform C-IE condition (4.2) requirement for parameter convergence). $\square$

Theorem 6. *Provided Assumption 12 holds, then the origin of the parameter estimation error dynamics (4.24) is asymptotically converging i.e., objective (4.8) is achieved.*

Proof. Consider the following Lyapunov candidate

$$V_c(\Delta\Theta) = \frac{1}{2}\mathrm{tr}(\Delta\Theta^{\mathrm{T}}\Gamma^{-1}\bar{W}\Delta\Theta) \tag{4.40}$$

Taking the time derivative of (4.40) and substituting the (4.24), yields

$$\dot{V}_c(\Delta\Theta) = -\ \mathrm{tr}\Big(\gamma\Delta\Theta^{\mathrm{T}}\bar{W}\Phi_P(t)\Delta\Theta + \gamma_I\Delta\Theta^{\mathrm{T}}\bar{W}\Phi_I(t)\Delta\Theta + \Delta\Theta^{\mathrm{T}}(\tilde{L}\otimes I_p)\Delta\Theta - \Delta\Theta^{\mathrm{T}}\bar{W}\Delta\delta\Big) \tag{4.41}$$

Utilizing the Lemma 28 (refer to Appendix A), invoking the Assumption 12 based positive-definiteness Lemma 15 in (4.41), $\forall t \geq t_0 + T$, following yields

$$\dot{V}_c(\Delta\Theta) \leq -\mathrm{tr}\Big(\Delta\Theta^{\mathrm{T}}M\Delta\Theta\Big) + \mathrm{tr}\Big(\Delta\Theta^{\mathrm{T}}\bar{W}\Delta\delta\Big) \leq 0 \tag{4.42}$$

Applying tr(.) operator property, Cauchy-Schwartz inequality in (4.42), $\forall t \geq t_0 + T$, yields

$$\dot{V}_c(\Delta\Theta) \leq -\Big(\lambda_{min}(M) - \frac{1}{2\xi}\Big)\|\Delta\Theta\|^2 + \underbrace{\frac{\xi\|\bar{W}\|^2}{2}}_{\zeta>0}\|\Delta\delta\|^2 \leq 0 \tag{4.43}$$

Where $\exists\xi > 0$ is a free-scale constant parameter such that $\Big(\lambda_{min}(M) - \frac{1}{2\xi} > 0\Big)$. Further using the (4.40), (4.43) can be upper bounded as, $\forall t \geq t_0 + T$

$$\dot{V}_c(\Delta\Theta) \leq -2\underbrace{\left\{\frac{\lambda_{min}(M) - \frac{1}{2\xi}}{\bar{w}\lambda_{max}(\Gamma^{-1})}\right\}}_{\mu>0}V_c + \zeta\|\Delta\delta\|^2 \leq 0 \tag{4.44}$$

where $\bar{w} = \max_i w_i$. Furthermore, above can be written as

$$\dot{V}_c(\Delta\Theta) \leq -2\mu V_c + \zeta\|\Delta\delta\|^2 \leq 0,\ \forall t \geq t_0 + T \tag{4.45}$$

Based on Lemma 14 (part-3) proof argument; $\Delta\delta \to 0$ as $t \to \infty$. Hence, using Comparison Lemma (Lemma 3.4 at page 103 (Khalil and Grizzle, 2002)), (4.45) implies that the origin of (4.24) is asymptotically converging i.e., objective (4.8) is achieved. □

Remark 15. *Above Lyapunov analysis ensures asymptotic convergence of parameter estimation error to zero in the presence of nonuniform delay among agents (i.e., different delays between all the different agents). Moreover, communication delay information is not available to the agents.* □

4.5 Distributed Parameter Estimation Algorithm without Communication Delay

In this section, a special case of the above problem is tackled, where communication delay between agents is not considered i.e., $T_{ji} = 0, \forall i = 1(1)q, j \in N_i$.

4.5.1 Stability/Convergence analysis

Based on (4.24), the distributed adaptive parameter estimation error dynamics with delay-free scenario, for all the q number of agents can be compactly represented as

$$\Delta\dot{\Theta}(t) = -\gamma\Gamma\Phi_P(t)\Delta\Theta(t) - \gamma_I\Gamma\Phi_I(t)\Delta\Theta(t) - \Gamma(\tilde{L} \otimes I_p)\Delta\Theta(t) \tag{4.46}$$

The following Corollary characterizes several properties of the above distributed adaptive parameter estimator dynamics without communication delay (4.46).

Corollary 6.1. *Utilizing MAS model* (4.4)*-(4.5), provided Assumptions 10-11 hold, then the dynamics (4.46) exhibits:*

(1) Origin of the parameter estimation error dynamics (4.46) *is UGS and all the auxiliary signals remain bounded,* $\forall t \geq t_0$, $\forall i = 1(1)q$.

(2) Asymptotic convergence of prediction errors, $\forall i = 1(1)q$

$$\big(\hat{y}_i(t) - y_i(t)\big) \to 0 \; as \; t \to \infty$$
$$\big(\hat{Z}_i(t) - Z_i(t)\big) \to 0 \; as \; t \to \infty$$

(3) Asymptotic convergence of parameter estimates consensus:

$$\hat{\Theta}_j(t) - \hat{\Theta}_i(t) \to 0,\ \forall i = 1(1)q,\ j \in N_i.$$

Proof. Consider the following Lyapunov candidate

$$V_d(\Delta\Theta) = \frac{1}{2}\mathrm{tr}\big(\Delta\Theta^{\mathrm{T}}\Gamma^{-1}\bar{W}\Delta\Theta\big) \tag{4.47}$$

Taking time derivative of (4.47) and substituting the parameter estimation error dynamics (4.46), yields

$$\dot{V}_d(\Delta\Theta) = -\,\mathrm{tr}\Big(\gamma\Delta\Theta^{\mathrm{T}}\bar{W}\Phi_P(t)\Delta\Theta + \gamma_I\Delta\Theta^{\mathrm{T}}\bar{W}\Phi_I(t)\Delta\Theta + \Delta\Theta^{\mathrm{T}}(\tilde{L}\otimes I_p)\Delta\Theta\Big) \leq 0 \tag{4.48}$$

From (4.48), utilizing the Lemma 10, Lemma 28 (refer to Appendix A), and part-1 to part-3 of Corollary 6.1 can be proofed in the same way as part-1 to part-3 of Lemma 14. □

4.5.2 Stability/Convergence with uniform C-IE

Corollary 6.2. *Utilizing the Lemma 15 proof argument which is based on Assumption 12, then origin of the parameter estimation error dynamics* (4.46) *is uniformly globally exponential stable (UGES), i.e.,*

$$\|\Delta\Theta(t)\| \leq \eta\|\Delta\Theta(t_0+T)\|e^{-\eta(t-t_0-T)},\ \forall t \geq t_0+T \tag{4.49}$$

where $\eta \in \mathbb{R}_{>0}$ is a constant positive scalar.

Proof. Invoking the Lemma 15 in (4.48), yields

$$\dot{V}_d(\Delta\Theta) = -\mathrm{tr}\big(\Delta\Theta^{\mathrm{T}}M\Delta\Theta\big) < 0,\ \forall t \geq t_0+T \tag{4.50}$$

using (4.47), the above inequality modifies as

$$\dot{V}_d \leq -\underbrace{\left(\frac{2\lambda_{min}(M)}{\bar{w}\lambda_{max}(\Gamma^{-1})}\right)}_{\eta\in\mathbb{R}_{>0}} V_d,\ \forall t \geq t_0+T \tag{4.51}$$

Hence using the comparison Lemma (Lemma 3.4 at page 103 (Khalil and Grizzle, 2002)), the differential inequality in (4.51) leads to the subsequent exponentially convergent bound

$$\|\Delta\Theta(t)\| \leq \eta \|\Delta\Theta(t_0 + T)\| e^{-\eta(\mathrm{t}-\mathrm{t}_0-\mathrm{T})}, \; \forall t \geq t_0 + T \tag{4.52}$$

Since the Lyapunov function in (4.47) is radially unbounded and the constant η are independent of the initial conditions, the algebraic inequality in (4.52) proves UGES of the origin of the parameter estimation error dynamics (4.46), $\forall t \geq t_0 + T$. $\square$

4.6 Simulation Results

Consider a MAS architecture with $q = 4$ agents/robots, where each agent/robot is modeled by a single integrator dynamics $\dot{x}_i(t) = u_i(t)$ where $u_i(t)$ is generated by the dynamics (4.9). $x_i(t) = [x_{i1}(t), x_{i2}(t)] \in \mathbb{R}^2$ is the position on the plane of the i^{th} agent/robot, $u_i(t) = [u_{i1}(t), u_{i2}(t)] \in \mathbb{R}^2$ is the controller which is designed based on (4.9). The output $y_i(t)$ for each i^{th} agent is chosen to be a generalized quadratic function of $x_i(t)$ $\big($i.e., $y_i(t) = \Theta^{\mathrm{T}}\phi(x_i(t))\big)$ which can be formulated by sensing the agent current position $x_i(t)$. The pictorial representation of the above simulation analogy is depicted in Figure 4.2.

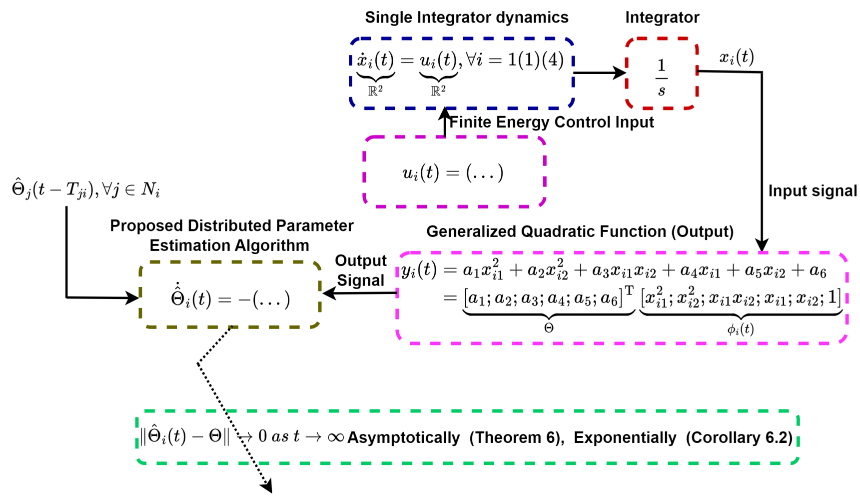

Figure 4.2: Block diagram for simulation analysis.

Chosen numerical data: True unknown constant parameter vector; $\Theta = [1, 1, 0.4, -4, -4, 4, 4] \in \mathbb{R}^6$, regressor vector signal $\phi_i(t) = [x_{i1}^2, x_{i2}^2, x_{i1}x_{i2}, x_{i1}, x_{i2}, 1] \in \mathbb{R}^6$. In this paper, weighting function $\beta(t)$ with Properties 10-12 is generated by the following dynamics

$$\dot{\beta}(t) = -k\beta(t),\ \beta(t_0) = \beta(0) \neq 0,\ \forall t \geq t_0$$

where $k \in \mathbb{R}_{>0}$. For simulation, $k = 0.2$, $\beta(0) = 1$ is chosen while all the other initial conditions are kept to be zero.

4.6.1 Spiral Motion of Robots

In this subsection for simulation, the controller $u_i(t) \in \mathbb{R}^2$ is chosen to generate spiral motion.

$$\begin{aligned}\dot{x}_1(t) =& \text{Proj}_{M_1}\Big(\underbrace{[0.2e^{-0.2(t-t_0)}cos(1.2t); 0.2e^{-0.2(t-t_0)}sin(1.2t)]}_{u_{1,ex}(t)} \\ &\underbrace{-K(x_1 - x_1^*)}_{u_{1,n}(t)}\Big),\ M_1 = [0, 0.5] \times [0, 0.5]\end{aligned}$$

$$\begin{aligned}\dot{x}_2(t) =& \text{Proj}_{M_2}\Big(\underbrace{[0.22e^{-0.3(t-t_0)}cos(1.3t); 0.22e^{-0.3(t-t_0)}sin(1.3t)]}_{u_{2,ex}(t)} \\ &\underbrace{-K(x_2 - x_2^*)}_{u_{2,n}(t)}\Big),\ M_2 = [0.5, 1] \times [0.5, 0.5]\end{aligned}$$

$$\begin{aligned}\dot{x}_3(t) =& \text{Proj}_{M_3}\Big(\underbrace{[0.18e^{-0.2(t-t_0)}cos(t); 0.18e^{-0.2(t-t_0)}sin(t)]}_{u_{3,ex}(t)} \\ &\underbrace{-K(x_3 - x_3^*)}_{u_{3,n}(t)}\Big),\ M_3 = [0.5, 1] \times [0.5, 1]\end{aligned}$$

$$\begin{aligned}\dot{x}_4(t) =& \text{Proj}_{M_4}\Big(\underbrace{[0.3e^{-0.35(t-t_0)}cos(2t); 0.3e^{-0.35(t-t_0)}sin(2t)]}_{u_{4,ex}(t)} \\ &\underbrace{-K(x_4 - x_4^*)}_{u_{4,n}(t)}\Big),\ M_4 = [0.5, 0.5] \times [0.5, 1]\end{aligned}$$

In the above controllers, $K = 0.1$ denotes the control gain, $x_1^* = [0.25; 0.25]$, $x_2^* = [0.75; 0.25]$, $x_3^* = [0.75; 0.75]$, $x_4^* = [0.25; 0.75]$, denote the final target position's of all the 4 agent/robot, respectively. Based on the above $u_i(t)$, it can be validated that each individual regressor $\phi_i(t) \in \mathbb{R}^6$ holds uniform IE condition (Roy *et al.*, 2017*b*) and hence, in cooperation they achieve uniform C-IE condition (4.2) as in Lemma 15, respectively i.e.,

$$
\begin{aligned}
&(rank(Y_1) = 6, Y_1 > 0),\ (rank(Y_2) = 6, Y_2 > 0),\\
&(rank(Y_3) = 6, Y_3 > 0),\ (rank(Y_4) = 6, Y_4 > 0),\\
&\Big(rank\,\underbrace{(\gamma_I \bar{W}\Phi_I(t) + \tilde{L} \otimes I_6)}_{M} = 24 \text{ or } M > 0\Big),\ \forall t \geq t_0 + T
\end{aligned}
$$

here, $\bar{W} = W \otimes I_6$.

Figure 4.3 demonstrates comparative simulation of the algorithm with various "uniform" delay, where convergence rate degrades with increasing the magnitude of delay. However, each time convergence is guaranteed as proved in the previous section.

Figure 4.4 illustrates a comparison with a C-PE based estimator (as used in (Wensing and Slotine, 2018) and others), where the update law only uses T_p and T_C as compared to (4.24). It is clear that due to decaying excitation, the C-PE based estimator cannot provide parameter convergence, however, the proposed algorithm is able to achieve parameter convergence.

Finally, Figure 4.5 shows the plot of concatenated parameter estimation error for the non-uniform delay case, validating the claims of Theorem 6.

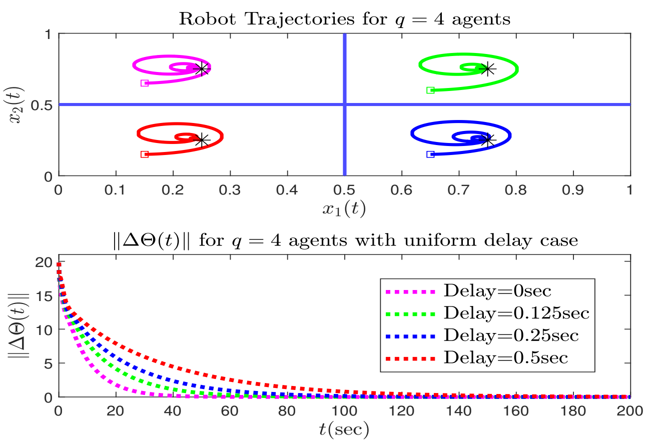

Figure 4.3: Comparative plot for norm of parameter estimation error with robot trajectories using uniform C-IE based proposed algorithm (4.19) or (4.24).

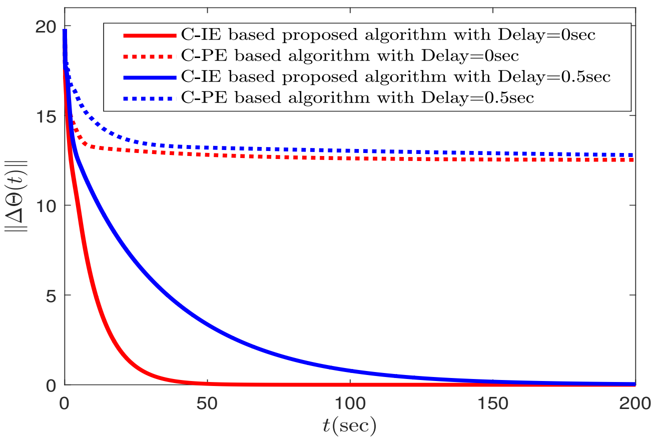

Figure 4.4: Comparative plot for proposed distributed parameter estimation algorithm (4.19) or (4.24) and (C-PE) based algorithm.

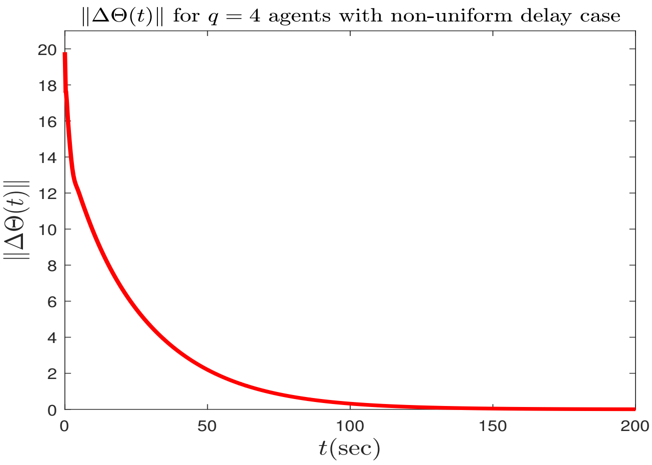

Figure 4.5: Norm of parameter estimation error using proposed distributed parameter estimation algorithm (4.19) or $\big($4.24) with non-uniform communication delay $(T_{12} = 0.4(\text{sec}), T_{21} = 0.2(\text{sec}), T_{43} = 0.3(\text{sec}), T_{32} = 0.55(\text{sec})\big)$ case.

