Character analogues of Cohen-type identities and related Voronoi summation formulas

A Ph.D. Thesis

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Dedicated to my family

Certificate

This is to certify that the thesis entitled **"Character analogues of Cohen-type identities and related Voronoi summation formulas"** being submitted by **"Ms. Khyati"** to the **Indraprastha Institute of Information Technology Delhi**, for the award of the Degree of **Doctor of Philosophy** is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
4 July, 2025

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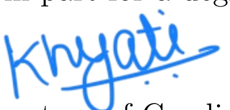
Candidate Declaration

The author hereby declares that the work presented in the thesis entitled **"Character analogues of Cohen-type identities and related Voronoi summation formulas"**, submitted as partial fulfilment for the degree of **Doctor of Philosophy** to Indraprastha Institute of Information Technology Delhi, has been carried out under the supervision of Dr. Debika Banerjee.
The work done in this thesis is original and has not been submitted earlier as a whole or in part for a degree or diploma at this or any other Institution or University.



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Abstract

Summation formulas play a vital role in analytic number theory. Several kinds of summation formulas exist, like the Poisson summation formula, Abel summation formula, Euler-Maclaurin formula, etc. In 1904, G. F. Voronoi proved that the error term in the Dirichlet divisor problem can be expressed in terms of infinite series involving Bessel functions. Additionally, he offered a broader version of the above summation formulas involving a test function f, where $f(t)$ is a function of bounded variation. Consequently, he deduced a better bound for the error term in the Dirichlet divisor problem at that time.

Following Voronoi's astounding discovery, other number theorists like A. L. Dixon, W. L. Ferrar, J. R. Wilton, Koshliakov, M. Jutila etc looked into the formula and offered proofs under different conditions on the function $f(x)$. Apart from its connection to different fields of mathematics, Voronoi-type summation formulas also have some applications in physics, especially in quantum graph theory.

In 2014, B. C. Berndt and A. Zaharescu introduced the twisted divisor sums associated with the Dirichlet character while studying Ramanujan's type identity involving finite trigonometric sums and doubly infinite series of Bessel functions. Later, S. Kim extended the definition of twisted divisor sums to twisted sums of divisor functions.

Here, we study identities associated with the aforementioned weighted divisor functions and the modified K-Bessel function in light of recent results obtained by D. Banerjee and B. Maji. Moreover, we provide a new expression for $L(1, \chi)$ from which the positivity of $L(1, \chi)$ for any real primitive character χ is established which is important is the proof of Prime number theorems in arithmetic progression.

In addition, we deduce Cohen-type identities and then exhibit the Voronoi-type summation formulas for them.

Additionally, we discuss an equivalent version of the aforementioned results in terms of identities involving finite sums of trigonometric functions and the doubly infinite series. As an application, we provide an identity for $r_6(n)$, which is analogous to Hardy's famous result where $r_6(n)$ denotes the number of representations of natural number n as a sum of six squares.

Contents

List of Symbols

Symbol	**Meaning**
$\mathbb{N}$	The set of *natural numbers*
$\mathbb{Z}$	The set of *integers*
$\mathbb{Q}$	The set of *rational numbers*
$\mathbb{R}$	The set of *real numbers*
$\mathbb{C}$	The set of *complex numbers*
$\min\{a,b\}$	*Minimum* of a and b
$\max\{a,b\}$	*Maximum* of a and b
$a \equiv b\ (n)$	The integers a and b are congruent modulo a positive integer n
$d(n)$	The *divisor* function
$r_k(n)$	the number of *representations* of n as a *sum of k-squares.*
$\zeta(s)$	The *Riemann zeta* function
$\zeta(s,q)$	The *Hurwitz zeta* function
$L(s,\chi)$	The *Dirichlet* L-function where χ is a *Dirichlet character*
$f = O(g)$	$\|f\| \leq C\|g\|$ for a suitable positive constant C
$f \sim g$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$
(c)	The vertical line $[c - i\infty, c + i\infty]$

Research Publications

Published Articles:

1. Debika Banerjee and **Khyati Khurana**: Trigonometric analogues of the identities associated with twisted sums of divisor functions, *The Ramanujan Journal*, 64, pp. 629–684, 2024.

2. Debika Banerjee and **Khyati Khurana**: Character analogues of Cohen-type identities and related Voronoï summation formulas, *Adv. in Appl. Math.*, 153, pp. 102601, 2024.

3. Debika Banerjee, Bittu Chahal, Sneha Chaubey and **Khyati Khurana**: Distribution of values of general Euler totient function, *Journal of Mathematical Analysis and Applications*, 530(2), pp. 127660., 2024.

4. Debika Banerjee and **Khyati Khurana**: On Wigert's type divisor problem, *Journal of Mathematical Analysis and Applications*, 517(2), pp. 126619., 2023.

Preprints and Communicated Articles:

1. Debika Banerjee and **Khyati Khurana**: Integral representation of the error term in the divisor problem with congruence conditions, *communicated.*

2. Debika Banerjee and **Khyati Khurana** and Arindam Roy: Sign changes of the error term in the generalised divisor problem with the congruence condition, *in preparation.*

3. Debika Banerjee, Shubham Gupta and **Khyati Khurana**: Approximate functional equation associated with a divisor function twisted by character, *in preparation.*

1

Introduction

In this chapter, we present some basic notation in special functions that are pertinent to my work. The first part of this chapter discusses the basic definitions and results associated with Bessel functions, while the other part provides some inspiration and a framework for the thesis's structure.

1.1 Some Definitions and Terminology

Let J_ν denote the ordinary Bessel function of the first kind of order ν [53, p. 40]

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2}z)^{\nu+2n}}{n!\Gamma(\nu+n+1)}, \ z \in \mathbb{C}, \tag{1.1}$$

and Y_ν denotes the Bessel function of the second kind of order ν [53, p. 64], which is defined in terms of J_ν,

$$Y_\nu(z) := \frac{J_\nu(z)\cos\pi\nu - J_{-\nu}(z)}{\sin\pi\nu}, \; z \in \mathbb{C}, \nu \notin \mathbb{Z}, \tag{1.2}$$

$$Y_n(z) := \lim_{\nu\to n} Y_\nu(z), \; n \in \mathbb{Z}, \tag{1.3}$$

and K_ν denotes the modified K-Bessel function of order ν [53, p. 78], which is defined as the following

$$K_\nu(z) := \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin\pi\nu}, \; z \in \mathbb{C}, \nu \notin \mathbb{Z}, \tag{1.4}$$

$$K_n(z) := \lim_{\nu\to n} K_\nu(z), \; n \in \mathbb{Z}. \tag{1.5}$$

with I_ν being the Bessel function of the imaginary argument [53, p. 77] given by

$$I_\nu(z) := \sum_{n=0}^{\infty} \frac{(\frac{1}{2}z)^{\nu+2n}}{n!\Gamma(\nu+n+1)}, \; z \in \mathbb{C}. \tag{1.6}$$

Since our main results involve the modified K-Bessel function, it is important to state some related results. The asymptotic estimate for the K-Bessel function defined in (1.4) is [53, p. 202]

$$K_\nu(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} + O\left(\frac{e^{-x}}{x^{\frac{3}{2}}}\right) \text{ as } x \to \infty. \tag{1.7}$$

Throughout the thesis, we will consider $\Re(\nu) \geq 0$ as $K_{-\nu}(x) = K_\nu(x)$. We recall that $K_{\frac{1}{2}}$ is equal to [53, p. 80]

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}, \tag{1.8}$$

and $K_0(x)$ is defined by (1.5). From the integral representation of $K_0(x)$ [53, p. 446]

$$K_0(x) = \int_0^\infty e^{-x\cosh t}dt,$$

one can see that $K_0(x)$ is positive and monotonically decreasing on the interval $(0, \infty)$. We also note the series representation of $K_0(x)$ [53, p. 80]

$$K_0(x) = -\log\left(\frac{x}{2}\right) I_0(x) + \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m}}{(m!)^2} \frac{\Gamma'(m+1)}{\Gamma(m+1)},$$

where $I_0(x)$ is defined in (1.6). From its series representation mentioned above, one can infer that $K_0(x)$ tends to $+\infty$ as x decreases to 0. Next, we define $M_\nu(z)$ by

$$M_\nu(z) := -Y_\nu(z) - \frac{2}{\pi} K_\nu(z). \tag{1.9}$$

1.2 Dirichlet divisor problem and Gauss Circle problem

Let $d(n) = \sum_{d|n} 1$ denote the divisor function. The Dirichlet divisor problem deals with the estimation of the error term $\Delta(x)$ that appears in the asymptotic formula for the summatory function $D(x) := \sum_{n \leq x} d(n)$. Dirichlet proved that

$$D(x) = x \log x + (2\gamma - 1)x + \Delta(x), \tag{1.10}$$

where γ is the Euler-Mascheroni constant. Dirichlet, in 1849, showed that $\Delta(x) = O(x^{\frac{1}{2}})$. The best estimate to date is $\Delta(x) = O(x^{\theta+\varepsilon})$ for every $\varepsilon > 0$, with $\theta \leq 131/416$ $(= 0.31490..)$ due to Huxley (2003). Recently, Li and Yang [38] improved the value of $\theta \leq 0.314483...$ at four decimal places, using the Bombieri–Iwaniec method. It is widely conjectured that $\theta = 1/4$ is admissible. In 1904, Voronoi [52] expressed the error term in (1.10) in terms of Bessel functions,

$$\Delta(x) = \frac{1}{4} + \sum_{n=1}^{\infty} d(n) \left(\frac{x}{n}\right)^{1/2} M_1(4\pi\sqrt{nx}), \tag{1.11}$$

where $M_\nu(z)$ is defined in (1.9). He also deduced the general summation formula, which reads as follows

$$\sum_{a\le n\le b}{}' d(n)f(n) = \int_a^b (\log(x)+2\gamma)f(x)dx$$
$$+\sum_{n=1}^{\infty} d(n)\int_a^b f(x)\ (4K_0(4\pi\sqrt{nx}) - 2\pi Y_0(4\pi\sqrt{nx}))dx, \quad (1.12)$$

where the prime $'$ on the summation on the left-hand side indicates that if a or b is an integer, then only half of the term corresponding to $n = a$ (respectively, $n = b$) is counted and $f(t)$ is a function of bounded variation in the interval (a, b). Voronoi employed (1.11) to prove the result $\Delta(x) = O(x^{1/3+\epsilon})$ for each fixed $\varepsilon > 0$. After Voronoi's remarkable discovery of (1.12), many number theorists examined the formula (1.12) and provided proofs assuming various conditions on the function $f(x)$, often involving moderate or strong restrictions. A. L. Dixon and W. L. Ferrar [24] gave proof for a bounded second differential coefficient function $f(x)$ in (a, b). Koshliakov [36] proved (1.12) for the analytic function f inside a closed contour strictly containing the interval $[a, b]$ with $0 < a < b$, $a, b \notin \mathbb{Z}$. J. R. Wilton [55] proved (1.12) for the function f, which has compact support in the interval $[a, b]$ such that $\lim_{\varepsilon\to 0} V_\alpha^{\beta-\varepsilon} f(x) = V_\alpha^{\beta-0} f(x)$ where V_α^β denotes the total variation of $f(x)$ over (α, β). In 1987, M. Jutila [34] gave a Voronoi-type summation formula involving an exponential factor. One can refer to [3, 4, 33, 40] for details and developments on Voronoi's summation formulas. Apart from its connection to different fields of mathematics, Voronoi-type summation formulas also have some applications in physics, especially in quantum graph theory [26].

Analogous to the classical divisor problem, there is another renowned open problem known as Gauss's circle problem, which concerns the magnitude of the error term $P(x)$ in the summatory function associated with the arithmetic function

$$r_2(n) := \{(n_1, n_2) \in \mathbb{Z} \mid {n_1}^2 + {n_2}^2 = n\}.$$

In 1834, Gauss proved the estimate

$$P(x) := \sum_{0\le n\le x} r_2(n) - \pi x = O(x^{1/2}). \tag{1.13}$$

Voronoi also proposed a formula similar to (1.12) for $r_2(n)$ under the same condition on f [18, p.182-183]

$$\sum_{a\le n\le b}{}' r_2(n) f(n) = \pi \int_a^b f(x)dx + \pi \sum_{n=1}^{\infty} r_2(n) \int_a^b f(x)\ J_0(2\pi\sqrt{nx})dx. \tag{1.14}$$

In 1915, Hardy [32, eq. (1.25)] proved a formula analogous to (1.11) for the error term in (1.13),

$$\sum_{n\le x} r_2(n) = \pi x - 1 + \sqrt{x} \sum_{q=1}^{\infty} \frac{r_2(q)}{\sqrt{q}} J_1(2\pi\sqrt{qx}), \tag{1.15}$$

using the following result due to Ramanujan [32, eq. (2.12)]

$$\sum_{n=1}^{\infty} r_2(n) e^{-s\sqrt{n}} = \frac{2\pi}{s^2} - 1 + 2\pi s \sum_{n=1}^{\infty} \frac{r_2(n)}{(s^2+4\pi^2 n)^{\frac{3}{2}}}, \text{ where } \Re(s) > 0. \tag{1.16}$$

In 1934, Dixon and Ferrar [25] provided a generalization of identity (1.16). They proved that, for $\Re(\nu) > 0$ and $x > 0$,

$$\sum_{n=0}^{\infty} r_2(n) n^{\nu/2} K_\nu(2\pi\sqrt{nx}) = \frac{\Gamma(\nu+1)}{2\pi^{\nu+1}} x^{\frac{\nu}{2}} \sum_{n=0}^{\infty} \frac{r_2(n)}{(n+x)^{\nu+1}}. \tag{1.17}$$

Plugging $\nu = 1/2$ in (1.17) and using the property of the Bessel function (1.8), we obtain (1.16). Employing Jacobi's formula $r_2(n) = 4\sum_{\substack{d|n \\ d\ odd}} (-1)^{(d-1)/2}$ in (1.15), we have

$$\sum_{0\le n\le x}{}' r_2(n) = \pi x + 2\sqrt{x} \lim_{N\to\infty} \sum_{mn\le N} \left\{ \frac{J_1\left(4\pi\sqrt{m(n+\frac{1}{4})x}\right)}{\sqrt{m(n+\frac{1}{4})}} \right.$$

$$-\frac{J_1\left(4\pi\sqrt{m(n+\frac{3}{4})x}\,\right)}{\sqrt{m(n+\frac{3}{4})}}\Bigg\}. \qquad (1.18)$$

In the following section, we see that Ramanujan explores the generalised result of (1.18).

1.3 Identities involving a finite trigonometric sum and a doubly infinite series of Bessel functions

The lost notebook [47] of Ramanujan contains several beautiful identities. Some are intimately connected with the famous *circle* and *divisor* problems. Among these results, on page 355 in his lost notebook, we encounter the following two important identities involving a finite trigonometric sum and a doubly infinite series of Bessel functions.

Entry 1.3.1. *If* $0 < \theta < 1$ *and* $x > 0$, *then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin(2\pi n\theta) = \pi x\left(\frac{1}{2}-\theta\right) - \frac{\cot(\pi\theta)}{4} + \frac{\sqrt{x}}{2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}} - \frac{J_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}}\right\}. \qquad (1.19)$$

Entry 1.3.2. *If* $0 < \theta < 1$ *and* $x > 0$, *then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2\pi n\theta) = \frac{1}{4} - x\log(2\sin(\pi\theta)) + \frac{\sqrt{x}}{2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{M_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}} + \frac{M_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}}\right\}, \qquad (1.20)$$

where

$$F(x) = \begin{cases} \lfloor x \rfloor, & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer;} \end{cases}$$

and $M_1(z)$ is defined in (1.9). These entries have three different interpretations. The double series in Entry 1.3.1 and Entry 1.3.2 can be interpreted as an iterated series in two possible ways. It can also be interpreted symmetrically, where the products of the indices tend to infinity. It is important to note that the identity (1.19) in Entry 1.3.1 with the order of the double sum interchanged was established by B. C. Berndt and A. Zaharescu in [16]. A few years later, B. C. Berndt, S. Kim and A. Zaharescu in [10] provided the proof of (1.19) as recorded by Ramanujan. On the other hand, B. C. Berndt, S. Kim and A. Zaherescu in [8] provided proof of (1.20) of Entry 1.3.2 with the order of the summation reversed and with additional conditions. Moreover, they gave the proof of (1.19) of Entry 1.3.1 and (1.20) of Entry 1.3.2 under the symmetric interpretation in [8]. Recently, in 2019, B. C. Berndt, J. Li and A. Zaharescu in [15] offered the proof of the identity (1.20) in Entry 1.3.2 as given by Ramanujan. As an application of Entry 1.3.1 in [16] with the order of the double sum reversed, B. C. Berndt and Zaharescu derived the following beautiful identity associated with $r_2(n)$,

$$\sum_{0\leq n\leq x}{}' r_2(n) = \pi x + 2\sqrt{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\frac{1}{4})x}\right)}{\sqrt{m(n+\frac{1}{4})}} - \frac{J_1\left(4\pi\sqrt{m(n+\frac{3}{4})x}\right)}{\sqrt{m(n+\frac{3}{4})}}\right\}. \tag{1.21}$$

The prime $'$ on the summation sign on the left-hand side of (1.21) implies that weight 1/2 is considered if x is an integer. Similar to Entry 1.3.1, Entry 1.3.2 is associated with the Dirichlet divisor problem. The occurrence of Bessel functions $M_1(z)$ in (1.11) indicates that there exists some correlation between Entry 1.3.2 and (1.10). Considering this fact, B. C. Berndt and A. Zaharescu derived an identity equivalent to Entry 1.3.2 in [16] by introducing twisted divisor sum $d_\chi(n)$ defined by

$$d_\chi(n) = \sum_{d|n}\chi(d), \tag{1.22}$$

where χ is a primitive Dirichlet character modulo q. Their identity reads as follows:

$$\sum_{n\le x}{}' d_\chi(n) = -\frac{x}{\tau(\bar\chi)}\sum_{h=1}^{q-1}\bar\chi(h)\log(2\sin(\pi h/q)) + \frac{\sqrt{q}}{\tau(\bar\chi)}\sum_{n=1}^{\infty} d_{\bar\chi}(n)\left(\frac{x}{n}\right)^{1/2} M_1(4\pi\sqrt{nx/q}), \tag{1.23}$$

where χ is a non-principal, even primitive character modulo q, $M_1(z)$ is defined in (1.9) and $\tau(\chi)$ is the Gauss sum of a Dirichlet character modulo q is defined by

$$\tau(\chi) = \sum_{h=1}^{q}\chi(h)e^{2\pi i h/q}. \tag{1.24}$$

Hence, (1.23) can be considered a character analogue of Entry 1.3.2. B. C. Berndt, S. Kim and A. Zaharescu [9, 13] generalised Ramanujan's entries by studying Riesz sums for twisted divisor sums.

In the following section, we analyse the identities associated with Bessel functions and divisor functions.

1.4 Identities involving divisor function and Bessel functions

We begin by reminiscing about another beautiful identity due to Ramanujan involving the K-Bessel function, which is recorded on page 253 of his lost notebook. If α and β are any two positive numbers such that $\alpha\beta = \pi^2$ and ν is any complex number, then

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)n^{\nu/2}K_{\nu/2}(2n\alpha) - \sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)n^{\nu/2}K_{\nu/2}(2n\beta) = \frac{1}{4}\Gamma\left(\frac{\nu}{2}\right)\zeta(\nu)\{\beta^{(1-\nu)/2}-\alpha^{(1-\nu)/2}\} + \frac{1}{4}\Gamma\left(-\frac{\nu}{2}\right)\zeta(-\nu)\{\beta^{(1+\nu)/2}-\alpha^{(1+\nu)/2}\}, \tag{1.25}$$

where $\sigma_\nu(n)$ is the general divisor function defined by

$$\sigma_\nu(n)=\sum_{d|n} d^\nu, \tag{1.26}$$

and $K_\nu(z)$ is defined in (1.4). Later in 1955, Guinand [30] derived a formula almost similar to (1.25) by appealing to a formula due to Watson [54] involving the K-Bessel function. One can use Ramanujan's formula (1.25) to derive Koshliakov's formula [36], given by

$$\begin{aligned}&\sqrt{\alpha}\left(\frac{1}{4}\gamma-\frac{1}{4}\log(4\beta)+\sum_{n=1}^{\infty}d(n)K_0(2n\alpha)\right)\\&=\sqrt{\beta}\left(\frac{1}{4}\gamma-\frac{1}{4}\log(4\alpha)+\sum_{n=1}^{\infty}d(n)K_0(2n\beta)\right),\end{aligned} \tag{1.27}$$

where γ denotes Euler's constant and $K_0(z)$ is defined in (1.5). Koshliakov, in 1929, proved the formula (1.27) by employing the Voronoi summation formula (1.12). In 1936, Ferrar [28] reproved (1.27) by appealing to the functional equation of $\zeta(s)$. Later, in 1966, K. Soni [48] showed that the functional equation of $\zeta^2(s)$ is equivalent to the Voronoi summation formula (1.12) and is equivalent to Koshliakov's formula (1.27). In 1972, Oberhettinger and Soni [44] established that the functional equation of $\zeta(s)$ and Koshliakov's formula are equivalent using the methods of Hamburger. In 2008, B. C. Berndt, Y. Lee, and J. Sohn [14] proved (1.25) by elaborating Guinand's method. They rediscovered Koshliakov's formula (1.27) by taking $\nu\to 0$ in (1.25). However, A. Dixit in [22] gave an extended version of Ramanujan's formula (1.25) by appealing to the Cauchy residue theorem and the theory of the Mellin transform. Further analysis of identities analogous to (1.25) and (1.27) have been done by B. C. Berndt, S. Kim and A. Zaharescu in [12]. They studied character analogues of Koshliakov's formula (1.27) for even characters. They replaced the classical divisor function $d(n)$ with the twisted divisor sums $d_\chi(n)$, which is defined in (1.22) and proved the following beautiful identity

$$\frac{qL(1,\chi)}{4\tau(\chi)}+\sum_{n=1}^{\infty}d_\chi(n)K_0\left(\frac{2\pi nz}{\sqrt{q}}\right)=\frac{\sqrt{q}L(1,\chi)}{4z}+\frac{\tau(\chi)}{z\sqrt{q}}\sum_{n=1}^{\infty}d_{\bar{\chi}}(n)K_0\left(\frac{2\pi n}{z\sqrt{q}}\right),$$

where χ is a non-principal even primitive character mod q, $\Re(z) > 0$, and $\tau(\chi)$ is the Gauss sum defined in (1.24), and $K_0(z)$ is defined in (1.5). In particular, for even real character χ, they established the positivity of $L(1, \chi)$, which is instrumental in proving Dirichlet's theorem on primes in arithmetic progressions. The weighted divisor sums defined in (1.22) were introduced by B. C. Berndt and A. Zaharescu [16], where they showed that the twisted or weighted divisor sums could be studied in connection with identities associated with $r_2(n)$. However, in 2017, S. Kim [35] extended the definition of twisted divisor sums to twisted sums of divisor functions, namely,

$$\sigma_{k,\chi}(n) := \sum_{d|n} d^k \chi(d), \ \bar{\sigma}_{k,\chi}(n) := \sum_{d|n} d^k \chi(n/d), \ \sigma_{k,\chi_1,\chi_2}(n) := \sum_{d|n} d^k \chi_1(d)\chi_2(n/d), \tag{1.28}$$

and studied Riesz sum-type identities associated with them. As a corollary of the main results, the author obtained a Riesz sum identity for $r_6(n)$ where $r_6(n)$ signifies the number of representations of n as a sum of six squares denoted by $r_6(n)$.

Recently, A. Dixit and A. Kesarwani [23] studied a new generalization of the modified Bessel function of the second kind. They derived a formula analogous to (1.25) associated with the generalized Bessel function. They proved that their formula is equivalent to the functional equation of a non-holomorphic Eisenstein series on $SL(2, \mathbb{Z})$. The study of the infinite series in (1.25) is of prime importance as it is intimately connected with the Fourier series expansion of non-holomorphic Eisenstein series on $SL(2, \mathbb{Z})$ or Maass wave forms [37, 39, 43, 51]. Motivated by this fact, Cohen, in 2010 [19], established the following result, similar to (1.25),

$$\begin{aligned} &4x^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\sigma_\nu(n)}{n^{\nu/2}} K_{\nu/2}(2\pi n x) \ + \ \Lambda(s)(x^{(1-\nu)/2} - x^{(\nu-1)/2}) \\ &= 4x^{-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\sigma_\nu(n)}{n^{\nu/2}} K_{\nu/2}(\frac{2\pi n}{x}) + \Lambda(-s)(x^{-(1+\nu)/2} - x^{(1+\nu)/2}), \end{aligned} \tag{1.29}$$

where $\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ and $K_\nu(z)$ is defined in (1.4). As an application, he obtained the following beautiful identity involving the divisor function $\sigma_s(n)$ and the modified K-Bessel function.

Proposition 1.4.1. *[19, p. 62, Theorem 3.4] For $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$ and any integer N such that $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, then*

$$8\pi x^{\frac{\nu}{2}} \sum_{n=1}^{\infty} \sigma_{-\nu}(n) n^{\nu/2} K_\nu(4\pi\sqrt{nx}) = -\frac{\Gamma(\nu)\zeta(\nu)}{(2\pi)^{\nu-1}} + \frac{\Gamma(1+\nu)\zeta(1+\nu)}{\pi^{\nu+1}2^\nu x} + \left\{ \frac{\zeta(\nu)x^{\nu-1}}{\sin\left(\frac{\pi\nu}{2}\right)} \right.$$
$$-\pi \frac{\zeta(\nu+1)x^\nu}{\cos(\frac{\pi\nu}{2})} + \frac{2}{\sin\left(\frac{\pi\nu}{2}\right)} \sum_{j=1}^{N} \zeta(2j)\ \zeta(2j-\nu) x^{2j-1}$$
$$\left. + \frac{2}{\sin\left(\frac{\pi\nu}{2}\right)} \sum_{n=1}^{\infty} \sigma_{-\nu}(n) \frac{x^{2N+1}}{(n^2-x^2)} \left(n^{\nu-2N} - x^{\nu-2N}\right) \right\}. \tag{1.30}$$

In addition to (1.30), he derived several interesting identities involving the divisor function $\sigma_s(n)$ and the modified K-Bessel function. Later, B. C. Berndt, A. Dixit, A. Roy, and A. Zaharescu [7], in their seminal work, showed that Cohen-type identity (1.30) can be used to derive the Voronoi-type summation formula for $\sigma_s(n)$.

Proposition 1.4.2. *[7, p. 841, Theorem 6.1] Let $0 < \alpha < \beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let f denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that $-\frac{1}{2} < \Re(\nu) < \frac{1}{2}$. Then*

$$\sum_{\alpha<j<\beta} \sigma_{-\nu}(j) f(j) = \int_\alpha^\beta f(t) \left(\zeta(1-\nu, \chi)\ t^{-\nu} + \zeta(\nu+1)\right) dt + 2\pi \sum_{n=1}^{\infty} \sigma_{-\nu}(n) n^{\nu/2}$$
$$\times \int_\alpha^\beta f(t)(t)^{-\frac{\nu}{2}} \left\{ \left(\frac{2}{\pi} K_\nu(4\pi\sqrt{nt}) - Y_\nu(4\pi\sqrt{nt}) \right) \cos\left(\frac{\pi\nu}{2}\right) \right.$$
$$\left. - J_\nu(4\pi\sqrt{nt}) \sin\left(\frac{\pi\nu}{2}\right) \right\} dt.$$

Inspired by Cohen's results [19], D. Banerjee and B. Maji [2] studied the infinite series involving the generalised divisor function and the modified K-Bessel functions. More precisely, they studied the following infinite series, for $r \in \mathbb{Z}, z \in \mathbb{C}$ and a and x be any two positive real numbers,

$$\sum_{n=1}^{\infty} \sigma_z^{(r)}(n) n^{\frac{\nu}{2}} K_\nu(a\sqrt{nx}), \tag{1.31}$$

where $\sigma_z^{(r)}(n) = \sum_{d^r|n} d^z$ and ν is a complex number with $\Re(\nu) \geq 0$. It is important to note that $\sigma_z^{(1)}(n) = \sigma_z(n)$. Hence, almost all the Cohen-type identities can be derived from their results. They provided the following identity as a direct consequence of their primary results:

Proposition 1.4.3. *Let a and x be two positive real numbers and $k \geq 1$ be an odd integer. For $\Re(\nu) > 0$, we have*

$$\left(\frac{a^2 x}{4}\right)^{\frac{\nu}{2}+k+1} \sum_{n=1}^{\infty} \sigma_k(n) n^{\frac{\nu}{2}} K_\nu(a\sqrt{nx})$$
$$= \frac{(-1)^{\frac{k+1}{2}}}{2} \Gamma(\nu+k+1)\,(2\pi)^{k+1} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{\left(\frac{16\pi^2}{a^2}\frac{n}{x}+1\right)^{\nu+k+1}} + Q_\nu(x),$$

where

$$Q_\nu(x) = -\frac{a^{2k+2}\Gamma(\nu)\zeta(-k)}{2^{2k+4}} x^{k+1} + \frac{a^{2k}\Gamma(1+\nu)\zeta(1-k)}{2^{2k+1}} x^k$$
$$+ \frac{1}{2}\Gamma(1+k+\nu)\Gamma(1+k)\zeta(1+k).$$

The above identity was also obtained by B. C. Berndt et al. in [6, equation (6.11)] as a particular case of their main result. Proposition 1.4.1, 1.4.2, and 1.4.3 can be regarded as identities corresponding to character modulo 1.

In this thesis, we are interested in the character analogues of (1.31). That is, we study the following infinite series

$$\sum_{n=1}^{\infty} \sigma_{z,\chi}(n) n^{\frac{\nu}{2}} K_\nu(a\sqrt{nx}), \qquad \sum_{n=1}^{\infty} \bar{\sigma}_{z,\chi}(n) n^{\frac{\nu}{2}} K_\nu(a\sqrt{nx}),$$
$$\sum_{n=1}^{\infty} \sigma_{z,\chi_1,\chi_2}(n) n^{\frac{\nu}{2}} K_\nu(a\sqrt{nx}), \tag{1.32}$$

where $\sigma_{z,\chi}(n)$, $\bar{\sigma}_{z,\chi}(n)$ and $\sigma_{z,\chi_1,\chi_2}(n)$ are defined in (1.28) for $z \in \mathbb{C}$, and ν is a complex number with $\Re(\nu) \geq 0$. The expression (1.7) ensures the absolute convergence of all the infinite series defined in (1.32). Afterwards, we focus on deriving its trigonometric analogues. More precisely, we offer the identities associated with the

K-Bessel function and the following weighted sums of divisor functions

$$\sum_{d|n} d^z \sin\left(2\pi d\theta\right), \quad \sum_{d|n} d^z \sin\left(\frac{2\pi n\theta}{d}\right), \quad \sum_{d|n} d^z \cos\left(2\pi d\theta\right), \quad \sum_{d|n} d^z \cos\left(\frac{2\pi n\theta}{d}\right),$$

etc. Moreover, we present formulas for the following two infinite series,

$$\sum_{n=1}^{\infty} r_6(n) n^{\nu/2} K_\nu(a\sqrt{nx}), \qquad \sum_{n=1}^{\infty} r_6(n) e^{-4\pi\sqrt{nx}}. \tag{1.33}$$

We also derive an identity from our two main results that gives rise to (1.33).

In Chapter 3, we study the character analogues for the case $z \in \mathbb{Z}_{\geq 0}$ and their equivalent versions in trigonometric forms. Equivalently, we study the infinite series (1.32) for $z = k$ and its trigonometric analogues. Next, we see that two of the series yield a generalisation of the identity associated with $r_6(n)$ and the Bessel function. In addition, we offer a novel representation for $L(1,\chi)$, allowing us to demonstrate the positivity of $L(1,\chi)$ for any real primitive character χ. It is worth noting that B. C. Berndt, S. Kim, and A. Zaharescu previously established the positivity of $L(1,\chi)$ for real primitive even characters in [12]. In this chapter, we provide a new proof that establishes the positivity of $L(1,\chi)$ for all real primitive Dirichlet characters χ, including both even and odd cases. This positivity is a crucial ingredient in the proof of Dirichlet's theorem on primes in arithmetic progressions.

In Chapter 4, we study Cohen-type identities associated with characters and their trigonometric analogues. Our main objective in this chapter is to extend identity in Proposition 1.4.1 to character modulo q.

In Chapter 5, we offer the Voronoi-type summation formula for twisted sums of divisor functions and its trigonometric analogues.

1.5 Conclusion and future work

In Chapter 6, we discuss some future research works related to the thesis.

2

Some Preliminaries

We begin this chapter by recalling and proving some important results that we will be using throughout the thesis.

2.1 Some basic results and definitions

We start this section by characterizing Dirichlet's characters.

Lemma 2.1.1. *[42, p. 118] If f is a multiplicative function, $f(n) = 0$ whenever $(n, q) > 1$, and f has period q, then f is a Dirichlet character modulo q.*

Next, we will see the definition of a primitive character.

Definition 2.1.1. *[50, Definition-8.4, p. 364] A Dirichlet character χ to modulus q is said to be primitive if there exists no character χ_1 to modulus $d < q$, such that $\chi(n) = \chi_1(n)$ for all integers n coprime to q.*

The Dirichlet L-function is defined by

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \Re(s) > 1, \tag{2.1}$$

where χ is a Dirichlet character modulo q. It can be meromorphically continued to the entire complex plane. Furthermore, if χ is principal, the corresponding Dirichlet L-function has a simple pole at $s = 1$. Otherwise, the L-function is entire.

The Hurwitz zeta function is defined by

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}, \ \Re(s) > 1 \text{ and } 0 < \alpha < 1. \tag{2.2}$$

It is well known that the Dirichlet L-function $L(s,\chi)$ for $\Re(s) > 1$ can be expressed in terms of the Hurwitz zeta function [20, p. 71, equation (16)]

$$L(s,\chi) = \frac{1}{q^s} \sum_{r=1}^{q-1} \zeta\left(s, \frac{r}{q}\right) \chi(r), \tag{2.3}$$

where χ is the Dirichlet character modulo q with $q > 2$. Conversely, we have

$$\zeta\left(s, \frac{h}{q}\right) = \frac{q^s}{\phi(q)} \sum_{\chi} \bar{\chi}(h) L(s,\chi), \tag{2.4}$$

for $(h,q) = 1$ and $0 < h < q$.

Next, we observe that the generating functions for $\sigma_{z,\chi}(n)$ and $\bar{\sigma}_{z,\chi}(n)$ and $\sigma_{z,\chi_1,\chi_2}(n)$ defined in (1.28) are the following

$$\zeta(s)L(s-z,\chi) = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{d=1}^{\infty} \frac{d^z \chi(d)}{d^s} = \sum_{n=1}^{\infty} \frac{\sigma_{z,\chi}(n)}{n^s}, \tag{2.5}$$

$$\zeta(s-z)L(s,\chi) = \sum_{m=1}^{\infty} \frac{1}{m^{s-z}} \sum_{d=1}^{\infty} \frac{\chi(d)}{d^s} = \sum_{n=1}^{\infty} \frac{\bar{\sigma}_{z,\chi}(n)}{n^s}, \tag{2.6}$$

$$L(s-z,\chi_1)L(s,\chi_2) = \sum_{d=1}^{\infty} \frac{d^z \chi_1(d)}{d^s} \sum_{m=1}^{\infty} \frac{\chi_2(m)}{m^s} = \sum_{n=1}^{\infty} \frac{\sigma_{z,\chi_1,\chi_2}(n)}{n^s}, \tag{2.7}$$

for $\Re(s) > \max(\Re(z)+1, 1)$, where $\zeta(s)$ denotes the the Riemann zeta function and

$L(s,\chi)$ denotes the Dirichlet L-function defined by (2.1) for $\Re(s) > 1$. We recall that the functional equation of $\zeta(s)$ [50, p. 234]

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s). \tag{2.8}$$

Replacing s by $1-s$ in (2.8), we obtain

$$\Gamma(s)\zeta(s) = \frac{\pi^s\zeta(1-s)}{2^{1-s}\cos\left(\frac{\pi s}{2}\right)}. \tag{2.9}$$

Next, we write the functional equation for $L(s,\chi)$ [20, p. 71]

$$L(s,\chi) = \frac{\tau(\chi)}{i^\kappa\sqrt{q}}\left(\frac{\pi}{q}\right)^{s-1/2}\frac{\Gamma(\frac{1-s+\kappa}{2})}{\Gamma(\frac{s+\kappa}{2})}L(1-s,\bar{\chi}), \tag{2.10}$$

where

$$\kappa = \kappa(\chi) = \begin{cases} 0, & \text{if } \ \chi(-1) = 1, \\ 1, & \text{if } \ \chi(-1) = -1. \end{cases}$$

The functional relations for $\Gamma(s)$ are given by [20, p. 73]

$$\Gamma(s+1) = s\Gamma(s), \qquad \Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\Gamma(2s), \tag{2.11}$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}. \tag{2.12}$$

Employing (2.11) and (2.12) in (2.10), we obtain [42, Corollary 10.9, p. 333]

$$L(s,\chi) = i^{-\kappa}\frac{\tau(\chi)}{\pi}\left(\frac{(2\pi)}{q}\right)^s\Gamma(1-s)\sin\frac{\pi(s+\kappa)}{2}L(1-s,\bar{\chi}). \tag{2.13}$$

Now replacing s by $s-z$ in (2.13), we get

$$L(s-z,\chi) = i^{-\kappa}\frac{\tau(\chi)}{\pi}\left(\frac{(2\pi)}{q}\right)^{s-z}\Gamma(1+z-s)\sin\frac{\pi(s+\kappa-z)}{2}L(1+z-s,\bar{\chi}).$$

So, we can rewrite the above equation as

$$\Gamma(1+z-s)L(1+z-s,\bar{\chi}) = i^{\kappa}\frac{\pi}{\tau(\chi)}\left(\frac{q}{2\pi}\right)^{s-z}\frac{L(s-z,\chi)}{\sin\pi(\frac{s+\kappa-z}{2})}. \tag{2.14}$$

Next, we recall the bounds for the L-function [20, p. 82, eq 14] for $\sigma \geq 1/2$,

$$|L(s,\chi)| \leq 2q|s|. \tag{2.15}$$

The Hurwitz zeta function satisfies the following functional equations [17, p. 587]

$$\begin{aligned}&\sum_{r=1}^{q}\zeta\left(s,\frac{r}{q}\right)\cos\left(\frac{2\pi rh}{q}\right)\\ &= \frac{q\Gamma(1-s)}{(2\pi q)^{1-s}}\sin\left(\frac{\pi s}{2}\right)\left\{\zeta\left(1-s,\frac{h}{q}\right)+\zeta\left(1-s,1-\frac{h}{q}\right)\right\}, \quad (2.16)\\ &\sum_{r=1}^{q}\zeta\left(s,\frac{r}{q}\right)\sin\left(\frac{2\pi rh}{q}\right)\\ &= \frac{q\Gamma(1-s)}{(2\pi q)^{1-s}}\cos\left(\frac{\pi s}{2}\right)\left\{\zeta\left(1-s,\frac{h}{q}\right)-\zeta\left(1-s,1-\frac{h}{q}\right)\right\}. \quad (2.17)\end{aligned}$$

Hurwitz zeta function also satisfies [1, p. 264]

$$\zeta(-n,\theta) = -\frac{B_{n+1}(\theta)}{n+1}, \tag{2.18}$$

for each $n \geq 0$; where $B_n(\theta)$ is a Bernoulli polynomial defined as follows [1, p. 264]

$$\frac{ze^{\theta z}}{e^z-1} = \sum_{n=0}^{\infty}\frac{B_n(\theta)}{n!}z^n, \text{ for } |z|<2\pi, \tag{2.19}$$

for any $\theta \in \mathbb{C}$ and we have the relation [1, p. 274]

$$B_n(1-\theta) = (-1)^n B_n(\theta) \quad \text{for every } n \geq 0. \tag{2.20}$$

We will also note that [20, p.69, p.71]

$$\tau(\chi)\tau(\bar{\chi}) = \begin{cases} -q, \text{ for odd primitive } \chi \; mod \; q, \\ \;\; q, \;\; \text{for even non principal primitive } \chi \; mod \; q, \end{cases} \tag{2.21}$$

where $\tau(\chi)$ is defined in (1.24). Now we see the orthogonality relations for characters

$$\sum_{\substack{\chi \; mod \; q \\ \chi \; odd}} \chi(a)\bar{\chi}(h) = \begin{cases} \pm\frac{\phi(q)}{2}, & \text{if } h \equiv \pm a \; (mod \; q) \\ 0, & \text{otherwise ;} \end{cases} \tag{2.22}$$

$$\sum_{\substack{\chi \; mod \; q \\ \chi \; even}} \chi(a)\bar{\chi}(h) = \begin{cases} \frac{\phi(q)}{2}, & \text{if } h \equiv \pm a \; (mod \; q) \\ 0, & \text{otherwise.} \end{cases} \tag{2.23}$$

Here we would like to mention another identity [11, Lemma 2.5] namely

$$\sin\left(\frac{2\pi hd}{q}\right) = \frac{1}{i\phi(q)} \sum_{\substack{\chi \; mod \; q \\ \chi \; odd}} \chi(d)\tau(\bar{\chi})\chi(h), \tag{2.24}$$

$$\cos\left(\frac{2\pi hd}{q}\right) = \frac{1}{\phi(q)} \sum_{\substack{\chi \; mod \; q \\ \chi \; even}} \chi(d)\tau(\bar{\chi})\chi(h), \tag{2.25}$$

whenever $(d, q) = (h, q) = 1$. The factorization theorem for Gauss sum $\tau(\chi)$ in (1.24) is as follows [20, p. 65]

$$\chi(n)\tau(\bar{\chi}) = \sum_{h=1}^{q-1} \bar{\chi}(h)e^{2\pi inh/q}, \tag{2.26}$$

for any character modulo χ modulo q.

2.2 Key Lemmas

The Mellin transform of a locally integrable function $f(x)$ on $(0,\infty)$ is defined by

$$\mathcal{M}[f;s] = F(s) = \int_0^\infty f(t)\ t^{s-1}dt, \tag{2.27}$$

provided the integral converges. The basic properties of the Mellin transform follow immediately from those of the Laplace transform since these transforms are intimately connected. The integral in (2.27) defines the Mellin transform in a vertical strip in the s plane whose boundaries are determined by the analytic structure of $f(x)$ as $x \to 0+$ and $x \to +\infty$. If we assume that $f(x)$ satisfies the following growth condition

$$f(x) = \begin{cases} O(x^{-a+\varepsilon}) & \text{as } x \to 0+, \\ O(x^{-b-\varepsilon}) & \text{as } x \to +\infty, \end{cases} \tag{2.28}$$

where $\varepsilon > 0$ and $a < b$, then the integral (2.27) converges absolutely in the strip $a < \Re(s) < b$ and defines an analytic function there in the strip. This strip is known as the strip of analyticity of $\mathcal{M}[f;s]$. Furthermore, the inversion formula for (2.27) follows directly from the corresponding inversion formula for the bilateral Laplace transform. Thus,

$$f(x) = \frac{1}{2\pi i}\int_{(c)} x^{-s}\mathcal{M}[f;s]ds \qquad (a < c < b), \tag{2.29}$$

which is valid at all points $x \geq 0$ where $f(x)$ is continuous. Here the notation (c) denotes the vertical line $[c-i\infty, c+i\infty]$. For example, $\mathcal{M}[e^x;s] = \Gamma(s)$ for $\Re(s) > 0$, and we have the corresponding Mellin's inversion formula

$$e^{-y} = \frac{1}{2\pi i}\int_{(c)} \Gamma(s)y^{-s}\mathrm{d}s,$$

valid for $\Re(y) > 0$. The following lemma states the asymptotic behaviour of $\Gamma(s)$.

Lemma 2.2.1. *[46, p. 38] In a vertical strip, for s=$\sigma + it$ with $a \leq \sigma \leq b$ and $|t| \geq 1$,*

$$|\Gamma(s)| = (2\pi)^{\frac{1}{2}} |t|^{\sigma - \frac{1}{2}} \exp^{-\frac{1}{2}\pi|t|} \left(1 + O\left(\frac{1}{|t|}\right)\right).$$

In our investigation, we shall require the following results related to the Mellin transform of derivatives of a function.

Lemma 2.2.2. *Let $n \in \mathbb{N}$. Assume that ϕ is n-times differentiable function and*

$$\mathcal{M}[\phi(t); s] = \int_0^\infty \phi(t) t^{s-1} dt = \Phi(s). \tag{2.30}$$

If ϕ satisfies (2.28)*, then*

$$\mathcal{M}[\phi^{(n)}(t)t^n; s] = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} \Phi(s), \tag{2.31}$$

where $s \in \{w \in \mathbb{C};\ a < \Re(w) < b\}$, provided

$$\lim_{t \to 0,\infty} t^{s+n-j-1} \phi^{(n-j-1)}(t) = 0 \quad j = 0, 1, \cdots, n-1. \tag{2.32}$$

Proof. The proof relies on mathematical induction. Using integration by parts, we have

$$\mathcal{M}[x\phi'(x); s] = \int_0^\infty \phi'(t)\ t^s dt = [t^s \phi(t)]_0^\infty - s \int_0^\infty \phi(t)\ t^{s-1} dt.$$

Noting $\phi(t)$ satisfies (2.28), we can claim that

$$\mathcal{M}[x\phi'(x); s] = -s\Phi(s) \ \text{ for } a < \Re(s) < b.$$

Suppose the statement of the theorem is true for $n = N$ and ϕ is $N+1$-times differentiable function and satisfies (2.32). Then

$$\begin{aligned} \mathcal{M}[x^{N+1}\phi^{(N+1)}(x); s] &= \int_0^\infty t^{N+1}\phi^{(N+1)}(t)\ t^{s-1} dt \\ &= \left[t^{s+N}\phi^{(N)}(t)\right]_0^\infty - (s+N) \int_0^\infty t^N \phi^{(N)}(t)\ t^{s-1} dt. \end{aligned}$$

As ϕ satisfies (2.32), we have

$$\mathcal{M}[\phi^{(N+1)}(t)t^{N+1}; s] = -(s+N)\int_0^\infty t^N\phi^{(N)}(t)\ t^{s-1}dt = (-1)^{N+1}\frac{\Gamma(s+N+1)}{\Gamma(s)}\Phi(s),$$

and this completes the proof. □

Lemma 2.2.3. *[45, p. 91, Formula (3.3.9)] We have*

$$\mathcal{M}[(1+x)^{-a}; s] = \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)},$$

for $0 < \Re(s) < \Re(a)$.

As an immediate consequence of Lemma 2.2.3 we get,

Lemma 2.2.4. *For any* $n \in \mathbb{N}$,

$$\mathcal{M}\left[\frac{a(a+1)\cdots(a+n-1)x^n}{(1+x)^{a+n}}; s\right] = \frac{\Gamma(s+n)\Gamma(a-s)}{\Gamma(a)},$$

whenever $0 < \Re(s) < \Re(a)$.

Proof. By Lemma 2.2.3, we can write

$$\mathcal{M}[(1+x)^{-a}; s] = \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)},$$

for $0 < \Re(s) < \Re(a)$. The function $\phi(x) = \frac{1}{(1+x)^a}$ for $x \geq 0$ is a continuous function and satisfies all the conditions of Lemma 2.2.2. Furthermore,

$$\phi^{(n)}(x) = (-1)^n\frac{a(a+1)\cdots(a+n-1)}{(1+x)^{(a+n)}}.$$

We have

$$\Phi(s) = \frac{\Gamma(a-s)\Gamma(s)}{\Gamma(a)} \quad \text{for } 0 < \Re(s) < \Re(a).$$

Hence by Lemma 2.2.2,

$$\mathcal{M}\left[\frac{a(a+1)\cdots(a+n-1)x^n}{(1+x)^{(a+n)}}; s\right] = \frac{\Gamma(s+n)}{\Gamma(s)}\Phi(s) = \frac{\Gamma(s+n)\Gamma(a-s)}{\Gamma(a)},$$

for $0 < \Re(s) < \Re(a)$. □

Lemma 2.2.5. *Let $n \geq 0$ be any integer and $t > 0$ be any real number. Then*

$$\frac{1}{2\pi i}\int_{(c)} \Gamma(s+n)\Gamma(a-s)t^{-s}\mathrm{d}s = \frac{\Gamma(a+n)}{(1+t)^{a+n}}t^n,$$

for $0 < c < \Re(a)$.

Proof. We get our desired result by combining Lemmas 2.2.3 and 2.2.4 and applying Mellin's inversion formula. □

Lemma 2.2.6. *[27, p. 346, Formula (20)] We have*

$$\mathcal{M}\left[\frac{\log t}{t-1}; s\right] = \frac{\pi^2}{\sin^2(\pi s)},$$

for $0 < \Re(s) < 1$. The integral is convergent in the sense of Cauchy's principal value.

Lemma 2.2.7. *We have*

$$\mathcal{M}\left[\frac{4\log\ x}{x^2-1}\ ; s\right] = \frac{\pi^2}{\sin^2\left(\frac{\pi s}{2}\right)}, \tag{2.33}$$

for $0 < \Re(s) < 2$. The integral is convergent in the sense of Cauchy's principal value.

Proof. This is a direct consequence of Lemma 2.2.6. □

Now, we record a result related to the modified K-Bessel function $K_\nu(x)$ defined by (1.4).

Lemma 2.2.8. *[2, p. 10, Lemma 3.3] Let $\nu \in \mathbf{C}$. For any $c > \max\{0, -\Re(\nu)\}$, we have*

$$t^{\frac{\nu}{2}}K_\nu(a\sqrt{tx}) = \frac{1}{2}\left(\frac{2}{a\sqrt{x}}\right)^\nu \frac{1}{2\pi i}\int_{(c)} \Gamma(s)\Gamma(s+\nu)\left(\frac{4}{a^2x}\right)^s t^{-s}ds.$$

Next, we study the infinite sum which involves the rising factorial.

Lemma 2.2.9. *For* $0 < \frac{W}{n} < 1$*, we define*

$$A_k := \sum_{m=1}^{\infty} m(m+1)...(m+k-1)\left(-\frac{W}{n}\right)^m.$$

Then we have for $k \geq 1$*,*

$$A_k = -k!\,\frac{W}{n}\left(\frac{n}{n+W}\right)^{k+1}.$$

Proof. We prove this result by induction on k. For $k = 1$, it is easy to see that

$$A_1 := \sum_{m=1}^{\infty} m\left(-\frac{W}{n}\right)^m = -\frac{W}{n}\left(\frac{n}{n+W}\right)^2.$$

Let the result hold for $k-1$. Now prove for k. Consider

$$B_M := \sum_{m=1}^{M} m(m+1)...(m+k-1)\left(-\frac{W}{n}\right)^m. \tag{2.34}$$

Now multiply B_M by $\left(-\frac{W}{n}\right)$, we obtain

$$\begin{aligned}\left(-\frac{W}{n}\right)B_M &= \sum_{m=1}^{M} m(m+1)...(m+k-1)\left(-\frac{W}{n}\right)^{m+1}\\ &= \sum_{s=2}^{M+1}(s-1)s\cdots(s+k-2)\left(-\frac{W}{n}\right)^s. \end{aligned} \tag{2.35}$$

Now subtracting (2.35) from (2.34) gives

$$\begin{aligned}\left(1+\frac{W}{n}\right)B_M =& k\sum_{m=1}^{M} m(m+1)...(m+k-2)\left(-\frac{W}{n}\right)^m\\ &+ M(M+1)...(M+k-1)\left(-\frac{W}{n}\right)^{M+1}. \end{aligned} \tag{2.36}$$

Taking $M \to \infty$ in (2.36), we obtain

$$\left(1+\frac{W}{n}\right)A_k = kA_{k-1}. \tag{2.37}$$

Substituting the expression for A_{k-1} from the induction hypothesis, we obtain the desired result. □

Corollary 2.2.1. *Let* $0 < \frac{W}{n} < 1$ *be a positive real number. Then we have for* $k \geq 1$,

$$\sum_{m=2}^{M} m(m+1)...(m+k-1)\left(-\frac{W}{n}\right)^{m} = k!\,\frac{W}{n}\left(1-\left(\frac{n}{n+W}\right)^{k+1}\right),$$

and

$$\sum_{m=2}^{M}\left(-\frac{W}{n}\right)^{m} = \left(-\frac{W}{n}\right)^{2}\frac{1}{1+\frac{W}{n}} = \frac{W}{n}\left(1-\left(\frac{n}{n+W}\right)\right).$$

Proof. The first assertion follows directly from Lemma 2.2.9, while the second follows from the standard result for the geometric series. □

3

Character analogues for the case $z \in \mathbb{Z}_{\geq 0}$ and their equivalent versions in trigonometric forms

3.1 Introduction

Let us begin this section by introducing the Dirichlet Theorem in Arithmetic progression. Let c, q be fixed integers such that $(c, q) = 1$, then there are infinitely many primes $p \equiv c(mod\, q)$. In reality, Dirichlet proved a more specific result that the primes are evenly distributed over $\phi(q)$ residue classes modulo q. He demonstrated that

$$\sum_{\substack{p \leq x \\ p \equiv c(mod \ q)}} \frac{1}{p} \sim \frac{\ln_2(x)}{\phi(q)} \tag{3.1}$$

where $\phi(q)$ denotes the Euler totient function, and $\ln_2(x)$ denotes two-fold iterated logarithm, i.e., $\ln_2(x) = \ln(\ln x)$. The estimate (3.1) is derived from the assumption that $L(1, \chi) \neq 0$ for a non-principal Dirichlet character χ modulo q. Our primary goal in this chapter is to study the character analogues of (1.31). In particular, we extend the identity of D. Banerjee and B. Maji as mentioned in Proposition 1.4.3. It is worth mentioning that as a consequence of our main results, we prove that $L(1, \chi) \neq 0$ for a real primitive character χ modulo q.

This chapter is organized as follows: In our next section, we provide the main results and proofs. We divide Section 3.2 into four subsections. In Section 3.2.1, we derive identities involving odd characters and their equivalent versions in the sine function, which is analogous to Ramanujan's Entry 1.3.1. As an application of Section 3.2.1, we offer the results corresponding to $r_6(n)$ in Section 3.2.2. Furthermore, Section 3.2.3 discusses the identities involving even primitive characters and their equivalent versions in the cosine function, which is analogous to Ramanujan's Entry 1.3.2. From two of the main results from Sections 3.2.1 and 3.2.3, we derive the non-vanishing of $L(1, \chi)$ in Section 3.2.4. Furthermore, identities involving two characters and their equivalent versions in two trigonometric functions are covered in Section 3.2.5.

3.2 Main Results and Proofs

In this subsection, we study the identities involving the arithmetic function $\sigma_{k,\chi}(n)$, $\bar{\sigma}_{k,\chi}(n)$, $\sigma_{k,\chi_1,\chi_2}(n)$ defined by (1.28) and the modified K-Bessel function.

3.2.1 Identities involving odd characters

In this subsection, we will consider k to be an even, non-negative integer and χ an odd primitive character.

Theorem 3.2.1. *Let k be an even, non-negative integer and χ be an odd primitive Dirichlet character modulo q. Then, for any $\Re(\nu) > 0$,*

$$\sum_{n=1}^{\infty} \sigma_{k,\chi}(n) n^{\frac{\nu}{2}} K_\nu(a\sqrt{nx}) = \delta_k \frac{2^{\nu+1}}{a^{\nu+2}} \Gamma(1+\nu) L(1,\chi) x^{-\frac{\nu}{2}-1}$$

$$+\frac{(-1)^{\frac{k}{2}}iq^k}{a^\nu 2^{k+2-\nu}\pi^{k+1}}\Gamma(\nu)\tau(\chi)\Gamma(k+1)L(k+1,\bar{\chi})\ x^{-\frac{\nu}{2}}$$
$$-\frac{(-1)^{\frac{k}{2}}ia^\nu q^{\nu+k}\ x^{\frac{\nu}{2}}}{2^{3\nu+k+2}\pi^{2\nu+k+1}}\Gamma(\nu+k+1)\tau(\chi)\sum_{n=1}^{\infty}\frac{\bar{\sigma}_{k,\bar{\chi}}(n)}{\left(n+\frac{a^2qx}{16\pi^2}\right)^{\nu+k+1}}, \tag{3.2}$$

where δ_k *is given by*

$$\delta_k=\begin{cases}1, & \text{if } k=0,\\ 0, & \text{else }.\end{cases} \tag{3.3}$$

We remark that Theorem 3.2.1 is equivalent to the following result.

Theorem 3.2.2. *Let* $k\geq 0$ *be an even integer. Then for any* $\Re(\nu)>0$ *we have*

$$\left(\frac{a^2x}{4}\right)^{\frac{\nu}{2}+k+1}\sum_{n=1}^{\infty}n^{\nu/2}K_\nu(a\sqrt{nx})\sum_{d|n}d^k\sin(2\pi d\theta)$$
$$=-\frac{(-1)^{\frac{k}{2}}a^{2k+2}k!}{2^{2k+4}(2\pi)^{k+1}}\Gamma(\nu)\left(\zeta(1+k,\theta)-\zeta(1+k,1-\theta)\right)x^{k+1}$$
$$+\delta_k\frac{\pi\Gamma(1+\nu)}{4}\left(\zeta(0,\theta)-\zeta(0,1-\theta)\right)$$
$$+\frac{(-1)^{\frac{k}{2}}}{4}(2\pi)^{k+1}\sum_{d=1}^{\infty}d^k\sum_{m=0}^{\infty}\left\{\frac{\Gamma(\nu+k+1)}{(1+\frac{16\pi^2d}{a^2x}(m+\theta))^{1+\nu+k}}\right.$$
$$\left.-\frac{\Gamma(\nu+k+1)}{(1+\frac{16\pi^2d}{a^2x}(m+1-\theta))^{1+\nu+k}}\right\}, \tag{3.4}$$

where δ_k *is defined in* (3.3).

Remark. It should be emphasized that all infinite series mentioned in this chapter are absolutely convergent. In particular, we demonstrate the absolute convergence of the doubly infinite series present on the right-hand side of (3.4). For this, we consider

$$\sum_{d=1}^{\infty}d^k\sum_{m=0}^{\infty}\left\{\frac{1}{(1+\frac{16\pi^2d}{a^2x}(m+\theta))^{1+\nu+k}}-\frac{1}{(1+\frac{16\pi^2d}{a^2x}(m+1-\theta))^{1+\nu+k}}\right\}$$
$$\leq\sum_{d=1}^{\infty}\sum_{m=0}^{\infty}\frac{d^k}{(1+\frac{16\pi^2d}{a^2x}(m+\theta))^{1+\nu+k}}+\sum_{d=1}^{\infty}\sum_{m=0}^{\infty}\frac{d^k}{(1+\frac{16\pi^2d}{a^2x}(m+1-\theta))^{1+\nu+k}}$$

$$\ll x^{1+\nu+k} \sum_{d=1}^{\infty} \frac{1}{d^{1+\nu}} \sum_{m=0}^{\infty} \frac{1}{m^{1+\nu+k}} \ll \infty.$$

Before proving these theorems, let us first consider a more general setup. Let χ be any Dirichlet character modulo q and $z \in \mathbb{C}$. Let $f_z(n)$ be one of the arithmetical functions $\sigma_{z,\chi}(n)$ or $\bar{\sigma}_{z,\chi}(n)$ or $\sigma_{z,\chi_1,\chi_2}(n)$ defined in (1.28). We denote

$$F_z(s) := \sum_{n=1}^{\infty} \frac{f_z(n)}{n^s}, \quad \Re(s) > 1. \tag{3.5}$$

Hence $F_z(s)$ is one of the Dirichlet series given in (2.5) or (2.6) or (2.7). As mentioned in the previous section, we will consider $\Re(\nu) > 0$ and $\nu = 0$. Employing Lemma 2.2.8 with $t = n$ and subsequently interchanging the summation and integration, we get

$$\begin{aligned}\sum_{n=1}^{\infty} f_z(n) n^{\nu/2} K_\nu(a\sqrt{nx}) &= \frac{1}{2}\left(\frac{2}{a\sqrt{x}}\right)^{\nu} \frac{1}{2\pi i} \int_{(c)} \Gamma(s)\Gamma(s+\nu) \left(\frac{4}{a^2 x}\right)^{s} \sum_{n=1}^{\infty} f_z(n) n^{-s} \mathrm{d}s \\ &= \frac{1}{2} X^{\nu/2} \frac{1}{2\pi i} \int_{(c)} \Gamma(s)\Gamma(s+\nu) F_z(s) X^s \mathrm{d}s, \end{aligned} \tag{3.6}$$

where $c > \Re(z) + 1$ and $X = \frac{4}{a^2 x}$. Next, we investigate the following integral

$$I_z^{(\nu)}(X) := \frac{1}{2\pi i} \int_{(c)} \Gamma(s+\nu)\Gamma(s) F_z(s) X^s \mathrm{d}s. \tag{3.7}$$

We shall use the Cauchy residue theorem to evaluate this line integral in (3.7). Let us consider the positively oriented rectangular contour $\mathcal{C}$: consisting of the line segments $[c - iT, c + iT], [c + iT, -d + iT], [-d + iT, -d - iT]$ and $[-d - iT, c - iT]$ where the choice for d is as follows: $0 < d < \min\{1, \Re(\nu)\}$ whenever $\Re(\nu) > 0$ and $0 < d < 1$ otherwise. Here, T is taken to be a large positive number. The possible poles of the integrand function in (3.7) are at $s = 0, 1$ and $z + 1$. Now letting $T \to \infty$ and invoking Lemma 2.2.1, one can show that the integrals along the horizontal segments $[c + iT, -d + iT]$ and $[-d - iT, c - iT]$ vanish and get

$$I_z^{(\nu)}(X) = R_{z+1} + R_1 + R_0 + \frac{1}{2\pi i} \int_{(-d)} \Gamma(s+\nu)\Gamma(s) F_z(s) X^s \mathrm{d}s, \tag{3.8}$$

where R_{z+1}, R_1 and R_0 are the residues at $s = z+1, 1$ and $s = 0$, respectively. It is easy to see that $R_{z+1} = 0$ whenever $z = 0$. Hence combining (3.6) and (3.7) together with (3.8), we obtain

$$\sum_{n=1}^{\infty} f_z(n) n^{\nu/2} K_\nu(a\sqrt{nx}) = \frac{1}{2} X^{\nu/2} \left(R_{z+1} + R_1 + R_0 + J_z^{(\nu)}(X) \right), \tag{3.9}$$

where $J_z^{(\nu)}(X)$ is defined by

$$J_z^{(\nu)}(X) := \frac{1}{2\pi i} \int_{(-d)} \Gamma(s+\nu)\Gamma(s) F_z(s) X^s \mathrm{d}s. \tag{3.10}$$

Next, we will offer the proofs of the theorems corresponding to $z = k$, where k is a non-negative integer.

Proof of Theorem 3.2.1 and its equivalence with Theorem 3.2.2. We begin by looking at the proof of each theorem individually. Following that, we will demonstrate its equivalence.

Proof of Theorem 3.2.1 Letting $f_k(n) = \sigma_{k,\chi}(n)$ where χ being an odd primitive character modulo q and k an even, non-negative integer in (3.9), we obtain

$$\sum_{n=1}^{\infty} \sigma_{k,\chi}(n) n^{\nu/2} K_\nu(a\sqrt{nx}) = \frac{1}{2} X^{\nu/2} \left(R_{k+1} + R_1 + R_0 + J_z^{(\nu)}(X) \right), \tag{3.11}$$

where $\Re(\nu) > 0$ and $J_k^{(\nu)}(X)$ is defined in (3.10) with $F_k(s) = \zeta(s)L(s-k,\chi)$. It is easy to see that $R_{k+1} = 0$ as the integrand function in (3.10) does not have any pole at $s = k+1$. Here, one can notice that $L(s-k,\chi)$ has a zero at $s = 1$ when $k \geq 2$ is an even integer and χ is odd. Therefore, we will not get any contribution from the pole of $\zeta(s)$ at $s = 1$. However, if $k = 0$, the integrand in (3.10) will encounter a pole at $s = 1$. Therefore, we can get

$$R_1 = \begin{cases} 0, & \text{if } k > 0, \\ \Gamma(1+\nu)L(1,\chi)X, & \text{if } k = 0. \end{cases} \tag{3.12}$$

The integrand also has a pole at $s = 0$ with residue R_0 given by

$$R_0 = -\frac{\Gamma(\nu)L(-k,\chi)}{2} = \frac{(-1)^{\frac{k}{2}} i\tau(\chi)}{2\pi}\left(\frac{(2\pi)}{q}\right)^{-k}\Gamma(1+k)\Gamma(\nu)L(1+k,\bar{\chi}), \quad (3.13)$$

where in the last step, we have applied the functional equation (2.13). Collecting (3.12) and (3.13) and $R_{k+1} = 0$ and then substituting them in (3.11), we get

$$X^{-\frac{\nu}{2}}\sum_{n=1}^{\infty}\sigma_{k,\chi}(n)n^{\frac{\nu}{2}}K_\nu(a\sqrt{nx}) = \frac{(-1)^{\frac{k}{2}} i\tau(\chi)}{4\pi}\left(\frac{(2\pi)}{q}\right)^{-k}\Gamma(1+k)\Gamma(\nu)L(1+k,\bar{\chi}) + \delta_k\frac{\Gamma(1+\nu)L(1,\chi)}{2}X + \frac{1}{2}J_k^{(\nu)}(X), \quad (3.14)$$

where δ_k is defined in (3.3). To evaluate $J_k^{(\nu)}(X)$ defined in (3.10), we invoke the functional equations (2.9) and (2.13) assuming that χ is odd and k is even. We obtain

$$\begin{aligned}
J_k^{(\nu)}(X) &= \frac{h_k}{2\pi i}\int_{(-d)}\Gamma(s+\nu)\Gamma(1+k-s)\zeta(1-s)L(1-s+k,\bar{\chi})Y^s\mathrm{d}s \\
&= \frac{Yh_k}{2\pi i}\int_{(1+d)}\Gamma(1-s+\nu)\Gamma(k+s)\zeta(s)L(s+k,\bar{\chi})Y^{-s}\mathrm{d}s \\
&= Yh_k\sum_{n=1}^{\infty}\sigma_{-k,\bar{\chi}}(n)\ \frac{1}{2\pi i}\int_{(1+d)}\Gamma(1-s+\nu)\Gamma(k+s)(nY)^{-s}\mathrm{d}s,
\end{aligned}$$

where $h_k = \frac{(-1)^{1+\frac{k}{2}} i\tau(\chi)}{2\pi}\left(\frac{q}{2\pi}\right)^k$ and $Y = \frac{4\pi^2}{q}X$ with $X = \frac{4}{a^2x}$. As $0 < d < \Re(\nu)$, we can apply Lemma 2.2.5 with $n = k$ and $a = 1+\nu$ to obtain

$$\begin{aligned}
J_k^{(\nu)}(X) &= Y^{k+1}\Gamma(1+\nu)h_k\sum_{n=1}^{\infty}\sigma_{-k,\bar{\chi}}(n)\frac{(\nu+1)\cdots(\nu+k)n^k}{(1+nY)^{1+\nu+k}} \\
&= Y^{k+1}\Gamma(1+\nu+k)h_k\sum_{n=1}^{\infty}\frac{\bar{\sigma}_{k,\chi}(n)}{(1+nY)^{1+\nu+k}}, \quad (3.15)
\end{aligned}$$

where in the penultimate step we have used the fact $n^k\sigma_{-k,\chi}(n) = \bar{\sigma}_{k,\chi}(n)$. Therefore, remarking $Y = \frac{16\pi^2}{a^2qx}$ and inserting (3.15) in (3.14) and simplifying, we get (3.2).

Proof of Theorem 3.2.2 First, we again assume that $\Re(\nu) > 0$. The double series on the right-hand side of the identity (3.4) converges absolutely and uniformly on any compact interval for $\theta \in (0,1)$. Since the summands in the right-hand side of (3.4) are continuous functions, the series converges to a continuous function of θ. Therefore, it is sufficient to prove the identity (3.4) for $\theta = h/q$, where q is prime and $0 < h < q$. If the identity holds on the dense subset of fractions, then by continuity, the identity holds for all values of θ. Employing Lemma 2.2.8 with $t = n$ and subsequently interchanging the summation and integration, we get for an even integer $k \geq 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{d|n} d^k \sin\left(\frac{2\pi dh}{q}\right) n^{\nu/2}K_\nu(a\sqrt{nx})\\
&= \frac{1}{2}\left(\frac{2}{a\sqrt{x}}\right)^{\nu}\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\Gamma(s+\nu)\left(\frac{4}{a^2x}\right)^{s}\sum_{n=1}^{\infty}\sum_{d|n} d^k \sin\left(\frac{2\pi dh}{q}\right) n^{-s}\mathrm{d}s\\
&= \frac{1}{2}\left(\frac{2}{a\sqrt{x}}\right)^{\nu}\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\Gamma(s+\nu)\left(\frac{4}{a^2x}\right)^{s}\sum_{m=1}^{\infty} m^{-s}\sum_{d=1}^{\infty} d^{k-s}\sin\left(\frac{2\pi dh}{q}\right)\mathrm{d}s\\
&= \frac{q^k}{2}X^{\nu/2}\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\Gamma(s+\nu)\zeta(s)\sum_{r=1}^{q}\zeta\left(s-k,\frac{r}{q}\right)\sin\left(\frac{2\pi rh}{q}\right)(q^{-1}X)^s\mathrm{d}s,
\end{aligned} \tag{3.16}$$

where $c > k+1$ and $X = \frac{4}{a^2x}$. Next, we investigate the following integral

$$\begin{aligned}
\mathcal{G}_k^{(\nu)}(X) &:= \frac{1}{2\pi i}\int_{(c)}\Gamma(s+\nu)\Gamma(s)\zeta(s)\sum_{r=1}^{q}\zeta\left(s-k,\frac{r}{q}\right)\sin\left(\frac{2\pi rh}{q}\right)(q^{-1}X)^s\mathrm{d}s\\
&= \frac{(-1)^{k/2}q^{-k}}{2(2\pi)^{k+1}}\frac{1}{2\pi i}\int_{(c)}\Gamma(s+\nu)\zeta(1-s)\Gamma(k+1-s)\\
&\qquad \times\left\{\zeta\left(k+1-s,\frac{h}{q}\right)-\zeta\left(k+1-s,1-\frac{h}{q}\right)\right\}(4\pi^2X)^s\mathrm{d}s,
\end{aligned} \tag{3.17}$$

where in the last step, we used (2.9), (2.17). Next, we consider the following integral

$$\begin{aligned}
\mathcal{H}_T :=&\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\Gamma(s+\nu)\zeta(1-s)\Gamma(k+1-s)\\
&\qquad \times\left\{\zeta\left(k+1-s,\frac{h}{q}\right)-\zeta\left(k+1-s,1-\frac{h}{q}\right)\right\}(4\pi^2X)^s\mathrm{d}s,
\end{aligned} \tag{3.18}$$

for some large positive number T. We consider the contour $\mathcal{C}$ that we have defined in the general setup. One can note that the simple pole of $\Gamma(k+1-s)$ at $s = k+1, k+3, k+5, \cdots$ will get cancelled by the trivial zeroes of $\zeta(1-s)$ except at $s = k+1$ when $k=0$. By employing (2.18) and (2.20), one can easily derive that $\left(\zeta(s, \frac{h}{q}) - \zeta(s, 1-\frac{h}{q})\right)$ encounters zeros at negative odd integers. Hence the remaining poles of $\Gamma(k+1-s)$ at $s = k+2, k+4, k+6, \cdots$ will get cancelled by the simple zeroes of $\left(\zeta(k+1-s, \frac{h}{q}) - \zeta(k+1-s, 1-\frac{h}{q})\right)$. Inside the contour $\mathcal{C}$, the integrand has a pole at $s=0$ and possibly at $s=1$. By employing Cauchy's residue theorem, the integral in (3.18) can be rewritten as

$$\mathcal{H}_T = R_0 + \delta_k\ R_1 + \frac{1}{2\pi i}\left(\int_{-d-iT}^{-d+iT} \pm \int_{-d\pm iT}^{c\pm iT}\right)\Gamma(s+\nu)\zeta(1-s)\Gamma(k+1-s)$$
$$\times\left\{\zeta\left(k+1-s, \frac{h}{q}\right) - \zeta\left(k+1-s, 1-\frac{h}{q}\right)\right\}(4\pi^2X)^s \mathrm{d}s, \quad (3.19)$$

where δ_k is defined in (3.3); and R_0 and R_1 are the residues at $s=0$ and $s=1$, respectively which are given as

$$R_0 = -\Gamma(k+1)\Gamma(\nu)\left\{\zeta\left(k+1, \frac{h}{q}\right) - \zeta\left(k+1, 1-\frac{h}{q}\right)\right\}, \quad (3.20)$$
$$R_1 = \frac{1}{2}\Gamma(\nu+1)\left\{\zeta\left(0, \frac{h}{q}\right) - \zeta\left(0, 1-\frac{h}{q}\right)\right\}(4\pi^2X). \quad (3.21)$$

From (2.15), we have $|L(s,\chi)| \leq 2q|s|$ for $\sigma \geq 1/2$. Then by Lemma 2.2.1 together with functional equation (2.14), we find that in a bounded vertical strip $L(s,\chi) \ll_{q,\sigma} |t|^{O_{\sigma,q}(1)}$ with $|t| > 1$. Now by (2.4) we can easily deduce that $\zeta\left(s, \frac{h}{q}\right) \ll_{q,\sigma} |t|^{O_{\sigma,q}(1)}$ in a bounded vertical strip. Since $-1 < -d < \sigma < c < k+2$, employing Lemma 2.2.1 and the aforementioned bound for Hurwitz zeta function, we can conclude that the integrals along the horizontal segments, i.e., the last two integrals in (3.19), vanish as $T \to \infty$. Hence, letting $T \to \infty$ and then substituting back the expression (3.19) in (3.17), we get

$$\mathcal{G}_k^{(\nu)}(X) = \frac{(-1)^{k/2}q^{-k}}{2(2\pi)^{k+1}}\ \left(R_0 + \delta_k\ R_1 + \mathcal{A}_k^{(\nu)}(X)\right), \quad (3.22)$$

and $\mathcal{A}_k^{(\nu)}(X)$ is defined by

$$\begin{aligned}
\mathcal{A}_k^{(\nu)}(X) &:= \frac{1}{2\pi i}\int_{(-d)} \Gamma(s+\nu)\zeta(1-s)\Gamma(k+1-s) \\
&\quad \times \left\{\zeta\left(k+1-s,\frac{h}{q}\right)-\zeta\left(k+1-s,1-\frac{h}{q}\right)\right\}(4\pi^2X)^s \mathrm{d}s \\
&= 4\pi^2 X\frac{1}{2\pi i}\int_{(1+d)} \Gamma(\nu+1-s)\zeta(s)\Gamma(k+s) \\
&\quad \times \left\{\zeta\left(k+s,\frac{h}{q}\right)-\zeta\left(k+s,1-\frac{h}{q}\right)\right\}(4\pi^2X)^{-s} \mathrm{d}s \\
&= 4\pi^2 X\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\left\{(m+h/q)^{-k}\frac{1}{2\pi i}\int_{(1+d)}\Gamma(\nu+1-s)\Gamma(k+s)\right. \\
&\quad \times \left(4\pi^2Xr(m+\frac{h}{q})\right)^{-s}\mathrm{d}s \\
&\left. -(m+1-h/q)^{-k}\frac{1}{2\pi i}\int_{(1+d)}\Gamma(\nu+1-s)\Gamma(k+s)\left(4\pi^2Xr(m+1-\frac{h}{q})\right)^{-s}\mathrm{d}s\right\}.
\end{aligned} \tag{3.23}$$

Employing Lemma 2.2.5 for $n=k$ and $a=1+\nu$ in 3.23, we get

$$\begin{aligned}
&\mathcal{A}_k^{(\nu)}(X) = (4\pi^2X)^{k+1}\Gamma(\nu+k+1) \\
&\times\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\left\{\frac{r^k}{(1+4\pi^2Xr(m+h/q))^{\nu+k+1}}-\frac{r^k}{(1+4\pi^2Xr(m+1-h/q))^{\nu+k+1}}\right\}.
\end{aligned} \tag{3.24}$$

Inserting (3.20), (3.21) and (3.24) in (3.22), and then combining with (3.16) and (3.17), we obtain the identity (3.4).

Next, we demonstrate that Theorem 3.2.1 is equivalent to Theorem 3.2.2.