4.6.2 Linear Motion of a subset of Robots

In this subsection for simulation, the controller $u_i(t) \in \mathbb{R}^2$ is chosen to execute linear motion of two robots, while the other two remain static.

$$\begin{aligned}
\dot{x}_1(t) &= \text{Proj}_{M_1}\Big(-K(x_1 - x_1^*)\Big),\ M_1 = [0, 0.5] \times [0, 0.5] \\
\dot{x}_2(t) &= \mathbf{0}_2,\ M_2 = [0.5, 1] \times [0.5, 0.5] \\
\dot{x}_3(t) &= \text{Proj}_{M_3}\Big(-K(x_3 - x_3^*)\Big),\ M_3 = [0.5, 1] \times [0.5, 1] \\
\dot{x}_4(t) &= \mathbf{0}_2,\ M_4 = [0.5, 0.5] \times [0.5, 1]
\end{aligned}$$

Based on the above controller's design, it can be validated that each individual regressor $\phi_i(t) \in \mathbb{R}^6$ does not satisfy uniform IE condition (Roy *et al.*, 2017*b*), however, in cooperation they achieve uniform C-IE (4.2) as in Lemma 15, respectively, i.e.,

$$(rank(Y_1) \neq 6, Y_1 \geq 0),\ (rank(Y_2) \neq 6, Y_2 \geq 0),$$
$$(rank(Y_3) \neq 6, Y_3 \geq 0),\ (rank(Y_4) \neq 6, Y_4 \geq 0),$$
$$\Big(rank\,\underbrace{(\gamma_I \bar{W}\Phi_I(t) + \tilde{L} \otimes I_6)}_{M} = 24 \text{ or } M > 0\Big),\ \forall t \geq t_0 + T$$

Based on the above data, the proposed distributed adaptive parameter estimation algorithm (4.19) or (4.24) validates the Theorem 6 even if none of the individual regressor's hold uniform IE condition (Roy *et al.*, 2017*b*) which is depicted in Figure 4.6.

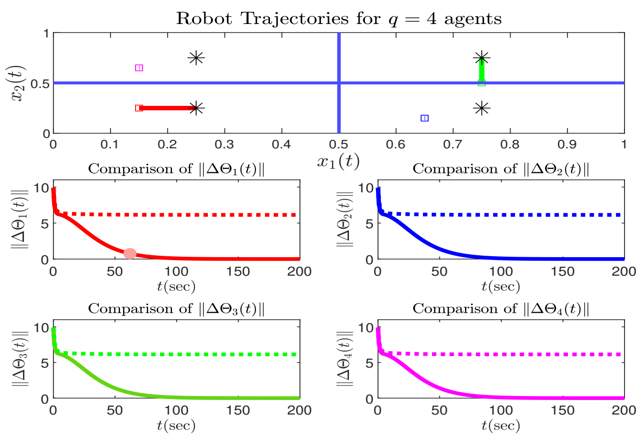

Figure 4.6: Comparative plot for norm of parameter estimation error with robot trajectories using proposed algorithm (4.19) or (4.24), if none of the individual regressor $\phi_i(t)$'s are uniform IE. Here dotted curve represent $\|\Delta\Theta_i(t)\|$ without consensus term, solid curve represent $\|\Delta\Theta_i(t)\|$ with consensus term, $\forall i = 1(1)4$.

4.7 Conclusion

This chapter proposes a novel distributed adaptive parameter estimation algorithm for multi-agent systems, where parameter convergence is ensured under a relaxed mathematical condition of uniform C-IE in the presence of communication delay. The C-IE condition can be achieved by using finite-energy exploration, unlike the restrictive C-PE condition used in past literature. The proposed algorithm utilizes a new set of integrator dynamics, which omits computationally involved switching or internal instability of recent works on C-IE based designs, while still ensuring parameter convergence. The proposed algorithm provides - 1> UGS and asymptotic convergence (with communication delay), 2> UGES (without communication delay) of the origin of the parameter estimation error dynamics, respectively.

Chapter 5

Distributed Adaptive Parameter Estimation over Weakly Connected Digraphs using a Relaxed Excitation Condition

5.1 Introduction

In this chapter, a novel distributed adaptive parameter estimation (DAPE) algorithm is proposed for a MAS architecture over a weakly connected digraph network, where parameter convergence is ensured under a newly coined relaxed excitation condition, called generalized cooperative initial excitation (gC-IE). This is in contrast to the past literature, where such DAPE algorithms demand C-PE and generalized cooperative persistent of excitation (gC-PE) for strongly connected digraph, and weakly connected digraph networks, respectively, for parameter convergence. The gC-PE and C-PE conditions are restrictive in the sense that they require the richness/excitation of information over the entire time-span of the signal/data, unlike the gC-IE condition where excitation is needed only in the initial time-span. The newly coined gC-IE condition is an extension of C-IE condition. While the C-IE condition applies to a strongly connected digraph, the newly proposed gC-IE condition extends the concept to weakly connected digraph. The proposed algorithm utilizes a novel set of weighted integrator dynamics, which omits the requirement of computationally involved multiples switching mechanisms in past literature, while still ensuring parameter convergence. The proposed algorithm provides GES of the origin of the parameter estimation error dynamics under gC-IE condition. Furthermore, a robustness analysis for the proposed DAPE algorithm in the presence of additive disturbance is carried out where the parameter estimation error dynamics exhibits ISS. Simulation results further validate the efficacy of the proposed algorithm.

5.2 Notations & Preliminaries

5.2.1 Notations

Given a matrix C, $\rho(C)$ denotes the spectral radius of C. A M-matrix is a square matrix of the form; $A = sI - C$ with non-positive off-diagonal entries, where C is a non-negative matrix (i.e., all the elements are equal to or grater than zero), and $s \geq \rho(C)$. This matrix is non-singular when $s > \rho(C)$. Let matrices $P, Q \in \mathbb{R}^{n \times n}$, $P > Q$ ($P \geq Q$) means that $P - Q$ is positive-definite (positive semi-definite), respectively. A pair of two matrices (A_1, A_2) holds commutative property iff ($A_1 A_2 = A_2 A_1$).

5.2.2 Preliminaries on Algebraic Graph Theory

A directed graph or digraph $G \triangleq (V, E)$ is characterized by the set of n vertices $V \triangleq \{1, \ldots, n\}$ and the set of directed edges $E \subseteq V \times V$. In a directed graph, an edge $(i, j) \in E$ signifies that j^{th} agent can get information from i^{th} agent but the converse is not true. In this case, i^{th} agent is called a neighbor of j^{th} agent. Here, N_i is defined as the set of neighbors for the i^{th} agent, defined as: $N_i \triangleq \big\{j \in V \mid (j, i) \in E\big\}$. Note that self edges (i, i) are not allowed, thus $(i, i) \notin E$ and $i \notin N_i$. A directed path connecting nodes i_0 and i_n is a sequence of edges of the form ($i_{\alpha-1}$, i_α), $\alpha = \{1, 2, \ldots, n\}$. If there is a path between every pair of nodes in a graph, then the graph is connected. A node i is reachable from j if there is a directed path from j to i. A strongly connected component (SCC) is a subgraph of G in which any pair of nodes are reachable from each other. Any digraph G can be partitioned into a series of SCCs that do not share any common nodes. If an SCC has only outflows, i.e., there are no edges from other components to this SCC, it is then called a root strongly connected component (RSCC). A digraph is weakly connected if each couple of distinct nodes (i, j), $i \neq j$ is connected with a path that does not account for the direction of the edges.

The Laplacian matrix L of the weakly connected digraph G can be written as a lower block triangular form (Maghenem *et al.*, 2022; Cheng and Scherpen, 2019; Javed *et al.*, 2021; Di Cairano *et al.*, 2008).

$$L = \begin{bmatrix} L_a & 0 \\ L_{ab} & L_b \end{bmatrix} \tag{5.1}$$

. The matrix L_a is a block diagonal matrix which is defined as

$$L_a = \text{diag}\{L_{a1}, \dots, L_{aQ}\} = \text{blkdiag}\{L_{aq}\}, \forall q = 1, \dots, Q \tag{5.2}$$

where Q denotes the total no of RSCCs in graph G. In (5.2), each diagonal block is an irreducible, singular M-matrix, corresponding to a certain RSCC. The lower left block L_{ab} of L is a non-negative matrix and the lower right block L_b is a non-singular M-matrix.

In this chapter, for subsequent formulation the notation h_q is used to denote the number of agents corresponding to each RSCC q and $h = \sum_{q=1}^{Q} h_q$ denotes the no of agents corresponding to total number of RSCCs in graph G (as Q in (5.2)) and $h^{'}$ denotes the number of remaining agents in the graph G. Hence ($h + h^{'} = n$).

5.2.3 Definitions on Signal Excitation Conditions

This section introduces several signal excitation definitions, which are used for commenting on the convergence properties of the subsequently proposed DAPE algorithm.

Unlike the C-PE (Papusha *et al.*, 2014; Chen *et al.*, 2013), and C-IE (Garg and Roy, 2020*c*; Goel *et al.*, 2022; Garg and Roy, 2020*a*) based frameworks, which are restricted to only undirected graphs and strongly connected digraphs, a generalized cooperative persistence of excitation (gC-PE) condition (Javed *et al.*, 2021; Maghenem *et al.*, 2022) over generalized digraph (for example weakly connected digraph) is defined below.

Definition 7. *Let G be a weakly connected graph withn number of nodes/agents. A group of bounded vector-valued functions $\phi_i(t) \in \mathbb{R}^m$, where $t \in [t_0, \infty)$, $t_0 \geq 0$, $\forall i = 1(1)n$, is said to satisfy the "gC-PE" condition for G if for each RSCC q, with total agents h_q, the following inequality holds*

$$\int_{t}^{t+T_q} \sum_{i=1}^{h_q} \phi_i(r)\phi_i^T(r)dr \geq \Upsilon_q I_m, \; \forall t \geq t_0 \tag{5.3}$$

for some T_q[1]$, \Upsilon_q \in \mathbb{R}_{>0}$. □

Inspired from (Javed *et al.*, 2021; Maghenem *et al.*, 2022; Garg and Roy, 2019*a*, 2020*a*,*c*; Goel *et al.*, 2022; Garg and Basu Roy, 2023), a generalized cooperative initial excitation (gC-IE) condition is proposed in this work as follows.

Definition 8. *Let G be a weakly connected graph withn n number of nodes/agents. A group of bounded vector-valued functions $\phi_i(t) \in \mathbb{R}^m$, where $t \in [t_0, \infty)$, $t_0 \geq 0$, $\forall i = 1(1)n$, are said to satisfy the gC-IE condition for G if for each RSCC q, with total agents h_q, the following inequality holds*

$$\int_{t_0}^{t_0+T_q} \sum_{i=1}^{h_q} \phi_i(r)\phi_i^T(r)dr \geq \Upsilon_q I_m. \tag{5.4}$$

for some $T_q, \Upsilon_q > 0$. □

The gC-PE condition (5.3) and newly proposed gC-IE condition (5.4) implies that each RSCC in a weakly connected digraph network G should hold the C-PE and C-IE condition, respectively.

The newly proposed gC-IE condition generalizes existing C-IE condition, such as those presented in (Garg and Roy, 2019*a*) for undirected graphs (bidirectional), and in (Garg and Roy, 2020*c*; Goel *et al.*, 2022; Garg and Basu Roy, 2023) for strongly connected and balanced digraphs. In particular, when the digraph is strongly connected, it follows that ($h_q = n$), which implies the C-IE condition (2.3). Similarly, when the digraph is fully disconnected, (5.4) reduces to the standard IE condition (1.8) applied to every agent of the network.

Remark 16. *Based on* (5.3)*, it can be inferred that for each RSCC q with total agents h_q, the excitation is needed for the entire time-span i.e., $[t, t+T_q]$, $\forall t \geq t_0$, which demands that infinite amount of energy (for example; $\phi_i(t) = sin(t)$, $\phi_i(t) = 1$, etc) are needed for excitation. However, in* (5.4)*, the excitation is needed only in the initial finite time-window i.e., $[t_0, t_0+T_q]$, which can be satisfied with finite amount of energy (for example; $\phi_i(t) = e^{-t}sin(t)$, $\phi_i(t) = e^{-t}$, etc). Hence, from the above discussion, it can be argued*

[1] T_q is the time where mathematical rank condition (5.3) holds, however T_q is not known to us.

that the proposed gC-IE condition is milder than the restrictive gC-PE condition (Javed et al.*, 2021; Maghenem* et al.*, 2022).* □

5.2.4 Definition on Stability Properties

Here, the stability properties of the systems corresponding to input will be defined based on Lemma 4.6 of(Khalil and Grizzle, 2002).

Definition 9. *Suppose a system $\dot{x}(t) = f(t, x, u)$ is continuously differentiable and globally Lipschitz in (x, u), uniformly in t. If the unforced system $\dot{x}(t) = f(t, x, 0)$ has a globally exponentially stable equilibrium point at the origin $x = 0$, then the system is input-to-state stable (ISS)*[2]. □

5.3 Problem Formulation

5.3.1 Agent's Communication Topology and Model

In this chapter, an online DAPE problem for an MAS architecture comprising$n$ number of agents is tackled. The information flow among agents is described by a digraph G. Here, the aim of the agents is to cooperatively learn a global unknown constant parameter by using individual real-time measurements of input-output signals, satisfying the following form (Maghenem *et al.*, 2022; Javed *et al.*, 2021; Ortega *et al.*, 2020).

$$y_i(t) = \Theta^T \phi_i(t),\ \forall t \geq t_0,\ \forall i = 1(1)n \tag{5.5}$$

where $y_i(t) \in \mathbb{R}^p$ denotes the output, $\Theta \in \mathbb{R}^{m \times p}$ denotes the matrix of unknown constant parameters and $\phi_i(t) \in \mathbb{R}^m$ is the input basis functions/feature vectors (also known as "regressor" in adaptive control literature).

Remark 17. *The above posed problem with model* (5.5) *or equivalent models are used in adaptive control, composite adaptive control, distributed adaptive control literature (see (Schwager* et al.*, 2009; Abdul Razak* et al.*, 2018; Papusha* et al.*, 2014; Chen* et al.*, 2013;*

[2]For ISS definition refer to the Definition 4.7 of (Khalil and Grizzle, 2002).

Wensing and Slotine, 2018; Stegagno and Yuan, 2019; Yuan et al.*, 2021; Basu Roy* et al.*, 2018; Chowdhary* et al.*, 2013) for further details). While this work is entirely focused on parameter estimation, it's application in distributed control problems, such as distributed model reference adaptive control (MRAC) (Peng* et al.*, 2013; Yuan* et al.*, 2019, 2021), distributed adaptive coverage control (Schwager* et al.*, 2008*a*; Li* et al.*, 2012), distributed adaptive extremum seeking control (Poveda and Quijano, 2013; Poveda* et al.*, 2019; Guay* et al.*, 2015, 2018), etc, are kept as promising future directions.* □

Assumption 13. *Graph G, which represents the communication topology among n number of agents, is a weakly connected digraph.* □

Assumption 14. $\|\Theta\| < \bar{\Theta}$*, for some constant* $\bar{\Theta} \in \mathbb{R}_{>0}$. □

5.3.2 Objective

The objective of this work is to develop a DAPE algorithm, using the online measurements of input $\big(\phi_i(t)\big)$, output $\big(y_i(t)\big)$ of the model (5.5) while collaborating (sharing instantaneous estimate) with the neighboring agents, such that

$$\|\hat{\Theta}_i(t) - \Theta\| \to 0 \; as \; t \to \infty, \; \forall t \geq t_0, \; \forall i = 1(1)n, \tag{5.6}$$

without requiring the restrictive "gC-PE" condition.

Here, $\hat{\Theta}_i(t) \in \mathbb{R}^{m \times p}$ denotes the estimate of unknown constant parameter matrix Θ by i^{th} agent. Furthermore, the subsequent assumption is considered to facilitate the DAPE algorithm design.

Assumption 15. $\phi_i(t), \; \dot{\phi}_i(t) \in \mathcal{L}_\infty, \; \forall i = 1(1)n.$ □

5.4 DAPE Algorithm Design

This section elaborates on the implementation procedure for the subsequently designed DAPE algorithm. To omit the requirement of restrictive gC-PE condition (used in (Javed *et al.*, 2021; Maghenem *et al.*, 2022)), a novel set of weighted integrator dynamics is proposed, which is inspired from (Garg *et al.*, 2022)[3].

5.4.1 Weighted Integrator Dynamics

Consider the following weighted integrator dynamics, $\forall t \geq t_0$.

$$\dot{\Phi}_i(t) = \beta(t)\phi_i(t)\phi_i^T(t),\ \Phi_i(t_0) = 0,\ \forall i = 1(1)n \tag{5.7}$$

$$\dot{\Psi}_i(t) = \beta(t)y_i(t)\phi_i^T(t),\ \Psi_i(t_0) = 0,\ \forall i = 1(1)n \tag{5.8}$$

where $\Phi_i(t) \in \mathbb{R}^{m\times m}$ denotes the weighted integrated regressor, $\Psi_i(t) \in \mathbb{R}^{p\times m}$ is known as the weighted integrated output, and $\beta(t) \in \mathbb{R}$ is strategically introduced weighting function, which has the following properties, $\forall i = 1(1)n$.

Property 13. $\beta(t) > 0,\ \forall t \in [t_0, \infty)$. □

Property 14. $\beta(t) < \bar{\beta} < \infty,\ \forall t \in [t_0, \infty) \implies \beta(t) \in \mathcal{L}_\infty$. □

Property 15. $\beta(t) \in \mathcal{L}_1$. □

whereas $\bar{\beta} \in \mathbb{R}_{>0}$ is the upper-bound of $\beta(t)$. Based on the above Properties 13-15, examples of $\beta(t)$ candidates can be - e^{-kt}, te^{-kt}, t^2e^{-kt}, etc. With $k > 0$.

Analytically solving (5.7), (5.8), along with (5.5), it can be deduced that

$$\Psi_i(t) = \Theta^T\Phi_i(t),\ \forall t \geq t_0,\ \forall i = 1(1)n \tag{5.9}$$

The matrix $\Phi_i(t) \in \mathbb{R}^{m\times m}$ has the following properties.

Lemma 16. *$\Phi_i(t)$ is a positive semi-definite function of time i.e., $\Phi_i(t) \geq 0$, $\forall t \geq t_0$, $\forall i = 1(1)n$.*

Proof. For proof refer to the Lemma 11 proof argument from chapter 4. □

Lemma 17. *$\Phi_i(t)$ is a non-decreasing function of time in the sense of matrix inequality i.e., $\Phi_i(t_2) \geq \Phi_i(t_1)$, $\forall t_2 \geq t_1 \geq t_0$, $\forall i = 1(1)n$.*

[3] While the work in (Garg *et al.*, 2022) has introduced the weighted integrator in the context of single-agent parameter estimation, the current work extends the idea to a multi-agent setting.

Proof. For proof refer to the Lemma 12 proof argument from chapter 4. □

Lemma 18. $\Phi_i(t), \Psi_i(t) \in \mathcal{L}_\infty$, $\forall t \geq t_0$, $\forall i = 1(1)n$.

Proof. For proof refer to the Lemma 13 proof argument from chapter 4. □

5.4.2 Local Objective Function for i^{th} agent

Consider the following local objective function for i^{th} agent, $\forall t \geq t_0, \forall i = 1(1)n$

$$\begin{aligned}\underbrace{L_i\big(\hat{\Theta}_i(t), t\big)}_{\mathbb{R}^{m\times p}\times\mathbb{R}_{\geq 0}\to\mathbb{R}} =& \frac{\gamma}{2}\underbrace{\|\hat{\Theta}_i^T(t)\phi_i(t) - y_i(t)\|^2}_{L_{i,P}} + \frac{\gamma_1}{2}\underbrace{\int_{t_0}^{t}\beta(r)\|\hat{\Theta}_i^T(t)\phi_i(r) - y_i(r)\|^2 dr}_{L_{i,I}} \\ &+ \frac{\gamma_2}{2}\underbrace{\sum_{j\in N_i} a_{ij}\|\hat{\Theta}_j(t) - \hat{\Theta}_i(t)\|^2}_{L_{i,C}} \end{aligned} \tag{5.10}$$

where $\gamma, \gamma_1, \gamma_2 \in \mathbb{R}_{>0}$ are constant positive tuning parameters. Each i^{th} agent has its local objective or cost function, which is being minimized using the following estimation algorithm.

5.4.3 Proposed DAPE Algorithm for i^{th} agent

The DAPE algorithm is designed as gradient-descent of (5.10) with respect to $\hat{\Theta}_i(t)$, $\forall t \geq t_0$, $\forall i = 1(1)n$

$$\begin{aligned}\dot{\hat{\Theta}}_i(t) =& -\nabla_{\hat{\Theta}_i(t)} L_i\big(\hat{\Theta}_i(t), t\big) \\ =& -\gamma\underbrace{\phi_i(t)\big(\hat{y}_i(t) - y_i(t)\big)^T}_{E_{i,P}} - \gamma_1\underbrace{\big(\hat{\Psi}_i(t) - \Psi_i(t)\big)^T}_{E_{i,I}} + \gamma_2\sum_{j\in N_i}\underbrace{a_{ij}\big(\hat{\Theta}_j(t) - \hat{\Theta}_i(t)\big)}_{E_{i,C}}\end{aligned} \tag{5.11}$$

where $\hat{y}_i(t) \triangleq \hat{\Theta}_i^T(t)\phi_i(t)$, $\hat{\Psi}_i(t) \triangleq \hat{\Theta}_i^T(t)\Phi_i(t)$.

Remark 18. *Different components in the above local objective function* (5.10) - $L_{i,P}$, $L_{i,I}$, $L_{i,C}$ *are having a one-to-one correspondence with different components of the distributed adaptive parameter estimator* (5.11) - $E_{i,P}$, $E_{i,I}$, $E_{i,C}$. □

In (5.11), $E_{i,P}$ component is a proportional type of local prediction error, $E_{i,I}$[4] component is a weighted integral type of local prediction error, whereas the last term $E_{i,C}$ represents neighboring interaction among the agents using communication graph topology based on Assumption 13. Define parameter estimation error as

$$\Delta\Theta(t) \triangleq [\Delta\Theta_1^T(t), \Delta\Theta_2^T(t), \cdots, \Delta\Theta_n^T(t)]^T \in \mathbb{R}^{mn\times p} \tag{5.12}$$

$$\Delta\Theta_i(t) \triangleq \hat{\Theta}_i(t) - \Theta \in \mathbb{R}^{m\times p}, \ \forall i = 1(1)n \tag{5.13}$$

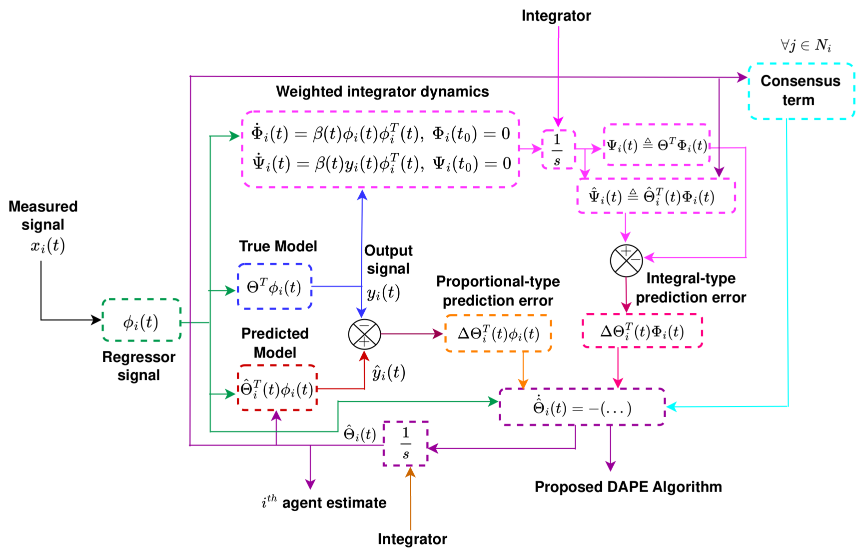

Figure 5.1: Block diagram of the proposed DAPE algorithm.

5.5 Stability/Convergence Analysis

5.5.1 Compact form representation of the DAPE Algorithm

Utilizing Assumption 13 and its preliminaries, compact representation of an online parameter estimation error dynamics (5.11) (for more details refer to figure 5.1) for the entire group can be structured based on (5.1) as, $\Delta\Theta \triangleq [\Delta\Theta_a^T, \Delta\Theta_b^T]^T \in \mathbb{R}^{m(h+h')\times p}$, where $\Delta\Theta_a \in \mathbb{R}^{mh\times p}$ is associated with agents corresponding to RSCCs, and $\Delta\Theta_b \in \mathbb{R}^{mh'\times p}$ is associated with the remaining agents. Here, $h = \sum_{q=1}^{Q} h_q$ (based on (5.1)) and $(h + h' = n)$. Utilizing (5.1), the parameter estimation error dynamics (5.11) for all the n

[4]The intuitive notion is to capture past information through the integral-action since it reduces the steady state bias to zero.

number of agents can be compactly represented as

$$\Delta\dot{\Theta}_a(t) = -\Big(\gamma\Omega_{a,P}(t) + \gamma_1\Omega_{a,I}(t) + \gamma_2(L_a \otimes I_m)\Big)\Delta\Theta_a(t) \tag{5.14}$$

$$\Delta\dot{\Theta}_b(t) = -\Big(\gamma\Omega_{b,P}(t) + \gamma_1\Omega_{b,I}(t) + \gamma_2(L_b \otimes I_m)\Big)\Delta\Theta_b(t) - \gamma_2(L_{ab} \otimes I_m)\Delta\Theta_a(t) \tag{5.15}$$

where $\Omega_{a,P}(t)$, $\Omega_{a,I}(t)$, $\Omega_{b,P}(t)$, $\Omega_{b,I}(t)$, are block diagonal matrices, defined as

$$\begin{aligned}
\Omega_{a,P}(t) &\triangleq \mathrm{diag}\Big\{\phi_{a1}(t)\phi_{a1}^T(t), \ldots, \phi_{ah}(t)\phi_{ah}^T(t)\Big\} \in \mathbb{R}^{mh\times mh} \\
\Omega_{a,I}(t) &\triangleq \mathrm{diag}\Big\{\Phi_{a1}(t), \ldots, \Phi_{ah}(t)\Big\} \in \mathbb{R}^{mh\times mh} \\
\Omega_{b,P}(t) &\triangleq \mathrm{diag}\Big\{\phi_{b1}(t)\phi_{b1}^T(t), \ldots, \phi_{bh'}(t)\phi_{bh'}^T(t)\Big\} \in \mathbb{R}^{mh'\times mh'} \\
\Omega_{b,I}(t) &\triangleq \mathrm{diag}\Big\{\Phi_{b1}(t), \ldots, \Phi_{bh'}(t)\Big\} \in \mathbb{R}^{mh'\times mh'}.
\end{aligned}$$

Here, $L_a \in \mathbb{R}^{h\times h}, L_a \otimes I_m \in \mathbb{R}^{mh\times mh}, L_b \in \mathbb{R}^{h'\times h'}, L_b \otimes I_m \in \mathbb{R}^{mh'\times mh'}, L_{ab} \in \mathbb{R}^{h'\times h}, L_{ab} \otimes I_m \in \mathbb{R}^{mh'\times mh}$.

Based on (5.1), L_a is a block diagonal matrix with Q blocks in total, one for each RSCC.

To comment on the stability/convergence properties of (5.14)-(5.15), few subsequent lemmas are formulated.

Lemma 19. *Let $L_a \in \mathbb{R}^{h\times h}$ be an irreducible, singular M-matrix. Then,*

(1) L_a has rank $(h-1)$.

(2) There exists a positive diagonal matrix $Z_a \in \mathbb{R}^{h\times h}$; $Z_a = diag\{z\} > 0$ where $z = [z_1, z_2, \ldots, z_h]^T \in \mathbb{R}^h > 0$ such that $\tilde{L}_a = Z_aL_a + L_a^TZ_a$ is positive semi-definite matrix.

Proof. For proof refer to the Theorem 4.31 from (Qu, 2009). □

Lemma 20. *Consider a matrix $\bar{Z}_a = Z_a \otimes I_m \in \mathbb{R}^{mh\times mh}$, where $Z_a > 0$ is the diagonal matrix defined above. Then, the following holds:*

(1) $\bar{Z}_a$ is block diagonal and positive-definite matrix.

(2) Pairs of matrices $\big(\bar{Z}_a, \Omega_{a,P}(t)\big)$, $\big(\bar{Z}_a, \Omega_{a,I}(t)\big)$ hold commutative property, respectively, $\forall t \geq t_0$.

(3) $\bar{Z}_a\Omega_{a,P}(t)$, $\bar{Z}_a\Omega_{a,I}(t)$, are block diagonal positive semi-definite matrix, $\forall t \geq t_0$.

Proof. Since $\bar{Z}_a = Z_a \otimes I_m$ and $Z_a = \text{diag}\{z_1, \dots, z_h\} > 0$, then

$$\bar{Z}_a = \text{diag}\{z_1 I_m, \dots, z_h I_m\} > 0 \tag{5.16}$$

Based on (5.16), it can be concluded that $\bar{Z}_a$ is block diagonal and positive-definite matrix.

Furthermore, utilizing (5.14), (5.15) with the $\Omega_{a,P}(t)$, $\Omega_{a,I}(t)$ structure, then $\bar{Z}_a\Omega_{a,P}(t)$ yields

$$\bar{Z}_a\Omega_{a,P}(t) = \text{diag}\{z_1\phi_{a1}(t)\phi_{a1}^T(t), \dots, z_h\phi_{ah}(t)\phi_{ah}^T(t)\} \tag{5.17}$$

Whereas, $\Omega_{a,P}(t)\bar{Z}_a$ yields

$$\Omega_{a,P}(t)\bar{Z}_a = \text{diag}\{z_1\phi_{a1}(t)\phi_{a1}^T(t), \dots, z_h\phi_{ah}(t)\phi_{ah}^T(t)\} \tag{5.18}$$

Based on (5.17), (5.18), it can be concluded that the pair of matrices $\left(\bar{Z}_a, \Omega_{a,P}(t)\right)$ is commutative i.e., $\left(\bar{Z}_a\Omega_{a,P}(t) = \Omega_{a,P}(t)\bar{Z}_a\right)$, $\forall t \geq t_0$. Since $\Omega_{a,I}(t)$ has the same block diagonal structure as $\Omega_{a,P}(t)$. Hence, it can be concluded that, the pair of matrices $\left(\bar{Z}_a, \Omega_{a,I}(t)\right)$ is commutative in nature i.e., $\left(\bar{Z}_a\Omega_{a,I}(t) = \Omega_{a,I}(t)\bar{Z}_a\right)$, $\forall t \geq t_0$.

Furthermore, based on (5.17), (5.18), and utilizing the fact that $\Omega_{a,P}(t) \geq 0$ and $\Omega_{a,I}(t) \geq 0$ (from Lemma 16), it can be concluded that $\bar{Z}_a\Omega_{a,P}(t)$, $\bar{Z}_a\Omega_{a,I}(t)$ are block diagonal positive semi-definite matrix, $\forall t \geq t_0$. □

5.5.2 Stability/Convergence with gC-IE

Theorem 7. *Suppose that the Assumptions 13-15 hold, then the origin of the parameter estimation error $\Delta\Theta_a(t)$ dynamics (5.14) is globally exponentially stable (GES) if the gC-IE condition* (5.4) *holds i.e.,*

$$\|\Delta\Theta_a(t)\| \leq \|\Delta\Theta_a(t_0 + T)\|e^{-\eta(t-t_0-T)}, \ \forall t \geq t_0 + T \tag{5.19}$$

where $\eta \in \mathbb{R}_{>0}$ is the constant positive scalar.

Proof. Consider the following Lyapunov candidate

$$V_1(\Delta\Theta_a) = \frac{1}{2}\text{tr}\big(\Delta\Theta_a^T \bar{Z}_a \Delta\Theta_a\big) \tag{5.20}$$

Taking time derivative of (5.20) along with the (5.14), yields

$$\dot{V}_1(\Delta\Theta_a) = -\,\text{tr}\Big(\gamma\Delta\Theta_a^T \bar{Z}_a\Omega_{a,P}\Delta\Theta_a + \gamma_1\Delta\Theta_a^T \bar{Z}_a\Omega_{a,I}\Delta\Theta_a + \gamma_2\Delta\Theta_a^T(\tilde{L}_a \otimes I_m)\Delta\Theta_a\Big) \leq 0 \tag{5.21}$$

which is negative semi-definite; holds from the fact (Lemma 19, Lemma 20) that $\tilde{L}_a$, $\bar{Z}_a\Omega_{a,P}$, $\bar{Z}_a\Omega_{a,I}$, are the positive semi-definite matrices, which implies that the origin of the parameter estimation error dynamics (5.14) is Lyapunov stable, $\forall t \geq t_0$. Based on the Assumption 15 and Lemma 18, it can be concluded that $\big(\bar{Z}_a\Omega_{a,P}, M_a, \bar{Z}_a\dot{\Omega}_{a,P}, \dot{M}_a \in \mathcal{L}_\infty\big)$. Using the above arguments, it can also be concluded that $V_1(\Delta\Theta_a(t))$ is uniformly bounded above by its initial value, which implies that all the local prediction errors tends to zero as $t \to \infty$, $\forall i = 1(1)h$.

Invoking the gC-IE condition based Lemma 29 (refer to Appendix A) in (5.21), yields

$$\dot{V}_1(\Delta\Theta_a) = -\text{tr}\big(\Delta\Theta_a^T\Gamma(t)\Delta\Theta_a\big) < 0,\ \forall t \geq t_0 + T \tag{5.22}$$

Using (5.20), the above inequality modifies as

$$\dot{V}_1 \leq -\underbrace{\frac{2\lambda_{min}(\Gamma(t))}{\lambda_{max}(\bar{Z}_a)}}_{\eta>0} V_1,\ \forall t \geq t_0 + T \tag{5.23}$$

Hence using the comparison Lemma from (Khalil, 2015), the differential inequality in (5.23) leads to the subsequent exponentially convergent bound

$$\|\Delta\Theta_a(t)\| \leq \|\Delta\Theta_a(t_0 + T)\| e^{-\eta(t-t_0-T)},\ \forall t \geq t_0 + T \tag{5.24}$$

Since the Lyapunov function in (5.20) is radially unbounded, the algebraic inequality in (5.24), proves GES of the error dynamics $\Delta\Theta_a(t)$, $\forall t \geq t_0 + T$. □

To comment on the stability/convergence properties of (5.15), the subsequent lemma along with a corollary is provided.

Lemma 21. *Let $L_b \in \mathbb{R}^{h' \times h'}$ be a non-singular M-matrix, then there exists a positive diagonal matrix $Z_b \in \mathbb{R}^{h' \times h'} > 0$ such that $\tilde{L}_b = L_b^T Z_b + Z_b L_b$ is a positive-definite matrix.*

Proof. For proof refer to the Theorem 4.25 of (Qu, 2009). □

Corollary 7.1. *Suppose that the Assumptions 13-15 hold, then the parameter estimation error $\Delta\Theta_b(t)$ dynamics* (5.15) *exhibits ISS with respect to input $\Delta\Theta_a(t)$, $\forall t \geq t_0$.*

Proof. From (5.1) and Lemma 21, there exists a positive diagonal matrix $Z_b = \text{diag}(L_b^{-1}\mathbf{1}_{h'}) \in \mathbb{R}^{h' \times h'}$ such that $\tilde{L}_b = L_b^T Z_b + Z_b L_b > \eta_1 I_{h'}$, for some $\eta_1 > 0$.

Now, consider the following Lyapunov candidate

$$V_2(\Delta\Theta_b) = \frac{1}{2}\text{tr}\big(\Delta\Theta_b^T \bar{Z}_b \Delta\Theta_b\big) \tag{5.25}$$

where $\bar{Z}_b = Z_b \otimes I_m \in \mathbb{R}^{mh' \times mh'}$.

Taking the time derivative of (5.25) along with the (5.15), yields

$$\begin{aligned}\dot{V}_2(\Delta\Theta_b) = &- \text{tr}\Big(\gamma\Delta\Theta_b^T \bar{Z}_b \Omega_{b,P} \Delta\Theta_b + \gamma_1 \Delta\Theta_b^T M_b \Delta\Theta_b + \gamma_2 \Delta\Theta_b^T (\tilde{L}_b \otimes I_m)\Delta\Theta_b \\ &+ 2\gamma_2 \Delta\Theta_b^T \bar{Z}_b (L_{ab} \otimes I_m)\Delta\Theta_a\Big),\ \forall t \geq t_0 \end{aligned} \tag{5.26}$$

where $M_b = \bar{Z}_b \Omega_{b,I}$. Based on the above facts and utilizing Lemma 20[5], 21, (5.26) can be written as

$$\dot{V}_2 \leq -\gamma_2 \eta_1 \|\Delta\Theta_b\|^2 + 2\gamma_2 \|\Delta\Theta_b\| \|\bar{Z}_b (L_{ab} \otimes I_m)\| \|\Delta\Theta_a\| \tag{5.27}$$

Utilizing (5.27), Lyapunov Theorem 4.19, Lemma 4.6 form (Khalil and Grizzle, 2002), it can be concluded that dynamics (5.15) exhibits ISS to input $\Delta\Theta_a(t)$, $\forall t \geq t_0$, as Definition 9. □

[5] Since $\bar{Z}_b$, $\Omega_{b,P}(t)$, and $\Omega_{b,I}(t)$ matrices have the same structure and properties as matrices $\bar{Z}_a$, $\Omega_{a,P}(t)$, and $\Omega_{a,I}(t)$, respectively. Hence, all the three properties of Lemma 20 will hold here.

Remark 19. *In Corollary 7.1, if input* $\Delta\Theta_a(t) = \mathbf{0}_{mh}$*, Corollary 7.1 exhibits GES for origin of the parameter estimator dynamics* (5.15)*,* $\forall t \geq t_0$ *(for more details refer to the Definition 9).* $\square$

Remark 20. *Theorem 7 for* (5.14) *shows GES if gC-IE condition* (5.4) *holds. Corollary 7.1 for* (5.15) *shows ISS with respect to input* $\Delta\Theta_a(t)$*. Hence, the cascade of parameter estimator dynamics* (5.14)-(5.15) *exhibits GES (for proof see the Lemma 4.7 of (Khalil and Grizzle, 2002)).* $\square$

5.6 Robustness Analysis

In this section, the above MAS architecture based regression problem (5.5) with objective (5.6) is tackled under an external additive disturbance. Consider the following online perturbed regression model.

$$y_i(t) = \Theta^T \phi_i(t) + d_i(t),\ \forall t \geq t_0,\ \forall i = 1(1)n \tag{5.28}$$

where $d_i(t) \in \mathbb{R}^p \in \mathcal{L}_\infty$, $\forall i = 1(1)n$ represent an additive perturbation signal, satisfying the following bound

$$\|d_i(t)\| \leq \bar{d}_i < +\infty,\ \forall i = 1(1)n \tag{5.29}$$

where $\bar{d}_i$ is an upper-bound, $\forall i = 1(1)n$.