Theorem 3.2.1 ⇒ Theorem 3.2.2 We will prove the theorem for $\theta=h/q$, where q is prime and $0<h<q$. Now we multiply the identity (3.2) in Theorem 3.2.1 with $\frac{1}{i\phi(q)}\chi(h)\tau(\bar{\chi})$ and then take the sum on odd primitive character χ modulo q. Hence, the left-hand side of the identity in (3.2) becomes

$$\frac{1}{i\phi(q)}\sum_{\chi\ odd}\chi(h)\tau(\bar{\chi})\sum_{n=1}^{\infty}\sigma_{k,\chi}(n)n^{\frac{\nu}{2}}K_\nu(a\sqrt{nx})$$

$$= \frac{1}{i\phi(q)} \sum_{n=1}^{\infty} n^{\frac{\nu}{2}} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k \sum_{\chi \ odd} \chi(d)\chi(h)\tau(\bar{\chi})$$

$$= \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k \sin\left(\frac{2\pi dh}{q}\right), \tag{3.25}$$

where in the last step, we have used (2.24). We now consider the summation over χ in the right-hand side of (3.2). For the first term, we have

$$\frac{1}{i\phi(q)} \sum_{\chi \ odd} \chi(h)\tau(\bar{\chi})L(1,\chi)$$
$$= -\frac{\pi}{q\phi(q)} \sum_{\chi \ odd} \chi(h)\tau(\chi)\tau(\bar{\chi})L(0,\bar{\chi}) = \frac{\pi}{2}\left(\zeta(0,h/q) - \zeta(0,1-h/q)\right), \tag{3.26}$$

where we have used (2.3), (2.10),(2.21) and (2.22). Similarly, to evaluate the second term in (3.2), we find that

$$\frac{1}{i\phi(q)} \sum_{\chi \ odd} \chi(h)\tau(\chi)\tau(\bar{\chi})L(1+k,\bar{\chi})$$
$$= -\frac{q^{-k}}{2i}\left(\zeta(1+k,h/q) - \zeta(1+k,1-h/q)\right). \tag{3.27}$$

Finally, we evaluate the infinite sum appearing in the last term on the right-hand side of (3.2). We see that

$$\frac{1}{i\phi(q)} \sum_{\chi \ odd} \chi(h)\tau(\chi)\tau(\bar{\chi}) \sum_{n=1}^{\infty} \bar{\sigma}_{k,\bar{\chi}}(n) \frac{\Gamma(\nu+k+1)}{\left(\frac{16\pi^2}{a^2 q}\frac{n}{x}+1\right)^{\nu+k+1}}$$

$$= -\frac{q}{i\phi(q)} \sum_{n=1}^{\infty} \frac{\Gamma(\nu+k+1)}{\left(\frac{16\pi^2}{a^2 q}\frac{n}{x}+1\right)^{\nu+k+1}} \sum_{d|n} d^k \sum_{\chi \ odd} \chi(h)\bar{\chi}(n/d)$$

$$= -\frac{q}{i\phi(q)} \sum_{d=1}^{\infty} d^k \sum_{r=1}^{\infty} \frac{\Gamma(\nu+k+1)}{\left(\frac{16\pi^2}{a^2 q}\frac{dr}{x}+1\right)^{\nu+k+1}} \sum_{\chi \ odd} \chi(h)\bar{\chi}(r)$$

$$= -\frac{q}{2i} \sum_{d=1}^{\infty} d^k \left(\sum_{\substack{r=1 \\ r\equiv h(q)}}^{\infty} \frac{\Gamma(\nu+k+1)}{\left(\frac{16\pi^2}{a^2 q}\frac{dr}{x}+1\right)^{\nu+k+1}} - \sum_{\substack{r=1 \\ r\equiv -h(q)}}^{\infty} \frac{\Gamma(\nu+k+1)}{\left(\frac{16\pi^2}{a^2 q}\frac{dr}{x}+1\right)^{\nu+k+1}} \right)$$

$$
\begin{aligned}
&= -\frac{q}{2i}\sum_{d=1}^{\infty} d^k \sum_{m=0}^{\infty}\left(\frac{\Gamma(\nu+k+1)}{(1+\frac{16\pi^2(mq+h)d}{qa^2x})^{1+\nu+k}} - \frac{\Gamma(\nu+k+1)}{(1+\frac{16\pi^2(mq+q-h)d}{qa^2x})^{1+\nu+k}}\right)\\
&= -\frac{q}{2i}\sum_{d=1}^{\infty} d^k \sum_{m=0}^{\infty}\left(\frac{\Gamma(\nu+k+1)}{(1+\frac{16\pi^2 d}{a^2x}(m+h/q))^{1+\nu+k}}\right.\\
&\qquad\qquad \left. - \frac{\Gamma(\nu+k+1)}{(1+\frac{16\pi^2 d}{a^2x}(m+1-h/q))^{1+\nu+k}}\right), \qquad (3.28)
\end{aligned}
$$

where in the penultimate step, we have used (2.22). Employing (3.25), (3.27), (3.26), and (3.28) in (3.2), we get (3.4).

Theorem 3.2.2 $\Rightarrow$ Theorem 3.2.1 Let $\theta = h/q$, and χ be an odd primitive character modulo q. We first multiply the identity (3.4) in Theorem 3.2.2 by $\bar{\chi}(h)/\tau(\bar{\chi})$, and then take summation on h, $0 < h < q$. We then examine (3.4), focusing specifically on summation over h. The left-hand side of the identity (3.4) becomes

$$
\begin{aligned}
&\frac{1}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\sum_{n=1}^{\infty} n^{\nu/2}K_\nu(a\sqrt{nx})\sum_{d|n} d^k \sin(2\pi dh/q)\\
&= \frac{1}{2i\tau(\bar{\chi})}\sum_{n=1}^{\infty} n^{\nu/2}K_\nu(a\sqrt{nx})\sum_{d|n} d^k \sum_{h=1}^{q-1}\bar{\chi}(h)\left(e^{2\pi i dh/q} - e^{-2\pi i dh/q}\right)\\
&= \frac{1}{2i}\sum_{n=1}^{\infty} n^{\nu/2}K_\nu(a\sqrt{nx})\sum_{d|n} d^k\left(\chi(d)-\chi(-d)\right)\\
&= i^{-1}\sum_{n=1}^{\infty}\sigma_{k,\chi}(n) n^{\frac{\nu}{2}}K_\nu(a\sqrt{nx}), \qquad (3.29)
\end{aligned}
$$

where in the penultimate step, we have used (2.26). With the help of (2.3) and (2.21), the right-hand side of (3.4) transforms into

$$
\frac{1}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\left(\zeta(1+k,h/q) - \zeta(1+k,1-h/q)\right) = -2q^k\tau(\chi)L(1+k,\bar{\chi}), \quad (3.30)
$$

and employing the functional equation for L-function (2.13), the second term becomes

$$\frac{1}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\left(\zeta(0,h/q)-\zeta(0,1-h/q)\right)=-\frac{2\tau(\chi)}{q}L(0,\bar{\chi})=i^{-1}L(1,\chi). \tag{3.31}$$

The infinite series on the right-hand side of (3.4) takes the form

$$\begin{aligned}
&\frac{1}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\sum_{d=1}^{\infty}d^k\sum_{m=0}^{\infty}\Bigg(\frac{1}{(1+\frac{16\pi^2 d}{a^2x}(m+h/q))^{1+\nu+k}}\\
&\qquad\qquad -\frac{1}{(1+\frac{16\pi^2 d}{a^2x}(m+1-h/q))^{1+\nu+k}}\Bigg)\\
&=\frac{1}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\sum_{d=1}^{\infty}d^k\sum_{\substack{r=1\\ r\equiv h(q)}}^{\infty}\frac{1}{(1+\frac{16\pi^2 dr}{a^2xq})^{1+\nu+k}}\\
&-\frac{1}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\sum_{d=1}^{\infty}d^k\sum_{\substack{r=1\\ r\equiv -h(q)}}^{\infty}\frac{1}{(1+\frac{16\pi^2 dr}{a^2xq})^{1+\nu+k}}\\
&=\frac{2}{\tau(\bar{\chi})}\sum_{d=1}^{\infty}\sum_{r=1}^{\infty}\frac{d^k\bar{\chi}(r)}{(1+\frac{16\pi^2 dr}{a^2xq})^{1+\nu+k}}=-\frac{2\tau(\chi)}{q}\sum_{n=1}^{\infty}\frac{\bar{\sigma}_{k,\bar{\chi}}(n)}{(1+\frac{16\pi^2 dr}{a^2xq})^{1+\nu+k}}.
\end{aligned} \tag{3.32}$$

Inserting (3.29), (3.30), (3.31) and (3.32) into (3.4), we obtain (3.2). □

Our next result, corresponding to $\nu = 0$, is as follows:

Theorem 3.2.3. *Let k be an even, non-negative integer and χ be an odd primitive Dirichlet character modulo q. Then*

$$\begin{aligned}
\sum_{n=1}^{\infty}\sigma_{k,\chi}(n)K_0(a\sqrt{nx})&=\delta_k\frac{2}{a^2x}L(1,\chi)-\frac{L(-k,\chi)}{4}\left(\log\left(\frac{8\pi}{a^2}\right)+\frac{L'(-k,\chi)}{L(-k,\chi)}-2\gamma\right)\\
&+\frac{L(-k,\chi)}{4}\log x+(-1)^{\frac{k}{2}}\frac{ik!q^k}{2(2\pi)^{k+1}}\tau(\chi)\sum_{n=1}^{\infty}\bar{\sigma}_{k,\bar{\chi}}(n)\left(\frac{1}{n^{k+1}}-\frac{1}{(n+\frac{a^2qx}{16\pi^2})^{k+1}}\right),
\end{aligned} \tag{3.33}$$

where δ_k is defined in (3.3).

Proof. Let us begin the proof by taking $f_k(n) = \sigma_{k,\chi}(n)$ with χ being an odd primitive character modulo q and $k \geq 0$ an even integer and $\nu = 0$ in (3.9). The corresponding Dirichlet series, in this case, is $F_k(s) = \zeta(s)L(s-k,\chi)$. Therefore, we obtain

$$\sum_{n=1}^{\infty} \sigma_{k,\chi}(n) K_0(a\sqrt{nx}) = \frac{1}{2}(R_{k+1} + R_1 + R_0 + J_k^{(0)}(X)), \tag{3.34}$$

where $J_k^{(0)}(X)$ is defined in (3.10). It is clear that $R_{k+1} = 0$ for $k \geq 0$. $L(s-k,\chi)$ has a zero at $s = 1$ in case $k \geq 2$ is an even integer, and χ is odd. So, we will not get any contribution from the pole of $\zeta(s)$ at $s = 1$. However, in the case of $k = 0$, the integrand in (3.10) will encounter a pole at $s = 1$. Hence, we can write

$$R_1 = \begin{cases} 0, & \text{if } k > 0, \\ L(1,\chi)X, & \text{if } k = 0, \end{cases} \tag{3.35}$$

and the integrand in (3.10) encounters a double pole at $s = 0$ with residue R_0 given by

$$R_0 = -\frac{L(-k,\chi)}{2}\left(\log(2\pi X) + \frac{L'(-k,\chi)}{L(-k,\chi)} - 2\gamma\right). \tag{3.36}$$

Now using (3.35) and (3.36) and the fact $R_{k+1} = 0$ in (3.34), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \sigma_{k,\chi}(n) K_0(a\sqrt{nx}) =& \frac{\delta_k}{2} L(1,\chi)X - \frac{L(-k,\chi)}{4}\left(\log(2\pi X) + \frac{L'(-k,\chi)}{L(-k,\chi)} - 2\gamma\right) \\ &+ \frac{1}{2} J_k^{(0)}(X),\end{aligned} \tag{3.37}$$

where δ_k is defined in (3.3). For $J_k^{(0)}(X)$, we employ the functional equations (2.9) and (2.13) to obtain

$$\begin{aligned} J_k^{(0)}(X) &= \frac{h_k}{2\pi i}\int_{(-d)} \Gamma(s)\Gamma(1+k-s)\zeta(1-s)L(1-s+k,\bar{\chi})Y^s \mathrm{d}s \\ &= \frac{Yh_k}{2\pi i}\int_{(1+d)} \Gamma(1-s)\Gamma(k+s)\zeta(s)L(s+k,\bar{\chi})Y^{-s} \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= Y h_k \sum_{n=1}^{\infty} \sigma_{-k,\bar{\chi}}(n)\ \frac{1}{2\pi i} \int_{(1+d)} \Gamma(1-s)\Gamma(k+s)(nY)^{-s} \mathrm{d}s \\
&= \pi Y h_k \sum_{n=1}^{\infty} \sigma_{-k,\bar{\chi}}(n)\ \frac{1}{2\pi i} \int_{(1+d)} \frac{\Gamma(k+s)}{\Gamma(s)\sin(\pi s)} (nY)^{-s} \mathrm{d}s \\
&= \pi Y h_k \left(\sum_{n \leq Y^{-1}} + \sum_{n > Y^{-1}} \right) \sigma_{-k,\bar{\chi}}(n) \\
&\qquad \times \frac{1}{2\pi i} \int_{(1+d)} \frac{\Gamma(k+s)}{\Gamma(s)\sin(\pi s)} (nY)^{-s} \mathrm{d}s, \qquad (3.38)
\end{aligned}
$$

where $h_k = \frac{(-1)^{1+\frac{k}{2}} i\tau(\chi)}{2\pi} \left(\frac{q}{2\pi}\right)^k$ and $Y = \frac{4\pi^2}{q} X$ with $X = \frac{4}{a^2 x}$. In the second last step, we have used the reflection formula (2.12). We will first investigate the infinite sum $\sum_{n>Y^{-1}}$. To evaluate this inner line integral in (3.38), we shall use the Cauchy residue theorem with the contour consisting of the line segments $[1+d-iT, 1+d+iT], [1+d+iT, M+\frac{1}{2}+iT], [M+\frac{1}{2}+iT, M+\frac{1}{2}-iT], [M+\frac{1}{2}-iT, 1+d-iT]$ where $M \in \mathbb{N}$ is a large number, and T is a large positive number. The poles of the integrand function in (3.38) are at 2, 3, $\cdots$, M, and they are simple. The residue at $s = m$ is given by

$$
\mathcal{R}_m := \begin{cases} \frac{1}{\pi}(-1)^m m(m+1)...(m+k-1)(nY)^{-m} & \text{for } k \geq 1, \\ \frac{1}{\pi}(-1)^m (nY)^{-m} & \text{for } k = 0, \end{cases} \qquad (3.39)
$$

where $m = 2, 3, \cdots, M$. Employing Lemma 2.2.1, we can show that both the integrals along the horizontal lines $[1+d+iT, M+\frac{1}{2}+iT]$ and $[M+\frac{1}{2}-iT, 1+d-iT]$ vanish as $T \to \infty$. From (3.39), we arrive at

$$
\begin{aligned}
&\frac{1}{2\pi i} \int_{(1+d)} \frac{\Gamma(k+s)}{\Gamma(s)\sin(\pi s)} (nY)^{-s} \mathrm{d}s \\
&= -\sum_{m=2}^{M} \mathcal{R}_m + \frac{1}{2\pi i} \int_{(M+\frac{1}{2})} \frac{\Gamma(k+s)}{\Gamma(s)\sin(\pi s)} (nY)^{-s} \mathrm{d}s \\
&= -\sum_{m=2}^{M} \mathcal{R}_m + O_k\left(\frac{M^k}{(nY)^{M+1/2}} \right),
\end{aligned}
$$

where we have used $|\sin \pi(\sigma + it)| \gg e^{\pi|t|}$ and Lemma 2.2.1 for $|t| \geq 1$ to bound the integral $\int_{(M+\frac{1}{2})}$ and the implied constant depends on k. Next, allowing $M \to \infty$, the error term goes to 0 as $n > Y^{-1}$. Employing Corollary 2.2.1 for $W = 1/Y$, we readily obtain that

$$\frac{1}{2\pi i}\int_{(1+d)} \frac{\Gamma(k+s)}{\Gamma(s)\sin(\pi s)}(nY)^{-s}\mathrm{d}s = -\frac{k!}{\pi}\left(\frac{1}{nY}\right)\left(1-\frac{n^{k+1}}{(Y^{-1}+n)^{k+1}}\right),$$

and we easily deduce from the above expression that

$$\begin{aligned}&\sum_{n>Y^{-1}} \sigma_{-k,\bar{\chi}}(n)\frac{1}{2\pi i}\int_{(1+d)} \frac{\Gamma(k+s)}{\Gamma(s)\sin(\pi s)}(nY)^{-s}\mathrm{d}s \\ &= -\frac{k!}{\pi Y}\sum_{n>Y^{-1}} \frac{\sigma_{-k,\bar{\chi}}(n)}{n}\left(1-\frac{n^{k+1}}{(Y^{-1}+n)^{k+1}}\right). \end{aligned} \tag{3.40}$$

Similarly, by shifting the line of integration to the left, we obtain

$$\begin{aligned}&\sum_{n\leq Y^{-1}} \sigma_{-k,\bar{\chi}}(n)\frac{1}{2\pi i}\int_{(1+d)} \frac{\Gamma(k+s)}{\Gamma(s)\sin(\pi s)}(nY)^{-s}\mathrm{d}s \\ &= -\frac{k!}{\pi Y}\sum_{n\leq Y^{-1}} \frac{\sigma_{-k,\bar{\chi}}(n)}{n}\left(1-\frac{n^{k+1}}{(Y^{-1}+n)^{k+1}}\right). \end{aligned} \tag{3.41}$$

Inserting (3.40) and (3.41) in (3.38),

$$J_k^{(0)}(X) = -k!h_k\sum_{n=1}^{\infty} \frac{\sigma_{-k,\bar{\chi}}(n)}{n}\left(1-\frac{n^{k+1}}{(Y^{-1}+n)^{k+1}}\right). \tag{3.42}$$

We finish the proof by noting $Y = \frac{16\pi^2}{a^2qx}$, substituting (3.42) in (3.37) and then simplifying. □

Theorem 3.2.4. *Let $k \geq 2$ be an even integer and χ be an odd primitive Dirichlet character modulo q. Then, for any $\Re(\nu) > 0$,*

$$\begin{aligned}\sum_{n=1}^{\infty} \bar{\sigma}_{k,\chi}(n)n^{\frac{\nu}{2}}K_\nu(a\sqrt{nx}) &= \frac{2^{\nu+2k+1}}{a^{\nu+2k+2}}\Gamma(k+1)\Gamma(\nu+k+1)L(1+k,\chi)\ x^{-\frac{\nu}{2}-k-1} \\ &- \frac{(-1)^{\frac{k}{2}}i(aq)^\nu\ x^{\frac{\nu}{2}}}{2^{3\nu+k+2}\pi^{2\nu+k+1}}\Gamma(\nu+k+1)\tau(\chi)\sum_{n=1}^{\infty} \frac{\sigma_{k,\bar{\chi}}(n)}{\left(n+\frac{a^2qx}{16\pi^2}\right)^{\nu+k+1}}.\end{aligned}$$

Analogous to Theorem 3.2.1, one can show that Theorem 3.2.4 is equivalent to the following result.

Theorem 3.2.5. *Let $k \geq 2$ be an even integer. Then for any $\Re(\nu) > 0$ we have*

$$\left(\frac{a^2x}{4}\right)^{\frac{\nu}{2}+k+1} \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k \sin\left(\frac{2\pi n\theta}{d}\right)$$
$$= \frac{(-1)^{\frac{k}{2}} 2^k \pi^{k+1}}{4} \Gamma(\nu+k+1)\left(\zeta(-k,\theta) - \zeta(-k,1-\theta)\right)$$
$$+ \frac{(-1)^{\frac{k}{2}}}{4}(2\pi)^{k+1}\Gamma(\nu+k+1)\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\left\{\frac{(m+\theta)^k}{(1+\frac{16\pi^2 r}{a^2x}(m+\theta))^{1+\nu+k}}\right.$$
$$\left. - \frac{(m+1-\theta)^k}{(1+\frac{16\pi^2 r}{a^2x}(m+1-\theta))^{1+\nu+k}}\right\}. \tag{3.43}$$

Proof of Theorem 3.2.4 and its equivalence with Theorem 3.2.5. The proofs of Theorems 3.2.4 and 3.2.5 are similar to the proofs of Theorems 3.2.1 and 3.2.2, respectively. The equivalence of Theorems 3.2.4 and 3.2.5 can be derived similarly. To avoid repetitions, we skip the details of the proof. □

Our next result, corresponding to $\nu = 0$ is as follows:

Theorem 3.2.6. *Let $k \geq 2$ be an even integer and χ be an odd primitive Dirichlet character modulo q. Then*

$$\sum_{n=1}^{\infty} \bar{\sigma}_{k,\chi}(n) K_0(a\sqrt{nx}) = \frac{2^{2k+1}}{a^{2k+2}}\Gamma^2(k+1)L(k+1,\chi)\frac{1}{x^{k+1}} + \frac{1}{2}\zeta'(-k)L(0,\chi)$$
$$+ \frac{(-1)^{\frac{k}{2}} i k! \tau(\chi)}{2(2\pi)^{k+1}} \sum_{n=1}^{\infty} \sigma_{k,\bar{\chi}}(n)\left(\frac{1}{n^{k+1}} - \frac{1}{(n+\frac{a^2qx}{16\pi^2})^{k+1}}\right).$$

Proof. Here we will consider $f_k(n) = \bar{\sigma}_{k,\chi}(n)$ with χ being an odd primitive character modulo q and $k \geq 2$ an even integer and $\nu = 0$ in (3.9). We skip the detail of the proof because of its similarity with the proof of Theorem 3.2.3. □

3.2.2 Identity related to $r_6(n)$

In this subsection, we study the identities associated with $r_6(n)$ and the modified K-Bessel functions. Let us first see the representation of $r_6(n)$ as follows [5, p. 63]

$$\begin{aligned} r_6(n) =& 16 \sum_{\substack{d|n \\ \frac{n}{d} \ odd}} (-1)^{(n/d-1)/2} d^2 - 4 \sum_{\substack{d|n \\ d \ odd}} (-1)^{(d-1)/2} d^2 \\ =& \sum_{d|n} d^2 \left\{ 16 \sin\left(\frac{\pi n}{2d}\right) - 4 \sin\left(\frac{\pi d}{2}\right) \right\}. \end{aligned} \tag{3.44}$$

Theorem 3.2.2, together with Theorem 3.2.5, gives rise to the following beautiful identity corresponding to $r_6(n)$.

Corollary 3.2.1. *For any* $\Re(\nu) > 0$ *we have*

$$\begin{aligned} &\left(\frac{a^2x}{4}\right)^{\frac{\nu}{2}+3} \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^2 \left\{ 16 \sin\left(\frac{2\pi n\theta}{d}\right) - 4 \sin(2\pi d\theta) \right\} \\ &= \frac{16}{3}\pi^3 \Gamma(\nu+3)(\theta - 3\theta^2 + 2\theta^3) - \frac{a^6}{256}\Gamma(\nu)(\cot(\pi\theta) + \cot^3(\pi\theta))x^3 \\ &+ (2\pi)^3\Gamma(\nu+3) \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{n^2 - 4(m+\theta)^2}{(1 + \frac{16\pi^2 n}{a^2 x}(m+\theta))^{\nu+3}} - \frac{n^2 - 4(m+1-\theta)^2}{(1 + \frac{16\pi^2 n}{a^2 x}(m+1-\theta))^{\nu+3}} \right\}. \end{aligned} \tag{3.45}$$

Proof. Multiplying (3.4) of Theorem 3.2.2 by -4 and (3.43) of Theorem 3.2.5 by 16, and then adding both the expressions, putting $k = 2$ yield

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^2 \left\{ 16 \sin\left(\frac{2\pi n\theta}{d}\right) - 4 \sin(2\pi d\theta) \right\} \\ &= -16\pi^3 \Gamma(\nu+3) X^{\frac{\nu}{2}+3}(\zeta(-2,\theta) - \zeta(-2,1-\theta)) \\ &- \frac{1}{4\pi^3} X^{\frac{\nu}{2}} \Gamma(\nu)(\zeta(3,\theta) - \zeta(3,1-\theta)) \\ &+ X^{\frac{\nu}{2}}(2\pi X)^3 \Gamma(\nu+3) \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{n^2 - 4(m+\theta)^2}{(1 + \frac{16\pi^2 r}{a^2 x}(m+\theta))^{\nu+3}} \right. \\ &\qquad \left. - \frac{n^2 - 4(m+1-\theta)^2}{(1 + \frac{16\pi^2 r}{a^2 x}(m+1-\theta))^{\nu+3}} \right\}. \end{aligned} \tag{3.46}$$

Using (2.18), we have

$$\zeta(-2,\theta)-\zeta(-2,1-\theta)=-\frac{1}{3}(B_3(\theta)-B_3(1-\theta))=-\frac{1}{3}(\theta-3\theta^2+2\theta^3). \tag{3.47}$$

By using partial fraction expansion for $\cot(\pi\theta)$, we can easily get

$$\zeta(n,1-\theta)+(-1)^n\zeta(n,\theta)=-\frac{\pi}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\cot(\pi\theta).$$

Hence by above, we get

$$(\zeta(3,\theta)-\zeta(3,1-\theta))=\pi^3(\cot(\pi\theta)+\cot^3(\pi\theta)). \tag{3.48}$$

Substituting (3.47), (3.48) in (3.46), we get the result. □

One can also obtain an interesting identity analogous to Hardy's result in (1.17) by substituting $\theta = 1/4$.

Corollary 3.2.2. *For any $\Re(\nu) > 0$ we have*

$$\left(\frac{a^2x}{4}\right)^{\frac{\nu}{2}+3}\sum_{n=1}^{\infty} r_6(n)n^{\nu/2}K_\nu(a\sqrt{nx})=\frac{\pi^3}{2}\Gamma(\nu+3)-\frac{a^6}{128}\Gamma(\nu)x^3$$
$$+(2\pi)^3\Gamma(\nu+3)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left\{\frac{n^2-4(m+1/4)^2}{(1+\frac{16\pi^2 n}{a^2x}(m+1/4))^{\nu+3}}-\frac{n^2-4(m+3/4)^2}{(1+\frac{16\pi^2 n}{a^2x}(m+3/4))^{\nu+3}}\right\}. \tag{3.49}$$

Proof. Proof directly follows from (3.44) and Corollary 3.2.1. □

In particular, $\nu = 1/2, a = 4\pi$ yields an identity analogous to Hardy's result (1.16).

Corollary 3.2.3. *We have*

$$\sum_{n=1}^{\infty} r_6(n)e^{-4\pi\sqrt{nx}}=\frac{15}{512\pi^3}x^{-3}-1$$
$$+\frac{15}{32\pi^3}x^{-3}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left\{\frac{n^2-4(m+1/4)^2}{(1+\frac{n}{x}(m+1/4))^{\frac{7}{2}}}-\frac{n^2-4(m+3/4)^2}{(1+\frac{n}{x}(m+3/4))^{\frac{7}{2}}}\right\}. \tag{3.50}$$

Proof. Substituting $\nu = 1/2, a = 4\pi$ in Corollary 3.2.2 and using (1.8), we get (3.50). Next, we demonstrate the absolute convergence of infinite series present on the right-hand side of (3.50). For this, we consider

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left|\frac{n^2-4(m+1/4)^2}{(1+\frac{n}{x}(m+1/4))^{\frac{7}{2}}}-\frac{n^2-4(m+3/4)^2}{(1+\frac{n}{x}(m+3/4))^{\frac{7}{2}}}\right|$$
$$\leq \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left\{\left|\frac{n^2-4(m+1/4)^2}{(1+\frac{n}{x}(m+1/4))^{\frac{7}{2}}}\right|+\left|\frac{n^2-4(m+3/4)^2}{(1+\frac{n}{x}(m+3/4))^{\frac{7}{2}}}\right|\right\}$$
$$\leq 2x^{\frac{7}{2}}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}\sum_{m<n}\frac{1}{(m+1/4)^{\frac{7}{2}}}+8x^{\frac{7}{2}}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{7}{2}}}\sum_{m\geq n}\frac{1}{(m+1/4)^{\frac{3}{2}}}$$
$$\ll x^{\frac{7}{2}}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}\ll\infty.$$

□

3.2.3 Identities involving even characters

In this subsection, we present similar results when k is an odd positive integer, and χ is a non-principal even primitive character. In the last result, we take $k = 0$.

Theorem 3.2.7. *Let $k \geq 1$ be an odd integer and χ be a non-principal even primitive Dirichlet character modulo q. Then, for any $\Re(\nu) > 0$,*

$$\sum_{n=1}^{\infty}\sigma_{k,\chi}(n)n^{\frac{\nu}{2}}K_\nu(a\sqrt{nx}) = \frac{(-1)^{\frac{k-1}{2}}q^k}{a^\nu 2^{k+2-\nu}\pi^{k+1}}\Gamma(\nu)\tau(\chi)\Gamma(k+1)L(1+k,\bar{\chi})\ x^{-\frac{\nu}{2}}$$
$$+\frac{(-1)^{\frac{k+1}{2}}a^\nu q^{\nu+k}\ x^{\frac{\nu}{2}}}{2^{3\nu+k+2}\pi^{2\nu+k+1}}\Gamma(\nu+k+1)\tau(\chi)\sum_{n=1}^{\infty}\frac{\sigma_{k,\bar{\chi}}(n)}{\left(n+\frac{a^2qx}{16\pi^2}\right)^{\nu+k+1}}.$$

Theorem 3.2.7 is equivalent to the following theorem, mentioned below.

Theorem 3.2.8. *Let $k \geq 1$ be an odd integer. Then for any $\Re(\nu) > 0$ we have*

$$\left(\frac{a^2x}{4}\right)^{\frac{\nu}{2}+k+1}\sum_{n=1}^{\infty}n^{\nu/2}K_\nu(a\sqrt{nx})\sum_{d|n}d^k\cos(2\pi d\theta)$$
$$=\frac{(-1)^{\frac{k-1}{2}}a^{2k+2}k!}{2^{2k+4}(2\pi)^{k+1}}\Gamma(\nu)\left\{\zeta(k+1,\theta)+\zeta(k+1,1-\theta)\right\}x^{k+1}$$

$$+\frac{(-1)^{\frac{k+1}{2}}(2\pi)^{k+1}\Gamma(\nu+k+1)}{4}\sum_{d=1}^{\infty} d^k \sum_{m=0}^{\infty}\left\{\frac{1}{\left(\frac{16\pi^2}{a^2}\frac{d(m+\theta)}{x}+1\right)^{\nu+k+1}}\right.$$
$$\left.+\frac{1}{\left(\frac{16\pi^2}{a^2}\frac{d(m+1-\theta)}{x}+1\right)^{\nu+k+1}}\right\} - \delta_{k,1}\ \frac{a^2}{16}\Gamma(1+\nu)x, \tag{3.51}$$

where $\delta_{k,1}$ *is given by*

$$\delta_{k,1} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{else }. \end{cases} \tag{3.52}$$

Proof of Theorem 3.2.7 and its equivalence with Theorem 3.2.8. We now outline the idea behind the proof of Theorem 3.2.7, and highlight the key differences between the proofs of Theorem 3.2.1 and Theorem 3.2.7. The main distinction in proving Theorem 3.2.7 lies in the use of the functional equation for the $L-$function. Specifically, for an even character χ, we take $\kappa = 0$, whereas for an odd character, $\kappa = 1$. Another critical difference arises from the locations of the zeros of $L(s, \chi) = 0$. When χ is an even primitive character, the nontrivial zeros of $L(s, \chi)$ occur at $s = 0, -2, -4, -6, \cdots$. In contrast, for an odd primitive character, the zeros are at $s = -1, -3, -5, -7, \cdots$. Consequently, in the proof of Theorem 3.2.7, we obtain $L(s-k, \chi) = 0$ at $s = 1$. Hence, there is no contribution from the pole of $\zeta(s)$ at $s = 1$. It is easy to see that $R_{k+1} = 0$. The integrand has a pole at $s = 0$ due to the Gamma function with residue R_0 given by

$$R_0 = -\frac{\Gamma(\nu)L(-k,\chi)}{2} = \frac{(-1)^{\frac{k-1}{2}}\tau(\chi)}{2\pi}\left(\frac{(2\pi)}{q}\right)^{-k}\Gamma(1+k)\Gamma(\nu)L(1+k,\bar{\chi}), \tag{3.53}$$

where in the last step, we have applied the functional equation (2.13) for $\kappa = 0$, since χ is an even character. The computation of $J_k^{(\nu)}(X)$ proceeds similarly to the proof of Theorem 3.2.1, with the only difference being that we use the functional equations (2.9) and (2.13) for the even character χ, where k is an odd integer.

The proof of Theorem 3.2.8 is similar to the proof of Theorem 3.2.2. Here, we use (2.16) instead of (2.17) and $k \geq 1$ to be an odd integer. Next, we demonstrate that Theorem 3.2.7 and Proposition 1.4.3 are equivalent to Theorem 3.2.8.

Theorem 3.2.7 ⇒ Theorem 3.2.8 It is sufficient to prove the theorem for $\theta = h/q$, where q is prime and $0 < h < q$. We begin our proof by considering the expression on the left-hand side of the identity (3.51) in Theorem 3.2.8. Employing (2.25), we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k \cos\left(\frac{2\pi dh}{q}\right) \\
&= \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \left(\sum_{\substack{d|n \\ q|d}} d^k + \sum_{\substack{d|n \\ q\nmid d}} d^k \cos\left(\frac{2\pi dh}{q}\right) \right) \\
&= \sum_{m=1}^{\infty} (qm)^{\nu/2} K_\nu(a\sqrt{qmx}) \sum_{d|m} (qd)^k \\
&\qquad + \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{\substack{d|n \\ q\nmid d}} \frac{d^k}{\phi(q)} \sum_{\chi \ even} \chi(d)\chi(h)\tau(\bar{\chi}) \\
&= q^{\frac{\nu}{2}+k} \sum_{m=1}^{\infty} m^{\nu/2} K_\nu(a\sqrt{qmx}) \sum_{d|m} d^k - \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{\substack{d|n \\ q\nmid d}} \frac{d^k}{\phi(q)} \chi_0(d) \\
&\qquad + \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{\substack{d|n \\ q\nmid d}} \frac{d^k}{\phi(q)} \sum_{\substack{\chi \neq \chi_0 \\ \chi even}} \chi(d)\chi(h)\tau(\bar{\chi}) \\
&= q^{\frac{\nu}{2}+k} \sum_{m=1}^{\infty} m^{\nu/2} K_\nu(a\sqrt{qmx}) \sum_{d|m} d^k - \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \frac{1}{\phi(q)} \left(\sum_{d|n} d^k - \sum_{\substack{d|n \\ q|d}} d^k \right) \\
&\qquad + \frac{1}{\phi(q)} \sum_{\substack{\chi \neq \chi_0 \\ \chi even}} \chi(h)\tau(\bar{\chi}) \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k \chi(d) \\
&= \frac{q^{\frac{\nu}{2}+k+1}}{\phi(q)} \sum_{m=1}^{\infty} \sigma_k(m) m^{\nu/2} K_\nu(a\sqrt{qmx}) - \frac{1}{\phi(q)} \sum_{n=1}^{\infty} \sigma_k(n) n^{\nu/2} K_\nu(a\sqrt{nx}) \\
&\qquad + \frac{1}{\phi(q)} \sum_{\substack{\chi \neq \chi_0 \\ \chi even}} \chi(h)\tau(\bar{\chi}) \sum_{n=1}^{\infty} \sigma_{k,\chi}(n) n^{\nu/2} K_\nu(a\sqrt{nx}). \qquad (3.54)
\end{aligned}
$$

Now, we first evaluate the first two sums on the right-hand side of (3.54). By Proposition 1.4.3, we have

$$
\begin{aligned}
&\frac{q^{\frac{\nu}{2}+k+1}}{\phi(q)}\sum_{m=1}^{\infty}\sigma_k(m)m^{\nu/2}K_\nu(a\sqrt{qmx})-\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\sigma_k(n)n^{\nu/2}K_\nu(a\sqrt{nx})\\
&=-\frac{\Gamma(\nu)\zeta(-k)}{4\phi(q)}X^{\frac{\nu}{2}}\left(q^{k+1}-1\right)-\frac{\Gamma(1+\nu)}{4}X^{1+\frac{\nu}{2}}\delta_{k,1}+\frac{(-1)^{\frac{k+1}{2}}}{2\phi(q)}X^{\frac{\nu}{2}}\Gamma(\nu+k+1)\\
&\qquad\times(2\pi X)^{k+1}\left(\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{\left(\frac{16\pi^2}{a^2q}\frac{n}{x}+1\right)^{\nu+k+1}}-\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{\left(\frac{16\pi^2}{a^2}\frac{n}{x}+1\right)^{\nu+k+1}}\right), \qquad (3.55)
\end{aligned}
$$

where $\delta_{k,1}$ is defined in (3.52). Now, we examine the last sum on the right-hand side of (3.54). Invoking the identity in Theorem (3.2.7), we have

$$
\begin{aligned}
&\frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\tau(\bar{\chi})\sum_{n=1}^{\infty}\sigma_{k,\chi}(n)n^{\nu/2}K_\nu(a\sqrt{nx})\\
&=\frac{(-1)^{\frac{k-1}{2}}k!q^{k+1}}{2(2\pi)^{k+1}\phi(q)}X^{\frac{\nu}{2}}\Gamma(\nu)\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)L(1+k,\bar{\chi})\\
&+\frac{(-1)^{\frac{k+1}{2}}}{2\phi(q)}X^{\frac{\nu}{2}}(2\pi X)^{k+1}\Gamma(\nu+k+1)\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\sum_{n=1}^{\infty}\frac{\bar{\sigma}_{k,\bar{\chi}}(n)}{\left(\frac{16\pi^2}{a^2q}\frac{n}{x}+1\right)^{\nu+k+1}}. \qquad (3.56)
\end{aligned}
$$

We consider

$$
\begin{aligned}
&\sum_{\substack{\chi\neq\chi_0\\ \chi\ even}}\chi(h)L(1+k,\bar{\chi})\\
&=\sum_{\chi\ even}\chi(h)L(1+k,\bar{\chi})-L(1+k,\chi_0)\\
&=\sum_{\chi\ even}\chi(h)\frac{1}{q^{k+1}}\sum_{r=1}^{q-1}\bar{\chi}(r)\zeta(k+1,r/q)-\left(\sum_{n=1}^{\infty}\frac{1}{n^{k+1}}-\sum_{n=1}^{\infty}\frac{1}{(nq)^{k+1}}\right)\\
&=\frac{1}{q^{k+1}}\sum_{r=1}^{q-1}\zeta(k+1,r/q)\sum_{\chi\ even}\chi(h)\bar{\chi}(r)-\left(1-\frac{1}{q^{k+1}}\right)\zeta(k+1)\\
&=\frac{\phi(q)}{2q^{k+1}}\left\{\zeta(k+1,h/q)+\zeta(k+1,1-h/q)\right\}-\left(1-\frac{1}{q^{k+1}}\right)\zeta(k+1), \qquad (3.57)
\end{aligned}
$$

where in the penultimate step, we used (2.3) and (2.23). Now, we examine the last expression in (3.56), and we obtain

$$\begin{aligned}
&\frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\sum_{n=1}^{\infty}\frac{\bar{\sigma}_{k,\bar{\chi}}(n)}{\left(\frac{16\pi^2}{a^2q}\frac{n}{x}+1\right)^{\nu+k+1}}\\
&=\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\frac{\sum_{d/n}d^k}{\left(\frac{16\pi^2}{a^2q}\frac{n}{x}+1\right)^{\nu+k+1}}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\bar{\chi}(n/d)\\
&=\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\frac{\sum_{d/n}d^k}{\left(\frac{16\pi^2}{a^2q}\frac{n}{x}+1\right)^{\nu+k+1}}\left\{\sum_{\chi even}\chi(h)\bar{\chi}(n/d)-\chi_0(n/d)\right\}\\
&=\frac{1}{2}\sum_{d=1}^{\infty}d^k\sum_{\substack{r=1\\ r\equiv\pm h(mod q)}}^{\infty}\frac{1}{\left(\frac{16\pi^2}{a^2q}\frac{dr}{x}+1\right)^{\nu+k+1}}-\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\frac{(\sigma_k(n)-\sigma_k(n/q))}{\left(\frac{16\pi^2}{a^2q}\frac{n}{x}+1\right)^{\nu+k+1}}\\
&=\frac{1}{2}\sum_{d=1}^{\infty}d^k\sum_{m=0}^{\infty}\left\{\frac{1}{\left(\frac{16\pi^2}{a^2q}\frac{d(mq+h)}{x}+1\right)^{\nu+k+1}}+\frac{1}{\left(\frac{16\pi^2}{a^2q}\frac{d(mq+q-h)}{x}+1\right)^{\nu+k+1}}\right\}\\
&-\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{\left(\frac{16\pi^2}{a^2q}\frac{n}{x}+1\right)^{\nu+k+1}}+\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\frac{\sigma_k(n/q)}{\left(\frac{16\pi^2}{a^2q}\frac{n}{x}+1\right)^{\nu+k+1}}\\
&=\frac{1}{2}\sum_{d=1}^{\infty}d^k\sum_{m=0}^{\infty}\left\{\frac{1}{\left(\frac{16\pi^2}{a^2}\frac{d(m+h/q)}{x}+1\right)^{\nu+k+1}}+\frac{1}{\left(\frac{16\pi^2}{a^2}\frac{d(m+1-h/q)}{x}+1\right)^{\nu+k+1}}\right\}\\
&-\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{\left(\frac{16\pi^2}{a^2q}\frac{n}{x}+1\right)^{\nu+k+1}}+\frac{1}{\phi(q)}\sum_{r=1}^{\infty}\frac{\sigma_k(r)}{\left(\frac{16\pi^2}{a^2}\frac{r}{x}+1\right)^{\nu+k+1}}.
\end{aligned}\tag{3.58}$$

Substituting (3.57), (3.58) in (3.56), we obtain

$$\begin{aligned}
&\frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\tau(\bar{\chi})\sum_{n=1}^{\infty}\sigma_{k,\chi}(n)n^{\nu/2}K_\nu(a\sqrt{nx})\\
&=\frac{(-1)^{\frac{k-1}{2}}k!}{2(2\pi)^{k+1}}X^{\frac{\nu}{2}}\Gamma(\nu)\left\{\frac{1}{2}\left(\zeta(k+1,h/q)+\zeta(k+1,1-h/q)\right)-\frac{q^{k+1}-1}{\phi(q)}\zeta(k+1)\right\}
\end{aligned}$$

$$+\frac{(-1)^{\frac{k+1}{2}}}{4}X^{\frac{\nu}{2}}(2\pi X)^{k+1}\Gamma(\nu+k+1)\sum_{d=1}^{\infty}d^k\sum_{m=0}^{\infty}\Bigg\{\frac{1}{\left(\frac{16\pi^2}{a^2}\frac{d(m+h/q)}{x}+1\right)^{\nu+k+1}}$$
$$+\frac{1}{\left(\frac{16\pi^2}{a^2}\frac{d(m+1-h/q)}{x}+1\right)^{\nu+k+1}}\Bigg\}$$
$$-\frac{(-1)^{\frac{k+1}{2}}}{2\phi(q)}X^{\frac{\nu}{2}}(2\pi X)^{k+1}\Gamma(\nu+k+1)\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{\left(\frac{16\pi^2}{a^2q}\frac{n}{x}+1\right)^{\nu+k+1}}$$
$$+\frac{(-1)^{\frac{k+1}{2}}}{2\phi(q)}X^{\frac{\nu}{2}}(2\pi X)^{k+1}\Gamma(\nu+k+1)\sum_{r=1}^{\infty}\frac{\sigma_k(r)}{\left(\frac{16\pi^2}{a^2}\frac{r}{x}+1\right)^{\nu+k+1}}. \tag{3.59}$$

Inserting (3.55) and (3.59) into (3.54), we get (3.51).
Theorem 3.2.8 ⇒ Theorem 3.2.7 Let $\theta = h/q$, and χ be an even primitive non-principal character modulo q. Multiplying the identity (3.51) in Theorem 3.2.8 by $\bar{\chi}(h)/\tau(\bar{\chi})$, and then summing on h, $0 < h < q$. The remaining steps are similar to the proof of Theorem 3.2.1. □

Our next result, corresponding to $\nu = 0$, is as follows:

Theorem 3.2.9. *Let $k \geq 1$ be an odd integer and χ be a non-principal even primitive Dirichlet character modulo q. Then*

$$\sum_{n=1}^{\infty}\sigma_{k,\chi}(n)K_0(a\sqrt{nx}) = -\frac{L(-k,\chi)}{4}\left(\log\left(\frac{8\pi}{a^2}\right)+\frac{L'(-k,\chi)}{L(-k,\chi)}-2\gamma\right)$$
$$+\frac{L(-k,\chi)}{4}\log x+(-1)^{\frac{k-1}{2}}\frac{k!q^k}{2(2\pi)^{k+1}}\tau(\chi)\sum_{n=1}^{\infty}\bar{\sigma}_{k,\bar{\chi}}(n)\left(\frac{1}{n^{k+1}}-\frac{1}{(n+\frac{a^2qx}{16\pi^2})^{k+1}}\right).$$

Proof. Here, we will take $f_k(n) = \sigma_{k,\chi}(n)$ and χ being a non-principal even primitive Dirichlet character modulo q and $k \geq 1$ an odd integer in (3.9). Proceeding by almost identically the same argument as in the proof of Theorems 3.2.1 and 3.2.3, one can deduce the result. We leave the details of the proofs for the reader. □

Theorem 3.2.10. *Let $k \geq 1$ be an odd integer and χ be a non-principal even* *primitive Dirichlet character modulo q. Then, for any $\Re(\nu) > 0$,*

$$\sum_{n=1}^{\infty} \bar{\sigma}_{k,\chi}(n) n^{\frac{\nu}{2}} K_\nu(a\sqrt{nx}) = \frac{2^{\nu+2k+1}}{a^{\nu+2k+2}} \Gamma(k+1)\Gamma(\nu+k+1) L(1+k,\chi)\ x^{-\frac{\nu}{2}-k-1}$$
$$+ \frac{(-1)^{\frac{k+1}{2}} (aq)^\nu\ x^{\frac{\nu}{2}}}{2^{3\nu+k+2}\pi^{2\nu+k+1}} \Gamma(\nu+k+1)\tau(\chi) \sum_{n=1}^{\infty} \sigma_{k,\bar{\chi}}(n) \frac{1}{\left(n + \frac{a^2 qx}{16\pi^2}\right)^{\nu+k+1}}.$$

Analogous to Theorem 3.2.7, Theorem 3.2.10 is equivalent to the following result.