Analytically solving (5.7), (5.8), along with (5.28), yields

$$\Psi_i(t) = \Theta^T \Phi_i(t) + d_{if}(t),\ \forall t \geq t_0,\ \forall i = 1(1)n \tag{5.30}$$

where $d_{if}(t) \in \mathbb{R}^{p\times m}$ represent the filtered disturbance, which can be obtained by the following differential law, $\forall i = 1(1)n$

$$\dot{d}_{if}(t) = \beta(t) d_i(t) \phi_i^T(t),\ d_{if}(t_0) = 0,\ \forall t \geq t_0 \tag{5.31}$$

Corollary 7.2. *Filtered disturbance* $d_{if}(t) \in \mathcal{L}_\infty$*,* $\forall t \geq t_0$*,* $\forall i = 1(1)n$*.*

Proof. Since $d_i(t), \phi_i(t) \in \mathcal{L}_\infty$, using Properties 13-15 of weighting function $\beta(t)$, (5.29), and $\|d_{if}(t)\|$ can be expressed as

$$\begin{aligned}\|d_{if}(t)\| &= \| \int_{t_0}^{t} \beta(r) d_i(r) \phi_i^T(r) dr \| \\ &\leq \bar{d}_i \underbrace{\|\phi_i(t)\|_{max}}_{\bar{\phi}\in\mathbb{R}_{>0}} \underbrace{\int_{t_0}^{t} \|\beta(r)\| dr}_{\bar{\beta}\in\mathbb{R}_{>0}} < \infty \end{aligned} \tag{5.32}$$

which implies $d_{if}(t) \in \mathcal{L}_\infty, \forall t \geq t_0, \forall i = 1(1)n.$ $\square$

In the presence of disturbance, the proposed DAPE algorithm for i^{th} agent, $\forall t \geq t_0$, is modified as the following.

$$\begin{aligned}\dot{\hat{\Theta}}_i(t) = &- \gamma \underbrace{\phi_i(t)\phi_i^T(t)\Delta\Theta_i(t)}_{E_{i,P}} - \gamma_1 \underbrace{\Psi_i(t)\Delta\Theta_i(t)}_{E_{i,I}} + \underbrace{\gamma\phi_i(t)d_i^T(t) + \gamma_1 d_{if}^T(t)}_{E_{i,D}} \\ &+ \gamma_2 \sum_{j\in N_i} \underbrace{a_{ij}\big(\hat{\Theta}_j(t) - \hat{\Theta}_i(t)\big)}_{E_{i,C}}, \ \forall i = 1(1)n \end{aligned} \tag{5.33}$$

whereas the extra term $E_{i,D}$ in (5.33) in contrast to (5.11), occurs due to the presence of disturbance.

In the same way as in subsection 5.5, the compact form representation of the DAPE algorithm for the n number of agents in the presence of disturbance can also be modified as.

$$\Delta\dot{\Theta}_a(t) = -\Big(\gamma\Omega_{a,P}(t) + \gamma_1\Omega_{a,I}(t) + \gamma_2(L_a \otimes I_m)\Big)\Delta\Theta_a(t) + \gamma D_a(t) + \gamma_1 D_{af}(t) \tag{5.34}$$

$$\begin{aligned}\Delta\dot{\Theta}_b(t) = &-\Big(\gamma\Omega_{b,P}(t) + \gamma_1\Omega_{b,I}(t) + \gamma_2(L_b \otimes I_m)\Big)\Delta\Theta_b(t) - \gamma_2(L_{ab} \otimes I_m)\Delta\Theta_a(t) \\ &+ \gamma D_b(t) + \gamma_1 D_{bf}(t)\end{aligned} \tag{5.35}$$

where the disturbance based column vectors $D_a(t)$, $D_{af}(t)$, $D_b(t)$, $D_{bf}(t)$, are defined as the following.

$$
\begin{aligned}
D_a(t) &\triangleq \left[\phi_{a1}(t)d_{a1}^T(t), \ldots, \phi_{ah}(t)d_{ah}^T(t)\right]^T \in \mathbb{R}^{mh\times p}\\
D_{af}(t) &\triangleq \left[d_{a1f}^T(t), \ldots, d_{ahf}^T(t)\right]^T \in \mathbb{R}^{mh\times p}\\
D_b(t) &\triangleq \left[\phi_{b1}(t)d_{b1}^T(t), \ldots, \phi_{bh'}(t)d_{ah'}^T(t)\right]^T \in \mathbb{R}^{mh'\times p}\\
D_{bf}(t) &\triangleq \left[d_{b1f}^T(t), \ldots, d_{bh'f}^T(t)\right]^T \in \mathbb{R}^{mh'\times p}.
\end{aligned}
$$

Based on the Assumption 15, (5.29), and Corollary 7.2, it can be concluded that the $D_a(t), D_{af}(t), D_b(t), D_{bf}(t) \in \mathcal{L}_\infty$.

Theorem 8. *Suppose that the Assumptions 13-15 along with the gC-IE condition* (5.4) *hold, then the parameter estimation error $\Delta\Theta_a(t)$ dynamics (5.34) exhibits ISS with respect to input disturbance $\big(D_a(t) + D_{af}(t)\big)$, $\forall t \geq t_0 + T$.*

Proof. Consider the following Lyapunov candidate

$$V_3(\Delta\Theta_a) = \frac{1}{2}\text{tr}\big(\Delta\Theta_a^T \bar{Z}_a \Delta\Theta_a\big) \tag{5.36}$$

Taking time derivative of (5.36) along with the (5.34), yields

$$
\begin{aligned}
\dot{V}_3(\Delta\Theta_a) = &- \text{tr}\Big(\gamma\Delta\Theta_a^T\bar{Z}_a\Omega_{a,P}\Delta\Theta_a + \gamma_1\Delta\Theta_a^T\bar{Z}_a\Omega_{a,I}\Delta\Theta_a + \gamma_2\Delta\Theta_a^T(\tilde{L}_a\otimes I_m)\Delta\Theta_a\\
&- \gamma\Theta_a^T\bar{Z}_aD_a - \gamma_1\Theta_a^T\bar{Z}_aD_{af}\Big),\ \forall t\geq t_0
\end{aligned}
\tag{5.37}
$$

Invoking the gC-IE condition based Lemma 29 (refer to Appendix A) in (5.37), following yields

$$\dot{V}_3(\Delta\Theta_a) \leq -\lambda_{\text{min}}(\Gamma(t))\|\Delta\Theta_a\|^2 + \bar{\gamma}_a\|\Delta\Theta_a\|(\|D_a + D_{af})\|,\ \forall t \geq t_0 + T \tag{5.38}$$

where $\bar{\gamma}_a \in \mathbb{R}_{>0}$. Utilizing (5.38), Lyapunov Theorem 4.19, Lemma 4.6 form (Khalil and Grizzle, 2002), it can be concluded that dynamics (5.34) exhibits ISS with respect to input disturbance $\big(D_a(t) + D_{af}(t)\big)$, $\forall t \geq t_0 + T$. □

Corollary 8.1. *Suppose that the Assumptions 13-15 hold, then the parameter estimation error $\Delta\Theta_b(t)$ dynamics* (5.35) *exhibits ISS with respect to input $\Delta\Theta_a(t)$*[6]*,* $\big(D_b(t) +$

[6]Here, $\Delta\Theta_a(t)$ is for (5.34).

$D_{bf}(t)\big)$, *respectively,* $\forall t \geq t_0$.

Proof. Consider the following Lyapunov candidate

$$V_4(\Delta\Theta_b) = \frac{1}{2}\text{tr}\big(\Delta\Theta_b^T \bar{Z}_b \Delta\Theta_b\big) \tag{5.39}$$

Invoking the Corollary 7.1 proof argument, taking the time derivative of (5.39) along with the (5.35), yields, $\forall t \geq t_0$

$$\dot{V}_4(\Delta\Theta_b) \leq -\gamma_2\eta_1\|\Delta\Theta_b\|^2 + \gamma_2\|\Delta\Theta_b\|\|\bar{Z}_b(L_{ab}\otimes I_m)\|\|\Delta\Theta_a\| + \bar{\gamma}_b\|\Delta\Theta_b\|(\|D_b + D_{bf}\|) \tag{5.40}$$

where $\bar{\gamma}_b \in \mathbb{R}_{>0}$. Utilizing (5.40), Lyapunov Theorem 4.19, Lemma 4.6 form (Khalil and Grizzle, 2002), it can be concluded that dynamics (5.35) exhibits ISS with respect to input $\Delta\Theta_a(t)$, $\big(D_b(t) + D_{bf}(t)\big)$, respectively $\forall t \geq t_0$. □

5.7 Simulation Results

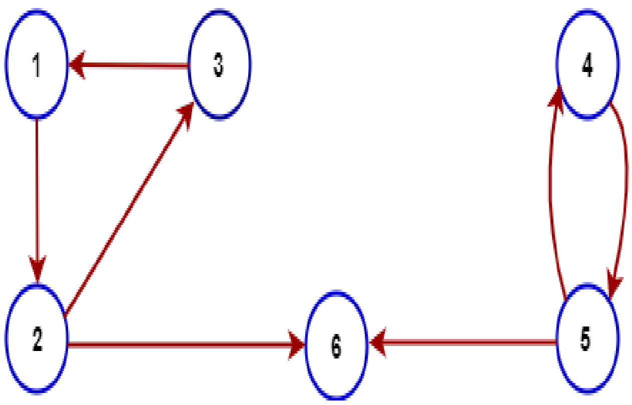

Figure 5.2: Weakly connected digraph topology for $n = 6$ agents.

Consider the MAS architecture comprising $n = 6$ agents, based on Assumption 13 as in Figure 5.2. Here, the agents of the network aim to cooperatively learn a global unknown constant parameter $\Theta = [1, -1, 0.5] \in \mathbb{R}^3$ by using individual real-time measurements of a signal of the form; $y_i(t) = \Theta^T\phi_i(t),\ \forall t \geq t_0,\ \forall i = 1(1)6$. Furthermore, each of the agent regressor $\phi_i(t)s \in \mathbb{R}^3$ is chosen such that, its neither IE nor PE [7]

[7]Note that practical real-time control applications mostly encounter situations, where information gradually dies out over time. Hence the choice of the exponentially decaying regressors in this simulation is reasonably realistic.

i.e.,

$$\phi_1(t) = [1, exp(-t), 0]$$

$$\phi_2(t) = [1, 0, 2exp(-0.6t)]$$

$$\phi_3(t) = [0, 0, cos(8t)]$$

$$\phi_4(t) = [0.1, 0, 0.1exp(-3t)cos(4t)]$$

$$\phi_5(t) = [0, 0.2, exp(-2.7t)]$$

$$\phi_6(t) = [2exp(-1.7t)sin(2t), 0, 0].$$

This simulation study considers all the initial conditions as zero, the constant positive tuning parameters are chosen as $\gamma = 2, \gamma_1 = 5, \gamma_2 = 7.5$, and the weighting function is $\beta(t) = exp(-0.2t)$. Based on the above data, the proposed DAPE algorithm (5.11) outperforms the conventional gC-PE based algorithm (equation (3) of (Javed *et al.*, 2021)), which are depicted in Figure 5.3. In Figure 5.2; there are two RSCCs named as; $q_1 = \{1, 2, 3\}$, $q_2 = \{4, 5\}$. If excitation is removed from $4th$ or $5th$ agent, then the norm of parameter estimation error will not converge in both the cases which is depicted in Figure 5.4. Hence, Figure 5.4 highlight the necessity of gC-IE/gC-PE condition for weakly connected digraph networks. Moreover, it can also be concluded that even after removal the excitation from $5th$ agent, proposed gC-IE based algorithm (5.11) has less steady-state bias in comparison to gC-PE based algorithm which implies C-IE condition is milder than C-PE condition. Hence, based on the Figures 5.3-5.4, it can be concluded that the gC-IE based proposed algorithm is beneficial in the context of decaying finite-energy regressor signals. Furthermore, to robustifying our proposed online DAPE algorithm, the disturbances $d_i(t)s \in \mathbb{R}$ are chosen as $d_1(t) = 0.22sin(3t)$, $d_2(t) = 0.24sin(2.5t)$, $d_3(t) = 0.1sin(2t)$, $d_4(t) = 0.13sin(1.5t)$, $d_5(t) = 0.2cos(1.3t)$, $d_1(t) = 0.15sin(t)$.

Based on the above data, the disturbance based proposed DAPE algorithm (5.33) validates the ISS claims as in Theorem 8, Corollary 8.1, respectively, and hence (5.33) algorithm outperforms the conventional gC-PE based algorithm which is depicted in Figure 5.5.

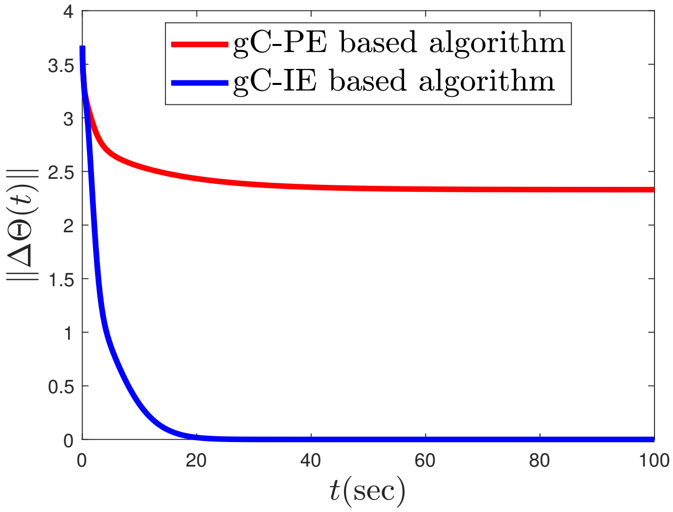

Figure 5.3: Comparative plot for proposed algorithm (5.11) and gC-PE based algorithm.

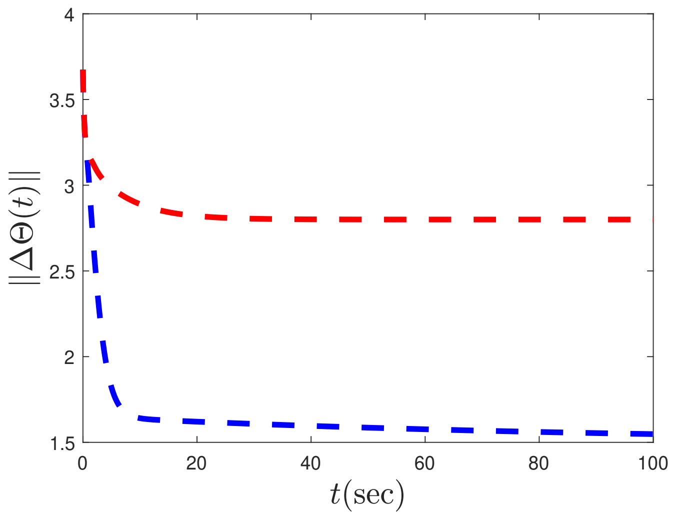

Figure 5.4: Comparative plot for proposed algorithm (5.11) and gC-PE based algorithm where excitation from $5th$ agent is removed.

5.8 Conclusion

This chapter designs a novel DAPE algorithm over weakly connected digraph networks, where parameter convergence is ensured under the newly coined gC-IE condition. The gC-IE condition is milder than the previously utilised gC-PE condition in a similar context since the gC-IE condition does not demand the excitation to persist beyond the initial time-window, unlike gC-PE. The proposed gC-IE condition is an extension of the recently introduced C-IE condition. While the C-IE condition is applicable for strongly connected digraphs, the proposed gC-IE condition extends the concept to weakly con-

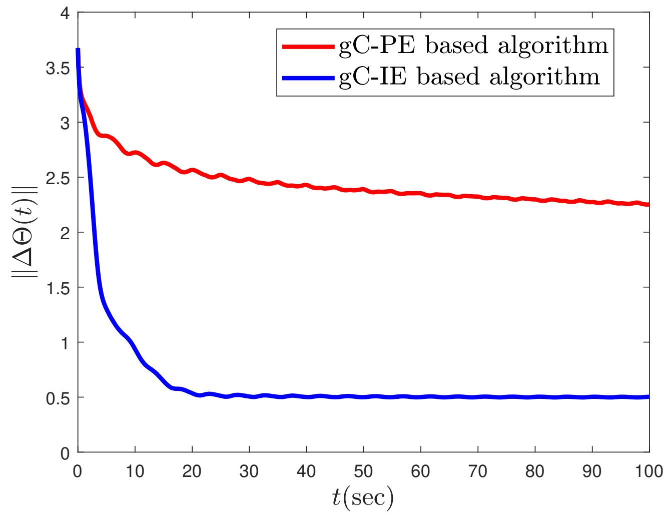

Figure 5.5: Comparative plot for proposed algorithm (5.33) and gC-PE based algorithm in the presence of disturbance.

nected digraphs. The proposed algorithm utilizes a novel set of weighted integrator dynamics, which omits the requirement of computationally involved multiples switching mechanisms in past literature, while still ensuring parameter convergence. The proposed algorithm provides the GES of origin of the parameter estimation error dynamics under the gC-IE condition. Furthermore, in the presence of unmodeled disturbance, the proposed DAPE algorithm exhibits ISS of the parameter estimation error dynamics under the gC-IE condition.

Chapter 6

Distributed Adaptive Extremum Seeking Control in Multi-Agent Systems Over Weakly Connected Digraph

6.1 Introduction

In this chapter, a novel distributed adaptive extremum seeking control (DAdESC) algorithm for a multi-agent system (MAS) over a weakly connected digraph network is proposed, where parameter convergence is ensured under the newly coined relaxed excitation condition from the previous chapter, called generalized cooperative initial excitation (gC-IE). Here, a zeroth-order optimization framework is used, where each agent can only query the numerical value of its cost function at the current coordinate, and it is assumed that only the parameter estimates information is shared among the agents, not the individual cost. Parameter estimation plays a decisive role in the stability and convergence properties of the DAdESC algorithm and it is also well established in the existing literature that to ensure parameter convergence in such context a restrictive generalized cooperative persistent of excitation (gC-PE) condition is required. Here, we eliminate the need for a restrictive gC-PE condition by utilizing a novel set of weighted integral filter dynamics, while ensuring sufficient richness using a milder condition, called gC-IE. The gC-PE condition is restrictive in the sense that it requires the richness/excitation of information over the entire time-span of the signal/data, unlike the gC-IE condition where excitation is needed only in the initial time-span. Here, a detailed Lyapunov function based analysis is performed to establish the closed-loop stability and convergence in the form of exponential stability. Moreover, to validate the robustness guarantees towards the unmodeled bounded disturbance, again a detailed Lyapunov function based analysis is performed to establish the closed-loop stability and convergence in the form of input-to-state stability (ISS).