Theorem 3.2.11. *Let $k \geq 1$ be an odd integer. Then for any $\Re(\nu) > 0$ we have*

$$\left(\frac{a^2 x}{4}\right)^{\frac{\nu}{2}+k+1} \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k \cos\left(\frac{2\pi n\theta}{d}\right)$$
$$= \frac{(-1)^{\frac{k+1}{2}} (2\pi)^{k+1}}{8} \Gamma(\nu+k+1) \left\{\zeta(-k,\theta) + \zeta(-k,1-\theta)\right\}$$
$$+ \frac{(-1)^{\frac{k+1}{2}} (2\pi)^{k+1}}{4} \Gamma(\nu+k+1) \sum_{r=1}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{(m+\theta)^k}{\left(\frac{16\pi^2}{a^2} \frac{r(m+\theta)}{x} + 1\right)^{\nu+k+1}} \right.$$
$$\left. + \frac{(m+1-\theta)^k}{\left(\frac{16\pi^2}{a^2} \frac{r(m+1-\theta)}{x} + 1\right)^{\nu+k+1}} \right\} - \frac{a^{2k+2}}{2^{2k+4}} \zeta(-k)\Gamma(\nu) x^{k+1}.$$

Proof of Theorem 3.2.10 and its equivalence with Theorem 3.2.11. The equivalence of Theorems 3.2.10 and 3.2.11 can be derived in a similar way as in the proof of Theorems 3.2.7 and 3.2.8, respectively. To avoid repetitions, we skip the details of the proof. □

The result corresponding to $\nu = 0$ is as follows:

Theorem 3.2.12. *Let $k \geq 1$ be an odd integer and χ be a non-principal even primitive Dirichlet character modulo q. Then*

$$\sum_{n=1}^{\infty} \bar{\sigma}_{k,\chi}(n) K_0(a\sqrt{nx}) = \frac{2^{2k+1}}{a^{2k+2}} \Gamma^2(k+1) L(k+1,\chi) \frac{1}{x^{k+1}} + \frac{1}{2}\zeta'(-k) L(0,\chi)$$
$$+ \frac{(-1)^{\frac{k-1}{2}} k!}{2(2\pi)^{k+1}} \tau(\chi) \sum_{n=1}^{\infty} \sigma_{k,\bar{\chi}}(n) \left(\frac{1}{n^{k+1}} - \frac{1}{(n + \frac{a^2 qx}{16\pi^2})^{k+1}}\right).$$

Proof. The proof is similar to the proof of Theorem 3.2.6. Thus, we omit the details. □

The next result corresponds to the case $\nu = 0$ and $k = 0$. We can also claim the positivity of $L(1, \chi)$ for even real character χ from the following identity.

Theorem 3.2.13. *Let χ be a non-principal even primitive Dirichlet character modulo q. Then we have*

$$\sum_{n=1}^{\infty} d_{\chi}(n) K_0(a\sqrt{nx}) = \frac{2}{a^2 x} L(1, \chi) - \frac{\tau(\chi)}{8} L(1, \bar{\chi}) + \frac{a^2 q\ x}{32\pi^4} \tau(\chi) \sum_{n=1}^{\infty} d_{\bar{\chi}}(n) \frac{\log\left(\frac{16\pi^2 n}{a^2 qx}\right)}{n^2 - \left(\frac{a^2 qx}{16\pi^2}\right)^2}, \tag{3.60}$$

provided $\frac{a^2 qx}{16\pi^2} \notin \mathbb{Z}_+$.

Proof. It deals with the special case $k = 0$ and $\nu = 0$ when χ is a non-principal even primitive character modulo q. Thus setting $f_0(n) = d_{\chi}(n)$ in (3.9), we obtain

$$\sum_{n=1}^{\infty} d_{\chi}(n) K_0(a\sqrt{nx}) = \frac{1}{2}(R_1 + R_0 + J_0^{(0)}(X)), \tag{3.61}$$

where the residues R_1 and R_0 are given by

$$R_1 = L(1, \chi) X, \tag{3.62}$$

$$R_0 = -\frac{1}{2} L'(0, \chi) = -\frac{\tau(\chi)}{4} L(1, \bar{\chi}), \tag{3.63}$$

where in (3.63), we have used [21, p. 181, equation (3.2)]. Next, we evaluate $J_0^{(0)}(X)$ defined in (3.10) with $F_0(s) = \zeta(s) L(s, \chi)$. Utilizing the functional equations (2.9) and (2.13), one can get

$$J_0^{(0)}(X) = \frac{\tau(\chi)}{4} Y \sum_{n=1}^{\infty} d_{\bar{\chi}}(n) \frac{1}{2\pi i} \int_{(1+d)} \frac{(nY)^{-s}}{\sin^2(\pi s/2)} \mathrm{d}s, \tag{3.64}$$

where $Y = \frac{4\pi^2 X}{q}$. As $0 < d < 1$, applying inverse Mellin transform to (2.33) of

Lemma 2.2.7 and then employing the formula in (3.64), we deduce that

$$J_0^{(0)}(X) = \frac{\tau(\chi)}{\pi^2} Y \sum_{n=1}^{\infty} d_{\bar{\chi}}(n) \frac{\log(nY)}{(nY)^2 - 1}. \tag{3.65}$$

Inserting (3.62), (3.63) and (3.65) in (3.61) and noting $Y = \frac{16\pi^2}{a^2 q x}$, one can complete the proof of (3.60). □

3.2.4 Non-vanishing of $L(1, \chi)$ for real character

In this subsection, we provide a new proof of the positivity of $L(1, \chi)$ for all real primitive characters χ, based on two of our main results- Theorem 3.2.3 and Theorem 3.2.13. This result plays a key role in the proof of Dirichlet's theorem on the infinitude of primes in arithmetic progressions.

- **Case 1. When χ is an odd real primitive character**

Suppose that χ is a real odd primitive Dirichlet character modulo q and setting $k = 0$ and then employing the functional equation (2.13) in (3.33), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} d_{\chi}(n) K_0(a\sqrt{nx}) &= \frac{L(1,\chi)}{x}\left(\frac{2}{a^2} - \frac{i\tau(\chi)}{4\pi} x \log x\right) \\ - \frac{L(0,\chi)}{4}\left(\log\left(\frac{8\pi}{a^2}\right) + \frac{L'(0,\chi)}{L(0,\chi)} - 2\gamma\right) &+ \frac{ia^2 q\, x}{64\pi^3}\tau(\chi) \sum_{n=1}^{\infty} \frac{d_{\bar{\chi}}(n)}{n(n + \frac{a^2 q x}{16\pi^2})}. \end{aligned} \tag{3.66}$$

Now, we can easily show that $d_\chi(n)$ is non-negative for each n from the Euler product on the left-hand side of (2.5). More precisely, the factors in its Euler product are of the forms

$$\left(1 - \frac{1}{p^s}\right)^{-1}, \left(1 - \frac{1}{p^s}\right)^{-2}, \text{ or } \left(1 - \frac{1}{p^{2s}}\right)^{-1},$$

according as to whether $\chi(p) = 0, 1$ or -1 respectively. Therefore, by rewriting the Euler product as a Dirichlet series, one can easily notice that $d_\chi(n) \geq 0$ for all n. In addition, it is clear from (1.22) that $d_\chi(n) \geq 1$ whenever n is a perfect square. We have already mentioned the fact that $K_0(x)$ tends to $+\infty$ as x decreases to 0 at the beginning of Section 1.1 in Chapter 1. Therefore, the left-hand side of (3.66) approaches $+\infty$ as x decreases to 0. Let us examine the right-hand side of (3.66).

Noting that $i\tau(\chi)$ is real for real odd primitive Dirichlet character [42, Theorem 9.9, p. 288]

$$L(1,\chi) = \begin{cases} -\frac{\tau(\chi)}{q}\sum_{a=1}^{q-1}\bar{\chi}(a)\log\left(\sin\frac{\pi a}{q}\right), & \text{if } \chi \text{ is even primitive chracter} \\ \frac{i\pi\tau(\chi)}{q^2}\sum_{a=1}^{q-1}a\bar{\chi}(a), & \text{if } \chi \text{ is odd primitive character,} \end{cases} \tag{3.67}$$

we can easily deduce that the infinite series on the right-hand side of (3.66) tends to 0 as x decreases to 0. Next noting that $i\tau(\chi)$ is real and $x\log x$ tends to 0 as x decreases to 0, we infer that $\frac{L(1,\chi)}{x}$ tends to $+\infty$ as x decreases to 0, which ensures the strict positivity of $L(1,\chi)$.

- **Case 2. When χ is an even real primitive character**

Let us rewrite (3.60) for real even primitive Dirichlet character modulo q,

$$\sum_{n=1}^{\infty} d_\chi(n)K_0(a\sqrt{nx}) = \frac{2}{a^2x}L(1,\chi) - \frac{\tau(\chi)}{8}L(1,\bar{\chi}) + \frac{a^2q\ x}{32\pi^4}\tau(\chi)\sum_{n=1}^{\infty}d_{\bar{\chi}}(n)\frac{\log\left(\frac{16\pi^2 n}{a^2qx}\right)}{n^2-\left(\frac{a^2qx}{16\pi^2}\right)^2},$$

provided $\frac{a^2qx}{16\pi^2} \notin \mathbb{Z}_+$. In this case, we can show that $d_\chi(n)$ is non-negative for every n using the same justifications as in Case 1. From (1.22), it can be easily seen that $d_\chi(n) \geq 1$ whenever n is a perfect square. As $K_0(x)$ tends to $+\infty$ as x decreases to 0, the left-hand side of (3.60) approaches $+\infty$ as x decreases to 0. Now, the infinite series in the right-hand side of (3.60) decreases rapidly as x decreases to 0. Therefore, we arrive at the conclusion that $\frac{2}{a^2x}L(1,\chi)$ tends to $+\infty$ as x decreases to 0 which proves the strict positivity of $L(1,\chi)$.

Next, we are going to investigate the identities involving two characters and their equivalent version in two trigonometric functions, which are the following:

3.2.5 Identities involving two characters

In this subsection, we provide the identities corresponding to

$$\sigma_{k,\chi_1,\chi_2}(n)=\sum_{d/n} d^k\chi_1(d)\chi_2(n/d),$$

where χ_1 and χ_2 are Dirichlet characters modulo p and q, respectively.

Theorem 3.2.14. *Let $k\geq 1$ be an odd integer. Let χ_1 and χ_2 be primitive characters modulo p and q, respectively, such that either both are non-principal even characters or both are odd characters. Then, for any $\Re(\nu)>0$,*

$$\sum_{n=1}^{\infty}\sigma_{k,\chi_1,\chi_2}(n)n^{\frac{\nu}{2}}K_\nu(a\sqrt{nx})$$
$$=\frac{(-1)^{\frac{k+1}{2}}(aq)^\nu p^{\nu+k}\ x^{\frac{\nu}{2}}}{2^{3\nu+k+2}\pi^{2\nu+k+1}}\tau(\chi_1)\tau(\chi_2)\Gamma(\nu+k+1)\sum_{n=1}^{\infty}\frac{\sigma_{k,\bar{\chi_2},\bar{\chi_1}}(n)}{\left(n+\frac{a^2pqx}{16\pi^2}\right)^{\nu+k+1}}. \tag{3.68}$$

Theorem 3.2.14 is the equivalent version of the following result.

Theorem 3.2.15. *Let $k\geq 1$ be an odd integer. Then for any $\Re(\nu)>0$ we have*

$$\left(\frac{a^2x}{4}\right)^{\frac{\nu}{2}+k+1}\sum_{n=1}^{\infty}n^{\nu/2}K_\nu(a\sqrt{nx})\sum_{d|n}d^k\sin(2\pi d\theta)\sin\left(\frac{2\pi n\psi}{d}\right)$$
$$=-\frac{(-1)^{\frac{k+1}{2}}}{8\,(2\pi)^{-k-1}}\Gamma(\nu+k+1)\sum_{m,n\geq 0}\left\{\frac{(n+\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+\psi)(m+\theta)}{x}+1\right)^{\nu+k+1}}\right.$$
$$-\frac{(n+1-\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+1-\psi)(m+\theta)}{x}+1\right)^{\nu+k+1}}-\frac{(n+\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+\psi)(m+1-\theta)}{x}+1\right)^{\nu+k+1}}$$
$$\left.+\frac{(n+1-\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+1-\psi)(m+1-\theta)}{x}+1\right)^{\nu+k+1}}\right\}. \tag{3.69}$$

For deriving the equivalence of our next result with Theorem 3.2.14, we need the help of Theorem 3.2.8 and Theorem 3.2.11.

Theorem 3.2.16. *Let $k \geq 1$ be an odd integer. Then for any $\Re(\nu) > 0$ we have*

$$\left(\frac{a^2x}{4}\right)^{\frac{\nu}{2}+k+1} \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k \cos(2\pi d\theta) \cos\left(\frac{2\pi n\psi}{d}\right)$$
$$= \frac{(-1)^{\frac{k-1}{2}} a^{2k+2} k!}{2^{2k+4}(2\pi)^{k+1}} \Gamma(\nu) \left\{\zeta(k+1,\theta) + \zeta(k+1,1-\theta)\right\} x^{k+1}$$
$$+ \frac{(-1)^{\frac{k+1}{2}}}{8} (2\pi)^{k+1} \Gamma(\nu+k+1) \sum_{n,m\geq 0} \left\{ \frac{(n+\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(m+\theta)(n+\psi)}{x}+1\right)^{\nu+k+1}} \right.$$
$$+ \frac{(n+1-\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(m+\theta)(n+1-\psi)}{x}+1\right)^{\nu+k+1}} + \frac{(n+\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(m+1-\theta)(n+\psi)}{x}+1\right)^{\nu+k+1}}$$
$$\left. + \frac{(n+1-\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(m+1-\theta)(n+1-\psi)}{x}+1\right)^{\nu+k+1}} \right\}.$$

Proof of Theorem 3.2.14 and its equivalence with Theorem 3.2.15. Here, we see the individual proofs of both results.

Proof of Theorem 3.2.14 We will take $f_k(n) = \sigma_{k,\chi_1,\chi_2}(n)$ and $k \geq 1$ an odd integer and $\Re(\nu) > 0$ in (3.9). By assumption, χ_1 and χ_2 are primitive characters modulo p and q, respectively, such that either both are non-principal even characters or both are odd characters. In the notation of (3.5), $F_k(s) = L(s-k,\chi_1)L(s,\chi_2)$. We get

$$\sum_{n=1}^{\infty} \sigma_{k,\chi_1,\chi_2}(n) K_\nu(a\sqrt{nx}) = \frac{1}{2} X^{\nu/2}(R_{k+1} + R_1 + R_0 + J_k^{(\nu)}(X)), \tag{3.70}$$

where $J_k^{(\nu)}(X)$ is defined in (3.10). It is easy to see that $R_{k+1} = 0$ and $R_1 = 0$. When both χ_1 and χ_2 are non-principal even primitive characters, $L(s,\chi_2)$ has a zero at $s = 0$. Hence, we will not be getting any contribution from the pole of $\Gamma(s)$ at $s = 0$. As a result, we will get $R_0 = 0$. If both χ_1 and χ_2 are odd primitive characters, $L(s-k,\chi_1)$ has a zero at $s = 0$ since k is an odd integer. Again, there will be no contribution of the pole from $\Gamma(s)$ at $s = 0$. Therefore $R_0 = 0$. Now utilizing the facts $R_{k+1}=0$, $R_1=0$ and $R_0=0$ in (3.70), we obtain

$$\sum_{n=1}^{\infty}\sigma_{k,\chi_1,\chi_2}(n)K_\nu(a\sqrt{nx})=\frac{1}{2}X^{\nu/2}J_k^{(\nu)}(X). \tag{3.71}$$

To evaluate $J_k^{(\nu)}(X)$, we utilize the functional equations (2.13), (2.14) with (2.21)

$$J_k^{(\nu)}(X)=Y_2 g_k\sum_{n=1}^{\infty}\sigma_{-k,\bar{\chi_1},\bar{\chi_2}}(n)\ \frac{1}{2\pi i}\int_{(1+d)}\Gamma(1-s+\nu)\Gamma(k+s)(nY_2)^{-s}\mathrm{d}s, \tag{3.72}$$

where $g_k=\frac{(-1)^{\frac{k+1}{2}}p^k\tau(\chi_1)\tau(\chi_2)}{(2\pi)^{k+1}}$ and $Y_2=\frac{4\pi^2}{pq}X$. As $0<d<1$, appealing to Lemma 2.2.5 with $n=k$ and $a=1+\nu$, we deduce

$$J_k^{(\nu)}(X)={Y_2}^{k+1}g_k\Gamma(1+\nu+k)\sum_{n=1}^{\infty}\frac{\sigma_{k,\bar{\chi_2},\bar{\chi_1}}(n)}{(1+nY_2)^{1+\nu+k}}, \tag{3.73}$$

where we have used the fact $\sigma_{-k,\bar{\chi_1},\bar{\chi_2}}(n)=n^{-k}\sigma_{k,\bar{\chi_2},\bar{\chi_1}}(n)$. We complete the proof of (3.68) by substituting (3.73) in (3.71) and remarking $Y_2=\frac{16\pi^2}{a^2pqx}$.

Proof of Theorem 3.2.15 It is sufficient to prove the theorem for rationals $\theta=h_1/p$ and $\psi=h_2/q$ where p and q are primes, $0<h_1<p$ and $0<h_2<q$. Employing Lemma 2.2.8 with $t=n$ and subsequently interchanging the summation and integration, we get for odd integer $k\geq 1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{d|n}d^k\sin\left(\frac{2\pi dh_1}{p}\right)\sin\left(\frac{2\pi nh_2}{dq}\right)n^{\nu/2}K_\nu(a\sqrt{nx})\\
&=\frac{1}{2}\left(\frac{2}{a\sqrt{x}}\right)^{\nu}\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\Gamma(s+\nu)\left(\frac{4}{a^2x}\right)^{s}\sum_{n=1}^{\infty}\sum_{d|n}d^k\sin\left(\frac{2\pi dh_1}{p}\right)\\
&\qquad\times\sin\left(\frac{2\pi nh_2}{dq}\right)n^{-s}\mathrm{d}s\\
&=\frac{1}{2}\left(\frac{2}{a\sqrt{x}}\right)^{\nu}\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\Gamma(s+\nu)\left(\frac{4}{a^2x}\right)^{s}\sum_{d=1}^{\infty}d^{k-s}\sin\left(\frac{2\pi dh_1}{p}\right)\\
&\qquad\times\sum_{m=1}^{\infty}m^{-s}\sin\left(\frac{2\pi mh_2}{q}\right)\mathrm{d}s
\end{aligned}$$

$$= \frac{p^k}{2} X^{\nu/2} \frac{1}{2\pi i} \int_{(c)} \Gamma(s)\Gamma(s+\nu) \sum_{r_1=1}^{p} \zeta\left(s-k, \frac{r_1}{p}\right) \sin\left(\frac{2\pi r_1 h_1}{p}\right)$$
$$\times \sum_{r_2=1}^{q} \zeta\left(s, \frac{r_2}{q}\right) \sin\left(\frac{2\pi r_2 h_2}{q}\right) (p^{-1}q^{-1}X)^s \mathrm{d}s, \tag{3.74}$$

where $c > k+1$ and $X = \frac{4}{a^2 x}$. Next, we invoke the functional relations (2.12) and (2.17) in (3.74), we obtain

$$\sum_{n=1}^{\infty} \sum_{d|n} d^k \sin\left(\frac{2\pi d h_1}{p}\right) \sin\left(\frac{2\pi n h_2}{dq}\right) n^{\nu/2} K_\nu(a\sqrt{nx})$$
$$= \frac{(-1)^{\frac{k-1}{2}}}{8(2\pi)^{k+1}} X^{\nu/2} \frac{1}{2\pi i} \int_{(c)} \Gamma(s+\nu)\Gamma(k+1-s) \left\{\zeta\left(k+1-s, \frac{h_1}{p}\right)\right.$$
$$\left.-\zeta\left(k+1-s, 1-\frac{h_1}{p}\right)\right\} \left\{\zeta\left(1-s, \frac{h_2}{q}\right) - \zeta\left(1-s, 1-\frac{h_2}{q}\right)\right\} (4\pi^2 X)^s \mathrm{d}s. \tag{3.75}$$

Now proceeding in the same way as in the proof of Theorem 3.2.2, we shift the line of integration to $\Re(s) = -d$ (with $0 < d < \min\{1, \Re(\nu)\}$), then replace $1-s$ by s to obtain

$$\sum_{n=1}^{\infty} \sum_{d|n} d^k \sin\left(\frac{2\pi d h_1}{p}\right) \sin\left(\frac{2\pi n h_2}{dq}\right) n^{\nu/2} K_\nu(a\sqrt{nx})$$
$$= \frac{(-1)^{\frac{k-1}{2}} \pi^2}{2(2\pi)^{k+1}} X^{1+\nu/2} \frac{1}{2\pi i} \int_{(1+d)} \Gamma(\nu+1-s)\Gamma(k+s) \left\{\zeta\left(k+s, \frac{h_1}{p}\right)\right.$$
$$\left.-\zeta\left(k+s, 1-\frac{h_1}{p}\right)\right\} \left\{\zeta\left(s, \frac{h_2}{q}\right) - \zeta\left(s, 1-\frac{h_2}{q}\right)\right\} (4\pi^2 X)^{-s} \mathrm{d}s$$
$$= \frac{(-1)^{\frac{k-1}{2}} \pi^2}{2(2\pi)^{k+1}} X^{1+\nu/2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \{$$
$$(n+h_1/p)^{-k} \frac{1}{2\pi i} \int_{(1+d)} \Gamma(\nu+1-s)\Gamma(k+s) \left(4\pi^2 X(n+h_1/p)(m+h_2/q)\right)^{-s} \mathrm{d}s$$
$$-(n+1-h_1/p)^{-k} \frac{1}{2\pi i} \int_{(1+d)} \Gamma(\nu+1-s)\Gamma(k+s)$$
$$\times \left(4\pi^2 X(n+1-h_1/p)(m+h_2/q)\right)^{-s} \mathrm{d}s$$
$$-(n+h_1/p)^{-k} \frac{1}{2\pi i} \int_{(1+d)} \Gamma(\nu+1-s)\Gamma(k+s)$$

$$\times \left(4\pi^2 X(n+h_1/p)(m+1-h_2/q)\right)^{-s} \mathrm{d}s$$
$$+(n+1-h_1/p)^{-k}\frac{1}{2\pi i}\int_{(1+d)} \Gamma(\nu+1-s)\Gamma(k+s)$$
$$\times \left(4\pi^2 X(n+1-h_1/p)(m+1-h_2/q)\right)^{-s} \mathrm{d}s\Big\}. \tag{3.76}$$

We get (3.69) by applying Lemma 2.2.5 for $n = k$ and $a = 1+\nu$ to each of these integrals in (3.76).

The equivalence of Theorems 3.2.14 and 3.2.15 can be easily deduced. So, we leave the details for the reader. □

Proof of Theorem 3.2.14 and its equivalence with Theorem 3.2.16. The proof of Theorem 3.2.16 is similar to Theorem 3.2.15, so we skip the proof. But we would show its equivalence with Theorem 3.2.14.

Theorem 3.2.14 ⇒ Theorem 3.2.16 It is sufficient to prove the theorem for rationals $\theta = h_1/p$ and $\psi = h_2/q$ where p and q are primes, $0 < h_1 < p$ and $0 < h_2 < q$. We multiply both sides of identity (3.68) in Theorem 3.2.14 by $\chi_1(h_1)\tau(\bar{\chi_1})/\phi(p)$ and $\chi_2(h_2)\tau(\bar{\chi_2})/\phi(q)$, then sum on non-principal primitive even character χ_1 modulo p and χ_2 modulo q. Using (2.25), the left-hand side of (3.68) becomes

$$\frac{1}{\phi(p)\phi(q)}\sum_{\substack{\chi_2\neq\chi_0\\ \chi_2\ even}}\chi_2(h_2)\tau(\bar{\chi_2})\sum_{\substack{\chi_1\neq\chi_0\\ \chi_1\ even}}\chi_1(h_1)\tau(\bar{\chi_1})\sum_{n=1}^{\infty}\sigma_{k,\chi_1,\chi_2}(n)n^{\nu/2}K_\nu(a\sqrt{nx})$$
$$=\frac{1}{\phi(p)\phi(q)}\sum_{n=1}^{\infty}n^{\nu/2}K_\nu(a\sqrt{nx})\sum_{d|n}d^k\left\{\sum_{\substack{\chi_2\neq\chi_0\\ \chi_2\ even}}\chi_2(h_2)\chi_2(n/d)\tau(\bar{\chi_2})\right\}$$
$$\times\left\{\sum_{\substack{\chi_1\neq\chi_0\\ \chi_1\ even}}\chi_1(h_1)\chi_1(d)\tau(\bar{\chi_1})\right\}$$
$$=\sum_{n=1}^{\infty}n^{\nu/2}K_\nu(a\sqrt{nx})\sum_{d|n}d^k\left\{\frac{1}{\phi(q)}\sum_{\chi_2\ even}\chi_2(h_2)\chi_2(n/d)\tau(\bar{\chi_2})+\frac{\chi_0(n/d)}{\phi(q)}\right\}$$
$$\times\left\{\frac{1}{\phi(p)}\sum_{\chi_1\ even}\chi_1(h_1)\chi_1(d)\tau(\bar{\chi_1})+\frac{\chi_0(d)}{\phi(p)}\right\}$$

$$= \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{\substack{d|n \\ (p,d)=(q,n/d)=1}} d^k \left\{ \cos\left(\frac{2\pi n h_2}{dq}\right) \cos\left(\frac{2\pi d h_1}{p}\right) \right.$$

$$\left. + \frac{1}{\phi(p)} \cos\left(\frac{2\pi n h_2}{dq}\right) + \frac{1}{\phi(q)} \cos\left(\frac{2\pi d h_1}{p}\right) + \frac{1}{\phi(p)\phi(q)} \right\}$$

$$= \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k \left\{ \cos\left(\frac{2\pi n h_2}{dq}\right) \cos\left(\frac{2\pi d h_1}{p}\right) + \frac{1}{\phi(p)} \cos\left(\frac{2\pi n h_2}{dq}\right) \right.$$

$$\left. + \frac{1}{\phi(q)} \cos\left(\frac{2\pi d h_1}{p}\right) + \frac{1}{\phi(p)\phi(q)} \right\}$$

$$- \frac{p}{\phi(p)} \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{\substack{d|n \\ p|d}} d^k \left\{ \cos\left(\frac{2\pi n h_2}{dq}\right) + \frac{1}{\phi(q)} \right\}$$

$$- \frac{q}{\phi(q)} \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{\substack{d|n \\ q|\frac{n}{d}}} d^k \left\{ \cos\left(\frac{2\pi d h_1}{p}\right) + \frac{1}{\phi(p)} \right\}$$

$$+ \frac{pq}{\phi(p)\phi(q)} \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{\substack{d|n \\ p|d,\ q|\frac{n}{d}}} d^k$$

$$= \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k \cos\left(\frac{2\pi n h_2}{dq}\right) \cos\left(\frac{2\pi d h_1}{p}\right)$$

$$+ \left\{ \frac{1}{\phi(p)} \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k \cos\left(\frac{2\pi n h_2}{dq}\right) \right.$$

$$\left. - \frac{p^{\frac{\nu}{2}+k+1}}{\phi(p)} \sum_{m=1}^{\infty} m^{\nu/2} K_\nu(a\sqrt{mpx}) \sum_{d|m} d^k \cos\left(\frac{2\pi m h_2}{dq}\right) \right\}$$

$$+ \left\{ \frac{1}{\phi(q)} \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k \cos\left(\frac{2\pi d h_1}{p}\right) \right.$$

$$\left. - \frac{q^{\frac{\nu}{2}+1}}{\phi(q)} \sum_{m=1}^{\infty} m^{\nu/2} K_\nu(a\sqrt{mqx}) \sum_{d|m} d^k \cos\left(\frac{2\pi d h_1}{p}\right) \right\}$$

$$+ \left\{ \frac{1}{\phi(p)\phi(q)} \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k - \frac{p^{\frac{\nu}{2}+k+1}}{\phi(p)\phi(q)} \sum_{m=1}^{\infty} m^{\nu/2} K_\nu(a\sqrt{mpx}) \sum_{d|m} d^k \right.$$

$$-\frac{q^{\frac{\nu}{2}+1}}{\phi(p)\phi(q)}\sum_{m=1}^{\infty}m^{\nu/2}K_\nu(a\sqrt{mqx})\sum_{d|m}d^k$$
$$\left.+\frac{p^{\frac{\nu}{2}+k+1}q^{\frac{\nu}{2}+1}}{\phi(p)\phi(q)}\sum_{m=1}^{\infty}m^{\nu/2}K_\nu(a\sqrt{mpqx})\sum_{d|m}d^k\right\}. \tag{3.77}$$

Employing Theorem 3.2.11 with $\theta = h_2/q$, we evaluate the second and third terms on the right-hand side of (3.77) as follows:

$$\frac{1}{\phi(p)}\sum_{n=1}^{\infty}n^{\nu/2}K_\nu(a\sqrt{nx})\sum_{d/n}d^k\cos\left(\frac{2\pi nh_2}{dq}\right)$$
$$-\frac{p^{\frac{\nu}{2}+k+1}}{\phi(p)}\sum_{m=1}^{\infty}m^{\nu/2}K_\nu(a\sqrt{mpx})\sum_{d|m}d^k\cos\left(\frac{2\pi mh_2}{dq}\right)$$
$$=\frac{(-1)^{\frac{k+1}{2}}(2\pi X)^{k+1}}{4q^k\phi(p)}X^{\frac{\nu}{2}}\Gamma(\nu+k+1)\left\{\sum_{r=1}^{\infty}\sum_{\substack{d=1\\ d\equiv\pm h_2(q)}}^{\infty}\frac{d^k}{\left(\frac{16\pi^2}{a^2q}\frac{rd}{x}+1\right)^{\nu+k+1}}\right.$$
$$\left.-\sum_{r=1}^{\infty}\sum_{\substack{d=1\\ d\equiv\pm h_2(q)}}^{\infty}\frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{rd}{x}+1\right)^{\nu+k+1}}\right\}+\frac{\Gamma(\nu)\zeta(-k)}{4\phi(p)}X^{\frac{\nu}{2}}\left(p^{k+1}-1\right). \tag{3.78}$$

Using Theorem 3.2.8 with $\theta = h_1/p$, we evaluate the fourth and fifth terms on the right-hand side of (3.77) as follows:

$$\frac{1}{\phi(q)}\sum_{n=1}^{\infty}n^{\nu/2}K_\nu(a\sqrt{nx})\sum_{d|n}d^k\cos\left(\frac{2\pi dh_1}{p}\right)$$
$$-\frac{q^{\frac{\nu}{2}+1}}{\phi(q)}\sum_{m=1}^{\infty}m^{\nu/2}K_\nu(a\sqrt{mqx})\sum_{d|m}d^k\cos\left(\frac{2\pi dh_1}{p}\right)$$
$$=\frac{(-1)^{\frac{k+1}{2}}k!}{4(2\pi)^{k+1}}X^{\frac{\nu}{2}}\Gamma(\nu)\left\{\zeta(k+1,h_1/p)+\zeta(k+1,1-h_1/p)\right\}$$
$$+\frac{(-1)^{\frac{k+1}{2}}}{4\phi(q)}X^{\frac{\nu}{2}}(2\pi X)^{k+1}\Gamma(\nu+k+1)\left\{\sum_{d=1}^{\infty}\sum_{\substack{r=1\\ r\equiv\pm h_1(p)}}^{\infty}\frac{d^k}{\left(\frac{16\pi^2}{a^2p}\frac{rd}{x}+1\right)^{\nu+k+1}}\right.$$

$$-\frac{1}{q^k}\sum_{d=1}^{\infty}\sum_{\substack{r\equiv 1\\ r=\pm h_1(p)}}^{\infty}\frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{rd}{x}+1\right)^{\nu+k+1}}\Bigg\}. \tag{3.79}$$

Using Proposition 1.4.3, we evaluate the last four terms on the right-hand side of (3.77) as follows:

$$\begin{aligned}
&\frac{1}{\phi(p)\phi(q)}\sum_{n=1}^{\infty}n^{\nu/2}K_\nu(a\sqrt{nx})\sum_{d|n}d^k-\frac{p^{\frac{\nu}{2}+k+1}}{\phi(p)\phi(q)}\sum_{m=1}^{\infty}m^{\nu/2}K_\nu(a\sqrt{mpx})\sum_{d|m}d^k\\
&-\frac{q^{\frac{\nu}{2}+1}}{\phi(p)\phi(q)}\sum_{m=1}^{\infty}m^{\nu/2}K_\nu(a\sqrt{mqx})\sum_{d|m}d^k+\frac{p^{\frac{\nu}{2}+k+1}q^{\frac{\nu}{2}+1}}{\phi(p)\phi(q)}\sum_{m=1}^{\infty}m^{\nu/2}K_\nu(a\sqrt{mpqx})\sum_{d|m}d^k\\
&=\frac{(-1)^{\frac{k+1}{2}}}{2\phi(p)\phi(q)}X^{\frac{\nu}{2}}\Gamma(\nu+k+1)\,(2\pi X)^{k+1}\Bigg\{\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{\left(\frac{16\pi^2}{a^2}\frac{n}{x}+1\right)^{\nu+k+1}}\\
&-\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{\left(\frac{16\pi^2}{a^2p}\frac{n}{x}+1\right)^{\nu+k+1}}-\frac{1}{q^k}\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{\left(\frac{16\pi^2}{a^2q}\frac{n}{x}+1\right)^{\nu+k+1}}+\frac{1}{q^k}\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{\left(\frac{16\pi^2}{a^2pq}\frac{n}{x}+1\right)^{\nu+k+1}}\Bigg\}\\
&-\frac{\Gamma(\nu)\zeta(-k)}{4\phi(p)}X^{\frac{\nu}{2}}(p^{k+1}-1).
\end{aligned} \tag{3.80}$$