6.2 Problem Formulation

Here, we are solving the following optimization problem which is subsequently defined.

$$\text{minimize} \quad L(z) \quad \text{subject} \quad z \in M \tag{6.1}$$

where $L(z) : \mathbb{R}^n \to \mathbb{R}$ is an unknown continuously differentiable objective function or performance index defined over an open set containing the closed set M[1] $\subset \mathbb{R}^n$. The objective function is assumed to be accessible only by measurements.

For further subsequent formulation, the unknown performance index $L(z)$ requires following assumptions.

Assumption 16. *The performance index $L(z)$ is linearly parameterized as follows.*

$$L(z) \triangleq \Theta^{\mathrm{T}}\phi(z) \tag{6.2}$$

where $\Theta \in \mathbb{R}^p$ is an unknown constant parameter vector and $\phi(z)$[2]: $\mathbb{R}^n \to \mathbb{R}^p$ is a known continuously differentiable regressor/basis function. □

Remark 21. *The fair choice of Assumption 16 has a wide range of applicability in many domains such as signal processing, system identification, adaptive control, parameter estimation, etc. For more details please refer to the following works (Abdul Razak* et al.*, 2021; La and Sheng, 2013; Chen* et al.*, 2013; Ortega* et al.*, 2021; Schwager* et al.*, 2009; Roy* et al.*, 2016; Poveda* et al.*, 2021).* □

Assumption 17. *$L(z)$ is a strongly convex function with constant $\mu > 0$ over an open set $D \supset M$. Moreover, the unique minima $z^\star \in M$.* □

Here, Assumption 17 implies that the following inequalities hold[3].

$$\big(\nabla L(p) - \nabla L(q)\big)^{\mathrm{T}}(p - q) \geq \mu\|p - q\|^2, \ \forall p, q \tag{6.3}$$

$$\nabla L(z^\star) = 0. \tag{6.4}$$

[1]It is assumed that the set M is compact, convex, and nonempty.
[2]Regressor function $\phi(z)$ has the property as; $z \in \mathcal{L}_\infty \implies \phi(z) \in \mathcal{L}_\infty$.
[3]For more details refer to the (Boyd *et al.*, 2004).

Here, the above optimization problem (6.1) is solved using a team of autonomous agents/robots in a distributed and online setting as described subsequently.

6.2.1 Agents' Communication Topology and Model

The communication topology for comprising n number of agents/robots, is represented by the graph G, where each vertex is treated as an agent and edge $(i, j) \in E$, denotes that j^{th} agent can get information from i^{th} agent. Each agent/robot can interact only with its local neighbors through existing communication links. Each robot is modelled as a single-integrator dynamics.

$$\dot{x}_i(t) = u_i(t),\ x_i(t_0) \in M_i \subset M, \tag{6.5}$$

$$y_i(t) = L(x_i(t)),\ \forall i = 1(1)n. \tag{6.6}$$

where $x_i(t), u_i(t) \in \mathbb{R}^n$ are the system states and control input of agent i, respectively, and $y_i(t) \in \mathbb{R}$ denotes the output performance. $L(x_i(t)) : \mathbb{R}^n \to \mathbb{R}$ denotes the measured performance index. Here, it is assumed that each i^{th} agent has the ability to access the performance index $L(x_i(t))$ at its own coordinate $x_i(t)$ in real-time, $\forall i = 1(1)n$. The precise knowledge of unique minima, $L(\cdot)$ is unknown to each agent.

In (6.5), M_i's are sub-regions (sub-sets) of the region of interest M, such that the following is satisfied.

$$\bigcup_{i=1}^{n} M_i = M \tag{6.7}$$

$$M_i \cap M_j = \emptyset,\ \forall i, j \in \{1, 2, \cdots, n\}, i \neq j \tag{6.8}$$

Note that the first condition (6.7) ensures that the entire area is spanned by the team of robots/agents and the second condition (6.8) obviates the possibility of inter-robot collision.

Assumption 18. *Graph G, which represents the communication topology among$n$ number of agents, is a weakly connected digraph.* □

Assumption 19. $\|\Theta\| < \bar{\Theta}$, *for some constant* $\bar{\Theta} \in \mathbb{R}_{>0}$. □

6.2.2 Objective

Here, the objective is to design an online distributed AdESC algorithm such that the minimization problem (6.1) is solved in a sense that the following arguments hold, provided $x_i(t_0) \in M_i \in M, \forall i = 1(1)n$.

$$x_i(t) \in M_i, \tag{6.9}$$

$$\|x_i(t) - x_i^\star\| \to 0 \; as \; t \to \infty, \; \forall t \geq t_0. \tag{6.10}$$

where $x_i^\star =$ arg min $L(x_i)$, subject to $x_i \in M_i \subset M$.

Based on Assumption 16, (6.6), and Θ is needed to be estimated in a distributed fashion which is also the part of designing distributed AdESC algorithm. Hence the following arguments along with (6.9)-(6.10) should also hold, $\forall i = 1(1)n$

$$\|\hat{\Theta}_i(t) - \Theta\| \to 0 \; as \; t \to \infty, \; \forall t \geq t_0, \tag{6.11}$$

without the restrictive “gC-PE” excitation condition.

6.3 Distributed AdESC Algorithm Design

This section elaborates on the implementation procedure for the subsequently designed online distributed AdESC algorithm. To omit the requirement of restrictive gC-PE condition (used in (Javed *et al.*, 2021; Maghenem *et al.*, 2022)), a novel set of weighted integrator dynamics is proposed, which is inspired from (Garg *et al.*, 2022). The filter structure facilitates to capture the weighted integral-type of prediction error in the parameter update law along with the proportional-type of prediction error.

6.3.1 Weighted Integrator Dynamics

Consider the following weighted integrator filter dynamics, $\forall t \geq t_0$

$$\dot{\Phi}_i(t) = \alpha(t)\phi_i(t)\phi_i^{\mathsf{T}}(t),\ \Phi_i(t_0) = 0,\ \forall i = 1(1)n \tag{6.12}$$

$$\dot{\Psi}_i(t) = \alpha(t)\phi_i(t)y_i^{\mathsf{T}}(t),\ \Psi_i(t_0) = 0,\ \forall i = 1(1)n \tag{6.13}$$

where $\Phi_i(t)$[4] $\in \mathbb{R}^{p\times p}$ is the weighted filtered regressor[5], $\Psi_i(t) \in \mathbb{R}^p$ is the weighted filtered output, and $\alpha(t) \in \mathbb{R}$ is a weighting function having the following properties.

Property 16. $\alpha(t) > 0,\ \forall t \in [t_0, \infty)$. □

Property 17. $\alpha(t) < \bar{\alpha} < \infty,\ \forall t \in [t_0, \infty),\ \Rightarrow \alpha(t) \in \mathcal{L}_\infty$. □

Property 18. $\alpha(t) \in \mathcal{L}_1$. □

where $\bar{\alpha} \in \mathbb{R}_{>0}$ is an upper-bound of $\alpha(t)$.

Analytically solving the (6.12)-(6.13) along with (6.2), (6.6), and following yields

$$\Psi_i(t) \triangleq \Phi_i(t)\Theta,\ \forall i = 1(1)n \tag{6.14}$$

Here, weighted filtered regressor $\Phi_i(t)$ based few subsequent Lemmas are portrayed which are essential for the design and analysis of an online distributed AdESC algorithm.

Lemma 22. *The weighted filtered regressor $\Phi_i(t)$ is a positive semi-definite function of time i.e., $\Phi_i(t) \geq 0,\ \forall t \geq t_0,\ \forall i = 1(1)n$.*

Proof. For proof refer to Lemma 11 proof argument from chapter 4. □

Lemma 23. *The weighted filtered regressor $\Phi_i(t)$ holds the following matrix inequality i.e.,*

$$\Phi_i(t_2) \geq \Phi_i(t_1),\ \forall t_2 \geq t_1 \geq t_0,\ \forall i = 1(1)n.$$

.

[4]Without loss of generality, in this paper $\Phi(x_i(t))$ is written as $\Phi_i(t)$, and same for others.

[5]While the work in (Garg *et al.*, 2022) has introduced the weighted integrator in the context of single-agent parameter estimation, the current work extends the idea to an MAS setting.

Proof. For proof refer to Lemma 12 proof argument from chapter 4. □

6.3.2 Proposed Distributed AdESC Algorithm

In the subsection, we will design the distributed AdESC algorithm which contains the parameter estimator and control dynamics duo.

Proposed DAPE Algorithm for i^{th} agent

Consider the following online DAPE algorithm, $\forall t \geq t_0$, $\forall i = 1(1)n$.

$$\dot{\hat{\Theta}}_i(t) = -\gamma \underbrace{\phi_i(t)\big(\hat{y}_i(t) - y_i(t)\big)^{\mathrm{T}}}_{E_{i,P}(t)} - \gamma_1 \underbrace{\big(\hat{\Psi}_i(t) - \Psi_i(t)\big)}_{E_{i,I}(t)} + \gamma_2 \sum_{j \in N_i} \underbrace{a_{ij}\big(\hat{\Theta}_j(t) - \hat{\Theta}_i(t)\big)}_{E_{i,C}(t)} \tag{6.15}$$

where $\gamma, \gamma_1, \gamma_2 \in \mathbb{R}_{>0}$ are constant positive tuning parameters and $\hat{y}_i(t) \triangleq \hat{\Theta}_i^{\mathrm{T}}(t)\phi_i(t)$, $\hat{\Psi}_i(t) \triangleq \Phi_i(t)\hat{\Theta}_i(t)$.

In (6.15), $E_{i,P}(t)$ component is a proportional type of local prediction error, $E_{i,I}(t)$[6] component is a weighted integral type of local prediction error, whereas the last term $E_{i,C}(t)$ represents neighboring interaction among the agents using communication graph topology based on Assumption 18.

Proposed Controller for i^{th} agent

Since one of our objectives is to keep the each i^{th} robot/agent trajectory in a predefined convex, compact, non-empty set M_i $\forall i = 1(1)n$, the controller dynamics is designed using the gamma-Projection operator $\big(\mathrm{Proj}_{\Gamma}(\cdot)\big)$ as mentioned in Definition 11 of (Lavretsky and Gibson, 2011).

Consider the following controller, $\forall t \geq t_0$, $\forall i = 1(1)n$.

$$u_i(t) = \mathrm{Proj}_{\Gamma_x}(x_i, \psi_i^x, M_i) \tag{6.16}$$

[6]The introduction of $E_{i,I}(t)$ in the design of an online distributed parameter estimation algorithm (6.15), which obviates the need for the restrictive gC-PE condition, as revealed in the subsequent analysis.

where $\Gamma_x \in \mathbb{R}^{n\times n} > 0$ is a controller gain (Lavretsky and Gibson, 2011).

Here, $\psi_i^x(t) \in \mathbb{R}^n$ is defined as, $\forall t \geq t_0$

$$\psi_i^x(t) = -\epsilon\eta\nabla\phi_i^{\mathrm{T}}(t)\hat{\Theta}_i(t) + u_{\text{i,ex}}(t) \tag{6.17}$$

where $\epsilon \in \mathbb{R}_{>0}$ is a tuning parameter, $\eta \in \mathbb{R}_{>0}$ is an another design parameter. The exploratory signal $u_{\text{i,ex}}(t)$ facilitates to satisfy excitation condition, however, C-PE or gC-PE conditions demands an infinite energy (non-square-integrable) persistent exploration. In contrast, gC-IE or C-IE condition can be satisfied by an appropriately designed decaying exploration. Hence, $u_{\text{i,ex}}(t)$ for each robot is chosen to be uniformly continuously differentiable and have the property - $u_{\text{i,ex}}(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, which implies $u_{\text{i,ex}}(t) \to 0$ as $t \to \infty$, $\forall i = 1(1)n$. In fact, $u_{\text{i,ex}}(t)$ can be generated using the following dynamics.

$$a_{i,N}\frac{d^N u_{\text{i,ex}}(t)}{dt^N} + a_{i,(N-1)}\frac{d^{N-1}u_{\text{i,ex}}(t)}{dt^{N-1}} + \cdots + a_{i,1}u_{\text{i,ex}}(t) = g_{\text{i,ex}}(t) \tag{6.18}$$

where $g_{\text{i,ex}}(t)$ is an auxiliary input satisfying $g_{\text{i,ex}}(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\forall i = 1(1)n$; the constants $a_{i,1}, \ldots, a_{i,N}$, and initial conditions and the order N can be suitably chosen for the above properties of $u_{\text{i,ex}}(t)$ to hold. For instance, $u_{\text{i,ex}}(t)$ can take the forms of e^{-kt}, $e^{-k_1 t}sin(k_2 t)$, $e^{-k_3 t}cos(k_4 t)$ or $e^{-k_5 t}\big(sin(k_6 t) + cos(k_7(t))\big)$, etc. where all the different constants $k_{(.)} \in \mathbb{R}_{>0}$.

6.4 Stability/Convergence Analysis

6.4.1 Stability/Convergence of the Proposed DAPE Algorithm

Compact form representation

Utilizing Assumption 18 and its preliminaries, compact representation of an online parameter estimation error dynamics (6.15) for the entire group can be structured based on (5.1) as, $\Delta\Theta \triangleq [\Delta\Theta_a^{\mathrm{T}}, \Delta\Theta_b^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(h+h')p}$, where $\Delta\Theta_a \in \mathbb{R}^{hp}$ is associated with agents corresponding to RSCCs in graph G, and $\Delta\Theta_b \in \mathbb{R}^{h'p}$ is associated with the remaining agents. Here, $h = \sum_{q=1}^{Q} h_q$ (based on (5.2)) and ($h + h' = n$). Utilizing (5.1)-(5.2), the parameter estimation error dynamics (6.15) for all the n number of agents can be compactly represented as

$$\Delta\dot{\Theta}_a(t) = -\Big(\gamma\Omega_{a,P}(t) + \gamma_1\Omega_{a,I}(t) + \gamma_2(L_a \otimes I_p)\Big)\Delta\Theta_a(t) \tag{6.19}$$

$$\Delta\dot{\Theta}_b(t) = -\Big(\gamma\Omega_{b,P}(t) + \gamma_1\Omega_{b,I}(t) + \gamma_2(L_b \otimes I_p)\Big)\Delta\Theta_b(t) - \gamma_2(L_{ab} \otimes I_p)\Delta\Theta_a(t) \tag{6.20}$$

where $\Omega_{a,P}(t)$, $\Omega_{a,I}(t)$, $\Omega_{b,P}(t)$, $\Omega_{b,I}(t)$, are the block diagonal matrices, defined as

$$\begin{aligned}
\Omega_{a,P}(t) &\triangleq \text{diag}\Big[\phi_{a1}(t)\phi_{a1}^{\mathrm{T}}(t), \ldots, \phi_{ah}(t)\phi_{ah}^{\mathrm{T}}(t)\Big] \in \mathbb{R}^{hp\times hp}\\
\Omega_{a,I}(t) &\triangleq \text{diag}\Big[\Phi_{a1}(t), \ldots, \Phi_{ah}(t)\Big] \in \mathbb{R}^{hp\times hp}\\
\Omega_{b,P}(t) &\triangleq \text{diag}\Big[\phi_{b1}(t)\phi_{b1}^{\mathrm{T}}(t), \ldots, \phi_{bh'}(t)\phi_{bh'}^{\mathrm{T}}(t)\Big] \in \mathbb{R}^{h'p\times h'p}\\
\Omega_{b,I}(t) &\triangleq \text{diag}\Big[\Phi_{b1}(t), \ldots, \Phi_{bh'}(t)\Big] \in \mathbb{R}^{h'p\times h'p}.
\end{aligned}$$

Here, $L_a \in \mathbb{R}^{h\times h}$, $L_a \otimes I_p \in \mathbb{R}^{hp\times hp}$, $L_b \in \mathbb{R}^{h'\times h'}$, $L_b \otimes I_p \in \mathbb{R}^{h'p\times h'p}$, $L_{ab} \in \mathbb{R}^{h'\times h}$, $L_{ab} \otimes I_p \in \mathbb{R}^{h'p\times hp}$.

Based on (5.1), L_a is a block diagonal matrix with Q blocks in total, one for each RSCC.

To comment on the stability/convergence properties of (6.19)-(6.20), few subsequent lemmas are formulated.

Lemma 24. *Let $L_a \in \mathbb{R}^{h\times h}$ be an irreducible, singular M-matrix. Then,*

(1) L_a has rank $(h-1)$.

(2) There exists a positive diagonal matrix $Z_a \in \mathbb{R}^{h\times h}$; $Z_a = diag[z] > 0$ where $z = [z_1, z_2, \ldots, z_h]^T \in \mathbb{R}^h > 0$ such that $\tilde{L}_a = Z_a L_a + L_a^T Z_a$ is a positive semi-definite matrix.

Proof. For proof refer to the Theorem 4.31 from (Qu, 2009). □

Lemma 25. *Consider a matrix $\bar{Z}_a = Z_a \otimes I_p \in \mathbb{R}^{hp\times hp}$, where $Z_a > 0$ is the diagonal matrix defined above. Then, the following holds:*

(1) $\bar{Z}_a$ is a block diagonal and positive-definite matrix.

(2) Pairs of matrices $\big(\bar{Z}_a, \Omega_{a,P}(t)\big)$, $\big(\bar{Z}_a, \Omega_{a,I}(t)\big)$ hold commutative property, respectively, $\forall t \geq t_0$.

(3) $\bar{Z}_a \Omega_{a,P}(t)$, $\bar{Z}_a \Omega_{a,I}(t)$, *are block diagonal positive semi-definite matrix, respectively,* $\forall t \geq t_0$.

Proof. Proof is similar to the Lemma 20 proof argument. $\square$

Assumption 20. *The block diagonal matrix* $\Omega_{a,P}(t)$ *which contains the regressors or information corresponding to the total no of RSCCs in graph G, holds the gC-IE condition as in* (5.4). $\square$

Lemma 26. *Provided the gC-IE condition based Assumption 20 holds, the matrix* $\Gamma(t) \triangleq \left(\gamma_2\left(\tilde{L}_a \otimes I_p\right) + \gamma_1 M_a\right)$, *is positive-definite over the time window* $[t_0 + T, \infty)$ *i.e.,*

$$X^{\mathrm{T}}\Gamma(t)X \geq \varrho\|X\|^2, \ \forall t \geq t_0 + T, \ \forall X \in \mathbb{R}^{hp} - \boldsymbol{0}_{hp} \tag{6.21}$$

where $\varrho \in \mathbb{R}_{>0}$ *is a positive constant scalar,* $T = \max_{q \in \{1,\cdots,Q\}} \{T_q\}$, *and* $M_a = \bar{Z}_a \Omega_{a,I}(t)$.