Substitute (3.78), (3.79) and (3.80) into the right-hand side of (3.77), we deduce the left-hand side of (3.68) as folllows:

$$\begin{aligned}
&\frac{1}{\phi(p)\phi(q)}\sum_{\substack{\chi_2\neq\chi_0\\ \chi_2\ even}}\chi_2(h_2)\tau(\bar{\chi_2})\sum_{\substack{\chi_1\neq\chi_0\\ \chi_1\ even}}\chi_1(h_1)\tau(\bar{\chi_1})\sum_{n=1}^{\infty}\sigma_{k,\chi_1,\chi_2}(n)n^{\nu/2}K_\nu(a\sqrt{nx})\\
&=\sum_{n=1}^{\infty}n^{\nu/2}K_\nu(a\sqrt{nx})\sum_{d|n}d^k\cos\left(\frac{2\pi nh_2}{dq}\right)\cos\left(\frac{2\pi dh_1}{p}\right)\\
&+\frac{(-1)^{\frac{k+1}{2}}(2\pi X)^{k+1}}{4q^k\phi(p)}X^{\frac{\nu}{2}}\Gamma(\nu+k+1)\Bigg\{\sum_{r=1}^{\infty}\sum_{\substack{d=1\\ d\equiv\pm h_2(q)}}^{\infty}\frac{d^k}{\left(\frac{16\pi^2}{a^2q}\frac{rd}{x}+1\right)^{\nu+k+1}}\\
&-\sum_{r=1}^{\infty}\sum_{\substack{d=1\\ d\equiv\pm h_2(q)}}^{\infty}\frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{rd}{x}+1\right)^{\nu+k+1}}\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(-1)^{\frac{k+1}{2}}k!}{4(2\pi)^{k+1}}X^{\frac{\nu}{2}}\Gamma(\nu)\left\{\zeta(k+1,h_1/p)+\zeta(k+1,1-h_1/p)\right\}\\
&+\frac{(-1)^{\frac{k+1}{2}}}{4\phi(q)}X^{\frac{\nu}{2}}(2\pi X)^{k+1}\Gamma(\nu+k+1)\left\{\sum_{d=1}^{\infty}\sum_{\substack{r=1\\ r\equiv\pm h_1(p)}}^{\infty}\frac{d^k}{\left(\frac{16\pi^2}{a^2p}\frac{rd}{x}+1\right)^{\nu+k+1}}\right.\\
&\left.-\frac{1}{q^k}\sum_{d=1}^{\infty}\sum_{\substack{r=1\\ r\equiv\pm h_1(p)}}^{\infty}\frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{rd}{x}+1\right)^{\nu+k+1}}\right\}+\frac{(-1)^{\frac{k+1}{2}}}{2\phi(p)\phi(q)}X^{\frac{\nu}{2}}\Gamma(\nu+k+1)\left(2\pi X\right)^{k+1}\\
&\times\left\{\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{\left(\frac{16\pi^2}{a^2}\frac{n}{x}+1\right)^{\nu+k+1}}-\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{\left(\frac{16\pi^2}{a^2p}\frac{n}{x}+1\right)^{\nu+k+1}}\right.\\
&\left.-\frac{1}{q^k}\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{\left(\frac{16\pi^2}{a^2q}\frac{n}{x}+1\right)^{\nu+k+1}}+\frac{1}{q^k}\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{\left(\frac{16\pi^2}{a^2pq}\frac{n}{x}+1\right)^{\nu+k+1}}\right\}. \qquad (3.81)
\end{aligned}
$$

Next, we examine the right-hand side of (3.68) of Theorem 3.2.14. We find that

$$
\begin{aligned}
&\frac{1}{\phi(p)\phi(q)}\sum_{\substack{\chi_2\neq\chi_0\\ \chi_2\ even}}\chi_2(h_2)\tau(\bar{\chi_2})\sum_{\substack{\chi_1\neq\chi_0\\ \chi_1\ even}}\chi_1(h_1)\tau(\bar{\chi_1})\tau(\chi_1)\tau(\chi_2)\sum_{n=1}^{\infty}\frac{\sigma_{k,\bar{\chi_2},\bar{\chi_1}}(n)}{\left(\frac{16\pi^2}{a^2pq}\frac{n}{x}+1\right)^{\nu+k+1}}\\
&=\frac{pq}{\phi(p)\phi(q)}\sum_{n=1}^{\infty}\sum_{d|n}\frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{n}{x}+1\right)^{\nu+k+1}}\left\{\sum_{\chi_2\ even}\chi_2(h_2)\bar{\chi_2}(d)-\chi_0(d)\right\}\\
&\qquad\times\left\{\sum_{\chi_1\ even}\chi_1(h_1)\bar{\chi_1}(n/d)-\chi_0(n/d)\right\}\\
&=\frac{pq}{\phi(p)\phi(q)}\sum_{d,r\geq1}^{\infty}\frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{dr}{x}+1\right)^{\nu+k+1}}\left\{\sum_{\chi_2\ even}\chi_2(h_2)\bar{\chi_2}(d)-\chi_0(d)\right\}\\
&\qquad\times\left\{\sum_{\chi_1\ even}\chi_1(h_1)\bar{\chi_1}(r)-\chi_0(r)\right\}\\
&=\frac{pq}{4}\sum_{\substack{d=1\\ d\equiv\pm h_2(q)}}^{\infty}\sum_{\substack{r=1\\ r\equiv\pm h_1(p)}}^{\infty}\frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{dr}{x}+1\right)^{\nu+k+1}}-\frac{pq}{2\phi(p)}\sum_{\substack{d=1\\ d\equiv\pm h_2(q)}}^{\infty}\sum_{\substack{r=1\\ p\nmid r}}^{\infty}\frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{dr}{x}+1\right)^{\nu+k+1}}\\
&-\frac{pq}{2\phi(q)}\sum_{\substack{d=1\\ q\nmid d}}^{\infty}\sum_{\substack{r=1\\ r\equiv\pm h_1(p)}}^{\infty}\frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{dr}{x}+1\right)^{\nu+k+1}}+\frac{pq}{\phi(p)\phi(q)}\sum_{\substack{d=1\\ q\nmid d}}^{\infty}\sum_{\substack{r=1\\ p\nmid r}}^{\infty}\frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{dr}{x}+1\right)^{\nu+k+1}}
\end{aligned}
$$

$$= \frac{pq}{4} \sum_{\substack{d=1\\ d\equiv \pm h_2(q)}}^{\infty} \sum_{\substack{r=1\\ r\equiv \pm h_1(p)}}^{\infty} \frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{dr}{x}+1\right)^{\nu+k+1}}$$
$$+ \left\{ -\frac{pq}{2\phi(p)} \sum_{\substack{d=1\\ d\equiv \pm h_2(q)}}^{\infty} \sum_{r=1}^{\infty} \frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{dr}{x}+1\right)^{\nu+k+1}} + \frac{pq}{2\phi(p)} \sum_{\substack{d=1\\ d\equiv \pm h_2(q)}}^{\infty} \sum_{r=1}^{\infty} \frac{d^k}{\left(\frac{16\pi^2}{a^2q}\frac{dr}{x}+1\right)^{\nu+k+1}} \right\}$$

$$- \frac{pq}{2\phi(q)} \sum_{d=1}^{\infty} \sum_{\substack{r=1\\ r\equiv \pm h_1(p)}}^{\infty} \frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{dr}{x}+1\right)^{\nu+k+1}} + \frac{pq^{k+1}}{2\phi(q)} \sum_{d=1}^{\infty} \sum_{\substack{r=1\\ r\equiv \pm h_1(p)}}^{\infty} \frac{d^k}{\left(\frac{16\pi^2}{a^2p}\frac{dr}{x}+1\right)^{\nu+k+1}}$$
$$+ \left\{ \frac{pq}{\phi(p)\phi(q)} \sum_{d=1}^{\infty}\sum_{r=1}^{\infty} \frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{dr}{x}+1\right)^{\nu+k+1}} - \frac{pq^{k+1}}{\phi(p)\phi(q)} \sum_{d=1}^{\infty}\sum_{r=1}^{\infty} \frac{d^k}{\left(\frac{16\pi^2}{a^2p}\frac{dr}{x}+1\right)^{\nu+k+1}} \right.$$
$$\left. -\frac{pq}{\phi(p)\phi(q)} \sum_{d=1}^{\infty}\sum_{r=1}^{\infty} \frac{d^k}{\left(\frac{16\pi^2}{a^2q}\frac{dr}{x}+1\right)^{\nu+k+1}} + \frac{pq^{k+1}}{\phi(p)\phi(q)} \sum_{d=1}^{\infty}\sum_{r=1}^{\infty} \frac{d^k}{\left(\frac{16\pi^2}{a^2}\frac{dr}{x}+1\right)^{\nu+k+1}} \right\}. \tag{3.82}$$

Multiplying equation (3.82) by $\frac{(-1)^{\frac{k+1}{2}}}{2p} X^{\frac{\nu}{2}+k+1} (\frac{2\pi}{q})^{k+1} \Gamma(\nu+k+1)$, we obtain the right-hand side of (3.68). Now equating the resulting expression with (3.81), we get the identity in Theorem 3.2.16.

Theorem 3.2.16 $\Rightarrow$ Theorem 3.2.14 Let $\theta = h_1/p$, $\psi = h_2/q$, and assume that both χ_1 and χ_2 are non-principal even primitive characters modulo p and q, respectively. Multiplying the identity in Theorem 3.2.16 by $\bar{\chi_1}(h_1)\bar{\chi_2}(h_2)/\tau(\bar{\chi_1})\tau(\bar{\chi_2})$, and then summing on h_1 and h_2, $0 < h_1 < p$, $0 < h_2 < q$, one can deduce that Theorem 3.2.16 implies Theorem 3.2.14 .

□

The result corresponding to $\nu = 0$ is as follows:

Theorem 3.2.17. *Let $k \geq 1$ be an odd integer. Assume that χ_1 and χ_2 are primitive characters modulo p and q, respectively, such that either both are non-principal even* *characters or both are odd characters, then*

$$\sum_{n=1}^{\infty}\sigma_{k,\chi_1,\chi_2}(n)K_0(a\sqrt{nx})$$
$$=\frac{1}{2}c_{k,\chi_1,\chi_2}+\frac{(-1)^{\frac{k-1}{2}}k!p^k}{2(2\pi)^{k+1}}\tau(\chi_1)\tau(\chi_2)\sum_{n=1}^{\infty}\sigma_{k,\bar{\chi}_2,\bar{\chi}_1}(n)\left(\frac{1}{n^{k+1}}-\frac{1}{(n+\frac{a^2pqx}{16\pi^2})^{k+1}}\right),$$

where c_{k,χ_1,χ_2} *is a constant defined as*

$$c_{k,\chi_1,\chi_2}=\begin{cases}L(-k,\chi_1)L'(0,\chi_2), & \text{if both } \chi_1 \text{ and } \chi_2 \text{ are even,}\\ L'(-k,\chi_1)L(0,\chi_2), & \text{if both } \chi_1 \text{ and } \chi_2 \text{ are odd.}\end{cases} \tag{3.83}$$

Proof. We leave the proof to the reader for its similarity with the proofs of Theorems 3.2.3 and 3.2.9. □

Setting $\chi_1=\chi_2=\chi$ and observing $\sigma_{k,\chi,\chi}(n)=\chi(n)\sum_{d/n}d^k=\chi(n)\sigma_k(n)$ in Theorems 3.2.14 and 3.2.17, we obtain the following interesting identities.

Corollary 3.2.4. *Let $k\geq 1$ be an odd integer and χ be a non-principal primitive character modulo q. Then, for any $\Re(\nu)>0$,*

$$\sum_{n=1}^{\infty}\sigma_k(n)\chi(n)n^{\frac{\nu}{2}}K_\nu(a\sqrt{nx})$$
$$=\frac{(-1)^{\frac{k+1}{2}}a^\nu q^{2\nu+k}\ x^{\frac{\nu}{2}}}{2^{3\nu+k+2}\pi^{2\nu+k+1}}\tau^2(\chi)\Gamma(\nu+k+1)\sum_{n=1}^{\infty}\frac{\sigma_k(n)\ \bar{\chi}(n)}{\left(n+\frac{a^2q^2x}{16\pi^2}\right)^{\nu+k+1}}.$$

Corollary 3.2.5. *Let $k\geq 1$ be an odd integer and χ be a non-principal primitive character modulo q. For $\nu=0$, we have*

$$\sum_{n=1}^{\infty}\sigma_k(n)\chi(n)K_0(a\sqrt{nx})$$
$$=\frac{1}{2}c_{k,\chi,\chi}+\frac{(-1)^{\frac{k-1}{2}}k!q^k}{2(2\pi)^{k+1}}\tau^2(\chi)\sum_{n=1}^{\infty}\sigma_k(n)\ \bar{\chi}(n)\ \left(\frac{1}{n^{k+1}}-\frac{1}{(n+\frac{a^2q^2x}{16\pi^2})^{k+1}}\right),$$

where $c_{k,\chi,\chi}$ *is defined in* (3.83).

The results corresponding to $\nu = 0$ and $k = 0$ are as follows:

Theorem 3.2.18. *Let χ_1 and χ_2 be non-principal even primitive characters modulo p and q, respectively. Then*

$$\sum_{n=1}^{\infty} d_{\chi_1,\chi_2}(n) K_0(a\sqrt{nx}) = \frac{a^2pq\ x}{32\pi^4}\tau(\chi_1)\tau(\chi_2)\sum_{n=1}^{\infty} d_{\bar{\chi}_1,\bar{\chi}_2}(n)\frac{\log\left(\frac{16\pi^2 n}{a^2pqx}\right)}{n^2-\left(\frac{a^2pqx}{16\pi^2}\right)^2},$$

provided $\frac{a^2pqx}{16\pi^2} \notin \mathbb{Z}_+$.

Proof. We begin the proof by setting $k = 0$ and $\nu = 0$ and $f_0(n) = d_{\chi_1,\chi_2}(n)$ in (3.9) where χ_1 and χ_2 are even non-principal primitive characters modulo p and q, respectively. This will give

$$\sum_{n=1}^{\infty} d_{\chi_1,\chi_2}(n) K_0(a\sqrt{nx}) = \frac{1}{2}(R_0 + R_1 + J_0^{(0)}(X)), \tag{3.84}$$

where $J_0^{(0)}(X))$ is defined in (3.10) with $F_0(z) = L(s,\chi_1)L(s,\chi_2)$. Here, we will have $R_1 = 0$. Both $L(s,\chi_1)$ and $L(s,\chi_2)$ have simple zero at $s = 0$ which will get cancelled by the double pole of $\Gamma^2(s)$ at $s = 0$. Hence, we have $R_0 = 0$. Employing functional relation (2.13), we obtain

$$J_0^{(0)}(X) = \frac{\tau(\chi_1)\tau(\chi_2)}{4} Y \sum_{n=1}^{\infty} d_{\bar{\chi}_1,\bar{\chi}_2}(n)\frac{1}{2\pi i}\int_{(1+d)} \frac{(nY)^{-s}}{\sin^2\left(\frac{\pi s}{2}\right)} \mathrm{d}s, \tag{3.85}$$

where $Y = \frac{4\pi^2}{pq}X$. Note that integral in (3.85) can be treated similarly as in the proof of Theorem 3.2.13. To avoid repetitions, we skip the details. □

Theorem 3.2.19. *Let χ_1 and χ_2 be odd primitive characters modulo p and q, respectively. Then we have*

$$\sum_{n=1}^{\infty} d_{\chi_1,\chi_2}(n) K_0(a\sqrt{nx})$$
$$= \frac{1}{2}L(0,\chi_1)L(0,\chi_2)\left(-2\gamma + \log\left(\frac{4}{a^2x}\right) + \frac{L'(0,\chi_1)}{L(0,\chi_1)} + \frac{L'(0,\chi_2)}{L(0,\chi_2)}\right)$$

$$+\frac{a^4p^2q^2}{512\pi^4}\ x^2\tau(\chi_1)\tau(\chi_2)\sum_{n=1}^{\infty}\frac{d_{\bar{\chi}_1,\bar{\chi}_2}(n)\log\left(\frac{a^2pqx}{16\pi^2 n}\right)}{n\left(n^2-(\frac{a^2pqx}{16\pi^2})^2\right)},$$

provided $\frac{a^2pqx}{16\pi^2}\notin\mathbb{Z}_+$.

Proof. Taking $k=0$ and $\nu=0$ and $f_0(n)=d_{\chi_1,\chi_2}(n)$ in (3.9) where χ_1 and χ_2 are odd primitive characters modulo p and q, respectively. This will give

$$\sum_{n=1}^{\infty}d_{\chi_1,\chi_2}(n)K_0(a\sqrt{nx})=\frac{1}{2}(R_0+R_1+J_0^{(0)}(X)), \tag{3.86}$$

where $J_0^{(0)}(X))$ is defined in (3.10) with $F_0(z)=L(s,\chi_1)L(s,\chi_2)$. Here, the integrand will encounter a double pole at $s=0$. So, we will have $R_1=0$. Hence, the residue R_0 is given by

$$R_0=L(0,\chi_1)L(0,\chi_2)\left(-2\gamma+\log(X)+\frac{L'(0,\chi_1)}{L(0,\chi_1)}+\frac{L'(0,\chi_2)}{L(0,\chi_2)}\right). \tag{3.87}$$

Employing (3.87) and $R_1=0$ in (3.86), we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}d_{\chi_1,\chi_2}(n)K_0(a\sqrt{nx})\\&=\frac{1}{2}L(0,\chi_1)L(0,\chi_2)\left(-2\gamma+\log(X)+\frac{L'(0,\chi_1)}{L(0,\chi_1)}+\frac{L'(0,\chi_2)}{L(0,\chi_2)}\right)+\frac{1}{2}J_0^{(0)}(X).\end{aligned} \tag{3.88}$$

Now appealing to functional relation (2.13), we will have

$$\begin{aligned}J_0^{(0)}(X)&=-\frac{\tau(\chi_1)\tau(\chi_2)}{4}Y\sum_{n=1}^{\infty}d_{\bar{\chi}_1,\bar{\chi}_2}(n)\frac{1}{2\pi i}\int_{(1+d)}\frac{(nY)^{-s}}{\cos^2\left(\frac{\pi s}{2}\right)}\mathrm{d}s\\&=-\frac{\tau(\chi_1)\tau(\chi_2)Y}{4}\left(\sum_{n<Y^{-1}}+\sum_{n>Y^{-1}}\right)d_{\bar{\chi}_1,\bar{\chi}_2}(n)\frac{1}{2\pi i}\int_{(1+d)}\frac{(nY)^{-s}}{\cos^2\left(\frac{\pi s}{2}\right)}\mathrm{d}s,\end{aligned} \tag{3.89}$$

where $Y=\frac{16\pi^2}{a^2pqx}$ and $Y^{-1}\notin\mathbb{Z}_+$. We first evaluate the inner line integral on the sum $\sum_{n>Y^{-1}}$. We shall use the Cauchy residue theorem with the contour formed by the lines $[1+d-iT,1+d+iT],[1+d+iT,M+\frac{1}{2}+iT],[M+\frac{1}{2}+iT,M+\frac{1}{2}-iT],[M+$

$\frac{1}{2} - iT, 1 + d - iT]$ where $M \in \mathbb{N}$ is any odd large number, and T is a large positive number. The poles of the integrand function in (3.89) are at $3, 5, \cdots, M$, and they are double poles. The residue at $s = m$ is given by

$$\mathcal{R}_m := -\frac{4}{\pi^2} (nY)^{-m} \log (nY), \tag{3.90}$$

where $m = 3,\, 5, \cdots,\, M$. Employing Lemma 2.2.1, we can show both the integrals along the horizontal line segments $[1+d+iT, M+\frac{1}{2}+iT]$ and $[M+\frac{1}{2}-iT, 1+d-iT]$ vanish as $T \to \infty$. Utilising (3.90), we arrive at

$$\begin{aligned}\frac{1}{2\pi i}\int_{(1+d)} \frac{(nY)^{-s}}{\cos^2\left(\frac{\pi s}{2}\right)} \mathrm{d}s &= -\sum_{m=1}^{\frac{M-1}{2}} \mathcal{R}_{2m+1} + \frac{1}{2\pi i}\int_{(M+\frac{1}{2})} \frac{(nY)^{-s}}{\cos^2\left(\frac{\pi s}{2}\right)} \mathrm{d}s \\ &= \frac{4}{\pi^2}\sum_{m=1}^{\frac{M-1}{2}} \frac{\log (nY)}{(nY)^{2m+1}} + O\left(\frac{1}{(nY)^{M+1/2}}\right). \end{aligned} \tag{3.91}$$

Letting $M \to \infty$, the error term in (3.91) goes to 0 as $n > Y^{-1}$. Thus simplifying, we can readily deduce that

$$\frac{1}{2\pi i}\int_{(1+d)} \frac{(nY)^{-s}}{\cos^2\left(\frac{\pi s}{2}\right)} \mathrm{d}s = \frac{4}{\pi^2}\sum_{m=1}^{\infty} \frac{\log (nY)}{(nY)^{2m+1}} = \frac{4}{\pi^2 Y^3}\frac{1}{n(n^2 - Y^{-2})} \log (nY),$$

and subsequently, we get

$$\begin{aligned}&\sum_{n > Y^{-1}} d_{\bar{\chi}_1,\bar{\chi}_2}(n) \frac{1}{2\pi i}\int_{(1+d)} \frac{(nY)^{-s}}{\cos^2\left(\frac{\pi s}{2}\right)} \mathrm{d}s \\ &= \frac{4}{\pi^2}\sum_{n > Y^{-1}} d_{\bar{\chi}_1,\bar{\chi}_2}(n) \frac{Y^{-3}}{n(n^2 - Y^{-2})} \log (nY). \end{aligned} \tag{3.92}$$

Similarly, by shifting the integration line to the left, we get

$$\begin{aligned}&\sum_{n < Y^{-1}} d_{\bar{\chi}_1,\bar{\chi}_2}(n) \frac{1}{2\pi i}\int_{(1+d)} \frac{(nY)^{-s}}{\cos^2\left(\frac{\pi s}{2}\right)} \mathrm{d}s \\ &= \frac{4}{\pi^2}\sum_{n \leq Y^{-1}} d_{\bar{\chi}_1,\bar{\chi}_2}(n) \frac{Y^{-3}}{n(n^2 - Y^{-2})} \log (nY). \end{aligned} \tag{3.93}$$

Hence combining (3.92) and (3.93) with (3.89), we obtain

$$J_0^{(0)}(X) = -\frac{\tau(\chi_1)\tau(\chi_2)}{\pi^2}\sum_{n=1}^{\infty} d_{\bar{\chi}_1,\bar{\chi}_2}(n)\frac{Y^{-2}}{n(n^2-Y^{-2})}\log(nY). \tag{3.94}$$

Inserting (3.94) in (3.88) and remarking that $Y = \frac{16\pi^2}{a^2pqx}$, we get the desired result. □

Theorem 3.2.20. *Let k be an even, non-negative integer. Assume that χ_1 and χ_2 are primitive characters modulo p and q, respectively, such that one is a non-principal even character and the other is an odd character. Then, for any $\Re(\nu) > 0$,*

$$\sum_{n=1}^{\infty}\sigma_{k,\chi_1,\chi_2}(n)n^{\frac{\nu}{2}}K_\nu(a\sqrt{nx})$$
$$=\frac{(-1)^{\frac{k}{2}}\,(aq)^\nu p^{\nu+k}\,x^{\frac{\nu}{2}}}{i2^{3\nu+k+2}\pi^{2\nu+k+1}}\tau(\chi_1)\tau(\chi_2)\Gamma(\nu+k+1)\sum_{n=1}^{\infty}\frac{\sigma_{k,\bar{\chi}_2,\bar{\chi}_1}(n)}{\left(n+\frac{a^2pqx}{16\pi^2}\right)^{\nu+k+1}}. \tag{3.95}$$

With the aid of Theorem 3.2.5, one can show the equivalence of Theorem 3.2.20 and our next result.

Theorem 3.2.21. *Let $k \geq 2$ be an even integer. Then for any $\Re(\nu) > 0$ we have*

$$\left(\frac{a^2x}{4}\right)^{\frac{\nu}{2}+k+1}\sum_{n=1}^{\infty}n^{\nu/2}K_\nu(a\sqrt{nx})\sum_{d|n}d^k\cos(2\pi d\theta)\sin\left(\frac{2\pi n\psi}{d}\right)$$
$$=\frac{(-1)^{\frac{k}{2}}\,(2\pi)^{k+1}}{8}\Gamma(\nu+k+1)\sum_{m,n\geq 0}^{\infty}\left\{\frac{(n+\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+\psi)(m+\theta)}{x}+1\right)^{\nu+k+1}}\right.$$
$$-\frac{(n+1-\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+1-\psi)(m+\theta)}{x}+1\right)^{\nu+k+1}}-\frac{(n+1-\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+1-\psi)(m+1-\theta)}{x}+1\right)^{\nu+k+1}}$$
$$\left.+\frac{(n+\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+\psi)(m+1-\theta)}{x}+1\right)^{\nu+k+1}}\right\}.$$

Similarly, with the help of Theorem 3.2.2, one can show the equivalence of Theorem 3.2.20 and our next result.

Theorem 3.2.22. *Let $k \geq 0$ be an even integer. Then for any $\Re(\nu) > 0$ we have*

$$\left(\frac{a^2x}{4}\right)^{\frac{\nu}{2}+k+1}\sum_{n=1}^{\infty} n^{\nu/2}K_\nu(a\sqrt{nx})\sum_{d|n} d^k \sin\left(2\pi d\theta\right)\cos\left(\frac{2\pi n\psi}{d}\right)$$
$$= -\frac{(-1)^{\frac{k}{2}}a^{2k+2}k!}{2^{2k+4}(2\pi)^{k+1}}\Gamma(\nu)\left(\zeta(1+k,\theta)-\zeta(1+k,1-\theta)\right)x^{k+1}$$
$$+\frac{(-1)^{\frac{k}{2}}(2\pi)^{k+1}}{8}\Gamma(\nu+k+1)\sum_{m,n\geq 0}\left\{\frac{(n+\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+\psi)(m+\theta)}{x}+1\right)^{\nu+k+1}}\right.$$
$$+\frac{(n+1-\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+1-\psi)(m+\theta)}{x}+1\right)^{\nu+k+1}}-\frac{(n+1-\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+1-\psi)(m+1-\theta)}{x}+1\right)^{\nu+k+1}}$$
$$\left.-\frac{(n+\psi)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+\psi)(m+1-\theta)}{x}+1\right)^{\nu+k+1}}\right\}.$$

Proof of Theorem 3.2.20 and its equivalence with Theorem 3.2.21. The proofs are similar to the proofs of Theorems 3.2.14 and 3.2.15, so we skip the proof. But we would show the equivalence of Theorem 3.2.20 with Theorem 3.2.21.

Theorem 3.2.20 $\Rightarrow$ Theorem 3.2.21 Let us multiply the identity (3.95) of Theorem 3.2.20 by $\chi_1(h_1)\tau(\bar{\chi_1})/\phi(p)$ and $\chi_2(h_2)\tau(\bar{\chi_2})/i\phi(q)$, then take sum on non-principal even primitive characters χ_1 modulo p and odd primitive characters χ_2 modulo q. So, from the left-hand side, we get

$$\frac{1}{i\phi(p)\phi(q)}\sum_{\chi_2\ odd}\chi_2(h_2)\tau(\bar{\chi_2})\sum_{\substack{\chi_1\neq\chi_0\\ \chi_1\ even}}\chi_1(h_1)\tau(\bar{\chi_1})\sum_{n=1}^{\infty}\sigma_{k,\chi_1,\chi_2}(n)n^{\frac{\nu}{2}}K_\nu(a\sqrt{nx})$$
$$=\frac{1}{i\phi(p)\phi(q)}\sum_{n=1}^{\infty}n^{\frac{\nu}{2}}K_\nu(a\sqrt{nx})\sum_{d/n}d^k\sum_{\chi_2\ odd}\chi_2(h_2)\chi_2(n/d)\tau(\bar{\chi_2})$$
$$\times\sum_{\substack{\chi_1\neq\chi_0\\ \chi_1\ even}}\chi_1(h_1)\chi_1(d)\tau(\bar{\chi_1})$$
$$=\sum_{n=1}^{\infty}n^{\frac{\nu}{2}}K_\nu(a\sqrt{nx})\sum_{d|n}d^k\sin\left(\frac{2\pi nh_2}{dq}\right)\left\{\frac{1}{\phi(p)}\sum_{\chi_1\ even}\chi_1(h_1)\chi_1(d)\tau(\bar{\chi_1})+\frac{\chi_0(d)}{\phi(p)}\right\}$$

$$= \sum_{n=1}^{\infty} n^{\frac{\nu}{2}} K_\nu(a\sqrt{nx}) \sum_{\substack{d|n \\ p\nmid d}} d^k \sin\left(\frac{2\pi n h_2}{dq}\right) \left\{\cos\left(\frac{2\pi d h_1}{p}\right) + \frac{1}{\phi(p)}\right\}$$

$$= \sum_{n=1}^{\infty} n^{\frac{\nu}{2}} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k \sin\left(\frac{2\pi n h_2}{dq}\right) \left\{\cos\left(\frac{2\pi d h_1}{p}\right) + \frac{1}{\phi(p)}\right\}$$

$$- \sum_{n=1}^{\infty} n^{\frac{\nu}{2}} K_\nu(a\sqrt{nx}) \sum_{\substack{d|n \\ p|d}} d^k \sin\left(\frac{2\pi n h_2}{dq}\right) \left\{1 + \frac{1}{\phi(p)}\right\}$$

$$= \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k \cos\left(\frac{2\pi d h_1}{p}\right) \sin\left(\frac{2\pi n h_2}{dq}\right)$$

$$+ \frac{1}{\phi(p)} \sum_{n=1}^{\infty} n^{\frac{\nu}{2}} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k \sin\left(\frac{2\pi n h_2}{dq}\right)$$

$$- \frac{p^{\frac{\nu}{2}+k+1}}{\phi(p)} \sum_{m=1}^{\infty} m^{\frac{\nu}{2}} K_\nu(a\sqrt{pmx}) \sum_{d|m} d^k \sin\left(\frac{2\pi m h_2}{dq}\right), \tag{3.96}$$

where we used the (2.24), (2.25) in the penultimate step. Applying Theorem 3.2.5 for the last two terms of the right-hand side of (3.96). So, we rewrite the right-hand side of (3.96) as follows:

$$\sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^k \cos\left(\frac{2\pi d h_1}{p}\right) \sin\left(\frac{2\pi n h_2}{dq}\right)$$

$$+ \frac{(-1)^{\frac{k}{2}}}{4\phi(p)} X^{\frac{\nu}{2}} (2\pi X)^{k+1} \Gamma(\nu+k+1) \sum_{r=1}^{\infty} \sum_{m=0}^{\infty} \left(\frac{(m+h_2/q)^k}{(1+\frac{16\pi^2 r}{a^2 x}(m+h_2/q))^{1+\nu+k}} - \frac{(m+h_2/q)^k}{(1+\frac{16\pi^2 r}{a^2 x}(m+1-h_2/q))^{1+\nu+k}} \right)$$

$$- \frac{(-1)^{\frac{k}{2}}}{4\phi(p)} X^{\frac{\nu}{2}} (2\pi X)^{k+1} \Gamma(\nu+k+1) \sum_{r=1}^{\infty} \sum_{m=0}^{\infty} \left(\frac{(m+h_2/q)^k}{(1+\frac{16\pi^2 r}{a^2 p x}(m+h_2/q))^{1+\nu+k}} - \frac{(m+h_2/q)^k}{(1+\frac{16\pi^2 r}{a^2 p x}(m+1-h_2/q))^{1+\nu+k}} \right). \tag{3.97}$$

Using (2.21), (2.22) and (2.23) in the right-hand side of (3.95) of Theorem 3.2.20, we have

$$
\begin{aligned}
&-\frac{(-1)^{\frac{k}{2}} X^{\frac{\nu}{2}}}{2p\phi(p)\phi(q)} \left(\frac{2\pi X}{q}\right)^{k+1} \Gamma(\nu+k+1) \sum_{n=1}^{\infty} \frac{\sigma_{k,\bar{\chi_2},\bar{\chi_1}}(n)}{\left(\frac{16\pi^2}{a^2pq}\frac{n}{x}+1\right)^{\nu+k+1}} \\
&\qquad \times \sum_{\chi_2 \ odd} \chi_2(h_2)\tau(\chi_2)\tau(\bar{\chi_2}) \sum_{\substack{\chi_1 \neq \chi_0 \\ \chi_1 \ even}} \chi_1(h_1)\tau(\chi_1)\tau(\bar{\chi_1}) \\
&= \frac{(-1)^{\frac{k}{2}} pq}{2p\phi(p)\phi(q)} \left(\frac{2\pi X}{q}\right)^{k+1} X^{\frac{\nu}{2}} \Gamma(\nu+k+1) \sum_{n=1}^{\infty} \frac{\sum_{d/n} d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{n}{x}+1\right)^{\nu+k+1}} \\
&\qquad \times \sum_{\chi_2 \ odd} \chi_2(h_2)\bar{\chi}_2(d) \left\{ \sum_{\chi_1 \ even} \chi_1(h_1)\bar{\chi}_1(n/d) - \bar{\chi}_0(n/d) \right\} \\
&= \frac{(-1)^{\frac{k}{2}} q}{2\phi(p)\phi(q)} \left(\frac{2\pi X}{q}\right)^{k+1} X^{\frac{\nu}{2}} \Gamma(\nu+k+1) \sum_{d=1}^{\infty}\sum_{r=1}^{\infty} \frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{dr}{x}+1\right)^{\nu+k+1}} \\
&\qquad \times \sum_{\chi_2 \ odd} \chi_2(h_2)\bar{\chi}_2(d) \left\{ \sum_{\chi_1 \ even} \chi_1(h_1)\bar{\chi}_1(r) - \bar{\chi}_0(r) \right\} \\
&= \frac{(-1)^{\frac{k}{2}} (2\pi X)^{k+1}}{8} X^{\frac{\nu}{2}} \Gamma(\nu+k+1) \sum_{m,n\geq 0}^{\infty} \left\{ \frac{(n+h_2/q)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+h_2/q)(m+h_1/p)}{x}+1\right)^{\nu+k+1}} \right. \\
&\quad - \frac{(n+1-h_2/q)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+1-h_2/q)(m+h_1/p)}{x}+1\right)^{\nu+k+1}} \\
&\quad \left. - \frac{(n+1-h_2/q)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+1-h_2/q)(m+1-h_1/p)}{x}+1\right)^{\nu+k+1}} + \frac{(n+h_2/q)^k}{\left(\frac{16\pi^2}{a^2}\frac{(n+h_2/q)(m+1-h_1/p)}{x}+1\right)^{\nu+k+1}} \right\} \\
&- \frac{(-1)^{\frac{k}{2}} q}{2\phi(p)\phi(q)} \left(\frac{2\pi X}{q}\right)^{k+1} X^{\frac{\nu}{2}} \Gamma(\nu+k+1) \sum_{\substack{d,r\geq 1 \\ p\nmid r}} \frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{dr}{x}+1\right)^{\nu+k+1}} \sum_{\chi_2 \ odd} \chi_2(h_2)\bar{\chi}_2(d).
\end{aligned}
\tag{3.98}
$$

By (2.22), we examine the last term of right-hand side of (3.98), we find

$$
\frac{1}{\phi(q)} \sum_{\substack{d,r\geq 1 \\ p\nmid r}} \frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{dr}{x}+1\right)^{\nu+k+1}} \sum_{\chi_2 \ odd} \chi_2(h_2)\bar{\chi}_2(d)
$$

$$
\begin{aligned}
=&\frac{1}{\phi(q)}\sum_{d,r\geq 1}\frac{d^k}{\left(\frac{16\pi^2}{a^2pq}\frac{dr}{x}+1\right)^{\nu+k+1}}\sum_{\chi_2\ odd}\chi_2(h_2)\bar{\chi}_2(d)\\
&-\frac{1}{\phi(q)}\sum_{d,r\geq 1}\frac{d^k}{\left(\frac{16\pi^2}{a^2q}\frac{dr}{x}+1\right)^{\nu+k+1}}\sum_{\chi_2\ odd}\chi_2(h_2)\bar{\chi}_2(d)\\
=&\frac{q^k}{2}\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\left\{\frac{(m+h_2/q)^k}{\left(\frac{16\pi^2}{a^2p}\frac{(m+h_2/q)r}{x}+1\right)^{\nu+k+1}}-\frac{(m+1-h_2/q)^k}{\left(\frac{16\pi^2}{a^2p}\frac{(m+1-h_2/q)r}{x}+1\right)^{\nu+k+1}}\right\}\\
-&\frac{q^k}{2}\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\left\{\frac{(m+h_2/q)^k}{\left(\frac{16\pi^2}{a^2}\frac{(m+h_2/q)r}{x}+1\right)^{\nu+k+1}}-\frac{(m+1-h_2/q)^k}{\left(\frac{16\pi^2}{a^2}\frac{(m+1-h_2/q)r}{x}+1\right)^{\nu+k+1}}\right\}. \qquad (3.99)
\end{aligned}
$$

Equating (3.97), (3.98), and using (3.99) we get the proof of Theorem 3.2.21 .
Theorem 3.2.21 ⇒ Theorem 3.2.20 Let $\theta = h_1/p$, $\psi = h_2/q$, and let χ_1 be an even primitive character modulo p and χ_2 be an odd primitive character modulo q. Multiplying the identity in Theorem 3.2.21 by $\bar{\chi_1}(h_1)\bar{\chi_2}(h_2)/\tau(\bar{\chi_1})\tau(\bar{\chi_2})$, and then summing on h_1 and h_2, $0 < h_1 < p$, $0 < h_2 < q$, one can prove that Theorem 3.2.21 implies Theorem 3.2.20 . □

Proof of Theorem 3.2.20 and its equivalence with Theorem 3.2.22. The proof is similar to the previous one. □

The result corresponding to $\nu = 0$ is as follows:

Theorem 3.2.23. *Let k be an even, non-negative integer. If χ_1 and χ_2 are primitive characters modulo p and q, respectively, such that one is a non-principal even character and the other is an odd character, then*

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sigma_{k,\chi_1,\chi_2}(n)K_0(a\sqrt{nx})\\
=&\frac{1}{2}e_{k,\chi_1,\chi_2}+(-1)^{\frac{k}{2}}\frac{ik!p^k}{2(2\pi)^{k+1}}\tau(\chi_1)\tau(\chi_2)\sum_{n=1}^{\infty}\sigma_{k,\bar{\chi}_2,\bar{\chi}_1}(n)\left(\frac{1}{n^{k+1}}-\frac{1}{(n+\frac{a^2pqx}{16\pi^2})^{k+1}}\right),
\end{aligned}
$$

where

$$e_{k,\chi_1,\chi_2} = \begin{cases} L(-k,\chi_1)L'(0,\chi_2), & \text{if } \chi_1 \text{ is odd and } \chi_2 \text{ is even,} \\ L'(-k,\chi_1)L(0,\chi_2), & \text{if } \chi_1 \text{ is even and } \chi_2 \text{ is odd.} \end{cases} \tag{3.100}$$

Proof. The proof is similar to the proof of Theorem 3.2.17. □

In the next chapter, we will study the Cohen-type identities for twisted sums of divisor functions.