Proof. Proof is similar to the Lemma 29 proof argument (refer to Appendix A). $\square$

Theorem 9. *Suppose that the Assumptions 18-20 hold, then the origin of the parameter estimation error* $\Delta\Theta_a(t)$ *dynamics (6.19) is globally exponentially stable (GES) i.e.,*

$$\|\Delta\Theta_a(t)\| \leq \|\Delta\Theta_a(t_0 + T)\| e^{-\varepsilon(t - t_0 - T)}, \ \forall t \geq t_0 + T \tag{6.22}$$

where $\varepsilon \in \mathbb{R}_{>0}$ *is the constant positive scalar.*

Proof. For proof refer to the Theorem 7 proof argument from chapter 5. $\square$

Remark 22. *Motivated by the fact that an integral (I) action in conjunction with a proportional (P) control improves the steady-state accuracy in single-agent based control systems, an online distributed parameter estimation algorithm is proposed in* (6.15). *The integral-like term* $E_{i,I}(t)$ *along with the consensus term* $E_{i,C}(t)$ *is critical in proving exponential convergence of the parameter estimation error* $\Delta\Theta(t)$ *to zero without the gC-PE condition, whereas, only the proportional-like term* $E_{i,P}(t)$ *along with the* $E_{i,C}(t)$ *would result in a nonzero steady-state error in the absence of gC-PE (Javed* et al.*, 2021; Maghenem* et al.*, 2022) condition.* $\square$

To comment on the stability/convergence properties of (6.20), the subsequent lemma along with a corollary is provided.

Lemma 27. *Let $L_b \in \mathbb{R}^{h' \times h'}$ be a non-singular M-matrix, then there exists a positive diagonal matrix $Z_b \in \mathbb{R}^{h' \times h'} > 0$ such that $\tilde{L}_b = L_b^T Z_b + Z_b L_b$ is a positive-definite matrix.*

Proof. For proof refer to the Theorem 4.25 of (Qu, 2009). $\square$

Corollary 9.1. *Suppose that the Assumptions 18-19 hold, then the parameter estimation error $\Delta\Theta_b(t)$ dynamics* (6.20) *exhibits ISS with respect to input $\Delta\Theta_a(t)$, $\forall t \geq t_0$.*

Proof. For proof refer to the Corollary 7.1 proof argument from chapter 5. $\square$

Remark 23. *In Corollary 9.1, if input $\Delta\Theta_a(t) = \boldsymbol{0}_{hp}$, Corollary 9.1 exhibits GES for origin of the parameter estimator dynamics* (6.20)*, $\forall t \geq t_0$ (for more details refer to the Definition* (9)*).* $\square$

Remark 24. *Theorem 9 for* (6.19) *shows GES if gC-IE condition based Assumption 20 holds. Corollary 9.1 for* (6.20) *shows ISS with respect to input $\Delta\Theta_a(t)$. Hence, the cascade of parameter estimator dynamics* (6.19)-(6.20) *exhibits GES (for proof see the Lemma 4.7 of (Khalil and Grizzle, 2002)) which implies that the objective* (6.11) *is achieved.*

$\square$

6.4.2 Stability/Convergence of the Proposed Controller Dynamics

Theorem 10. *Suppose that the Assumptions 16-17, 19 hold, then, the closed-loop error dynamics $\big(x_i(t) - x_i^\star\big)$ exhibits ISS with respect to $\Delta\Theta_i(t)$, $\forall t \geq t_0$, provided that the following gain condition holds:*

$$0 < \epsilon < 2\mu\eta.$$

Proof. Consider the following Lyapunov function candidate,

$$V_3 = \frac{1}{2}(x_i - x_i^\star)^{\mathrm{T}} \Gamma_x^{-1} (x_i - x_i^\star) \tag{6.23}$$

Provided Assumption 16 holds, taking the time derivative of (6.23) along with the (6.16), following yields

$$\dot{V}_3 \leq \big(x_i - x_i^\star\big)^{\mathrm{T}} \mathrm{Proj}_{\Gamma_x}\Big(- \epsilon\eta\nabla\phi_i^{\mathrm{T}}(t)\hat{\Theta}_i(t) + \epsilon\eta\nabla L_i(t) - \epsilon\eta\nabla L_i(t) + u_{\text{i,ex}}(t)\Big),\ \forall t \geq t_0 \tag{6.24}$$

After further manipulations, (6.24) yields

$$\dot{V}_3 \leq -(x_i - x_i^\star)^{\mathrm{T}}\epsilon\eta\nabla\phi_i^{\mathrm{T}}(t)\Delta\Theta_i(t) - \epsilon\eta(x_i - x_i^\star)^{\mathrm{T}}\nabla L_i(t) + \big(x_i - x_i^\star\big)^{\mathrm{T}} u_{\text{i,ex}}(t),\ \forall t \geq t_0 \tag{6.25}$$

With in a compact set M_i, $\exists C > 0$ such that $\|\nabla\phi_i\| \leq C$, provided that the Assumption 17 based inequalities (6.3)-(6.4) hold, invoking the Cauchy-Schwarz inequality on the cross term, and the inequality (6.25) can be improvised as

$$\dot{V}_3 \leq \|x_i - x_i^\star\|\epsilon\eta C\|\Delta\Theta_i(t)\| - \epsilon\eta\mu\|x_i - x_i^\star\|^2 + \beta_1\|u_{\text{i,ex}}(t)\|,\ \forall t \geq t_0 \tag{6.26}$$

where $\mu > 0$ and $\beta_1 \geq 0$ is the upper-bound on $\|x_i - x_i^\star\|$ since $x_i \in M_i$ and $x_i^\star \in M$, $\forall t \geq t_0$. Further simplifying (6.26), yields

$$\dot{V}_3 \leq \frac{1}{2}\eta^2 C^2\|\Delta\Theta_i(t)\|^2 + \beta_2 e^{-\beta_3 t} - \Big(\epsilon\eta\mu - \frac{1}{2}\epsilon^2\Big)\|x_i - x_i^\star\|^2,\ \forall t \geq t_0 \tag{6.27}$$

Utilizing (6.23), then (6.27) can be modified as

$$\dot{V}_3 \leq -\underbrace{\min\Big(2\epsilon\eta\mu - \epsilon^2\Big)}_{\beta_4>0} V_3 + \frac{1}{2}\eta^2 C^2\|\Delta\Theta_i(t)\|^2 + \beta_2 e^{-\beta_3 t},\ \forall t \geq t_0 \tag{6.28}$$

Providing that the following necessary and sufficient gain condition is satisfied; $0 < \epsilon < 2\mu\eta$, then the (6.28) can be written as

$$\dot{V}_3 \leq -\beta_4 V_3 + \frac{1}{2}\eta^2 C^2\|\Delta\Theta_i(t)\|^2 + \beta_2 e^{-\beta_3 t},\ \forall t \geq t_0 \tag{6.29}$$

where $\beta_2, \beta_3, \beta_4 \in \mathbb{R}_{>0}$ are the constant positive scalars. Utilizing (6.51), Lyapunov Theorem 4.19, Lemma 4.6 form (Khalil and Grizzle, 2002), it can be concluded that dynamics (6.16) exhibits ISS with respect to input $\Delta\Theta_i(t)$, $\forall t \geq t_0$, as Definition 9. □

Remark 25. *Based on Remark 24, $\Delta\Theta_i(t)$ is GES if gC-IE condition based Assump-*

tion 20 holds. Theorem 10 for $(x_i(t) - x^\star)$ shows ISS with respect to input $\Delta\Theta_i(t)$. Hence, the cascade of $\big(x_i(t) - x^\star,\ \Delta\Theta_i(t)\big)$ dynamics exhibits GES (for proof see the Lemma 4.7 of (Khalil and Grizzle, 2002)) which implies that the objectives (6.9)-(6.11) *are achieved.* □

6.4.3 BIBO Stability of Weighted Integrator

Corollary 10.1. *Weighted integrator outputs signals $\Phi_i(t),\ \Psi_i(t) \in \mathcal{L}_\infty$, $\forall t \geq t_0$, $\forall i = 1(1)n$.*

Proof. Equation (6.16) implies that $x_i(t) \in \mathcal{L}_\infty$ which means that $\phi_i(t) \in \mathcal{L}_\infty$ from Assumption 16. Exploiting the Properties 16-18 in (6.12), $\|\Phi_i(t)\|$ can be upper-bounded as

$$\begin{aligned}
\|\Phi_i(t)\| &= \| \int_{t_0}^{t} \underbrace{\alpha(r)}_{>0}\ \underbrace{\phi_i(r)\phi_i^{\mathrm{T}}(r)}_{\geq 0} \mathrm{d}r \| \\
&\leq \int_{t_0}^{t} \|\alpha(r)\| \|\phi_i(r)\phi_i^{\mathrm{T}}(r)\| \mathrm{d}r \\
&\leq \underbrace{\|\phi_i(t)\phi_i^{\mathrm{T}}(t)\|_{\max}}_{\varsigma>0} \underbrace{\int_{t_0}^{t} \|\alpha(r)\| \mathrm{d}r}_{\bar{\alpha}>0} < \infty
\end{aligned} \tag{6.30}$$

which implies $\Phi_i(t) \in \mathcal{L}_\infty$. Based on algebraic relation $\Psi_i(t) \triangleq \Phi_i(t)\Theta$, Assumption 19, and it can be concluded that $\Psi_i(t) \in \mathcal{L}_\infty$, $\forall t \geq t_0$, $\forall i = 1(1)n$. □

6.5 Robustness Analysis

In this section, the above optimization problem (6.1) is tackled under an external additive disturbance. Consider the following perturbed dynamics:

$$\dot{x}_i(t) = u_i(t),\ \forall x_i(t_0) \in M_i \subset M \tag{6.31}$$

$$y_i(t) = L(x_i(t)) + \delta(x_i(t)),\ \forall i = 1(1)n \tag{6.32}$$

where $\delta_i(t)^7 : \mathbb{R}^n \to \mathbb{R} \in \mathcal{L}_\infty$ represent an additive perturbation signal, satisfying the following bound

$$\|\delta_i(t)\| \leq \bar{\delta}_i < +\infty, \; \forall i = 1(1)n \tag{6.33}$$

where $\bar{\delta}_i$ is an known upper-bound, $\forall i = 1(1)n$.

Analytically solving (6.12), (6.13), along with (6.32), following yields

$$\Psi_i(t) \triangleq \Phi_i(t)\Theta + \delta_{if}(t), \; \forall t \geq t_0, \; \forall i = 1(1)n \tag{6.34}$$

where $\delta_{if}(t) \in \mathbb{R}^p$ represent the filtered disturbance, which can be obtained by the following differential law.

$$\dot{\delta}_{if}(t) = \alpha(t)\phi_i(t)\delta_i(t), \; \delta_{if}(t_0) = 0, \; \forall t \geq t_0, \; \forall i = 1(1)n \tag{6.35}$$

Corollary 10.2. *Filtered disturbance* $\delta_{if}(t) \in \mathcal{L}_\infty$*,* $\forall t \geq t_0$*,* $\forall i = 1(1)n$*.*

Proof. Since $\delta_i(t), \phi_i(t) \in \mathcal{L}_\infty$, using Properties 16-18 of weighting function $\alpha(t)$, (6.33), and $\|\delta_{if}(t)\|$ can be expressed as

$$\begin{aligned}\|\delta_{if}(t)\| &= \| \int_{t_0}^{t} \alpha(r)\phi_i(r)\delta_i(r)dr\| \\ &\leq \bar{\delta}_i \underbrace{\|\phi_i(t)\|_{max}}_{\bar{\phi}\in\mathbb{R}_{>0}} \underbrace{\int_{t_0}^{t} \|\alpha(r)\|dr}_{\bar{\alpha}\in\mathbb{R}_{>0}} < \infty\end{aligned} \tag{6.36}$$

which implies $\delta_{if}(t) \in \mathcal{L}_\infty, \forall t \geq t_0, \forall i = 1(1)n$. □

[7]Without loss of generality, in this paper $\delta(x_i(t))$ is written as $\delta_i(t)$. Moreover, it is also assumed that, if $x_i(t) \in \mathcal{L}_\infty \implies \delta_i(t) \in \mathcal{L}_\infty$

Proposed DAPE Algorithm for i^{th} agent

Consider the following online DAPE algorithm, $\forall t \geq t_0$, $\forall i = 1(1)n$.

$$\begin{aligned}\dot{\hat{\Theta}}_i(t) = &-\gamma \underbrace{\phi_i(t)\phi_i^{\mathrm{T}}(t)\Delta\Theta_i(t)}_{E_{i,P}(t)} - \gamma_1 \underbrace{\Psi_i(t)\Delta\Theta_i(t)}_{E_{i,I}(t)} + \underbrace{\gamma\phi_i(t)\delta_i(t) + \gamma_1\delta_{if}(t)}_{E_{i,D}(t)} \\ &+ \gamma_2 \sum_{j\in N_i} \underbrace{a_{ij}\big(\hat{\Theta}_j(t) - \hat{\Theta}_i(t)\big)}_{E_{i,C}(t)} \end{aligned} \tag{6.37}$$

whereas the extra term $E_{i,D}(t)$ in (6.37) in contrast to (6.15), occurs due to the presence of disturbance.

Proposed controller for i^{th} agent

Here, the controller dynamics will be same as in section 6.3.2.

6.5.1 Stability/Convergence of the Proposed DAPE Algorithm

In the same way as in subsection 6.4.1, the compact form representation of the DAPE algorithm for the n number of agents in the presence of disturbance can also be modified as.

$$\Delta\dot{\Theta}_a(t) = -\Big(\gamma\Omega_{a,P}(t) + \gamma_1\Omega_{a,I}(t) + \gamma_2(L_a \otimes I_p)\Big)\Delta\Theta_a(t) + \gamma D_a(t) + \gamma_1 D_{af}(t) \tag{6.38}$$

$$\begin{aligned}\Delta\dot{\Theta}_b(t) = &-\Big(\gamma\Omega_{b,P}(t) + \gamma_1\Omega_{b,I}(t) + \gamma_2(L_b \otimes I_p)\Big)\Delta\Theta_b(t) - \gamma_2(L_{ab} \otimes I_p)\Delta\Theta_a(t) \\ &+ \gamma D_b(t) + \gamma_1 D_{bf}(t)\end{aligned} \tag{6.39}$$

where the disturbance based column vectors $D_a(t)$, $D_{af}(t)$, $D_b(t)$, $D_{bf}(t)$, are defined as the following.

$$\begin{aligned}
D_a(t) &\triangleq \big[\phi_{a1}(t)\delta_{a1}(t), \ldots, \phi_{ah}(t)\delta_{ah}(t)\big] \in \mathbb{R}^{hp} \\
D_{af}(t) &\triangleq \big[\delta_{a1f}(t), \ldots, \delta_{ahf}(t)\big] \in \mathbb{R}^{hp} \\
D_b(t) &\triangleq \big[\phi_{b1}(t)\delta_{b1}(t), \ldots, \phi_{bh'}(t)\delta_{ah'}(t)\big] \in \mathbb{R}^{h'p} \\
D_{bf}(t) &\triangleq \big[\delta_{b1f}(t), \ldots, \delta_{bh'f}(t)\big] \in \mathbb{R}^{h'p}.
\end{aligned}$$

Since $\delta_i(t), \phi_i(t) \in \mathcal{L}_\infty$, (5.29), and Corollary 10.2, it can be concluded that the $D_a(t), D_{af}(t), D_b(t), D_{bf}(t) \in \mathcal{L}_\infty$.

Theorem 11. *Suppose that the Assumptions 18-20 hold, then the parameter estimation error $\Delta\Theta_a(t)$ dynamics (6.38) exhibits ISS with respect to input disturbance $\big(D_a(t) + D_{af}(t)\big)$, $\forall t \geq t_0 + T$.*

Proof. Consider the following Lyapunov candidate

$$V_4(\Delta\Theta_a) = \frac{1}{2}\mathrm{tr}\big(\Delta\Theta_a^{\mathrm{T}} \bar{Z}_a \Delta\Theta_a\big) \tag{6.40}$$

Taking time derivative of (6.40) along with the (6.38), yields

$$\begin{aligned}
\dot{V}_4(\Delta\Theta_a) = &- \mathrm{tr}\Big(\gamma\Delta\Theta_a^{\mathrm{T}}\bar{Z}_a\Omega_{a,P}\Delta\Theta_a + \gamma_1\Delta\Theta_a^{\mathrm{T}}\bar{Z}_a\Omega_{a,I}\Delta\Theta_a + \gamma_2\Delta\Theta_a^{\mathrm{T}}(\tilde{L}_a \otimes I_p)\Delta\Theta_a \\
&- \gamma\Theta_a^{\mathrm{T}}\bar{Z}_a D_a - \gamma_1\Theta_a^{\mathrm{T}}\bar{Z}_a D_{af}\Big), \ \forall t \geq t_0
\end{aligned} \tag{6.41}$$

Invoking the Assumption 20 based Lemma 26 in (6.41), yields

$$\dot{V}_4(\Delta\Theta_a) \leq -\eta V_4(\Delta\Theta_a) + \bar{\gamma}_a\|\Delta\Theta_a\|(\|D_a + D_{af})\|, \ \forall t \geq t_0 + T \tag{6.42}$$

where $\eta, \bar{\gamma}_a \in \mathbb{R}_{>0}$. Utilizing (6.42), Lyapunov Theorem 4.19, Lemma 4.6 form (Khalil and Grizzle, 2002), it can be concluded that dynamics (6.38) exhibits ISS with respect to input disturbance $\big(D_a(t) + D_{af}(t)\big)$, $\forall t \geq t_0 + T$. □

Corollary 11.1. *Suppose that the Assumptions 18-19 hold, then the parameter estimation error $\Delta\Theta_b(t)$ dynamics* (6.39) *exhibits ISS with respect to input $\Delta\Theta_a(t)$*[8]*, $\big(D_b(t) +$*

[8]Here, $\Delta\Theta_a(t)$ is for (6.38).

$D_{bf}(t)\big)$, *respectively,* $\forall t \geq t_0$.

Proof. Consider the following Lyapunov candidate

$$V_5(\Delta\Theta_b) = \frac{1}{2}\text{tr}\big(\Delta\Theta_b^{\mathrm{T}}\bar{Z}_b\Delta\Theta_b\big) \tag{6.43}$$

Invoking the Corollary 9.1 proof argument, taking the time derivative of (6.43) along with the (6.39), yields, $\forall t \geq t_0$

$$\dot{V}_4(\Delta\Theta_b) \leq -\gamma_2\eta_1\|\Delta\Theta_b\|^2 + \gamma_2\|\Delta\Theta_b\|\|\bar{Z}_b(L_{ab}\otimes I_p)\|\|\Delta\Theta_a\| + \bar{\gamma}_b\|\Delta\Theta_b\|(\|D_b + D_{bf}\|) \tag{6.44}$$

where $\bar{\gamma}_b \in \mathbb{R}_{>0}$. Utilizing (6.44), Lyapunov Theorem 4.19, Lemma 4.6 form (Khalil and Grizzle, 2002), it can be concluded that dynamics (6.39) exhibits ISS with respect to input $\Delta\Theta_a(t)$, $\big(D_b(t) + D_{bf}(t)\big)$, respectively $\forall t \geq t_0$. □

6.5.2 Stability/Convergence of the Proposed Controller Dynamics

Theorem 12. *Suppose that the Assumptions 16-17, 19 hold, then, the closed-loop error dynamics* $\big(x_i(t) - x_i^\star\big)$ *exhibits ISS with respect to* $\Delta\Theta_i(t)$, $\forall t \geq t_0$, *provided that the following gain condition holds:*

$$0 < \epsilon < 2\mu\eta.$$

Proof. Consider the following Lyapunov function candidate,

$$V_6 = \frac{1}{2}(x_i - x_i^\star)^{\mathrm{T}}\Gamma_x^{-1}(x_i - x_i^\star) \tag{6.45}$$

Provided Assumption 16 holds, taking the time derivative of (6.45) along with the (6.16), following yields

$$\dot{V}_6 \leq \big(x_i - x_i^\star\big)^{\mathrm{T}}\text{Proj}_{\Gamma_x}\Big(- \epsilon\eta\nabla\phi_i^{\mathrm{T}}(t)\hat{\Theta}_i(t) + \epsilon\eta\nabla L_i(t) - \epsilon\eta\nabla L_i(t) + u_{\text{i,ex}}(t)\Big),\ \forall t \geq t_0 \tag{6.46}$$

After further manipulations, (6.46) yields

$$\dot{V}_6 \leq -(x_i - x_i^\star)^{\mathrm{T}} \epsilon\eta \nabla\phi_i^{\mathrm{T}}(t)\Delta\Theta_i(t) - \epsilon\eta(x_i - x_i^\star)^{\mathrm{T}}\nabla L_i(t) + \big(x_i - x_i^\star\big)^{\mathrm{T}} u_{\mathrm{i,ex}}(t),\ \forall t \geq t_0 \tag{6.47}$$

With in a compact set M_i, $\exists C > 0$ such that $\|\nabla\phi_i\| \leq C$, provided that the Assumption 17 based inequalities (6.3)-(6.4) hold, invoking the Cauchy-Schwarz inequality on the cross term, and the inequality (6.47) can be improvised as

$$\dot{V}_6 \leq \|x_i - x_i^\star\|\epsilon\eta C\|\Delta\Theta_i(t)\| - \epsilon\eta\mu\|x_i - x_i^\star\|^2 + \beta_1\|u_{\mathrm{i,ex}}(t)\|,\ \forall t \geq t_0 \tag{6.48}$$

where $\mu > 0$ and $\beta_1 \geq 0$ is the upper-bound on $\|x_i - x_i^\star\|$ since $x_i \in M_i$ and $x_i^\star \in M$, $\forall t \geq t_0$. Further simplifying (6.48), yields

$$\dot{V}_6 \leq \frac{1}{2}\eta^2 C^2\|\Delta\Theta_i(t)\|^2 + \beta_2 e^{-\beta_3 t} - \Big(\epsilon\eta\mu - \frac{1}{2}\epsilon^2\Big)\|x_i - x_i^\star\|^2,\ \forall t \geq t_0 \tag{6.49}$$

Utilizing (6.45), then (6.49) can be modified as

$$\dot{V}_3 \leq -\underbrace{\min\Big(2\epsilon\eta\mu - \epsilon^2\Big)}_{\beta_4 > 0} V_3 + \frac{1}{2}\eta^2 C^2\|\Delta\Theta_i(t)\|^2 + \beta_2 e^{-\beta_3 t},\ \forall t \geq t_0 \tag{6.50}$$

Providing that the following necessary and sufficient gain condition is satisfied; $0 < \epsilon < 2\mu\eta$, then the (6.50) can be written as

$$\dot{V}_3 \leq -\beta_4 V_3 + \frac{1}{2}\eta^2 C^2\|\Delta\Theta_i(t)\|^2 + \beta_2 e^{-\beta_3 t},\ \forall t \geq t_0 \tag{6.51}$$

where $\beta_2, \beta_3, \beta_4 \in \mathbb{R}_{>0}$ are the constant positive scalars. Utilizing (6.51), Lyapunov Theorem 4.19, Lemma 4.6 form (Khalil and Grizzle, 2002), it can be concluded that dynamics (6.16) exhibits ISS with respect to input $\Delta\Theta_i(t)$, $\forall t \geq t_0$, as Definition 9. □

Remark 26. *Based on Theorem 8 and Corollary 8.1, it can be concluded that $\Delta\Theta_i(t)$ is ISS if gC-IE condition based Assumption 20 holds. Theorem 12 for $(x_i(t) - x^\star)$ shows ISS with respect to input $\Delta\Theta_i(t)$. Hence, the cascade of $\big(x_i(t) - x^\star,\ \Delta\Theta_i(t)\big)$ dynamics exhibits ISS (for proof see the Lemma 4.7 of (Khalil and Grizzle, 2002)) which implies that the objectives 6.9-6.11 are achieved in ISS sense.* □

6.6 Simulation Results

Consider a MAS architecture (based on Assumption 18) with $n = 4$ agents, where the i^{th} agent objective/performance function have the form is $L(x_i) = (x_{i1} - 2)^2 + (x_{i2} - 2)^2$. Based on Assumption 16, $\Theta = [1, 1, 0, -4, -4, 8] \in \mathbb{R}^6$ and
$\phi(x_i(t)) = [x_{i1}^2, x_{i2}^2, x_{i1}x_{i2}, x_{i1}, x_{i2}, 1] \in \mathbb{R}^6$, $\forall i = 1(1)4$. Weighting function $\alpha(t) = \exp^{-0.2(t-t_0)}$ is picked in such a way that it meets all of the Properties 16-18. Probing/Exploratory signals are chosen as; $u_{1,\text{ex}}(t) = 0.35 \exp^{-0.35(t-t_0)}[1, \sin(4t)]^{\mathrm{T}}$, $u_{2,\text{ex}}(t) = 0.45 \exp^{-0.35(t-t_0)}[1, \sin(2t)]^{\mathrm{T}}$, $u_{3,\text{ex}}(t) = 0.25 \exp^{-0.35(t-t_0)}[1, \sin(3t)]^{\mathrm{T}}$,
$u_{4,\text{ex}}(t) = 0.55 \exp^{-0.35(t-t_0)}[1, \sin(1.5t)]^{\mathrm{T}}$ (finite-energy decaying signals (non-PE)). The control tuning parameters are set to $\epsilon = 0.022, \eta = 1$. $\gamma = 1.5$, $\gamma_1 = 5$, $\gamma_2 = 2$.

Figures 6.1-6.2 validates the claims of Theorem 10 i.e., objectives 6.9-6.10 $\big($for $x_i(t)\big)$, objective 6.11 $\big($for $\tilde{\Theta}_i(t)\big)$ achieved, respectively. Moreover, figures 6.3-6.4 does not validates the claims of Theorem 10 since these figures are based on the gC-PE based algorithm. Figures 6.5-6.6 validates the claims of Theorem 12 i.e., objectives 6.9-6.10 $\big($for $x_i(t)\big)$, objective 6.11 $\big($for $\tilde{\Theta}_i(t)\big)$ achieved in ISS sense, respectively.

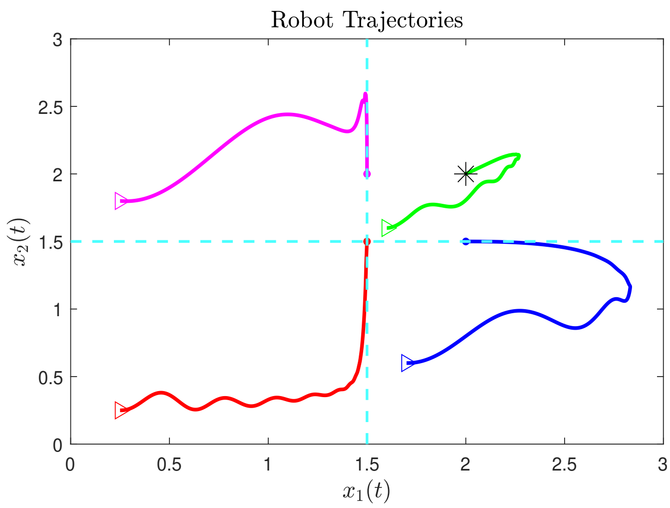

Figure 6.1: Robot trajectories based on proposed gC-IE based algorithm.

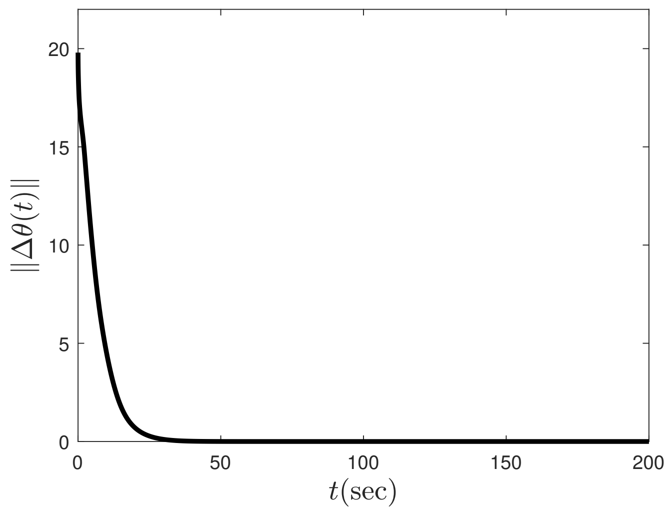

Figure 6.2: Norm of parameter estimation error based on proposed gC-IE based algorithm.

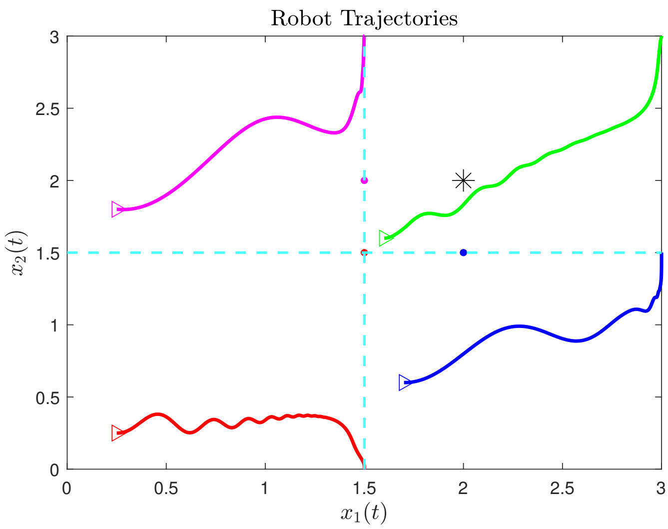

Figure 6.3: Robot trajectories based on gC-PE based algorithm.

6.7 Conclusion

In this chapter, a novel distributed adaptive extremum seeking control (DAdESC) algorithm for a multi-agent system (MAS) over a weakly connected digraph network is proposed, where parameter convergence is ensured under a newly coined relaxed excitation condition, called generalized cooperative initial excitation (gC-IE). Here, a zeroth-order optimization framework is used, where each agent can only query the numerical value of its cost function at the current coordinate, and it is assumed that only the parameter estimates information is shared among the agents, not the individual cost. Here, we eliminate

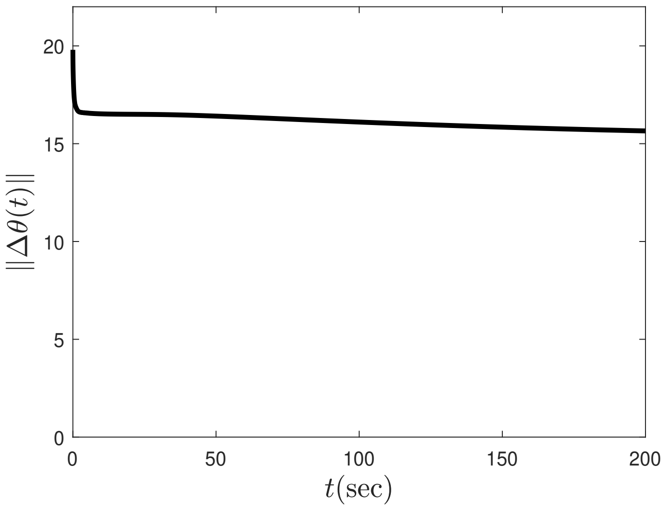

Figure 6.4: Norm of parameter estimation error based on gC-PE based algorithm.

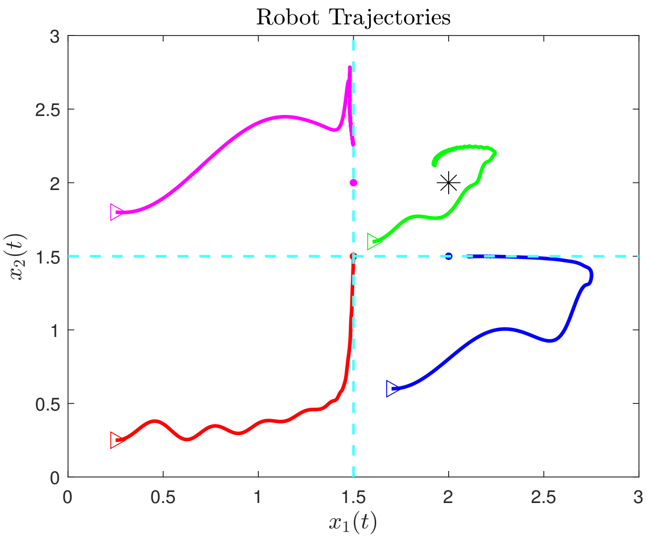

Figure 6.5: Robot Trajectories in the presence of disturbances while utilizing gC-IE based algorithm.

the need for a restrictive gC-PE condition by utilizing a novel set of weighted integral filter dynamics, while ensuring sufficient richness using a milder condition, called gC-IE. The gC-PE condition is restrictive in the sense that it requires the richness/excitation of information over the entire time-span of the signal/data, unlike the gC-IE condition where excitation is needed only in the initial time-span. Here, a detailed Lyapunov function based analysis is performed to establish the closed-loop stability and convergence in the form of exponential stability.

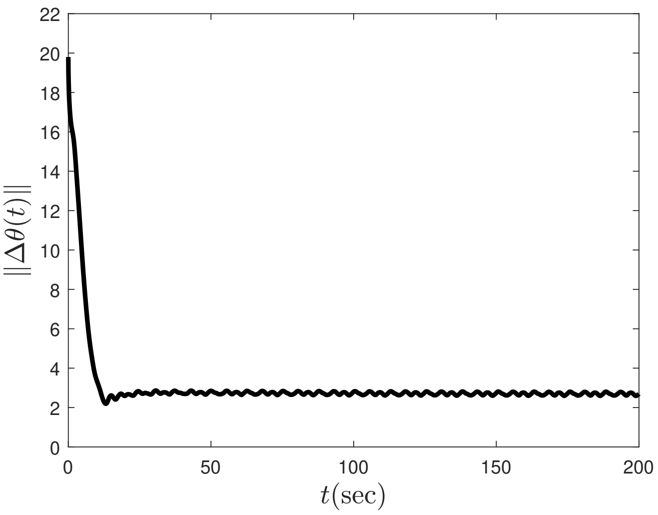

Figure 6.6: Norm of parameter estimation error in the presence of disturbances while utilizing gC-IE based algorithm.

Chapter 7

Conclusion and Future Directions of Research

7.1 Thesis Summary

This dissertation focuses on designing novel variants of distributed adaptive parameter estimation & control algorithms for multi-agent systems with improved performance in terms of transient response, parameter estimation, and robustness to unmodeled dynamics. The work conceptualized the concepts of cooperative initial excitation (C-IE), and generalized cooperative initial excitation (gC-IE), which are shown to be sufficient to ensure parameter convergence using the proposed distributed adaptive algorithms, unlike the conventional ones, which require the restrictive C-PE, gC-PE conditions for parameter convergence. The proposed distributed adaptive algorithms, which build on a novel set of low-pass filters or a set of weighted integrator dynamics, guarantee exponential convergence of tracking and parameter estimation error once the C-IE or gC-IE condition is satisfied by the regressor signal.

7.1.1 Summary of Chapter 2 (Part 1)

- In this part of the chapter, we design a novel collaborative parameter estimator dynamics, which, with the help of integral-like components, ensures parameter convergence under C-IE condition.
- The C-IE definition is a generalization of the concept of IE in multi-agent settings, where information sharing through connected graph guarantees consensus parameter convergence.
- It can be argued that the C-IE condition is milder than all of the other above mentioned conditions of PE, C-PE, and IE.

7.1.2 Summary of Chapter 2 (Part 2)

- In this part of the chapter, we design a combined cooperative adaptive cruise control (CACC) architecture for an uncertain homogeneous vehicle platoon.

- The combined CACC architecture is composed of a distributed parameter estimator of the uncertain vehicle dynamics parameters and a MRAC control law with a differential control parameter update routine.

- The control parameter estimator uses information from the vehicle dynamics parameter estimator, making the design analogous to combined MRAC.

- The distributed parameter estimator of the vehicle dynamics is designed based on a two-layer filtering mechanism and a consensus-based component using information from immediate preceding and following vehicles' instantaneous estimation.

- This distributed parameter estimator can ensure exponentially fast parameter convergence using the C-IE condition and thereby relaxes the need for excitation (information content regarding the unknown parameters) to persist for all time.

- Further, the designed MRAC law, along with the distributed parameter estimator ensures asymptotic convergence of the vehicle platoon to a string stable reference platoon, thus maintaining smooth and safe operation.

7.1.3 Summary of Chapter 3 (part 1)

- In this part of chapter, we develop a distributed consensus-based switched parameter estimation algorithm, where strategic multiple switching is incorporated to reflect the effect of C-IE condition, ensuring parameter convergence.

- It is analytically proved that the estimation error dynamics shows global exponential stability (GES) under the C-IE condition, which is in contrast to, where stringent C-PE condition is required to obtain a similar stability result.

- The concept of cooperative initial/finite excitation is introduced in a few recent works (Yuan *et al.*, 2021, 2018; Rezaei and Stefanovic, 2020; Poveda *et al.*, 2019; Garg and Roy, 2020*a*,*c*), however, all of them have considered a bidirectional communication among neighboring agents, i.e., an undirected graph topology is utilized.

- Unlike these results, the proposed work allows unidirectional communication among agents as long as the directed graph is balanced in nature. Further, the formulation is molded as an online optimization problem.

7.1.4 Summary of Chapter 3 (part 2)

- In this part of chapter, we propose a distributed composite adaptive synchronization algorithm for multiple uncertain Euler-Lagrange (EL) systems, where parameter convergence is achieved under a relaxed mathematical condition as compared to the state-of-the-art.

- Since the PE and C-PE conditions are restrictive in nature, these conditions are not satisfied in most practical applications.

- The proposed result extends the C-IE condition in distributed adaptive control architecture in the context of synchronizing multiple EL systems.
- A Two-tier, filter based parameter estimation algorithm with strategic switching ensures parameter convergence under the C-IE condition and thereby provides exponential convergence of tracking and parameter estimation error to zero.

7.1.5 Summary of Chapter 4

- In this chapter, we propose a novel distributed adaptive parameter estimation (DAPE) algorithm for a MAS architecture having a strongly connected digraph based communication topology in the presence of inter-agent communication delay.
- The proposed algorithm exhibits the following properties: (1) asymptotic consensus of parameter estimates is ensured without any restriction on the regressor or feature vectors and (2) parameter convergence is achieved under the uniform C-IE condition.
- Here, the notion of uniform C-IE is defined for the regressor concerning the agent dynamics, where each agent is modeled as single-integrator.
- Unlike previous results on C-IE, a novel weighting function based integrator is introduced here.
- The designed integrator dynamics do not have internal instability as well as online rank-checking based multiple switching requirements, which were the major concerns in open-loop and closed-loop filter architectures of C-IE based designs as in Chapters 2, 3, respectively.
- Moreover, the proposed algorithm utilizes a more generalized graph topology of a strongly connected digraph, unlike the previous C-IE based frameworks using an undirected graph as in Chapter 2 and a strongly connected and balanced digraph (without communication delay) as in Chapter 3 (part 1).