4
Cohen-type identities

4.1 Introduction

We first recall the definition of twisted sums of the divisor functions (1.28) by substituting the complex integer z for k,

$$\sigma_{z,\chi}(n) := \sum_{d|n} d^z \chi(d), \ \bar{\sigma}_{z,\chi}(n) := \sum_{d|n} d^z \chi(n/d), \ \sigma_{z,\chi_1,\chi_2}(n) := \sum_{d|n} d^z \chi_1(d)\chi_2(n/d), \tag{4.1}$$

In 2010, Cohen [19] proved the beautiful identity linked to the modified K-Bessel function and the generalised divisor function. In this chapter, we study the Cohen-type identities for these twisted sums of divisor functions defined in (4.1). More specifically, the objective of this chapter is to derive the character analogues of Cohen's identity (1.30). These types of identities are very crucial because, with the help of these identities, we will establish Voronoi-type summation formulas for them. In Section 4.2, we state our main results on Cohen-type identities for twisted divisor sums. Furthermore, in Section 4.3, we give proofs of our main results.

4.2 Main Results

This chapter deals with $z = -\nu$ with $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. We will assume that x is a strictly positive real number and $0 < \theta, \psi < 1$. Let us define

$$H_{\alpha,\beta} = \{(n-\alpha)(r-\beta),\ n, r \in \mathbb{N}\}, \tag{4.2}$$

where $0 \leq \alpha, \beta < 1$.

4.2.1 Identities involving even characters and specializations

In this subsection, we present the identities associated with $\sigma_{-\nu,\bar{\chi}}(n)$ and $\bar{\sigma}_{-\nu,\bar{\chi}}(n)$ when χ is a non-principal even primitive character.

Theorem 4.2.1. *Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Let χ be a non-principal, even primitive character modulo q. If N is any integer such that $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, then*

$$\begin{aligned}
8\pi x^{\nu/2} \sum_{n=1}^{\infty} \sigma_{-\nu,\bar{\chi}}(n) n^{\nu/2} K_{\nu}(4\pi\sqrt{nx}) &= -\frac{\Gamma(\nu)L(\nu,\bar{\chi})}{(2\pi)^{\nu-1}} + \frac{2\Gamma(1+\nu)L(1+\nu,\bar{\chi})}{(2\pi)^{\nu+1}} x^{-1} \\
+ \frac{2q^{1-\nu}}{\tau(\chi)\sin\left(\frac{\pi\nu}{2}\right)} &\left\{ \sum_{j=1}^{N} \zeta(2j)\ L(2j-\nu,\chi)(qx)^{2j-1} \right. \\
&\left. + (qx)^{2N+1} \sum_{n=1}^{\infty} \bar{\sigma}_{-\nu,\chi}(n) \left(\frac{n^{\nu-2N} - (qx)^{\nu-2N}}{n^2 - (qx)^2} \right) \right\},
\end{aligned} \tag{4.3}$$

provided $qx \notin \mathbb{Z}_+$.

With the aid of Proposition 1.4.1 and Theorem 4.2.1, one can prove our next result. Conversely, our next result will imply Theorem 4.2.1 independently.

Theorem 4.2.2. *Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Then, for any integer N such that*

$N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, *we have*

$$8\pi x^{\nu/2}\sum_{n=1}^{\infty} n^{\nu/2}K_{\nu}(4\pi\sqrt{nx})\sum_{d|n} d^{-\nu}\cos(2\pi d\theta) = -\frac{\pi}{\cos\left(\frac{\pi\nu}{2}\right)}\zeta(\nu+1)x^{\nu}$$
$$-\frac{\pi}{2\cos\left(\frac{\pi\nu}{2}\right)}\left(\zeta(1-\nu,\theta)+\zeta(1-\nu,1-\theta)\right)-\frac{1}{2x\sin\left(\frac{\pi\nu}{2}\right)}\left(\zeta(-\nu,\theta)+\zeta(-\nu,1-\theta)\right)$$
$$+\frac{1}{\sin\left(\frac{\pi\nu}{2}\right)}\sum_{j=1}^{N}\zeta(2j)\left(\zeta(2j-\nu,\theta)+\zeta(2j-\nu,1-\theta)\right)x^{2j-1}$$
$$+\frac{1}{\sin\left(\frac{\pi\nu}{2}\right)}x^{2N+1}\sum_{d=1}^{\infty}d^{-\nu}\sum_{m=0}^{\infty}\left\{\frac{(d(m+\theta))^{\nu-2N}-x^{\nu-2N}}{(d^2(m+\theta)^2-x^2)}\right.$$
$$\left.+\frac{(d(m+1-\theta))^{\nu-2N}-x^{\nu-2N}}{(d^2(m+1-\theta)^2-x^2)}\right\}, \tag{4.4}$$

provided $x \notin H_{\theta,0}$.

Theorem 4.2.3. *Let* $\nu \notin \mathbb{Z}$ *such that* $\Re(\nu) \geq 0$. *Let* χ *be a non-principal even primitive character modulo* q. *If* N *is any integer such that* $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, *then*

$$8\pi x^{\nu/2}\sum_{n=1}^{\infty}\bar{\sigma}_{-\nu,\bar{\chi}}(n)n^{\nu/2}K_{\nu}(4\pi\sqrt{nx}) = \frac{q}{\tau(\chi)}\left\{\frac{L(\nu,\chi)}{\sin\left(\frac{\pi\nu}{2}\right)}(qx)^{\nu-1}-\frac{\pi L(1+\nu,\chi)}{\cos\left(\frac{\pi\nu}{2}\right)}(qx)^{\nu}\right.$$
$$+\frac{2}{\sin\left(\frac{\pi\nu}{2}\right)}\sum_{j=1}^{N}\zeta(2j-\nu)L(2j,\chi)(qx)^{2j-1}$$
$$\left.+\frac{2}{\sin\left(\frac{\pi\nu}{2}\right)}(qx)^{2N+1}\sum_{n=1}^{\infty}\sigma_{-\nu,\chi}(n)\left(\frac{n^{\nu-2N}-(qx)^{\nu-2N}}{n^2-(qx)^2}\right)\right\},$$

provided $qx \notin \mathbb{Z}_+$.

Analogous to Theorem 4.2.1, the equivalent version of Theorem 4.2.3 is the following result.

Theorem 4.2.4. *Let* $\nu \notin \mathbb{Z}$ *such that* $\Re(\nu) \geq 0$. *Then, for any integer* N *such that* $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, *we have*

$$8\pi x^{\nu/2}\sum_{n=1}^{\infty} n^{\nu/2}K_{\nu}(4\pi\sqrt{nx})\sum_{d|n} d^{-\nu}\cos\left(\frac{2\pi n\theta}{d}\right)$$

$$
\begin{aligned}
&= -\frac{\Gamma(\nu)\zeta(\nu)}{(2\pi)^{\nu-1}} + \frac{x^{\nu-1}}{2\sin\left(\frac{\pi\nu}{2}\right)}\left\{\zeta(\nu,\theta)+\zeta(\nu,1-\theta)\right\}\\
&- \frac{\pi\ x^{\nu}}{2\cos\left(\frac{\pi\nu}{2}\right)}\left\{\zeta(1+\nu,\theta)+\zeta(1+\nu,1-\theta)\right\}\\
&+ \frac{1}{\sin\left(\frac{\pi\nu}{2}\right)}\sum_{j=1}^{N}\zeta(2j-\nu)x^{2j-1}\left\{\zeta(2j,\theta)+\zeta(2j,1-\theta)\right\}\\
&+ \frac{x^{2N+1}}{\sin\left(\frac{\pi\nu}{2}\right)}\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\left\{(m+\theta)^{-\nu}\frac{(r(m+\theta))^{\nu-2N}-x^{\nu-2N}}{r^2(m+\theta)^2-x^2}\right.\\
&\qquad\qquad\left.+(m+1-\theta)^{-\nu}\frac{(r(m+1-\theta))^{\nu-2N}-x^{\nu-2N}}{r^2(m+1-\theta)^2-x^2}\right\},
\end{aligned}
$$

provided $x \notin H_{\theta,0}$.

Setting $\nu = 1/2$ in Theorem 4.2.1 and Theorem 4.2.3, we obtain the following

Corollary 4.2.1. *We have*

$$
\begin{aligned}
2\pi\sum_{n=1}^{\infty}\sigma_{-\frac{1}{2},\bar{\chi}}(n)e^{-4\pi\sqrt{nx}} &= -\pi L(1/2,\bar{\chi}) + \frac{1}{4\pi}L(3/2,\bar{\chi})x^{-1}\\
&\quad + \frac{2q^{3/2}}{\tau(\chi)}x\sum_{n=1}^{\infty}\bar{\sigma}_{-\frac{1}{2},\chi}(n)\frac{1}{(n+qx)(\sqrt{n}+\sqrt{qx})}.
\end{aligned}
$$

Corollary 4.2.2. *We have*

$$
\begin{aligned}
2\pi\sum_{n=1}^{\infty}\bar{\sigma}_{-\frac{1}{2},\bar{\chi}}(n)e^{-4\pi\sqrt{nx}} =& \frac{q^{1/2}}{\tau(\chi)}L(1/2,\chi)x^{-\frac{1}{2}} - \frac{\pi q^{3/2}}{\tau(\chi)}L(3/2,\chi)x^{\frac{1}{2}}\\
&+ \frac{2q^2}{\tau(\chi)}x\sum_{n=1}^{\infty}\frac{\sigma_{-\frac{1}{2},\chi}(n)}{(n+qx)(\sqrt{n}+\sqrt{qx})}.
\end{aligned}
$$

4.2.2 Identities involving odd characters and specializations

In this subsection, we state the identities associated with $\sigma_{-\nu,\bar{\chi}}(n)$ and $\bar{\sigma}_{-\nu,\bar{\chi}}(n)$ when χ is an odd primitive character.

Theorem 4.2.5. *Let* $\nu \notin \mathbb{Z}$ *such that* $\Re(\nu) \geq 0$. *Let* χ *be an odd primitive character* *modulo q. If N is any integer such that* $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, *then*

$$\begin{aligned} 8\pi x^{\nu/2}\sum_{n=1}^{\infty}\sigma_{-\nu,\bar{\chi}}(n)n^{\nu/2}K_{\nu}(4\pi\sqrt{nx}) &= -\frac{\Gamma(\nu)L(\nu,\bar{\chi})}{(2\pi)^{\nu-1}} + \frac{2\Gamma(1+\nu)L(1+\nu,\bar{\chi})}{(2\pi)^{\nu+1}}x^{-1} \\ + \frac{2iq^{1-\nu}}{\tau(\chi)\cos\left(\frac{\pi\nu}{2}\right)}&\left\{\zeta(\nu+1)L(1,\chi)(qx)^{\nu} - \sum_{j=1}^{N}\zeta(2j)\ L(2j-\nu,\chi)(qx)^{2j-1} \right. \\ -(qx)^{2N+1}&\left.\sum_{n=1}^{\infty}\frac{\bar{\sigma}_{-\nu,\chi}(n)}{n}\left(\frac{n^{\nu+1-2N}-(qx)^{\nu+1-2N}}{n^2-(qx)^2}\right)\right\}, \end{aligned} \tag{4.5}$$

provided $qx \notin \mathbb{Z}_+$.

The equivalent version of Theorem 4.2.5 is the following result.

Theorem 4.2.6. *Let* $\nu \notin \mathbb{Z}$ *such that* $\Re(\nu) \geq 0$. *Then, for any integer N such that* $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, *we have*

$$\begin{aligned} &8\pi x^{\nu/2}\sum_{n=1}^{\infty}n^{\nu/2}K_{\nu}(4\pi\sqrt{nx})\sum_{d|n}d^{-\nu}\sin(2\pi d\theta) \\ &= \frac{1}{\cos\left(\frac{\pi\nu}{2}\right)}\zeta(\nu+1)\left(\zeta(1,\theta)-\zeta(1,1-\theta)\right)x^{\nu} \\ &- \frac{\pi}{2\sin\left(\frac{\pi\nu}{2}\right)}\left(\zeta(1-\nu,\theta)-\zeta(1-\nu,1-\theta)\right) + \frac{1}{2x\cos\left(\frac{\pi\nu}{2}\right)}\left(\zeta(-\nu,\theta)-\zeta(-\nu,1-\theta)\right) \\ &- \frac{1}{\cos\left(\frac{\pi\nu}{2}\right)}\sum_{j=1}^{N}\zeta(2j)\left(\zeta(2j-\nu,\theta)-\zeta(2j-\nu,1-\theta)\right)x^{2j-1} \\ &- \frac{1}{\cos\left(\frac{\pi\nu}{2}\right)}x^{2N+1}\sum_{d=1}^{\infty}d^{-\nu-1}\sum_{m=0}^{\infty}\left\{\frac{\left(d(m+\theta)\right)^{\nu+1-2N}-x^{\nu+1-2N}}{(m+\theta)\left(d^2(m+\theta)^2-x^2\right)}\right. \\ &\qquad\left. - \frac{\left(d(m+1-\theta)\right)^{\nu+1-2N}-x^{\nu+1-2N}}{(m+1-\theta)\left(d^2(m+1-\theta)^2-x^2\right)}\right\}, \end{aligned}$$

provided $x \notin H_{\theta,0}$.

Theorem 4.2.7. *Let* $\nu \notin \mathbb{Z}$ *such that* $\Re(\nu) \geq 0$. *Let* χ *be an odd primitive character modulo q. If N is any integer such that* $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, *then*

$$8\pi x^{\nu/2}\sum_{n=1}^{\infty}\bar{\sigma}_{-\nu,\bar{\chi}}(n)n^{\nu/2}K_{\nu}(4\pi\sqrt{nx}) = \frac{2\Gamma(\nu)\zeta(\nu)L(0,\bar{\chi})}{(2\pi)^{\nu-1}} + \frac{iq}{\tau(\chi)}\left\{\frac{L(\nu,\chi)}{\cos\left(\frac{\pi\nu}{2}\right)}(qx)^{\nu-1}\right.$$

$$+\frac{\pi L(1+\nu,\chi)}{\sin\left(\frac{\pi\nu}{2}\right)}(qx)^{\nu}+\frac{2}{\cos\left(\frac{\pi\nu}{2}\right)}\sum_{j=1}^{N-1}\zeta(2j+1-\nu)L(2j+1,\chi)(qx)^{2j}$$
$$+\frac{2}{\cos\left(\frac{\pi\nu}{2}\right)}(qx)^{2N}\sum_{n=1}^{\infty}\sigma_{-\nu,\chi}(n)\left(\frac{n^{\nu+1-2N}-(qx)^{\nu+1-2N}}{n^2-(qx)^2}\right)\Bigg\}, \tag{4.6}$$

provided $qx\notin\mathbb{Z}_+$.

Analogous to Theorem 4.2.5, the equivalent version of Theorem 4.2.7 is the following result.

Theorem 4.2.8. *Let* $\nu\notin\mathbb{Z}$ *such that* $\Re(\nu)\geq 0$. *Then, for any integer* N *such that* $N\geq\lfloor\frac{\Re(\nu)+1}{2}\rfloor$, *we have*

$$8\pi x^{\nu/2}\sum_{n=1}^{\infty}n^{\nu/2}K_{\nu}(4\pi\sqrt{nx})\sum_{d|n}d^{-\nu}\sin\left(\frac{2\pi n\theta}{d}\right)$$
$$=\frac{2}{(2\pi)^{\nu}}\Gamma(\nu)\zeta(\nu)\left(\zeta(1,\theta)-\zeta(1,1-\theta)\right)$$
$$+\frac{\pi}{2\sin\left(\frac{\pi\nu}{2}\right)}x^{\nu}\left(\zeta(1+\nu,\theta)-\zeta(1+\nu,1-\theta)\right)+\frac{x^{\nu-1}}{2\cos\left(\frac{\pi\nu}{2}\right)}\left(\zeta(\nu,\theta)-\zeta(\nu,1-\theta)\right)$$
$$+\frac{1}{\cos\left(\frac{\pi\nu}{2}\right)}\sum_{j=1}^{N-1}\zeta(2j+1-\nu)\left(\zeta(2j+1,\theta)-\zeta(2j+1,1-\theta)\right)x^{2j}$$
$$+\frac{x^{2N}}{\cos\left(\frac{\pi\nu}{2}\right)}\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\Bigg\{(m+\theta)^{-\nu}\frac{(r(m+\theta))^{\nu+1-2N}-x^{\nu+1-2N}}{r^2(m+\theta)^2-x^2}$$
$$-(m+1-\theta)^{-\nu}\frac{(r(m+1-\theta))^{\nu+1-2N}-x^{\nu+1-2N}}{r^2(m+1-\theta)^2-x^2}\Bigg\},$$

provided $x\notin H_{\theta,0}$.

Setting $\nu=1/2$ in Theorem 4.2.5 and Theorem 4.2.7, we obtain the following

Corollary 4.2.3. *We have*

$$2\pi\sum_{n=1}^{\infty}\sigma_{-\frac{1}{2},\bar{\chi}}(n)e^{-4\pi\sqrt{nx}}=-\pi L(1/2,\bar{\chi})+\frac{1}{4\pi}L(3/2,\bar{\chi})x^{-1}+\frac{2iq}{\tau(\chi)}\zeta(3/2)L(1,\chi)x^{1/2}$$
$$-\frac{2iq^{3/2}}{\tau(\chi)}x\sum_{n=1}^{\infty}\bar{\sigma}_{-\frac{1}{2},\chi}(n)\frac{(n+\sqrt{nqx}+qx)}{n(n+qx)(n^{\frac{1}{2}}+(qx)^{\frac{1}{2}})}.$$

Corollary 4.2.4. *We have*

$$2\pi \sum_{n=1}^{\infty} \bar{\sigma}_{-\frac{1}{2},\bar{\chi}}(n) e^{-4\pi\sqrt{nx}} = 2\pi\zeta(1/2)L(0,\bar{\chi}) + \frac{iq^{1/2}}{\tau(\chi)} L(1/2,\chi)x^{-\frac{1}{2}} + \frac{\pi i q^{3/2}}{\tau(\chi)} L(3/2,\chi)x^{\frac{1}{2}} + \frac{2iq}{\tau(\chi)} \sum_{n=1}^{\infty} \sigma_{-\frac{1}{2},\chi}(n) \frac{(n+\sqrt{nqx}+qx)}{(n+qx)(\sqrt{n}+\sqrt{qx})}.$$

4.2.3 Identities involving two characters and specializations

In this subsection, we state the identities corresponding to

$$\sigma_{-\nu,\chi_1,\chi_2}(n) = \sum_{d/n} d^{-\nu} \chi_1(d)\chi_2(n/d),$$

where χ_1 and χ_2 are the Dirichlet characters modulo p and q, respectively.

Theorem 4.2.9. *Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Both χ_1 and χ_2 are non-principal even primitive characters modulo p and q, respectively. If N is any integer such that $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, then*

$$\begin{aligned}
&8\pi x^{\frac{\nu}{2}} \sum_{n=1}^{\infty} \sigma_{-\nu,\bar{\chi_1},\bar{\chi_2}}(n) n^{\nu/2} K_\nu(4\pi\sqrt{nx}) \\
&= \frac{2p^{1-\nu}q}{\tau(\chi_1)\tau(\chi_2)\sin\left(\frac{\pi\nu}{2}\right)} \left\{ \sum_{j=1}^{N} L(2j,\chi_2)\ L(2j-\nu,\chi_1)(pqx)^{2j-1} \right. \\
&\left. +(pqx)^{2N+1} \sum_{n=1}^{\infty} \sigma_{-\nu,\chi_2,\chi_1}(n) \left(\frac{n^{\nu-2N} - (pqx)^{\nu-2N}}{n^2-(pqx)^2} \right) \right\},
\end{aligned}$$

provided $pqx \notin \mathbb{Z}_+$.

Theorem 4.2.10. *Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Then, for any integer N such that $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, we have*

$$\begin{aligned}
&8\pi x^{\frac{\nu}{2}} \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^{-\nu} \cos(2\pi d\theta) \cos\left(\frac{2\pi n\psi}{d}\right) \\
&= -\frac{\pi\ x^{\nu}}{2\cos\left(\frac{\pi\nu}{2}\right)} \left\{ \zeta(1+\nu,\psi) + \zeta(1+\nu,1-\psi) \right\}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\pi}{2\cos\left(\frac{\pi\nu}{2}\right)}\left(\zeta(1-\nu,\theta)+\zeta(1-\nu,1-\theta)\right)\\
&+\frac{1}{2\sin\left(\frac{\pi\nu}{2}\right)}\sum_{j=1}^{N}x^{2j-1}\left\{\zeta(2j,\psi)+\zeta(2j,1-\psi)\right\}\left\{\zeta(2j-\nu,\theta)+\zeta(2j-\nu,1-\theta)\right\}\\
&+\frac{1}{2\sin\left(\frac{\pi\nu}{2}\right)}x^{2N+1}\sum_{m,n\geq 0}^{\infty}\left\{(n+\psi)^{-\nu}\left(\frac{((n+\psi)(m+\theta))^{\nu-2N}-x^{\nu-2N}}{((n+\psi)(m+\theta))^{2}-x^{2}}\right)\right.\\
&\qquad+(n+1-\psi)^{-\nu}\left(\frac{((n+1-\psi)(m+\theta))^{\nu-2N}-x^{\nu-2N}}{((n+1-\psi)(m+\theta))^{2}-x^{2}}\right)\\
&\qquad+(n+\psi)^{-\nu}\left(\frac{((n+\psi)(m+1-\theta))^{\nu-2N}-x^{\nu-2N}}{((n+\psi)(m+1-\theta))^{2}-x^{2}}\right)\\
&\qquad\left.+(n+1-\psi)^{-\nu}\left(\frac{((n+1-\psi)(m+1-\theta))^{\nu-2N}-x^{\nu-2N}}{((n+1-\psi)(m+1-\theta))^{2}-x^{2}}\right)\right\},
\end{aligned}
$$

provided $x\notin H_{\theta,\psi}$.

To prove Theorem 4.2.10, one requires Proposition 1.4.1, Theorem 4.2.2, Theorem 4.2.4 and Theorem 4.2.9. But Theorem 4.2.10 implies Theorem 4.2.9 independently.

Theorem 4.2.11. *Let* $\nu\notin\mathbb{Z}$ *such that* $\Re(\nu)\geq 0$. *Both* χ_1 *and* χ_2 *are odd primitive characters modulo* p *and* q, *respectively. If* N *is any integer such that* $N\geq\lfloor\frac{\Re(\nu)+1}{2}\rfloor$, *then*

$$
\begin{aligned}
&8\pi x^{\nu/2}\sum_{n=1}^{\infty}\sigma_{-\nu,\bar{\chi_1},\bar{\chi_2}}(n)n^{\nu/2}K_{\nu}(4\pi\sqrt{nx})=\Gamma(\nu)L(\nu,\bar{\chi_1})L(0,\bar{\chi_2})\frac{2}{(2\pi)^{\nu-1}}\\
&-\frac{2p^{1-\nu}q}{\tau(\chi_1)\tau(\chi_2)\sin\left(\frac{\pi\nu}{2}\right)}\left\{-L(\nu+1,\chi_2)L(1,\chi_1)(pqx)^{\nu}\right.\\
&+\sum_{j=1}^{N-1}L(2j+1,\chi_2)\ L(2j+1-\nu,\chi_1)(pqx)^{2j}\\
&\left.+(pqx)^{2N}\sum_{n=1}^{\infty}\frac{\sigma_{-\nu,\chi_2,\chi_1}(n)}{n}\left(\frac{n^{\nu-2N+2}-(pqx)^{\nu-2N+2}}{n^{2}-(pqx)^{2}}\right)\right\},
\end{aligned}
$$

provided $pqx\notin\mathbb{Z}_{+}$.

The equivalent version of Theorem 4.2.11 is the following result.

Theorem 4.2.12. *Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Then, for any integer N such that $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, we have*

$$
\begin{aligned}
&8\pi x^{\frac{\nu}{2}} \sum_{n=1}^{\infty} n^{\nu/2} K_\nu(a\sqrt{nx}) \sum_{d|n} d^{-\nu} \sin(2\pi d\theta) \sin\left(\frac{2\pi n\psi}{d}\right) \\
&= \frac{1}{2\sin\left(\frac{\pi\nu}{2}\right)} \left(\zeta(1-\nu,\theta) - \zeta(1-\nu,1-\theta)\right)\left(\zeta(1,\psi) - \zeta(1,1-\psi)\right) \\
&- \frac{1}{2\sin\left(\frac{\pi\nu}{2}\right)} x^\nu \left(\zeta(1,\theta) - \zeta(1,1-\theta)\right)\left(\zeta(\nu+1,\psi) - \zeta(\nu+1,1-\psi)\right) \\
&+ \frac{1}{2\sin\left(\frac{\pi\nu}{2}\right)} \sum_{j=1}^{N-1} x^{2j} \left(\zeta(2j+1-\nu,\theta) - \zeta(2j+1-\nu,1-\theta)\right) \\
&\qquad\qquad \times \left(\zeta(2j+1,\psi) - \zeta(2j+1,1-\psi)\right) \\
&+ \frac{1}{2\sin\left(\frac{\pi\nu}{2}\right)} x^{2N} \sum_{m,n\geq 0}^{\infty} \left\{ \frac{(n+\psi)^{-\nu-1}}{(m+\theta)} \left(\frac{((n+\psi)(m+\theta))^{\nu-2N+2} - x^{\nu-2N+2}}{((n+\psi)(m+\theta))^2 - x^2} \right) \right. \\
&\qquad - \frac{(n+1-\psi)^{-\nu-1}}{(m+\theta)} \left(\frac{((n+1-\psi)(m+\theta))^{\nu-2N+2} - x^{\nu-2N+2}}{((n+1-\psi)(m+\theta))^2 - x^2} \right) \\
&\qquad - \frac{(n+\psi)^{-\nu-1}}{(m+1-\theta)} \left(\frac{((n+\psi)(m+1-\theta))^{\nu-2N+2} - x^{\nu-2N+2}}{((n+\psi)(m+1-\theta))^2 - x^2} \right) \\
&\qquad \left. + \frac{(n+1-\psi)^{-\nu-1}}{(m+1-\theta)} \left(\frac{((n+1-\psi)(m+1-\theta))^{\nu-2N+2} - x^{\nu-2N+2}}{((n+1-\psi)(m+1-\theta))^2 - x^2} \right) \right\},
\end{aligned}
$$

provided $x \notin H_{\theta,\psi}$.

Taking $\chi_1 = \chi_2 = \chi$ in Theorem 4.2.9 and Theorem 4.2.11, we get the following

Corollary 4.2.5. *Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Let χ be a non-principal even primitive character modulo q. If N is any integer such that $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, then*

$$
\begin{aligned}
&8\pi x^{\frac{\nu}{2}} \sum_{n=1}^{\infty} \sigma_{-\nu}(n)\bar{\chi}(n) n^{\nu/2} K_\nu(4\pi\sqrt{nx}) \\
&= \frac{2q^{2-\nu}}{\tau^2(\chi)\sin\left(\frac{\pi\nu}{2}\right)} \left\{ \sum_{j=1}^{N} L(2j,\chi)\ L(2j-\nu,\chi)(q^2x)^{2j-1} \right. \\
&\qquad \left. + (q^2x)^{2N+1} \sum_{n=1}^{\infty} \sigma_{-\nu}(n)\chi(n) \left(\frac{n^{\nu-2N} - (q^2x)^{\nu-2N}}{n^2 - (q^2x)^2} \right) \right\},
\end{aligned}
$$

provided $q^2x \notin \mathbb{Z}_+$.

Corollary 4.2.6. *Let* $\nu \notin \mathbb{Z}$ *such that* $\Re(\nu) \geq 0$. *Let* χ *be an odd primitive character modulo* p. *If* N *is any integer such that* $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, *then*

$$\begin{aligned}
8\pi x^{\frac{\nu}{2}} \sum_{n=1}^{\infty} \sigma_{-\nu}(n)\bar{\chi}(n) n^{\nu/2} K_{\nu}(4\pi\sqrt{nx}) &= \Gamma(\nu) L(\nu,\bar{\chi}) L(0,\bar{\chi}) \frac{2}{(2\pi)^{\nu-1}} \\
&- \frac{2p^{2-\nu}}{\tau^2(\chi)\sin\left(\frac{\pi\nu}{2}\right)} \Big\{ -L(\nu+1,\chi_2)L(1,\chi_1)(p^2x)^{\nu} \\
&+ \sum_{j=1}^{N-1} L(2j+1,\chi)\ L(2j+1-\nu,\chi)(p^2x)^{2j} \\
&+ (p^2x)^{2N} \sum_{n=1}^{\infty} \frac{\sigma_{-\nu}(n)\chi(n)}{n} \left(\frac{n^{\nu-2N+2} - (p^2x)^{\nu-2N+2}}{n^2-(p^2x)^2} \right) \Big\},
\end{aligned}$$

provided $p^2x \notin \mathbb{Z}_+$.

Theorem 4.2.13. *Let* $\nu \notin \mathbb{Z}$ *such that* $\Re(\nu) \geq 0$. *Let* χ_1 *be a non-principal even primitive character modulo* p *and* χ_2 *be an odd primitive character modulo* q. *If* N *is any integer such that* $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, *then*

$$\begin{aligned}
8\pi x^{\nu/2} \sum_{n=1}^{\infty} \sigma_{-\nu,\bar{\chi_1},\bar{\chi_2}}(n) n^{\nu/2} K_{\nu}(4\pi\sqrt{nx}) &= \frac{2}{(2\pi)^{\nu-1}} \Gamma(\nu) L(\nu,\bar{\chi_1}) L(0,\bar{\chi_2}) \\
&+ \frac{2ip^{1-\nu}q}{\tau(\chi_1)\tau(\chi_2)\cos\left(\frac{\pi\nu}{2}\right)} \left\{ \sum_{j=1}^{N-1} L(2j+1,\chi_2) L(2j+1-\nu,\chi_1)(pqx)^{2j} \right. \\
&\left. + (pqx)^{2N} \sum_{n=1}^{\infty} \sigma_{-\nu,\chi_2,\chi_1}(n) \left(\frac{n^{\nu-2N+1} - (pqx)^{\nu-2N+1}}{n^2-(pqx)^2} \right) \right\},
\end{aligned}$$

provided $pqx \notin \mathbb{Z}_+$.

Theorem 4.2.14. *Let* $\nu \notin \mathbb{Z}$ *such that* $\Re(\nu) \geq 0$. *Then, for any integer* N *such that* $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, *we have*

$$\begin{aligned}
&8\pi x^{\frac{\nu}{2}} \sum_{n=1}^{\infty} n^{\nu/2} K_{\nu}(4\pi\sqrt{nx}) \sum_{d|n} d^{-\nu} \cos(2\pi d\theta) \sin\left(\frac{2\pi n\psi}{d}\right) \\
&= \frac{\pi}{2\sin\left(\frac{\pi\nu}{2}\right)} x^{\nu} \left(\zeta(1+\nu,\psi) - \zeta(1+\nu,1-\psi) \right)
\end{aligned}$$

$$+\frac{1}{2\cos\left(\frac{\pi\nu}{2}\right)}\left(\zeta(1,\psi)-\zeta(1,1-\psi)\right)\left\{\zeta(1-\nu,\theta)+\zeta(1-\nu,1-\theta)\right\}$$
$$+\frac{1}{2\cos\left(\frac{\pi\nu}{2}\right)}\sum_{j=1}^{N-1}x^{2j}\left(\zeta(2j+1,\psi)-\zeta(2j+1,1-\psi)\right)$$
$$\times\left\{\zeta(2j+1-\nu,\theta)+\zeta(2j+1-\nu,1-\theta)\right\}$$
$$+\frac{1}{2\cos\left(\frac{\pi\nu}{2}\right)}x^{2N}\sum_{m,n\geq 0}^{\infty}\left\{(n+\psi)^{-\nu}\left(\frac{((n+\psi)(m+\theta))^{\nu-2N+1}-x^{\nu-2N+1}}{((n+\psi)(m+\theta))^2-x^2}\right)\right.$$
$$-(n+1-\psi)^{-\nu}\left(\frac{((n+1-\psi)(m+\theta))^{\nu-2N+1}-x^{\nu-2N+1}}{((n+1-\psi)(m+\theta))^2-x^2}\right)$$
$$+(n+\psi)^{-\nu}\left(\frac{((n+\psi)(m+1-\theta))^{\nu-2N+1}-x^{\nu-2N+1}}{((n+\psi)(m+1-\theta))^2-x^2}\right)$$
$$\left.-(n+1-\psi)^{-\nu}\left(\frac{((n+1-\psi)(m+1-\theta))^{\nu-2N+1}-x^{\nu-2N+1}}{((n+1-\psi)(m+1-\theta))^2-x^2}\right)\right\},$$

provided $x\notin H_{\theta,\psi}$.

Theorem 4.2.15. *Let* $\nu\notin\mathbb{Z}$ *such that* $\Re(\nu)\geq 0$. *Let* χ_1 *be an odd primitive character modulo* p *and* χ_2 *be a non-principal even primitive character modulo* q. *If* N *is any integer such that* $N\geq\lfloor\frac{\Re(\nu)+1}{2}\rfloor$, *then*

$$8\pi x^{\nu/2}\sum_{n=1}^{\infty}\sigma_{-\nu,\bar{\chi_1},\bar{\chi_2}}(n)n^{\nu/2}K_\nu(4\pi\sqrt{nx})$$
$$=\frac{2ip^{1-\nu}q}{\tau(\chi_1)\tau(\chi_2)\cos\left(\frac{\pi\nu}{2}\right)}\left\{L(\nu+1,\chi_2)L(1,\chi_1)(pqx)^\nu\right.$$
$$-\sum_{j=1}^{N}L(2j,\chi_2)\ L(2j-\nu,\chi_1)(pqx)^{2j-1}$$
$$\left.-(pqx)^{2N+1}\sum_{n=1}^{\infty}\frac{\sigma_{-\nu,\chi_2,\chi_1}(n)}{n}\left(\frac{n^{\nu-2N+1}-(pqx)^{\nu-2N+1}}{n^2-(pqx)^2}\right)\right\},$$

provided $pqx\notin\mathbb{Z}_+$.

Theorem 4.2.16. *Let* $\nu\notin\mathbb{Z}$ *such that* $\Re(\nu)\geq 0$. *Then, for any integer* N *such* *that* $N \geq \lfloor \frac{\Re(\nu)+1}{2} \rfloor$, *we have*

$$\begin{aligned}
&8\pi x^{\nu/2}\sum_{n=1}^{\infty} n^{\nu/2}K_\nu(4\pi\sqrt{nx})\sum_{d|n} d^{-\nu}\sin(2\pi d\theta)\cos\left(\frac{2\pi n\psi}{d}\right)\\
&= -\frac{\pi}{2\sin\left(\frac{\pi\nu}{2}\right)}\left(\zeta(1-\nu,\theta)-\zeta(1-\nu,1-\theta)\right)\\
&+\frac{x^\nu}{2\cos\left(\frac{\pi\nu}{2}\right)}\left(\zeta(1,\theta)-\zeta(1,1-\theta)\right)\left\{\zeta(1+\nu,\psi)+\zeta(1+\nu,1-\psi)\right\}\\
&-\frac{1}{2\cos\left(\frac{\pi\nu}{2}\right)}\sum_{j=1}^{N} x^{2j-1}\left(\zeta(2j-\nu,\theta)-\zeta(2j-\nu,1-\theta)\right)\left\{\zeta(2j,\psi)+\zeta(2j,1-\psi)\right\}\\
&-\frac{1}{2\cos\left(\frac{\pi\nu}{2}\right)}x^{2N+1}\sum_{m,n\geq 0}^{\infty}\left\{\frac{(n+\psi)^{-\nu-1}}{(m+\theta)}\left(\frac{((n+\psi)(m+\theta))^{\nu-2N+1}-x^{\nu-2N+1}}{((n+\psi)(m+\theta))^2-x^2}\right)\right.\\
&+\frac{(n+1-\psi)^{-\nu-1}}{(m+\theta)}\left(\frac{((n+1-\psi)(m+\theta))^{\nu-2N+1}-x^{\nu-2N+1}}{((n+1-\psi)(m+\theta))^2-x^2}\right)\\
&-\frac{(n+\psi)^{-\nu-1}}{(m+1-\theta)}\left(\frac{((n+\psi)(m+1-\theta))^{\nu-2N+1}-x^{\nu-2N+1}}{((n+\psi)(m+1-\theta))^2-x^2}\right)\\
&\left.-\frac{(n+1-\psi)^{-\nu-1}}{(m+1-\theta)}\left(\frac{((n+1-\psi)(m+1-\theta))^{\nu-2N+1}-x^{\nu-2N+1}}{((n+1-\psi)(m+1-\theta))^2-x^2}\right)\right\},
\end{aligned}$$

provided $x \notin H_{\theta,\psi}$.

To prove Theorem 4.2.14, one requires Theorem 4.2.13 and Theorem 4.2.8. On the other hand, Theorem 4.2.16 is based on Theorem 4.2.15 and Theorem 4.2.6. Conversely, Theorem 4.2.14 and Theorem 4.2.16 imply Theorem 4.2.13 and Theorem 4.2.15, respectively.

4.3 Proof of Cohen-type identities

This section is devoted to the proof of Cohen-type identities. Throughout this section, we will deal with $z=-\nu\notin\mathbb{Z}$ and $\Re(\nu)\geq 0$.
In general set up, if we consider $\Re(\nu)\geq 0$ with $\nu\notin\mathbb{Z}$ and $a=4\pi$, then (3.9) becomes

$$\sum_{n=1}^{\infty} f_{-\nu}(n)n^{\nu/2}K_\nu(4\pi\sqrt{nx})=\frac{1}{2}X^{\nu/2}\left(R_{1-\nu}+R_1+R_0+R_{-\nu}+J_{-\nu}^{(\nu)}(X)\right), \quad (4.7)$$

where $R_{-\nu}$ is the residue corresponding to the pole $s=-\nu$. It is easy to see that the pole at $s=-\nu$ appears if $\Re(\nu)=0$ and $\nu\notin\mathbb{Z}$. The expression $J_{-\nu}^{(\nu)}(X)$ defined in (3.10), can be rewritten as

$$J_{-\nu}^{(\nu)}(X):=\frac{1}{2\pi i}\int_{(-d)}\Gamma(s+\nu)\Gamma(s)F_{-\nu}(s)X^{s}\mathrm{d}s, \tag{4.8}$$

where $F_{-\nu}(s)$ defined in (3.5) is the Dirichlet series associated with the arithmetical function $f_{-\nu}(n)$. We will note that $X=\frac{1}{4\pi^2 x}$.

Proof of Theorem 4.2.1 and its equivalence with Theorem 4.2.2. We first see the proof of Theorem 4.2.1. Then, we shall provide proof of their equivalent identities.