7.1.6 Summary of Chapter 5

- In this chapter, we propose a novel DAPE algorithm for a multi-agent system (MAS) over a weakly connected digraph network, where parameter convergence is ensured under a newly coined relaxed excitation condition called generalized cooperative initial excitation (gC-IE).
- This is in contrast to the past literature, where such DAPE algorithms demand cooperative persistent of excitation (C-PE) and generalized cooperative persistent of excitation (gC-PE) for strongly connected digraph, and weakly connected digraph networks, respectively, for parameter convergence.
- The gC-PE and C-PE conditions are restrictive in the sense that they require the richness/excitation of information over the entire time-span of the signal/data, unlike the gC-IE condition, where excitation is needed only in the initial time-span.

- The gC-IE condition is an extension of the C-IE condition. While the C-IE condition is applicable to a strongly connected digraph, the proposed gC-IE condition extends the concept to a weakly connected digraph.

7.1.7 Summary of Chapter 6

- Chapter 6 presents a control application of the gC-IE condition in a novel distributed adaptive extremum seeking control (DAdESC) algorithm for a multi-agent system (MAS) over a weakly connected digraph network, where parameter convergence is ensured under gC-IE.
- Here, a zeroth-order optimization framework is used, where each agent can only query the numerical value of its cost function at the current coordinate. It is assumed that only the parameter estimates information is shared among the neighboring agents, not the individual cost.
- Parameter estimation plays a decisive role in the stability and convergence properties of the DAdESC algorithm, and it is also well established in the existing literature that, to ensure parameter convergence in such context, a restrictive generalized cooperative persistent of excitation (gC-PE) condition is required.
- Here, we eliminate the need for a restrictive gC-PE condition by utilizing a novel set of weighted integral filter dynamics, ensuring sufficient richness through the milder gC-IE condition.

7.2 Future Work

While the concept C-IE based distributed adaptive control has received early recognition in top-tier journals and conferences, it is still in a nascent stage and several open problems exist. Some of the possible extensions of the C-IE based distributed adaptive parameter estimation and control are proposed subsequently.

7.2.1 Extension to Output Feedback

C-IE based distributed adaptive control, which is fundamentally designed based on full-state measurements, can be extended to a more realistic scenario, where only partial-state (output) feedback is available for control design. The aim is to design an output-feedback distributed adaptive controller, which can guarantee parameter convergence under the C-IE assumption and retains the property of exponential stability, as obtained in state feedback case. As a stepping stone, distributed adaptive observer design problem can be solved while relaxing the restrictive C-PE condition using the C-IE formulation.

7.2.2 Extension to Event-Triggered Formulation

While the proposed distributed control algorithms in this dissertation assumes continuous communication with the neighbouring agents, future work can explore event-triggered mechanism for information exchange, where communication happens intermittently among neighboring agents. It would be interesting to analyze how C-IE needs to be improvised to accommodate intermittent communication without sacrificing the theoretical guarantees.

7.2.3 Extension to Time-Varying Graph Topology

Along the similar lines, a more general formulation can be devised using time-varying graph topology for distributed parameter estimation and control. Analytically establishing a connection between the nature of excitation of signals and the nature of time-variation of the graphs is an open area of research, which requires rigorous investigation.

Appendix A

A.1 Supporting Propositions along with Proofs for Chapter 3 (part-1).

Proposition 2. *If two positive semi-definite matrices* $P, Q \in \mathbb{R}^{s \times s}$ *satisfy the subsequent inequality*

$$\Gamma_1 P \leq Q \leq \Gamma_2 P, \text{ where } \Gamma_1, \Gamma_2 \in \mathbb{R}_{>0} \tag{A.1}$$

the nullity of both the matrices are same.

Proof. Let us consider $x \in N(Q)$, where $N(\cdot)$ denote the null-space of argument matrix. Then, the following can be written

$$\begin{aligned} y^T Q y = 0& \\ &\implies y^T \big(\Gamma_1 P\big) y \leq 0, \\ &\implies y^T P y \leq 0, \\ &\implies y^T P y = 0, \\ &\implies y \in N(P). \end{aligned}$$

Hence,

$$y \in N(Q) \implies y \in N(P). \tag{A.2}$$

Further, assume that $z \in N(P)$, which implies that

$$
\begin{aligned}
z^T P z &= 0 \\
&\implies z^T\big(\Gamma_2 P\big)z = 0, \\
&\implies z^T Q z \leq 0, \\
&\implies z^T Q z = 0, \\
&\implies z \in N(Q).
\end{aligned}
$$

Hence,

$$
z \in N(P) \implies z \in N(Q). \tag{A.3}
$$

Combining (A.2), (A.3), $N(P) = N(Q)$. □

Proposition 3. *If two positive semi-definite matrices $P_{11} \in \mathbb{R}^{s\times s}$ and $Q_{11} \in \mathbb{R}^{s\times s}$ have same null-space, then there exist positive scalars σ_1 and σ_2 such that*

$$
P_{11} \leq \sigma_1 Q_{11} \; and \; Q_{11} \leq \sigma_2 P_{11} \tag{A.4}
$$

Proof. Let us consider $x_{11} \in \bar{N}(P_{11})$, which implies that $x_{11} \in \bar{N}(Q_{11})$, $N(\cdot)$ being the nullspace and $\bar{N}(\cdot)$ being the complement of the nullspace. Then, the following can be written

$$
\lambda_{min,\bar{0}}(Q_{11})\|x_{11}\|^2 \leq x_{11}^T Q_{11} x_{11} \leq \lambda_{max}(Q_{11})\|x_{11}\|^2 \tag{A.5}
$$

$$
\lambda_{min,\bar{0}}(P_{11})\|x_{11}\|^2 \leq x_{11}^T P_{11} x_{11} \leq \lambda_{max}(P_{11})\|x_{11}\|^2 \tag{A.6}
$$

where $\lambda_{min,\bar{0}}(\cdot)$ denotes the minimum eigenvalue and $\lambda_{max}(\cdot)$ denotes the maximum eigenvalue. Hence based on the above, it can be shown that for $\sigma_1 > \frac{\lambda_{max}(P_{11})}{\lambda_{min,\bar{0}}(Q_{11})}$ and $\sigma_2 > \frac{\lambda_{max}(Q_{11})}{\lambda_{min,\bar{0}}(P_{11})}$, the relation (A.4) is thus satisfied. □

A.2 Supporting Lemma along with Proof for Chapter 4.

Lemma 28. *Consider a matrix $\bar{W} = W \otimes I_p \in \mathbb{R}^{pq\times pq}$, where $W > 0$ is a diagonal matrix defined in Lemma 10. Then, the following holds.*

(1) Pairs of matrices $\big(\bar{W},\ \Phi_P(t)\big)$, $\big(\bar{W},\ \Phi_I(t)\big)$, $\big(\bar{W},\ \Gamma\big)$ hold commutative property, respectively.

(2) $\bar{W}\Phi_P(t)$, $\bar{W}\Phi_I(t)$ are positive semi-definite matrices, respectively.

(3) $\bar{W}\Gamma$ is a positive-definite matrix.

Proof. Based on the defined structure of $\bar{W} = W \otimes I_p$, can be defined as

$$\bar{W} = \text{diag}\{w_1 I_p, \dots, w_q I_p\} > 0 \tag{A.7}$$

Since $W > 0$ is a diagonal matrix, defined in Lemma 10. Utilizing (4.24) with the $\Phi_P(t) \geq 0$, $\Phi_I(t) \geq 0$ structure, $\bar{W}\Phi_P(t)$ yields

$$\bar{W}\Phi_P(t) = \text{diag}\{w_1\phi_1(t)\phi_1^{\mathrm{T}}(t), \dots, w_q\phi_q(t)\phi_q^{\mathrm{T}}(t)\} \tag{A.8}$$

Whereas, $\Phi_P(t)\bar{W}$ yields

$$\Phi_P(t)\bar{W} = \text{diag}\{w_1\phi_1(t)\phi_1^{\mathrm{T}}(t), \dots, w_q\phi_q(t)\phi_q^{\mathrm{T}}(t)\} \tag{A.9}$$

Based on (A.8), (A.9), it can be concluded that the pair of matrices $\big(\bar{W}, \Phi_P(t)\big)$ is commutative i.e., $\big(\bar{W}\Phi_P(t) = \Phi_P(t)\bar{W}\big)$. Since $\Phi_I(t)$ has the same structure as $\Phi_P(t)$, it can be concluded that the pair of matrices $\big(\bar{W}, \Phi_I(t)\big)$ is commutative in nature i.e., $\big(\bar{W}\Phi_I(t) = \Phi_I(t)\bar{W}\big)$.

Utilizing (A.8), (A.9), Lemma 10, it can be concluded that $\bar{W}\Phi_P$, $\bar{W}\Phi_I \geq 0$ are positive semi-definite matrices, respectively.

Since $\Gamma = \text{diag}\{\Gamma_1, \dots, \Gamma_q\} \in \mathbb{R}^{pq \times pq}$, $\forall i = 1(1)q$ and each $\Gamma_i > 0$ as in (4.19), and utilizing (A.8), (A.9), it can be concluded that the pair of matrices $(\bar{W}, \Gamma)$ holds commutative property, and $\bar{W}\Gamma$ is a positive-definite matrix. $\square$

A.3 Supporting Lemma along with Proof for Chapter 5.

Lemma 29. *Provided the gC-IE condition* (5.4) *holds, the matrix* $\Gamma(t) \triangleq \Big(\gamma_2\big(\tilde{L}_a \otimes I_m\big) + \gamma_1 M_a\Big)$*, is positive-definite over the time window* $[t_0 + T, \infty)$ *i.e.,*

$$X^T\Gamma(t)X \geq \varrho\|X\|^2,\ \forall t \geq t_0 + T,\ \forall X \in \mathbb{R}^{mh} - \{\boldsymbol{0}_{mh}\} \tag{A.10}$$

where $\varrho \in \mathbb{R}_{>0}$ *is a positive constant scalar,* $T = \max\limits_{q\in\{1,\cdots,Q\}}\{T_q\}$*, and* $M_a = \bar{Z}_a\Omega_{a,I}(t)$*.*

Proof. Let (λ_i, p_i), $\forall i = 1(1)h$, be the eigen-value and eigen-vector pairs of positive semi-definite matrix $\tilde{L}_a = L_a^T Z_a + Z_a L_a$. Since $L_a\mathbf{1}_h = \mathbf{0}_h$, $Z_a\mathbf{1}_h = z$, and $L_a^T z = \mathbf{0}_h$ (based on Lemma 19), it then follows that $(L_a^T Z_a + Z_a L_a)\mathbf{1}_h = \mathbf{0}_h$. This implies that $\lambda_1 = 0$ and $\lambda_i > 0$, $\forall i = 2(1)h$.

Based on the above argument, the eigen-decomposition form of the matrix $(\tilde{L}_a \otimes I_m)$, for $i = 1$, $j = 1(1)m$, yields

$$(\tilde{L}_a \otimes I_m)\Big(\frac{1}{\sqrt{h}}\mathbf{1}_h \otimes f_j\Big) = \mathbf{0}_m \tag{A.11}$$

Next, the eigen-decomposition form of the matrix $(\tilde{L}_a \otimes I_m)$, $\forall i = 2(1)h$, $\forall j = 1(1)m$ is

$$(\tilde{L}_a \otimes I_m)(p_i \otimes f_j) = \lambda_i(p_i \otimes f_j) \tag{A.12}$$

where the right hand side of both the above equations can be verified based on the mixed product property $AB \otimes CD = (A \otimes C)(B \otimes D)$ for appropriately sized matrices A, B, C, and D. Here, $f_j \in \mathbb{R}^m$ is the j^{th} unit vector, $\forall j = 1(1)m$ i.e., f_j is the j^{th} column of the identity matrix I_m.

Considering $X \in \mathbb{R}^{mh}$ as a unit vector and expressing it on this basis as

$$X = \sum_{j=1}^{m}\nu_j\frac{1}{\sqrt{h}}\mathbf{1}_h \otimes f_j + \sum_{i=2}^{h}\sum_{j=1}^{m}\delta_{ij}p_i \otimes f_j \tag{A.13}$$

with $(\nu, \delta) \in \mathbb{R}^m \times \mathbb{R}^{(h-1)m}$ has unit norm. Here, $\nu = [\nu_1, \nu_2, \ldots, \nu_m]^T$ and $\delta = [\delta_2^T, \cdots, \delta_h^T]^T$, where $\delta_i = [\delta_{i1}, \cdots, \delta_{im}]^T$, for $i = 2(1)h$.

Define the quantity $\Omega_{bar}(t)$ from $\bar{Z}_a\Omega_{a,P}(t)$ over the time interval $[t_0, t_0+T]$ is

$$\Omega_{bar}(t) \triangleq \int_{t_0}^{t_0+T} \bar{Z}_a\Omega_{a,P}(r)dr \tag{A.14}$$

Substituting relation (A.13) into (A.10), using the term $\gamma_2 X^T(\tilde{L}_a \otimes I_m)X$, utilizing (A.11)-(A.12) yields

$$\gamma_2 X^T(\tilde{L}_a \otimes I_m)X = \sum_{i=2}^{h}\sum_{j=1}^{m} \lambda_i \delta_{ij}^2 \geq \gamma_2\lambda_2\big(1-||\nu||^2\big) > 0 \tag{A.15}$$

where $||\nu||^2 + ||\delta||^2 = 1$.

The second term $\gamma_1 X^T M_a(t)X$ is expanded as

$$\begin{aligned}\gamma_1 X^T M_a(t)X =& \frac{\gamma_1}{h}\sum_{i=1}^{m}\sum_{j=1}^{m} \nu_i\nu_j(\mathbf{1}_h \otimes f_i)^T M_a(t)(\mathbf{1}_h \otimes f_j)\\ &+ \frac{2\gamma_1}{\sqrt{h}}\sum_{i=2}^{h}\sum_{j=1}^{m}\sum_{s=1}^{m} \nu_s\delta_{ij}(\mathbf{1}_h \otimes f_s)^T M_a(t)(p_i \otimes f_j)\\ &+ \underbrace{\gamma_1 \sum_{i=2}^{h}\sum_{j=1}^{m}\sum_{s=2}^{h}\sum_{l=1}^{m} \delta_{ij}\delta_{sl}(p_i \otimes f_j)^T M_a(t)(p_s \otimes f_l)}_{\geq 0}\end{aligned} \tag{A.16}$$

Utilizing the Properties 13-15 of weighting function $\beta(t)$, (5.7), it can be concluded that

$$M_a(t) \geq \underline{\beta}\,\Omega_{bar}(t),\ \forall t \geq t_0 + T \tag{A.17}$$

where $\underline{\beta} \in \mathbb{R}_{>0}$ is a lower-bound of weighting function $\beta(t)$ over the initial time-interval $[t_0, t_0+T]$. Furthermore utilizing (5.4), a subsequent relation of $\Omega_{bar}(t)$ is defined as

$$(\mathbf{1}_h \otimes f_i)^T\Omega_{bar}(\mathbf{1}_h \otimes f_j) \triangleq \sum_{i=1}^{h} z_i \int_{t_0}^{t_0+T} \phi_{ai}(r)\phi_{ai}^T(r)dr \tag{A.18}$$

Substituting (A.17) into (A.16) and utilizing (A.18), $\gamma_1 X^T M_a(t)X$ can be lower-bounded as

$$\gamma_1 X^T M_a(t) X \geq \frac{\underline{\beta}\gamma_1 \underline{z}}{h}\nu^T\left(\int\limits_{t_0}^{t_0+T}\sum_{i=1}^{h}\phi_{ai}(r)\phi_{ai}^T(r)dr\right)\nu - \frac{2\underline{\beta}\gamma_1 h}{\sqrt{h}}\sum_{i=2}^{h}\sum_{j=1}^{m}\sum_{s=1}^{m}|\nu_s\delta_{ij}| \tag{A.19}$$

where $\underline{z} = \min_i z_i$, $\forall i = 1(1)h$. After further simplification and provided the gC-IE condition (5.4) holds, (A.19) yields

$$X^T M_a(t) X \geq \frac{||\nu||^2}{h}\underline{z}\underline{\beta}\underline{\Upsilon} - 2\bar{\Upsilon}\underline{\beta}h\sqrt{||\nu||^2(1-||\nu||^2)} \tag{A.20}$$

where $\bar{\Upsilon} = \|\Omega_{bar}(t_0+T)(\mathbf{1}_h \otimes f_s)\|_1$[1], $\underline{\Upsilon} = \min_{q\in\{1,\cdots,Q\}}\{\Upsilon_q\} > 0$ is the lower-bound, based on the gC-IE definition (5.4). By clubbing these two inequalities (i.e., (A.15) and (A.20)) together yields the following.

$$X^T\Gamma(t)X > \max\{\gamma_2 X^T(\tilde{L}_a \otimes I_m)X, \gamma_1 X^T M_a X\} \geq \varrho > 0$$

where ϱ is

$$\varrho = \inf_{||\nu||\leq 1}\max\left\{\underbrace{\gamma_2\lambda_2\Big(1-||\nu||^2\Big)}_{\Pi_1},\ \underbrace{\gamma_1\underline{\beta}\frac{||\nu||^2}{h}\underline{z}\,\underline{\Upsilon} - 2\bar{\Upsilon}\underline{\beta}\gamma_1 h\sqrt{||\nu||^2\Big(1-||\nu||^2\Big)}}_{\Pi_2}\right\} \tag{A.21}$$

by using continuity argument, infimum in (A.21) is attained and is strictly greater than zero as argued below.

Note that if ϱ is zero and the first term Π_1 is zero, then the second term Π_2 should also be zero because of the $\max\{\cdot\}$ operator in (A.21), which implies $\underline{\Upsilon}$ is zero since $||\nu|| = 1$ because of $\Pi_1 = 0$. This is a contradiction with the gC-IE condition (5.4), which ensures that $\underline{\Upsilon} > 0$. Hence, ϱ is greater than zero. $\square$

[1]Here, $\|\cdot\|_1$ denotes the 1-norm.

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LIST OF PAPERS BASED ON THESIS

Journal Publications

1. Raghavv Goel, Tushar Garg, and Sayan Basu Roy. “Closed-loop reference model based distributed MRAC using cooperative initial excitation and distributed reference input estimation.” *IEEE Transactions on Control of Network Systems* 9, no. 1 (2022): 37-49.

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Conference Publications

1. Tushar Garg and Sayan Basu Roy, and Shubhendu Bhasin. “Two-Tier Filter-based Experience Replay for Neuro-Adaptive Control with Inherent Robustness.” *IFAC-PapersOnLine (World Congress)* 56, no. 2 (2023): 6857-6864.

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