Proof of Theorem 4.2.1 Letting $f_{-\nu}(n)=\sigma_{-\nu,\bar{\chi}}(n)$ where χ being a non-principal even primitive character modulo q in (4.7), we obtain

$$\sum_{n=1}^{\infty}\sigma_{-\nu,\bar{\chi}}(n)n^{\nu/2}K_{\nu}(4\pi\sqrt{nx})=\frac{1}{2}X^{\nu/2}(R_{1-\nu}+R_1+R_0+R_{-\nu}+J_{-\nu}^{(\nu)}(X)), \tag{4.9}$$

where $J_{-\nu}^{(\nu)}(X)$ is given in (4.8) and $F_{-\nu}(s)=\zeta(s)L(s+\nu,\bar{\chi})$. The integrand in (4.8) will encounter simple poles at $s=1$ and $s=0$ with residues R_1 and R_0 given by

$$R_1=\Gamma(1+\nu)L(1+\nu,\bar{\chi})X, \quad \text{and} \quad R_0=-\frac{\Gamma(\nu)L(\nu,\bar{\chi})}{2}, \tag{4.10}$$

respectively. It is easy to see that $R_{1-\nu}=0$. As $\bar{\chi}$ is a non-principal even primitive character, $L(s+\nu,\bar{\chi})$ has a zero at $s=-\nu$. Therefore, we will not be getting any contribution from the pole of $\Gamma(s+\nu)$ at $s=-\nu$. Hence $R_{-\nu}=0$. Now employing (4.10) together with the facts $R_{1-\nu}=0$ and $R_{-\nu}=0$ in (4.9), we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}\sigma_{-\nu,\bar{\chi}}(n)n^{\nu/2}K_{\nu}(4\pi\sqrt{nx})\\&=\frac{1}{2}X^{\nu/2}\left(\Gamma(1+\nu)L(1+\nu,\bar{\chi})X-\frac{\Gamma(\nu)L(\nu,\bar{\chi})}{2}+J_{-\nu}^{(\nu)}(X)\right),\end{aligned} \tag{4.11}$$

where $X=\frac{1}{4\pi^2 x}$. Next, we evaluate the following integral $J_{-\nu}^{(\nu)}(X)$. Replacing s by

$1-s$ and then employing (2.9), (2.14), we obtain

$$\begin{aligned}J_{-\nu}^{(\nu)}(X) &= \frac{1}{2\pi i}\int_{(1+d)}\Gamma(1-s+\nu)\Gamma(1-s)\zeta(1-s)L(1-s+\nu,\bar{\chi})X^{1-s}\mathrm{d}s\\ &= \left(\frac{2\pi}{q}\right)^{\nu}\frac{\pi^2 X}{\tau(\chi)}\frac{1}{2\pi i}\int_{(1+d)}\frac{\zeta(s)L(s-\nu,\chi)}{((2\pi)^2q^{-1}X)^s\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi(s-\nu)}{2}\right)}\mathrm{d}s\\ &= \frac{(2\pi)^{\nu}q^{-\nu}}{4x\tau(\chi)}\frac{1}{2\pi i}\int_{(1+d)}\frac{\zeta(s)L(s-\nu,\chi)(qx)^s}{\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi(s-\nu)}{2}\right)}\mathrm{d}s. \qquad (4.12)\end{aligned}$$

To evaluate the integral in (4.12), we employ the Cauchy residue theorem with the rectangular contour formed by the lines $[1+d-iT, 1+d+iT], [1+d+iT, 2N+\delta+iT], [2N+\delta+iT, 2N+\delta-iT], [2N+\delta-iT, 1+d-iT]$ with $N \geq [\frac{\Re(\nu)+1}{2}]$ and $\{\Re(\nu)+1\} < \delta < 1$ and T is a large positive number. One can note that the simple poles of $\sin^{-1}(\frac{\pi(s-\nu)}{2})$ at $s=\nu,\nu-2,\cdots$ will get canceled by the simple zeroes of $L(s-\nu,\chi)$. Hence the poles of the integrand function in (4.12) are at $2,4,\cdots,2N$, and $\nu+2,\cdots,\nu+2b_N$ where $b_N = \lfloor\frac{2N+\delta-\nu}{2}\rfloor$, and they are simple. Utilising the fact $|\sin\pi(\sigma+it)| \gg e^{\pi|t|}$ for $|t|\geq 1$, one can see that the integrals along the horizontal lines $[1+d+iT, 2N+\delta+iT]$ and $[2N+\delta-iT, 1+d-iT]$ vanish as $T\to\infty$. Hence we get

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(1+d)}\frac{\zeta(s)L(s-\nu,\chi)(qx)^s}{\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi(s-\nu)}{2}\right)}\mathrm{d}s\\ &= -\sum_{j=1}^{N}H_{2j} - \sum_{r=1}^{b_N}\mathcal{H}_{2r} + \frac{1}{2\pi i}\int_{(2N+\delta)}\frac{\zeta(s)L(s-\nu,\chi)(qx)^s}{\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi(s-\nu)}{2}\right)}\mathrm{d}s, \qquad (4.13)\end{aligned}$$

where H_{2j} is the residue at $s=2j$ given by

$$H_{2j} = -\frac{2\zeta(2j)L(2j-\nu,\chi)(qx)^{2j}}{\pi\sin(\frac{\pi\nu}{2})},$$

for $j=1,2,\cdots,N$ and $\mathcal{H}_{2r}$ is the residue at $s=\nu+2r$ given by

$$\mathcal{H}_{2r} = 2\zeta(\nu+2r)L(2r,\chi)\frac{(qx)^{\nu+2r}}{\pi\sin(\frac{\pi\nu}{2})} = \frac{2}{\pi\sin(\frac{\pi\nu}{2})}\sum_{n=1}^{\infty}\sigma_{\nu,\chi}(n)(n^{-1}qx)^{\nu+2r},$$

for $r = 1, 2, \cdots, b_N$. In the above expression, we have applied the series representation of function $\zeta(\nu + 2r)L(2r, \chi)$ for $r \geq 1$. Now let us evaluate the integral in (4.13):

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(2N+\delta)} \frac{\zeta(s)L(s-\nu,\chi)(qx)^s}{\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi(s-\nu)}{2}\right)}\mathrm{d}s \\
&= \sum_{n=1}^{\infty}\sigma_{\nu,\chi}(n)\frac{1}{2\pi i}\int_{(2N+\delta)} \frac{(n^{-1}qx)^s}{\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi(s-\nu)}{2}\right)}\mathrm{d}s \\
&= \left(\sum_{n<qx}+\sum_{n>qx}\right)\sigma_{\nu,\chi}(n)\frac{1}{2\pi i}\int_{(2N+\delta)} \frac{(n^{-1}qx)^s}{\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi(s-\nu)}{2}\right)}\mathrm{d}s,
\end{aligned} \tag{4.14}$$

noting that $qx \notin \mathbb{Z}_+$. Next, we will first investigate the sum $\sum_{n>qx}$. To evaluate this inner line integral in (4.14), we shall use the Cauchy residue theorem with the contour consisting of the lines $[2N+\delta-iT, 2N+\delta+iT], [2N+\delta+iT, 2M+\frac{1}{2}+iT], [2M+\frac{1}{2}+iT, 2M+\frac{1}{2}-iT], [2M+\frac{1}{2}-iT, 2N+\delta-iT]$ where $M \in \mathbb{N}$ is a large number and T is a large positive number. The poles of the integrand function in (4.14) are at $2N+2, 2N+4, \cdots, 2M$ and $\nu+2b_N+2, \nu+2b_N+4, \cdots, \nu+2a_M$ where $a_M = \lfloor M + \frac{1}{4} - \frac{\nu}{2} \rfloor$, and they are simple. Now, taking into account the fact that both the integrals along the horizontal lines $[2M+\frac{1}{2}-iT, 2N+\delta-iT]$ and $[2N+\delta+iT, 2M+\frac{1}{2}+iT]$ vanish as $T \to \infty$, we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(2N+\delta)} \frac{(n^{-1}qx)^s}{\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi(s-\nu)}{2}\right)}\mathrm{d}s \\
&= \frac{2}{\pi\sin\left(\frac{\pi\nu}{2}\right)}\left(\sum_{r=N+1}^{M}(n^{-1}qx)^{2r} - \sum_{r=b_N+1}^{a_M}(n^{-1}qx)^{\nu+2r}\right) \\
&\quad + \frac{1}{2\pi i}\int_{(2M+\frac{1}{2})} \frac{(n^{-1}qx)^s}{\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi(s-\nu)}{2}\right)}\mathrm{d}s \\
&= \frac{2}{\pi\sin\left(\frac{\pi\nu}{2}\right)}\left(\sum_{r=N+1}^{M}(n^{-1}qx)^{2r} - \sum_{r=b_N+1}^{a_M}(n^{-1}qx)^{\nu+2r}\right) + O\left((n^{-1}qx)^{2M+\frac{1}{2}}\right),
\end{aligned}$$

where in the last step to evaluate integral, we have used the fact $|\sin(\sigma+it)| = \sqrt{\sin^2\sigma + \sinh^2 t}$ for $|t| < 1$ and $|\sin\pi(\sigma+it)| \gg e^{\pi|t|}$ for $|t| \geq 1$. Letting $M \to \infty$, the error term in the above expression goes to 0 since $n^{-1}qx < 1$. Therefore, we get

$$\frac{1}{2\pi i}\int_{(2N+\delta)} \frac{(n^{-1}qx)^s}{\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi(s-\nu)}{2}\right)}\mathrm{d}s$$
$$= \frac{2}{\pi\sin\left(\frac{\pi\nu}{2}\right)}\left(\sum_{r=N+1}^{\infty}(n^{-1}qx)^{2r} - \sum_{r=b_N+1}^{\infty}(n^{-1}qx)^{\nu+2r}\right),$$

which in turn will give

$$\sum_{n>qx}\sigma_{\nu,\chi}(n)\frac{1}{2\pi i}\int_{(2N+\delta)} \frac{(n^{-1}qx)^s}{\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi(s-\nu)}{2}\right)}\mathrm{d}s$$
$$= \frac{2}{\pi\sin\left(\frac{\pi\nu}{2}\right)}\sum_{n>qx}\sigma_{\nu,\chi}(n)\left(\sum_{r=N+1}^{\infty}(n^{-1}qx)^{2r} - \sum_{r=b_N+1}^{\infty}(n^{-1}qx)^{\nu+2r}\right).$$

From the above expression, one can deduce

$$\sum_{n>qx}\sigma_{\nu,\chi}(n)\frac{1}{2\pi i}\int_{(2N+\delta)} \frac{(n^{-1}qx)^s}{\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi(s-\nu)}{2}\right)}\mathrm{d}s$$
$$- \frac{2}{\pi\sin(\frac{\pi\nu}{2})}\sum_{n>qx}\sigma_{\nu,\chi}(n)\sum_{r=1}^{b_N}(n^{-1}qx)^{\nu+2r}$$
$$= \frac{2}{\pi\sin\left(\frac{\pi\nu}{2}\right)}\sum_{n>qx}\sigma_{\nu,\chi}(n)\left(\sum_{r=N+1}^{\infty}(n^{-1}qx)^{2r} - \sum_{r=1}^{\infty}(n^{-1}qx)^{\nu+2r}\right)$$
$$= \frac{2}{\pi\sin\left(\frac{\pi\nu}{2}\right)}\sum_{n>qx}\sigma_{\nu,\chi}(n)\frac{(qx)^{2N+2}}{n^{\nu}}\frac{\left(n^{\nu-2N} - (qx)^{\nu-2N}\right)}{n^2 - q^2x^2}$$
$$= \frac{2(qx)^{2N+2}}{\pi\sin\left(\frac{\pi\nu}{2}\right)}\sum_{n>qx}\bar{\sigma}_{-\nu,\chi}(n)\left(\frac{n^{\nu-2N} - (qx)^{\nu-2N}}{n^2 - q^2x^2}\right). \tag{4.15}$$

Similarly, by shifting the line of integration to the left, $\sum_{n\leq qx}$ can be evaluated as

$$\sum_{n<qx}\sigma_{\nu,\chi}(n)\frac{1}{2\pi i}\int_{(2N+\delta)} \frac{(n^{-1}qx)^s}{\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi(s-\nu)}{2}\right)}\mathrm{d}s$$
$$- \frac{2}{\pi\sin(\frac{\pi\nu}{2})}\sum_{n<qx}\sigma_{\nu,\chi}(n)\sum_{r=1}^{b_N}(n^{-1}qx)^{\nu+2r}$$

$$= -\frac{2}{\pi \sin\left(\frac{\pi\nu}{2}\right)} \sum_{n<qx} \sigma_{\nu,\chi}(n) \left(\sum_{r=-N}^{\infty} (n(qx)^{-1})^{2r} - \sum_{r=0}^{\infty} (n(qx)^{-1})^{-\nu+2r} \right)$$
$$= -\frac{2}{\pi \sin\left(\frac{\pi\nu}{2}\right)} \sum_{n<qx} \sigma_{\nu,\chi}(n) \left(\left(\frac{qx}{n}\right)^{2N} \frac{(qx)^2}{(qx)^2 - n^2} - \left(\frac{qx}{n}\right)^{\nu} \frac{(qx)^2}{(qx)^2 - n^2} \right)$$
$$= \frac{2(qx)^{2N+2}}{\pi \sin\left(\frac{\pi\nu}{2}\right)} \sum_{n<qx} \bar{\sigma}_{-\nu,\chi}(n) \left(\frac{n^{\nu-2N} - (qx)^{\nu-2N}}{n^2 - q^2x^2} \right). \tag{4.16}$$

Now substituting (4.15) and (4.16) in (4.14),

$$\frac{1}{2\pi i} \int_{(2N+\delta)} \frac{\zeta(s)L(s-\nu,\chi)(qx)^s}{\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi(s-\nu)}{2}\right)} ds = \frac{2}{\pi \sin(\frac{\pi\nu}{2})} \sum_{n=1}^{\infty} \sigma_{\nu,\chi}(n) \sum_{r=1}^{b_N} (n^{-1}qx)^{\nu+2r}$$
$$+ \frac{2(qx)^{2N+2}}{\pi \sin\left(\frac{\pi\nu}{2}\right)} \sum_{n=1}^{\infty} \bar{\sigma}_{-\nu,\chi}(n) \left(\frac{n^{\nu-2N} - (qx)^{\nu-2N}}{n^2 - q^2x^2} \right). \tag{4.17}$$

Inserting (4.17) in (4.13) and then simplifying, we obtain

$$\frac{1}{2\pi i} \int_{(1+d)} \frac{\zeta(s)L(s-\nu,\chi)(qx)^s}{\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi(s-\nu)}{2}\right)} ds = \frac{2}{\pi \sin\left(\frac{\pi\nu}{2}\right)} \sum_{j=1}^{N} \zeta(2j)\ L(2j-\nu,\chi)(qx)^{2j}$$
$$+ \frac{2}{\pi \sin\left(\frac{\pi\nu}{2}\right)} (qx)^{2N+2} \sum_{n=1}^{\infty} \bar{\sigma}_{-\nu,\chi}(n) \left(\frac{n^{\nu-2N} - (qx)^{\nu-2N}}{n^2 - q^2x^2} \right). \tag{4.18}$$

Combining (4.18) with (4.12), we deduce that

$$J_{-\nu}^{(\nu)}(X) = \frac{(2\pi)^{\nu} q^{-\nu}}{2\pi x \tau(\chi) \sin\left(\frac{\pi\nu}{2}\right)} \left\{ \sum_{j=1}^{N} \zeta(2j)\ L(2j-\nu,\chi)(qx)^{2j} \right.$$
$$\left. + (qx)^{2N+2} \sum_{n=1}^{\infty} \bar{\sigma}_{-\nu,\chi}(n) \left(\frac{n^{\nu-2N} - (qx)^{\nu-2N}}{n^2 - q^2x^2} \right) \right\}. \tag{4.19}$$

Next, by substituting (4.19) in (4.11), one can finish the proof of (4.3).

Next, we demonstrate that Theorem 4.2.1 is equivalent to Theorem 4.2.2.

Theorem 4.2.1 ⇒ Theorem 4.2.2 It is sufficient to prove the identity (4.4) in Theorem 4.2.2 for $\theta = h/q$, where q is prime and $0 < h < q$. Multiply the equation

(3.54) by $8\pi x^{\frac{\nu}{2}}$ on both sides, then substitute $k=-\nu$ and $a=4\pi$, we obtain

$$8\pi x^{\nu/2}\sum_{n=1}^{\infty} n^{\nu/2}K_\nu(4\pi\sqrt{nx})\sum_{d|n} d^{-\nu}\cos\left(\frac{2\pi dh}{q}\right)$$
$$=\frac{q^{1-\nu}}{\phi(q)}8\pi(qx)^{\nu/2}\sum_{m=1}^{\infty}\sigma_{-\nu}(m)\ m^{\nu/2}K_\nu(4\pi\sqrt{qmx})-\frac{8\pi x^{\nu/2}}{\phi(q)}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)n^{\nu/2}K_\nu(4\pi\sqrt{nx})$$
$$+\frac{8\pi x^{\nu/2}}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\tau(\bar{\chi})\sum_{n=1}^{\infty}\sigma_{-\nu,\chi}(n)n^{\nu/2}K_\nu(4\pi\sqrt{nx}). \tag{4.20}$$

Now, we first evaluate the first two sums on the right-hand side of (4.20). By using Proposition 1.4.1, we have

$$\frac{q^{1-\nu}}{\phi(q)}8\pi(qx)^{\nu/2}\sum_{m=1}^{\infty}\sigma_{-\nu}(m)\ m^{\nu/2}K_\nu(4\pi\sqrt{qmx})-\frac{8\pi x^{\nu/2}}{\phi(q)}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)n^{\nu/2}K_\nu(4\pi\sqrt{nx})$$
$$=-(2\pi)^{1-\nu}\Gamma(\nu)\zeta(\nu)\frac{(q^{1-\nu}-1)}{\phi(q)}+2(2\pi)^{-1-\nu}\Gamma(\nu+1)\zeta(\nu+1)x^{-1}\frac{(q^{-\nu}-1)}{\phi(q)}$$
$$-\frac{\pi}{\cos(\frac{\pi\nu}{2})}\zeta(\nu+1)x^{\nu}+\frac{2}{\sin(\frac{\pi\nu}{2})}\left\{\sum_{j=1}^{\infty}\zeta(2j)\zeta(2j-\nu)x^{2j-1}\frac{(q^{2j-\nu}-1)}{\phi(q)}\right.$$
$$\left.+\frac{x^{2N+1}}{\phi(q)}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)\left(q^{2N+2-\nu}\ \frac{n^{\nu-2N}-(qx)^{\nu-2N}}{n^2-(qx)^2}-\frac{n^{\nu-2N}-x^{\nu-2N}}{n^2-x^2}\right)\right\}$$
$$=-\frac{\pi}{\cos(\frac{\pi\nu}{2})}\zeta(1-\nu)\frac{(q^{1-\nu}-1)}{\phi(q)}-\frac{1}{\sin(\frac{\pi\nu}{2})}\zeta(-\nu)x^{-1}\frac{(q^{-\nu}-1)}{\phi(q)}-\frac{\pi}{\cos(\frac{\pi\nu}{2})}\zeta(\nu+1)x^{\nu}$$
$$+\ \frac{2}{\sin(\frac{\pi\nu}{2})}\left\{\sum_{j=1}^{\infty}\zeta(2j)\zeta(2j-\nu)x^{2j-1}\frac{(q^{2j-\nu}-1)}{\phi(q)}\right.$$
$$\left.+\ \frac{x^{2N+1}}{\phi(q)}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)\left(q^{2N+2-\nu}\ \frac{n^{\nu-2N}-(qx)^{\nu-2N}}{n^2-(qx)^2}-\frac{n^{\nu-2N}-x^{\nu-2N}}{n^2-x^2}\right)\right\},\tag{4.21}$$

in the last step, we used the functional equation of the zeta function (2.9). Now, we examine the last sum on the right-hand side of (4.20). By identity (4.3) in Theorem (4.2.1), we have

$$\frac{8\pi x^{\nu/2}}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\tau(\bar{\chi})\sum_{n=1}^{\infty}\sigma_{-\nu,\chi}(n)n^{\nu/2}K_\nu(4\pi\sqrt{nx})$$

$$
\begin{aligned}
&= -\frac{\pi}{q^{\nu-1}\cos(\frac{\pi\nu}{2})}\frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)L(1-\nu,\bar{\chi}) - \frac{1}{xq^{\nu}\sin(\frac{\pi\nu}{2})}\frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)L(-\nu,\bar{\chi})\\
&+\frac{2}{\sin\left(\frac{\pi\nu}{2}\right)}\ q^{1-\nu}\left\{\sum_{j=1}^{N}\zeta(2j)\ \frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)L(2j-\nu,\bar{\chi})(qx)^{2j-1}\right.\\
&\qquad\left.+(qx)^{2N+1}\frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\sum_{n=1}^{\infty}\bar{\sigma}_{-\nu,\bar{\chi}}(n)\frac{n^{\nu-2N}-(qx)^{\nu-2N}}{n^2-(qx)^2}\right\}. \qquad (4.22)
\end{aligned}
$$

We consider

$$
\begin{aligned}
\sum_{\substack{\chi\neq\chi_0\\ \chi\ even}}\chi(h)L(s,\bar{\chi}) &= \sum_{\chi\ even}\chi(h)L(s,\bar{\chi}) - L(s,\chi_0)\\
&= \frac{\phi(q)}{2q^s}\left\{\zeta(s,h/q)+\zeta(s,1-h/q)\right\} - \left(1-\frac{1}{q^s}\right)\zeta(s). \qquad (4.23)
\end{aligned}
$$

Now, we examine the last expression in (4.22), and we obtain

$$
\begin{aligned}
&\frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\sum_{n=1}^{\infty}\bar{\sigma}_{-\nu,\bar{\chi}}(n)\frac{n^{\nu-2N}-(qx)^{\nu-2N}}{n^2-(qx)^2}\\
&=\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\frac{n^{\nu-2N}-(qx)^{\nu-2N}}{n^2-(qx)^2}\sum_{d/n}d^{-\nu}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\bar{\chi}\left(\frac{n}{d}\right)\\
&=\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\frac{n^{\nu-2N}-(qx)^{\nu-2N}}{n^2-(qx)^2}\sum_{d/n}d^{-\nu}\left\{\sum_{\chi even}\chi(h)\bar{\chi}\left(\frac{n}{d}\right)-\chi_0\left(\frac{n}{d}\right)\right\}\\
&=\frac{1}{2}\sum_{d=1}^{\infty}d^{-\nu}\sum_{\substack{m=1\\ m\equiv\pm h(q)}}^{\infty}\frac{(dm)^{\nu-2N}-(qx)^{\nu-2N}}{d^2m^2-(qx)^2}\\
&-\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\frac{n^{\nu-2N}-(qx)^{\nu-2N}}{n^2-(qx)^2}\left(\sigma_{-\nu}(n)-\sigma_{-\nu}\left(\frac{n}{q}\right)\right)\\
&=\frac{1}{2}\sum_{d=1}^{\infty}d^{-\nu}\sum_{r=0}^{\infty}\left\{\frac{(d(rq+h))^{\nu-2N}-(qx)^{\nu-2N}}{d^2(rq+h)^2-(qx)^2}\right.\\
&\qquad\left.+\frac{(d(rq+q-h))^{\nu-2N}-(qx)^{\nu-2N}}{d^2(rq+q-h)^2-(qx)^2}\right\}
\end{aligned}
$$

$$-\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)\frac{n^{\nu-2N}-(qx)^{\nu-2N}}{n^2-(qx)^2}+\frac{1}{\phi(q)}\sum_{r=1}^{\infty}\sigma_{-\nu}(r)\frac{(qr)^{\nu-2N}-(qx)^{\nu-2N}}{(qr)^2-(qx)^2}$$
$$=\frac{q^{\nu-2-2N}}{2}\sum_{d=1}^{\infty}d^{-\nu}\sum_{r=0}^{\infty}\Bigg(\frac{(d(r+h/q))^{\nu-2N}-x^{\nu-2N}}{d^2(r+h/q)^2-x^2}$$
$$+\frac{(d(r+1-h/q))^{\nu-2N}-x^{\nu-2N}}{d^2(r+1-h/q)^2-x^2}\Bigg)$$
$$-\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)\left(\frac{n^{\nu-2N}-(qx)^{\nu-2N}}{n^2-(qx)^2}-q^{\nu-2N-2}\,\frac{n^{\nu-2N}-x^{\nu-2N}}{n^2-x^2}\right). \quad (4.24)$$

Employing (4.23), (4.24) in (4.22), we get

$$\frac{8\pi x^{\nu/2}}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\tau(\bar{\chi})\sum_{n=1}^{\infty}\sigma_{-\nu,\chi}(n)n^{\nu/2}K_{\nu}(4\pi\sqrt{nx})$$
$$=-\frac{\pi}{2\cos\left(\frac{\pi\nu}{2}\right)}\left(\zeta(1-\nu,h/q)+\zeta(1-\nu,1-h/q)-2\frac{(q^{1-\nu}-1)}{\phi(q)}\zeta(1-\nu)\right)$$
$$-\frac{1}{2x\sin\left(\frac{\pi\nu}{2}\right)}\left(\zeta(-\nu,h/q)+\zeta(-\nu,1-h/q)-2\frac{(q^{-\nu}-1)}{\phi(q)}\zeta(-\nu)\right)$$
$$+\frac{1}{\sin\left(\frac{\pi\nu}{2}\right)}\sum_{j=1}^{N}\zeta(2j)\,(\zeta(2j-\nu,h/q)+\zeta(2j-\nu,1-h/q)$$
$$-2\frac{(q^{2j-\nu}-1)}{\phi(q)}\zeta(2j-\nu)\Bigg)\,x^{2j-1}$$
$$+\frac{2}{\sin\left(\frac{\pi\nu}{2}\right)}x^{2N+1}\Bigg\{\frac{1}{2}\sum_{d=1}^{\infty}d^{-\nu}\sum_{r=0}^{\infty}\Bigg(\frac{(d(r+h/q))^{\nu-2N}-x^{\nu-2N}}{d^2(r+h/q)^2-x^2}$$
$$+\frac{(d(r+1-h/q))^{\nu-2N}-x^{\nu-2N}}{d^2(r+1-h/q)^2-x^2}\Bigg)$$
$$-\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)\left(q^{2+2N-\nu}\,\frac{n^{\nu-2N}-(qx)^{\nu-2N}}{n^2-(qx)^2}-\frac{n^{\nu-2N}-x^{\nu-2N}}{n^2-x^2}\right)\Bigg\}. \quad (4.25)$$

Combining (4.20), (4.21) and(4.25), we get (4.4).

Theorem 4.2.2 ⇒ Theorem 4.2.1 Let $\theta=h/q$, and χ be an even primitive character modulo q. We first multiply the identity (4.4) in Theorem 4.2.2 by $\bar{\chi}(h)/\tau(\bar{\chi})$, and then take summation on h, $0<h<q$. The left-hand side of the identity (4.4)

becomes

$$
\begin{aligned}
&\frac{8\pi x^{\nu/2}}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\sum_{n=1}^{\infty}n^{\nu/2}K_{\nu}(a\sqrt{nx})\sum_{d|n}d^{-\nu}\cos\left(2\pi dh/q\right)\\
&=\frac{8\pi x^{\nu/2}}{2\tau(\bar{\chi})}\sum_{n=1}^{\infty}n^{\nu/2}K_{\nu}(a\sqrt{nx})\sum_{d|n}d^{-\nu}\sum_{h=1}^{q-1}\bar{\chi}(h)\left(e^{2\pi idh/q}+e^{-2\pi idh/q}\right)\\
&=\frac{8\pi x^{\nu/2}}{2}\sum_{n=1}^{\infty}n^{\nu/2}K_{\nu}(a\sqrt{nx})\sum_{d|n}d^{-\nu}\left(\chi(d)+\chi(-d)\right)\\
&=8\pi x^{\nu/2}\sum_{n=1}^{\infty}\sigma_{-\nu,\chi}(n)n^{\frac{\nu}{2}}K_{\nu}(a\sqrt{nx}),
\end{aligned}
\tag{4.26}
$$

where in the penultimate step, we have used (2.26). Next, we analyse the right-hand side expression of the identity (4.4). The first term in the right-hand side of the identity (4.4) becomes

$$
-\frac{1}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\frac{\pi\zeta(\nu+1)}{\cos\left(\frac{\pi\nu}{2}\right)}x^{\nu}=0.
\tag{4.27}
$$

Next, we evaluate the second, third and fourth terms of the right-hand side of identity (4.4). With the help of (2.3), we have

$$
\frac{1}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\left\{\zeta(1-\nu,h/q)+\zeta(1-\nu,1-h/q)\right\}=\frac{2q^{1-\nu}}{\tau(\bar{\chi})}L(1-\nu,\bar{\chi}),
\tag{4.28}
$$

$$
\frac{1}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\left\{\zeta(-\nu,h/q)+\zeta(-\nu,1-h/q)\right\}=\frac{2q^{-\nu}}{\tau(\bar{\chi})}L(-\nu,\bar{\chi}),
\tag{4.29}
$$

$$
\begin{aligned}
&\frac{1}{\tau(\bar{\chi})\sin\left(\frac{\pi\nu}{2}\right)}\sum_{h=1}^{q-1}\bar{\chi}(h)\sum_{j=1}^{N}\zeta(2j)\left(\zeta(2j-\nu,h/q)+\zeta(2j-\nu,1-h/q)\right)x^{2j-1}\\
&\qquad=\frac{2q^{1-\nu}}{\tau(\bar{\chi})\sin\left(\frac{\pi\nu}{2}\right)}\sum_{j=1}^{N}\zeta(2j)L(2j-\nu,\bar{\chi})(qx)^{2j-1}.
\end{aligned}
\tag{4.30}
$$

By using (4.27), (4.28), (4.29), and (4.30), the right-hand side of the identity (4.4) becomes

$$\frac{q^{1-\nu}}{\tau(\bar{\chi})}\left\{-\frac{\pi}{\cos\left(\frac{\pi\nu}{2}\right)}L(1-\nu,\bar{\chi})-\frac{1}{qx\sin\left(\frac{\pi\nu}{2}\right)}L(-\nu,\bar{\chi})\right.$$
$$+\frac{2}{\sin\left(\frac{\pi\nu}{2}\right)}\sum_{j=1}^{N}\zeta(2j)L(2j-\nu,\bar{\chi})(qx)^{2j-1}$$
$$\left.+\frac{1}{\sin\left(\frac{\pi\nu}{2}\right)}(qx)^{2N+1}\sum_{h=1}^{q-1}\bar{\chi}(h)\sum_{d=1}^{\infty}d^{-\nu}\left\{\sum_{r\equiv\pm h(q)}^{\infty}\frac{(rd)^{\nu-2N}-(qx)^{\nu-2N}}{r^2d^2-(qx)^2}\right\}\right\}. \quad (4.31)$$

Next, we evaluate the last term

$$\sum_{h=1}^{q-1}\bar{\chi}(h)\sum_{d=1}^{\infty}d^{-\nu}\left\{\sum_{r\equiv\pm h(q)}^{\infty}\frac{(rd)^{\nu-2N}-(qx)^{\nu-2N}}{r^2d^2-(qx)^2}\right\}$$
$$=2\sum_{d=1}^{\infty}d^{-\nu}\sum_{r=1}^{\infty}\frac{(rd)^{\nu-2N}-(qx)^{\nu-2N}}{r^2d^2-(qx)^2}\bar{\chi}(r)$$
$$=2\sum_{n=1}^{\infty}\bar{\sigma}_{-\nu,\bar{\chi}}(n)\left(\frac{n^{\nu-2N}-(qx)^{\nu-2N}}{n^2-(qx)^2}\right). \quad (4.32)$$

Substituting (4.32) in (4.31), we obtain

$$8\pi x^{\nu/2}\sum_{n=1}^{\infty}\sigma_{-\nu,\chi}(n)n^{\frac{\nu}{2}}K_{\nu}(a\sqrt{nx})$$
$$=\frac{q^{1-\nu}}{\tau(\bar{\chi})}\left\{-\frac{\pi}{\cos\left(\frac{\pi\nu}{2}\right)}L(1-\nu,\bar{\chi})-\frac{1}{qx\sin\left(\frac{\pi\nu}{2}\right)}L(-\nu,\bar{\chi})\right.$$
$$+\frac{2}{\sin\left(\frac{\pi\nu}{2}\right)}\sum_{j=1}^{N}\zeta(2j)L(2j-\nu,\bar{\chi})(qx)^{2j-1}$$
$$\left.+\frac{2}{\sin\left(\frac{\pi\nu}{2}\right)}(qx)^{2N+1}\sum_{n=1}^{\infty}\bar{\sigma}_{-\nu,\bar{\chi}}(n)\left(\frac{n^{\nu-2N}-(qx)^{\nu-2N}}{n^2-(qx)^2}\right)\right\}. \quad (4.33)$$

Employing the functional equation (2.13) in the first two terms in (4.33), then replacing χ by $\bar{\chi}$, we get (4.3). □

Remark. The other proofs of this section will be similar. We skip the proofs to avoid repetitions.

In the upcoming chapter, we will establish Voronoi-type summation formulas for twisted sums of divisor functions by applying analytic continuation to the Cohen-type identities.

5

Connection with Voronoi type summation formula

5.1 Introduction

In 1904, Voronoi investigated two noteworthy formulas related to the error term in the divisor problem. In addition to providing an explicit formula for the error term in (1.11), Voronoi devised a general summation formula for the sums involving the divisor function $d(n)$. The general summation formula for the divisor function with weight f is as follows:

$$\sum_{a\le n\le b}{}' d(n)f(n) = \int_a^b (\log(x)+2\gamma)f(x)dx \\ + \sum_{n=1}^{\infty} d(n)\int_a^b f(x)\ (4K_0(4\pi\sqrt{nx}) - 2\pi Y_0(4\pi\sqrt{nx}))dx, \qquad (5.1)$$

Using the above results, he improved the error term for the Dirichlet divisor problem at that time. Therefore, we infer that the Voronoi formula has been used to examine the error term $\Delta(x)$ in the Dirichlet divisor problem.

These days, it is known that there are Voronoi summation formulas for the coefficients of various kinds of L-functions, such as the L-functions related to modular forms, Maass forms, and, more recently, automorphic forms. The outstanding survey on Voronoi summation formulas [41] is recommended for the reader. According to this survey, functional equations for different L-functions may be obtained using summation formulae, and the summation formulas can be derived using the properties of the L-functions.

It is important to mention that the Voronoi-type summation formula and Cohen-type identities are more closely related. In this chapter, we study Voronoi-type summation formulas for twisted sums of divisor functions defined in (4.1). Equivalently, we study the character analogues of the Voronoi summation formula as stated in Proposition 1.4.2. This chapter is organized as follows: we state our main results in Section 5.2. Furthermore, we derive proofs in Section 5.3.

5.2 Main results

In this section, we offer Voronoi-type summation formulas for $\sigma_{z,\chi}(n), \bar{\sigma}_{z,\chi}(n)$ and $\sigma_{z,\chi_1,\chi_2}(n)$ defined in (1.28) and their equivalent version in trigonometric forms.

5.2.1 Identities involving even characters

Theorem 5.2.1. *Let $0 < \alpha < \beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let f denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that χ is a non-principal even primitive character modulo q. For $0 < \Re(\nu) < \frac{1}{2}$, we have*

$$
\begin{aligned}
&\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} \sum_{\alpha<j<\beta} \bar{\sigma}_{-\nu,\chi}(j) f(j) \\
&= \frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} L(1-\nu,\chi) \int_\alpha^\beta \frac{f(t)}{t^\nu} \mathrm{d}t + 2\pi \sum_{n=1}^{\infty} \sigma_{-\nu,\bar{\chi}}(n)\ n^{\nu/2} \int_\alpha^\beta f(t)(t)^{-\frac{\nu}{2}}
\end{aligned}
$$

$$\times\left\{\left(\frac{2}{\pi}K_\nu\left(4\pi\sqrt{\frac{nt}{q}}\right)-Y_\nu\left(4\pi\sqrt{\frac{nt}{q}}\right)\right)\cos\left(\frac{\pi\nu}{2}\right)-J_\nu\left(4\pi\sqrt{\frac{nt}{q}}\right)\sin\left(\frac{\pi\nu}{2}\right)\right\}dt. \tag{5.2}$$

Our next result is proved using Theorem 5.2.1 and Proposition 1.4.2. But the following identity directly implies Theorem 5.2.1.

Theorem 5.2.2. *Let* $0<\alpha<\beta$ *and* $\alpha,\beta\notin\mathbb{Z}$. *Let* f *denote a function analytic inside a closed contour strictly containing* $[\alpha,\beta]$. *Assume* $0<\Re(\nu)<\frac{1}{2}$, *then*

$$\begin{aligned}
&\sum_{\alpha<j<\beta}\sum_{d/j}d^{-\nu}\cos\left(\frac{2\pi j\theta}{d}\right)f(j)\\
&=\frac{\Gamma(\nu)\cos\left(\frac{\pi\nu}{2}\right)}{(2\pi)^\nu}\{\zeta(\nu,\theta)+\zeta(\nu,1-\theta)\}\int_\alpha^\beta\frac{f(t)}{t^\nu}\mathrm{d}t+\pi\int_\alpha^\beta\frac{f(t)}{t^{\frac{\nu}{2}}}\sum_{r=1}^\infty r^{\frac{\nu}{2}}\\
&\times\sum_{m=0}^\infty\left[\left\{\left(\frac{2}{\pi}\frac{K_\nu\left(4\pi\sqrt{r\,(m+\theta)\,t}\right)}{(m+\theta)^{\frac{\nu}{2}}}-\frac{Y_\nu\left(4\pi\sqrt{r\,(m+\theta)\,t}\right)}{(m+\theta)^{\frac{\nu}{2}}}\right)\cos\left(\frac{\pi\nu}{2}\right)\right.\right.\\
&\qquad\left.-\frac{J_\nu\left(4\pi\sqrt{r\,(m+\theta)\,t}\right)}{(m+\theta)^{\frac{\nu}{2}}}\sin\left(\frac{\pi\nu}{2}\right)\right\}\\
&+\left\{\left(\frac{2}{\pi}\frac{K_\nu\left(4\pi\sqrt{r\,(m+1-\theta)\,t}\right)}{(m+1-\theta)^{\frac{\nu}{2}}}-\frac{Y_\nu\left(4\pi\sqrt{r\,(m+1-\theta)\,t}\right)}{(m+1-\theta)^{\frac{\nu}{2}}}\right)\cos\left(\frac{\pi\nu}{2}\right)\right.\\
&\qquad\left.\left.-\frac{J_\nu\left(4\pi\sqrt{r\,(m+1-\theta)\,t}\right)}{(m+1-\theta)^{\frac{\nu}{2}}}\sin\left(\frac{\pi\nu}{2}\right)\right\}\right]\mathrm{d}t. \qquad (5.3)
\end{aligned}$$

Theorem 5.2.3. *Let* $0<\alpha<\beta$ *and* $\alpha,\beta\notin\mathbb{Z}$. *Let* f *denote a function analytic inside a closed contour strictly containing* $[\alpha,\beta]$. *Assume that* χ *is a non-principal even primitive character modulo* q. *For* $0<\Re(\nu)<\frac{1}{2}$, *we have*

$$\begin{aligned}
&\frac{q^{1+\frac{\nu}{2}}}{\tau(\chi)}\sum_{\alpha<j<\beta}\sigma_{-\nu,\chi}(j)f(j)\\
&=\frac{q^{1+\frac{\nu}{2}}}{\tau(\chi)}L(1+\nu,\chi)\int_\alpha^\beta f(t)\mathrm{d}t+2\pi\sum_{n=1}^\infty\bar{\sigma}_{-\nu,\bar{\chi}}(n)\;n^{\nu/2}\int_\alpha^\beta f(t)(t)^{-\frac{\nu}{2}}
\end{aligned}$$

$$\times \left\{ \left(\frac{2}{\pi} K_\nu \left(4\pi \sqrt{\frac{nt}{q}} \right) - Y_\nu \left(4\pi \sqrt{\frac{nt}{q}} \right) \right) \cos\left(\frac{\pi\nu}{2}\right) - J_\nu \left(4\pi \sqrt{\frac{nt}{q}} \right) \sin\left(\frac{\pi\nu}{2}\right) \right\} dt. \tag{5.4}$$

Analogous to Theorem 5.2.1, the equivalent version of Theorem 5.2.3 is the following result.

Theorem 5.2.4. *Let $0 < \alpha < \beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let f denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume $0 < \Re(\nu) < \frac{1}{2}$, then*

$$\begin{aligned}
&\sum_{\alpha<j<\beta} f(j) \sum_{d|j} d^{-\nu} \cos(2\pi d\theta) \\
&= (2\pi)^\nu \Gamma(-\nu) \cos\left(\frac{\pi\nu}{2}\right) \{\zeta(-\nu, \theta) + \zeta(-\nu, 1-\theta)\} \int_\alpha^\beta f(t)\mathrm{d}t + \pi \int_\alpha^\beta \frac{f(t)}{t^{\frac{\nu}{2}}} \sum_{d=1}^\infty d^{-\frac{\nu}{2}} \\
&\times \sum_{m=0}^\infty \Bigg[(m+\theta)^{\frac{\nu}{2}} \left\{ \left(\frac{2}{\pi} K_\nu \left(4\pi\sqrt{d(m+\theta)t} \right) - Y_\nu \left(4\pi\sqrt{d(m+\theta)t} \right) \right) \right. \\
&\qquad \left. \times \cos\left(\frac{\pi\nu}{2}\right) - J_\nu \left(4\pi\sqrt{d(m+\theta)t} \right) \sin\left(\frac{\pi\nu}{2}\right) \right\} \\
&+ (m+1-\theta)^{\frac{\nu}{2}} \left\{ \left(\frac{2}{\pi} K_\nu \left(4\pi\sqrt{d(m+1-\theta)t} \right) - Y_\nu \left(4\pi\sqrt{d(m+1-\theta)t} \right) \right) \right. \\
&\qquad \left. \times \cos\left(\frac{\pi\nu}{2}\right) - J_\nu \left(4\pi\sqrt{d(m+1-\theta)t} \right) \sin\left(\frac{\pi\nu}{2}\right) \right\} \Bigg] \mathrm{d}t.
\end{aligned} \tag{5.5}$$

5.2.2 Identities involving odd characters

Theorem 5.2.5. *Let $0 < \alpha < \beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let f denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that χ is an odd primitive character modulo q. For $0 < \Re(\nu) < \frac{1}{2}$, we have*

$$\begin{aligned}
&\frac{q^{1+\frac{\nu}{2}}}{\tau(\chi)} \sum_{\alpha<j<\beta} \sigma_{-\nu,\chi}(j) f(j) \\
&= \frac{q^{1+\frac{\nu}{2}}}{\tau(\chi)} L(1+\nu, \chi) \int_\alpha^\beta f(t)\mathrm{d}t + 2\pi i \sum_{n=1}^\infty \bar{\sigma}_{-\nu,\bar{\chi}}(n)\ n^{\nu/2} \int_\alpha^\beta f(t)(t)^{-\frac{\nu}{2}} \\
&\times \left\{ \left(\frac{2}{\pi} K_\nu \left(4\pi \sqrt{\frac{nt}{q}} \right) + Y_\nu \left(4\pi \sqrt{\frac{nt}{q}} \right) \right) \sin\left(\frac{\pi\nu}{2}\right) - J_\nu \left(4\pi \sqrt{\frac{nt}{q}} \right) \cos\left(\frac{\pi\nu}{2}\right) \right\} dt.
\end{aligned} \tag{5.6}$$

We demonstrate that Theorem 5.2.5 is equivalent to the following theorem.

Theorem 5.2.6. *Let $0 < \alpha < \beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let f denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume $0 < \Re(\nu) < \frac{1}{2}$. Then, we have*

$$\sum_{\alpha<j<\beta} \sum_{d|j} d^{-\nu} \sin(2\pi d\theta) f(j)$$
$$= -(2\pi)^{\nu}\Gamma(-\nu) \sin\left(\frac{\pi\nu}{2}\right) \{\zeta(-\nu, \theta) - \zeta(-\nu, 1-\theta)\} \int_{\alpha}^{\beta} f(t)\mathrm{d}t - \pi \int_{\alpha}^{\beta} \frac{f(t)}{t^{\frac{\nu}{2}}} \sum_{d=1}^{\infty} d^{-\frac{\nu}{2}}$$
$$\times \sum_{m=0}^{\infty} \left[(m+\theta)^{\frac{\nu}{2}} \left\{ \left(\frac{2}{\pi} K_{\nu} \left(4\pi\sqrt{d\,(m+\theta)\,t} \right) + Y_{\nu} \left(4\pi\sqrt{d\,(m+\theta)\,t} \right) \right) \right. \right.$$
$$\left. \times \sin\left(\frac{\pi\nu}{2}\right) - J_{\nu}\left(4\pi\sqrt{d\,(m+\theta)\,t}\right) \cos\left(\frac{\pi\nu}{2}\right) \right\}$$
$$- (m+1-\theta)^{\frac{\nu}{2}} \left\{ \left(\frac{2}{\pi} K_{\nu} \left(4\pi\sqrt{d\,(m+1-\theta)\,t} \right) + Y_{\nu} \left(4\pi\sqrt{d\,(m+1-\theta)\,t} \right) \right) \right.$$
$$\left. \left. \times \sin\left(\frac{\pi\nu}{2}\right) - J_{\nu}\left(4\pi\sqrt{d\,(m+1-\theta)\,t}\right) \cos\left(\frac{\pi\nu}{2}\right) \right\} \right] \mathrm{d}t. \tag{5.7}$$

Theorem 5.2.7. *Let $0 < \alpha < \beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let f denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that χ is an odd primitive character modulo q. For $0 < \Re(\nu) < \frac{1}{2}$, we have*

$$\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} \sum_{\alpha<j<\beta} \frac{\bar{\sigma}_{-\nu,\chi}(j)}{j} f(j)$$
$$= \frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} L(1-\nu, \chi) \int_{\alpha}^{\beta} \frac{f(t)}{t^{\nu+1}} \mathrm{d}t - 2\pi i \sum_{n=1}^{\infty} \sigma_{-\nu,\bar{\chi}}(n)\ n^{\nu/2} \int_{\alpha}^{\beta} f(t)(t)^{-\frac{\nu}{2}-1}$$
$$\times \left\{ \left(\frac{2}{\pi} K_{\nu}\left(4\pi\sqrt{\frac{nt}{q}}\right) - Y_{\nu}\left(4\pi\sqrt{\frac{nt}{q}}\right) \right) \sin\left(\frac{\pi\nu}{2}\right) + J_{\nu}\left(4\pi\sqrt{\frac{nt}{q}}\right) \cos\left(\frac{\pi\nu}{2}\right) \right\} dt.$$

Similar to Theorem 5.2.5, one can show that the equivalent version of Theorem 5.2.7 is the following result.

Theorem 5.2.8. *Let $0 < \alpha < \beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let f denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume $0 < \Re(\nu) < \frac{1}{2}$. Then, we* *have*

$$\sum_{\alpha<j<\beta}\sum_{d|j} d^{-\nu}\sin\left(\frac{2\pi j\theta}{d}\right)\frac{f(j)}{j}=\frac{\Gamma(\nu)\sin\left(\frac{\pi\nu}{2}\right)}{(2\pi)^{\nu}}\{\zeta(\nu,\theta)-\zeta(\nu,1-\theta)\}\int_{\alpha}^{\beta}\frac{f(t)}{t^{\nu+1}}\mathrm{d}t$$

$$+\pi\int_{\alpha}^{\beta}\frac{f(t)}{t^{\frac{\nu}{2}+1}}\sum_{r=1}^{\infty}r^{\frac{\nu}{2}}\sum_{m=0}^{\infty}\left[\left\{\left(\frac{2}{\pi}\frac{K_\nu\left(4\pi\sqrt{r\,(m+\theta)\,t}\,\right)}{(m+\theta)^{\frac{\nu}{2}}}-\frac{Y_\nu\left(4\pi\sqrt{r\,(m+\theta)\,t}\,\right)}{(m+\theta)^{\frac{\nu}{2}}}\right)\right.\right.$$

$$\left.\times\sin\left(\frac{\pi\nu}{2}\right)+\frac{J_\nu\left(4\pi\sqrt{r\,(m+\theta)\,t}\,\right)}{(m+\theta)^{\frac{\nu}{2}}}\cos\left(\frac{\pi\nu}{2}\right)\right\}$$

$$-\left\{\left(\frac{2}{\pi}\frac{K_\nu\left(4\pi\sqrt{r\,(m+1-\theta)\,t}\,\right)}{(m+1-\theta)^{\frac{\nu}{2}}}-\frac{Y_\nu\left(4\pi\sqrt{r\,(m+1-\theta)\,t}\,\right)}{(m+1-\theta)^{\frac{\nu}{2}}}\right)\sin\left(\frac{\pi\nu}{2}\right)\right.$$

$$\left.\left.+\frac{J_\nu\left(4\pi\sqrt{r\,(m+1-\theta)\,t}\,\right)}{(m+1-\theta)^{\frac{\nu}{2}}}\cos\left(\frac{\pi\nu}{2}\right)\right\}\right]\mathrm{d}t.$$

5.2.3 Identities involving two characters

In this subsection, we deduce Voronoi-type summation formula associated with $\sigma_{-\nu,\chi_1,\chi_2}(n)=\sum_{d/n} d^{-\nu}\chi_1(d)\chi_2(n/d)$, where χ_1 and χ_2 are Dirichlet characters modulo p and q, respectively. Furthermore, we also evaluate the equivalency version in double trigonometric forms.

Theorem 5.2.9. *Let $0<\alpha<\beta$ and $\alpha,\beta\notin\mathbb{Z}$. Let f denote a function analytic inside a closed contour strictly containing $[\alpha,\beta]$. Assume that χ_1 and χ_2 are odd primitive characters modulo p and q, respectively. For $0<\Re(\nu)<\frac{1}{2}$, we have*

$$\frac{p^{1-\frac{\nu}{2}}q^{1+\frac{\nu}{2}}}{\tau(\chi_1)\tau(\chi_2)}\sum_{\alpha<j<\beta}\frac{\sigma_{-\nu,\chi_2,\chi_1}(j)f(j)}{j}=-2\pi\sum_{n=1}^{\infty}\sigma_{-\nu,\bar{\chi_1},\bar{\chi_2}}(n)\;n^{\nu/2}\int_{\alpha}^{\beta}f(t)(t)^{-\frac{\nu}{2}-1}$$

$$\times\left\{\left(\frac{2}{\pi}K_\nu\left(4\pi\sqrt{\frac{nt}{pq}}\,\right)+Y_\nu\left(4\pi\sqrt{\frac{nt}{pq}}\,\right)\right)\cos\left(\frac{\pi\nu}{2}\right)+J_\nu\left(4\pi\sqrt{\frac{nt}{pq}}\,\right)\sin\left(\frac{\pi\nu}{2}\right)\right\}dt.$$

We remark that Theorem 5.2.9 is equivalent to the following result.

Theorem 5.2.10. *Let $0<\alpha<\beta$ and $\alpha,\beta\notin\mathbb{Z}$. Let f denote a function analytic* *inside a closed contour strictly containing* $[\alpha, \beta]$. *Assume* $0 < \Re(\nu) < \frac{1}{2}$, *then*

$$\sum_{\alpha<j<\beta} \sum_{d/j} d^{-\nu} \sin(2\pi d\theta) \sin\left(\frac{2\pi j\psi}{d}\right) \frac{f(j)}{j}$$

$$= \frac{\pi}{2} \int_\alpha^\beta \frac{f(t)}{t^{\frac{\nu}{2}+1}} \sum_{m,n\geq 0}^{\infty} \Bigg[\left(\frac{m+\theta}{n+\psi}\right)^{\nu/2} \Bigg\{ \bigg(\frac{2}{\pi} K_\nu(4\pi\sqrt{(n+\psi)(m+\theta)t})$$
$$+ Y_\nu(4\pi\sqrt{(n+\psi)(m+\theta)t}) \bigg) \cos\left(\frac{\pi\nu}{2}\right)$$
$$+ J_\nu(4\pi\sqrt{(n+\psi)(m+\theta)t}) \sin\left(\frac{\pi\nu}{2}\right) \Bigg\}$$
$$- \left(\frac{m+\theta}{n+1-\psi}\right)^{\nu/2} \Bigg\{ \bigg(\frac{2}{\pi} K_\nu(4\pi\sqrt{(n+1-\psi)(m+\theta)t})$$
$$+ Y_\nu(4\pi\sqrt{(n+1-\psi)(m+\theta)t}) \bigg) \cos\left(\frac{\pi\nu}{2}\right)$$
$$+ J_\nu(4\pi\sqrt{(n+1-\psi)(m+\theta)t}) \sin\left(\frac{\pi\nu}{2}\right) \Bigg\}$$
$$- \left(\frac{m+1-\theta}{n+\psi}\right)^{\nu/2} \Bigg\{ \bigg(\frac{2}{\pi} K_\nu(4\pi\sqrt{(n+\psi)(m+1-\theta)t})$$
$$+ Y_\nu(4\pi\sqrt{(n+\psi)(m+1-\theta)t}) \bigg) \cos\left(\frac{\pi\nu}{2}\right)$$
$$+ J_\nu(4\pi\sqrt{(n+\psi)(m+1-\theta)t}) \sin\left(\frac{\pi\nu}{2}\right) \Bigg\}$$
$$+ \left(\frac{m+1-\theta}{n+1-\psi}\right)^{\nu/2} \Bigg\{ \bigg(\frac{2}{\pi} K_\nu(4\pi\sqrt{(n+1-\psi)(m+1-\theta)t})$$
$$+ Y_\nu(4\pi\sqrt{(n+1-\psi)(m+1-\theta)t}) \bigg) \cos\left(\frac{\pi\nu}{2}\right)$$
$$+ J_\nu(4\pi\sqrt{(n+1-\psi)(m+1-\theta)t}) \sin\left(\frac{\pi\nu}{2}\right) \Bigg\} \Bigg] \, \mathrm{d}t.$$

Theorem 5.2.11. *Let* $0 < \alpha < \beta$ *and* $\alpha, \beta \notin \mathbb{Z}$. *Let* f *denote a function analytic inside a closed contour strictly containing* $[\alpha, \beta]$. *Assume that* χ_1 *and* χ_2 *are non-principal even primitive characters modulo* p *and* q, *respectively. For* $0 < \Re(\nu) < \frac{1}{2}$, *we have*

$$\frac{p^{1-\frac{\nu}{2}} q^{1+\frac{\nu}{2}}}{\tau(\chi_1)\tau(\chi_2)} \sum_{\alpha<j<\beta} \sigma_{-\nu,\chi_2,\chi_1}(j) f(j) = 2\pi \sum_{n=1}^{\infty} \sigma_{-\nu,\bar{\chi_1},\bar{\chi_2}}(n)\ n^{\nu/2} \int_\alpha^\beta f(t)(t)^{-\frac{\nu}{2}}$$
$$\times \left\{ \left(\frac{2}{\pi} K_\nu\left(4\pi\sqrt{\frac{nt}{pq}}\right) - Y_\nu\left(4\pi\sqrt{\frac{nt}{pq}}\right)\right) \cos\left(\frac{\pi\nu}{2}\right) - J_\nu\left(4\pi\sqrt{\frac{nt}{pq}}\right) \sin\left(\frac{\pi\nu}{2}\right) \right\} dt.$$

For deriving our next result, one requires Proposition 1.4.2, Theorem 5.2.4, Theorem 5.2.2 and Theorem 5.2.11. But our next result implies Theorem 5.2.11 independently.

Theorem 5.2.12. *Let* $0 < \alpha < \beta$ *and* $\alpha, \beta \notin \mathbb{Z}$. *Let* f *denote a function analytic inside a closed contour strictly containing* $[\alpha, \beta]$. *Assume* $0 < \Re(\nu) < \frac{1}{2}$, *then*

$$\sum_{\alpha<j<\beta} \sum_{d/j} d^{-\nu} \cos(2\pi d\theta) \cos\left(\frac{2\pi j\psi}{d}\right) f(j)$$
$$= \frac{\pi}{2} \int_\alpha^\beta \frac{f(t)}{t^{\frac{\nu}{2}}} \sum_{m,n\geq 0}^{\infty} \left[\left(\frac{m+\theta}{n+\psi}\right)^{\nu/2} \left\{ \left(\frac{2}{\pi} K_\nu(4\pi\sqrt{(n+\psi)(m+\theta)t}) - Y_\nu(4\pi\sqrt{(n+\psi)(m+\theta)t})\right) \cos\left(\frac{\pi\nu}{2}\right) - J_\nu(4\pi\sqrt{(n+\psi)(m+\theta)t}) \sin\left(\frac{\pi\nu}{2}\right) \right\}\right.$$
$$+ \left(\frac{m+\theta}{n+1-\psi}\right)^{\nu/2} \left\{ \left(\frac{2}{\pi} K_\nu(4\pi\sqrt{(n+1-\psi)(m+\theta)t}) - Y_\nu(4\pi\sqrt{(n+1-\psi)(m+\theta)t})\right) \cos\left(\frac{\pi\nu}{2}\right) - J_\nu(4\pi\sqrt{(n+1-\psi)(m+\theta)t}) \sin\left(\frac{\pi\nu}{2}\right) \right\}$$
$$+ \left(\frac{m+1-\theta}{n+\psi}\right)^{\nu/2} \left\{ \left(\frac{2}{\pi} K_\nu(4\pi\sqrt{(n+\psi)(m+1-\theta)t}) - Y_\nu(4\pi\sqrt{(n+\psi)(m+1-\theta)t})\right) \cos\left(\frac{\pi\nu}{2}\right) - J_\nu(4\pi\sqrt{(n+\psi)(m+1-\theta)t}) \sin\left(\frac{\pi\nu}{2}\right) \right\}$$
$$\left. + \left(\frac{m+1-\theta}{n+1-\psi}\right)^{\nu/2} \left\{ \left(\frac{2}{\pi} K_\nu(4\pi\sqrt{(n+1-\psi)(m+1-\theta)t}) - Y_\nu(4\pi\sqrt{(n+1-\psi)(m+1-\theta)t})\right) \cos\left(\frac{\pi\nu}{2}\right) - J_\nu(4\pi\sqrt{(n+1-\psi)(m+1-\theta)t}) \sin\left(\frac{\pi\nu}{2}\right) \right\} \right] \mathrm{d}t. \tag{5.8}$$

Substituting $\chi_1 = \chi_2 = \chi$ in Theorem 5.2.9 and Theorem 5.2.11, we get the following results.

Corollary 5.2.1. *Let* $0 < \alpha < \beta$ *and* $\alpha, \beta \notin \mathbb{Z}$. *Let* f *denote a function analytic inside a closed contour strictly containing* $[\alpha, \beta]$. *Assume that* χ *is an odd primitive* *character modulo q. For* $0 < \Re(\nu) < \frac{1}{2}$*, we have*

$$\frac{q^2}{\tau^2(\chi)} \sum_{\alpha<j<\beta} \frac{\sigma_{-\nu}(j)\chi(j)f(j)}{j} = -2\pi \sum_{n=1}^{\infty} \sigma_{-\nu}(n)\bar{\chi}(j)\ n^{\nu/2} \int_{\alpha}^{\beta} f(t)(t)^{-\frac{\nu}{2}-1}$$
$$\times \left\{ \left(\frac{2}{\pi} K_\nu \left(4\pi\sqrt{\frac{nt}{q^2}} \right) + Y_\nu \left(4\pi\sqrt{\frac{nt}{q^2}} \right) \right) \cos\left(\frac{\pi\nu}{2}\right) + J_\nu \left(4\pi\sqrt{\frac{nt}{q^2}} \right) \sin\left(\frac{\pi\nu}{2}\right) \right\} dt.$$

Corollary 5.2.2. *Let* $0 < \alpha < \beta$ *and* $\alpha, \beta \notin \mathbb{Z}$*. Let* f *denote a function analytic inside a closed contour strictly containing* $[\alpha, \beta]$*. Assume that* χ *is a non-principal even primitive character modulo q. For* $0 < \Re(\nu) < \frac{1}{2}$*, we have*

$$\frac{q^2}{\tau^2(\chi)} \sum_{\alpha<j<\beta} \sigma_{-\nu}(j)\chi(j)f(j) = 2\pi \sum_{n=1}^{\infty} \sigma_{-\nu}(n)\bar{\chi}(j)\ n^{\nu/2} \int_{\alpha}^{\beta} f(t)(t)^{-\frac{\nu}{2}}$$
$$\times \left\{ \left(\frac{2}{\pi} K_\nu \left(4\pi\sqrt{\frac{nt}{q^2}} \right) - Y_\nu \left(4\pi\sqrt{\frac{nt}{q^2}} \right) \right) \cos\left(\frac{\pi\nu}{2}\right) - J_\nu \left(4\pi\sqrt{\frac{nt}{q^2}} \right) \sin\left(\frac{\pi\nu}{2}\right) \right\} dt.$$

Theorem 5.2.13. *Let* $0 < \alpha < \beta$ *and* $\alpha, \beta \notin \mathbb{Z}$*. Let* f *denote a function analytic inside a closed contour strictly containing* $[\alpha, \beta]$*. Assume that* χ_1 *is an odd primitive character modulo p and* χ_2 *is a non-principal even primitive character modulo q. For* $0 < \Re(\nu) < \frac{1}{2}$*, we have*

$$\frac{p^{1-\frac{\nu}{2}} q^{1+\frac{\nu}{2}}}{\tau(\chi_1)\tau(\chi_2)} \sum_{\alpha<j<\beta} \frac{\sigma_{-\nu,\chi_2,\chi_1}(j)f(j)}{j} = -2\pi i \sum_{n=1}^{\infty} \sigma_{-\nu,\bar{\chi_1},\bar{\chi_2}}(n)\ n^{\nu/2} \int_{\alpha}^{\beta} f(t)t^{-\frac{\nu}{2}-1}$$
$$\times \left\{ \left(\frac{2}{\pi} K_\nu \left(4\pi\sqrt{\frac{nt}{pq}} \right) - Y_\nu \left(4\pi\sqrt{\frac{nt}{pq}} \right) \right) \sin\left(\frac{\pi\nu}{2}\right) + J_\nu \left(4\pi\sqrt{\frac{nt}{pq}} \right) \cos\left(\frac{\pi\nu}{2}\right) \right\} dt.$$

One requires Theorem 5.2.8 and Theorem 5.2.13 to prove our next result. Conversely, our next result directly implies Theorem 5.2.13.

Theorem 5.2.14. *Let* $0 < \alpha < \beta$ *and* $\alpha, \beta \notin \mathbb{Z}$*. Let* f *denote a function analytic inside a closed contour strictly containing* $[\alpha, \beta]$*. Assume* $0 < \Re(\nu) < \frac{1}{2}$*, then*

$$\sum_{\alpha<j<\beta} \sum_{d/j} d^{-\nu} \cos(2\pi d\theta) \sin\left(\frac{2\pi j\psi}{d}\right) \frac{f(j)}{j}$$
$$= \frac{\pi}{2} \int_{\alpha}^{\beta} \frac{f(t)}{t^{\frac{\nu}{2}+1}} \sum_{m,n\geq 0}^{\infty} \left[\left(\frac{m+\theta}{n+\psi} \right)^{\nu/2} \left\{ \left(\frac{2}{\pi} K_\nu(4\pi\sqrt{(n+\psi)(m+\theta)t}) \right.\right.\right.$$

$$
\begin{aligned}
&\qquad -Y_\nu(4\pi\sqrt{(n+\psi)(m+\theta)t})\Big)\sin\left(\frac{\pi\nu}{2}\right)\\
&\qquad +J_\nu(4\pi\sqrt{(n+\psi)(m+\theta)t})\cos\left(\frac{\pi\nu}{2}\right)\Bigg\}\\
&-\left(\frac{m+\theta}{n+1-\psi}\right)^{\nu/2}\Bigg\{\Big(\frac{2}{\pi}K_\nu(4\pi\sqrt{(n+1-\psi)(m+\theta)t})\\
&\qquad -Y_\nu(4\pi\sqrt{(n+1-\psi)(m+\theta)t})\Big)\sin\left(\frac{\pi\nu}{2}\right)\\
&\qquad +J_\nu(4\pi\sqrt{(n+1-\psi)(m+\theta)t})\cos\left(\frac{\pi\nu}{2}\right)\Bigg\}\\
&+\left(\frac{m+1-\theta}{n+\psi}\right)^{\nu/2}\Bigg\{\Big(\frac{2}{\pi}K_\nu(4\pi\sqrt{(n+\psi)(m+1-\theta)t})\\
&\qquad -Y_\nu(4\pi\sqrt{(n+\psi)(m+1-\theta)t})\Big)\sin\left(\frac{\pi\nu}{2}\right)\\
&\qquad +J_\nu(4\pi\sqrt{(n+\psi)(m+1-\theta)t})\cos\left(\frac{\pi\nu}{2}\right)\Bigg\}\\
&-\left(\frac{m+1-\theta}{n+1-\psi}\right)^{\nu/2}\Bigg\{\Big(\frac{2}{\pi}K_\nu(4\pi\sqrt{(n+1-\psi)(m+1-\theta)t})\\
&\qquad -Y_\nu(4\pi\sqrt{(n+1-\psi)(m+1-\theta)t})\Big)\sin\left(\frac{\pi\nu}{2}\right)\\
&\qquad -J_\nu(4\pi\sqrt{(n+1-\psi)(m+1-\theta)t})\cos\left(\frac{\pi\nu}{2}\right)\Bigg\}\Bigg]\,\mathrm{d}t.
\end{aligned}
$$

Theorem 5.2.15. *Let $0<\alpha<\beta$ and $\alpha,\beta\notin\mathbb{Z}$. Let f denote a function analytic inside a closed contour strictly containing $[\alpha,\beta]$. Assume that χ_1 is a non-principal even primitive character modulo p and χ_2 is an odd primitive character modulo q. For $0<\Re(\nu)<\frac{1}{2}$, we have*

$$
\frac{p^{1-\frac{\nu}{2}}q^{1+\frac{\nu}{2}}}{\tau(\chi_1)\tau(\chi_2)}\sum_{\alpha<j<\beta}\sigma_{-\nu,\chi_2,\chi_1}(j)f(j)=2\pi i\sum_{n=1}^{\infty}\sigma_{-\nu,\bar{\chi_1},\bar{\chi_2}}(n)\ n^{\nu/2}\int_\alpha^\beta f(t)(t)^{-\frac{\nu}{2}}
$$
$$
\times\left\{\left(\frac{2}{\pi}K_\nu\left(4\pi\sqrt{\frac{nt}{pq}}\right)+Y_\nu\left(4\pi\sqrt{\frac{nt}{pq}}\right)\right)\sin\left(\frac{\pi\nu}{2}\right)-J_\nu\left(4\pi\sqrt{\frac{nt}{pq}}\right)\cos\left(\frac{\pi\nu}{2}\right)\right\}dt.
$$

The following result is based on Theorem 5.2.6 and Theorem 5.2.15. Conversely, our next result directly implies Theorem 5.2.15.

Theorem 5.2.16. *Let $0<\alpha<\beta$ and $\alpha,\beta\notin\mathbb{Z}$. Let f denote a function analytic* *inside a closed contour strictly containing* $[\alpha,\beta]$. *Assume* $0<\Re(\nu)<\frac{1}{2}$, *then*

$$\begin{aligned}
&\sum_{\alpha<j<\beta}\sum_{d/j} d^{-\nu}\sin(2\pi d\theta)\cos\left(\frac{2\pi j\psi}{d}\right)f(j)\\
&=-\frac{\pi}{2}\int_{\alpha}^{\beta}\frac{f(t)}{t^{\frac{\nu}{2}}}\sum_{m,n\geq 0}^{\infty}\left[\left(\frac{m+\theta}{n+\psi}\right)^{\nu/2}\left\{\left(\frac{2}{\pi}K_\nu(4\pi\sqrt{(n+\psi)(m+\theta)t})\right.\right.\right.\\
&\qquad\left.+Y_\nu(4\pi\sqrt{(n+\psi)(m+\theta)t})\right)\sin\left(\frac{\pi\nu}{2}\right)\\
&\qquad\left.-J_\nu(4\pi\sqrt{(n+\psi)(m+\theta)t})\cos\left(\frac{\pi\nu}{2}\right)\right\}\\
&+\left(\frac{m+\theta}{n+1-\psi}\right)^{\nu/2}\left\{\left(\frac{2}{\pi}K_\nu(4\pi\sqrt{(n+1-\psi)(m+\theta)t})\right.\right.\\
&\qquad\left.+Y_\nu(4\pi\sqrt{(n+1-\psi)(m+\theta)t})\right)\sin\left(\frac{\pi\nu}{2}\right)\\
&\qquad\left.-J_\nu(4\pi\sqrt{(n+1-\psi)(m+\theta)t})\cos\left(\frac{\pi\nu}{2}\right)\right\}\\
&-\left(\frac{m+1-\theta}{n+\psi}\right)^{\nu/2}\left\{\left(\frac{2}{\pi}K_\nu(4\pi\sqrt{(n+\psi)(m+1-\theta)t})\right.\right.\\
&\qquad\left.+Y_\nu(4\pi\sqrt{(n+\psi)(m+1-\theta)t})\right)\sin\left(\frac{\pi\nu}{2}\right)\\
&\qquad\left.-J_\nu(4\pi\sqrt{(n+\psi)(m+1-\theta)t})\cos\left(\frac{\pi\nu}{2}\right)\right\}\\
&-\left(\frac{m+1-\theta}{n+1-\psi}\right)^{\nu/2}\left\{\left(\frac{2}{\pi}K_\nu(4\pi\sqrt{(n+1-\psi)(m+1-\theta)t})\right.\right.\\
&\qquad\left.+Y_\nu(4\pi\sqrt{(n+1-\psi)(m+1-\theta)t})\right)\sin\left(\frac{\pi\nu}{2}\right)\\
&\qquad\left.\left.-J_\nu(4\pi\sqrt{(n+1-\psi)(m+1-\theta)t})\cos\left(\frac{\pi\nu}{2}\right)\right\}\right]\mathrm{d}t.
\end{aligned}$$

5.3 Proof of Voronoi summation formulas

This section is devoted to the proof of Voronoi-type summation formulas.

Proof of Theorem 5.2.1 and its equivalence with Theorem 5.2.2. To derive Theorem 5.2.1, we will adapt the method introduced by B. C. Berndt, A. Dixit, A. Roy, and A. Zaharescu in [7]. We have previously seen the proof of Theorem 4.2.1 in Section 4.3. Here, I would like to demonstrate the proof of Theorem 5.2.1 utilising

Theorem 4.2.1.

Proof of Theorem 5.2.1 One can see that identity (4.3) in Theorem 4.2.1 is valid not only for $x > 0$ but also for $-\pi < \arg x < \pi$ by analytic continuation. If we set $N = 1$ in (4.3), then the condition $\lfloor \frac{\Re(\nu)+1}{2} \rfloor \leq 1$ will imply that $0 \leq \Re(\nu) < 3$. We consider $0 < \Re(\nu) < \frac{1}{2}$. Replace x by iz/q in (4.3) for $-\pi < \arg z < \frac{\pi}{2}$ and then by $-iz/q$ for $-\frac{\pi}{2} < \arg z < \pi$. Now the common region of the resultant identities is $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$. So we add the resulting two identities and simplify, in the region $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$, to obtain

$$\Lambda(z,\nu) = \Psi_1(z,\nu), \tag{5.9}$$

where

$$\begin{aligned}\Lambda(z,\nu) = 2z^{-\frac{\nu}{2}} \sum_{n=1}^{\infty} \sigma_{-\nu,\bar{\chi}}(n)\, n^{\nu/2} \Bigg\{ & e^{\frac{i\pi\nu}{4}} K_\nu \left(4\pi e^{\frac{i\pi}{4}} \sqrt{\frac{nz}{q}} \right) \\ & + e^{\frac{-i\pi\nu}{4}} K_\nu \left(4\pi e^{\frac{-i\pi}{4}} \sqrt{\frac{nz}{q}} \right) \Bigg\},\end{aligned} \tag{5.10}$$

and

$$\Psi_1(z,\nu) = -\frac{q^{\frac{\nu}{2}}\Gamma(\nu)L(\nu,\bar{\chi})}{(2\pi)^\nu} z^{-\nu} + \frac{q^{1-\frac{\nu}{2}}}{\pi\tau(\chi)} \sum_{n=1}^{\infty} \frac{\bar{\sigma}_{-\nu,\chi}(n)}{n^2+z^2}\, z. \tag{5.11}$$

Note that $\Psi_1(z,\nu)$ is an analytic function of z in $\mathbb{C}$ except on negative real axis and at $z = in$ where $n \in \mathbb{Z}$. Hence, $\Psi_1(iz,\nu)$ is analytic in $\mathbb{C}$ except on the positive imaginary axis and at $z \in \mathbb{Z}$. Similarly, $\Psi_1(-iz,\nu)$ is analytic in $\mathbb{C}$ except on the negative imaginary axis and at $z \in \mathbb{Z}$. We deduce $\Psi_1(iz,\nu) + \Psi_1(-iz,\nu)$ is analytic in both the left and right half plane, except possibly when z is an integer. Since

$$\begin{aligned}\lim_{z\to\mp n} (z \pm n)\Psi_1(iz,\nu) &= \frac{q^{1-\frac{\nu}{2}}}{2\pi i\tau(\chi)} \bar{\sigma}_{-\nu,\chi}(n), \\ \lim_{z\to\mp n} (z \pm n)\Psi_1(-iz,\nu) &= -\frac{q^{1-\frac{\nu}{2}}}{2\pi i\tau(\chi)} \bar{\sigma}_{-\nu,\chi}(n),\end{aligned}$$

so we have

$$\lim_{z\to\mp n}(z\pm n)\left(\Psi_1(iz,\nu)+\Psi_1(-iz,\nu)\right)=0.$$

Hence, $\Psi_1(iz,\nu)+\Psi_1(-iz,\nu)$ is analytic in the entire right half plane. From (5.11), we observe that for z lying inside an interval (a,b) on the positive real line not containing an integer, we have

$$\Psi_1(iz,\nu)+\Psi_1(-iz,\nu)=-\frac{2q^{\frac{\nu}{2}}\Gamma(\nu)L(\nu,\bar{\chi})}{(2\pi)^{\nu}}\cos\left(\frac{\pi\nu}{2}\right)\frac{1}{z^{\nu}}. \tag{5.12}$$

Since both sides are analytic in the right half-complex plane as a function of z, by analytic continuation, the identity (5.12) holds for any z in the right half-plane. Next employing functional equation for L-function (2.13) in (5.12) and simplifying, we obtain for $-\frac{\pi}{2}<\arg z<\frac{\pi}{2}$,

$$\Psi_1(iz,\nu)+\Psi_1(-iz,\nu)=-\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)}L(1-\nu,\chi)\frac{1}{z^{\nu}}. \tag{5.13}$$

Next, Let f be an analytic function of z in a closed contour γ' intersecting the real axis in α and β where $0<\alpha<\beta,\ m-1<\alpha<m,\ n-1<\beta<n$ and $m,n\in\mathbb{Z}$. Now γ' consists of two parts γ_1 and γ_2 where γ_1 is the portion of the contour in the upper half-plane, and γ_2 is the portion corresponding to the lower half-plane. Now $\alpha\gamma_1\beta$ and $\alpha\gamma_2\beta$ denote the paths from α to β in the upper and lower half planes, respectively. By the Cauchy residue theorem, we have

$$\int_{\alpha\gamma_2\beta\gamma_1\alpha}f(z)\Psi_1(iz,\nu)dz=\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)}\sum_{\alpha<j<\beta}\bar{\sigma}_{-\nu,\chi}(j)f(j),$$

where $\frac{q^{1-\frac{\nu}{2}}}{2\pi i\tau(\chi)}\bar{\sigma}_{-\nu,\chi}(j)f(j)$ is the residue of $f(z)\Psi_1(iz,\nu)$ at each integer j where $\alpha<j<\beta$. Hence, the above expression can be rewritten as

$$\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)}\sum_{\alpha<j<\beta}\bar{\sigma}_{-\nu,\chi}(j)f(j)=\int_{\alpha\gamma_2\beta}f(z)\Psi_1(iz,\nu)dz-\int_{\alpha\gamma_1\beta}f(z)\Psi_1(iz,\nu)dz$$
$$=\int_{\alpha\gamma_2\beta}f(z)\Psi_1(iz,\nu)dz$$

$$+\int_{\alpha\gamma_1\beta} f(z)\left\{\Psi_1(-iz,\nu)+\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)}L(1-\nu,\chi)\frac{1}{z^\nu}\right\}dz, \tag{5.14}$$

where in the last step, we used (5.13). Again, we make use of the Cauchy residue theorem and obtain

$$\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)}L(1-\nu,\chi)\int_{\alpha\gamma_1\beta}\frac{f(z)}{z^\nu}\mathrm{d}z=\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)}L(1-\nu,\chi)\int_\alpha^\beta\frac{f(t)}{t^\nu}\mathrm{d}t. \tag{5.15}$$

From (5.9), $\Lambda(z,\nu)=\Psi_1(z,\nu)$ for $-\frac{\pi}{2}<\arg z<\frac{\pi}{2}$. So it is easy to see that $\Lambda(iz,\nu)=\Psi_1(iz,\nu)$ holds for $-\pi<\arg z<0$, and $\Lambda(-iz,\nu)=\Psi_1(-iz,\nu)$ holds for $0<\arg z<\pi$. Thus

$$\begin{cases}\int_{\alpha\gamma_2\beta} f(z)\Psi_1(iz,\nu)dz=\int_{\alpha\gamma_2\beta} f(z)\Lambda(iz,\nu)dz,\\ \int_{\alpha\gamma_1\beta} f(z)\Psi_1(-iz,\nu)dz=\int_{\alpha\gamma_1\beta} f(z)\Lambda(-iz,\nu)dz.\end{cases} \tag{5.16}$$

Here we notice that the series $\Lambda(iz,\nu)$ in (5.10) is uniformly convergent in compact subintervals of $-\pi<\arg z<0$, and series $\Lambda(-iz,\nu)$ is uniformly convergent in compact subintervals of $0<\arg z<\pi$. Thus, interchanging the order of summation and integration in (5.16) and inserting them in (5.14) together with (5.15), we get

$$\begin{aligned}\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)}\sum_{\alpha<j<\beta}\bar{\sigma}_{-\nu,\chi}(j)f(j)&=\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)}L(1-\nu,\chi)\int_\alpha^\beta\frac{f(t)}{t^\nu}\mathrm{d}t+2\sum_{n=1}^\infty\sigma_{-\nu,\bar{\chi}}(n)\ n^{\nu/2}\\
&\times\int_{\alpha\gamma_2\beta} f(z)(iz)^{-\frac{\nu}{2}}\left\{e^{\frac{i\pi\nu}{4}}K_\nu\left(4\pi e^{\frac{i\pi}{4}}\sqrt{\frac{inz}{q}}\right)+e^{\frac{-i\pi\nu}{4}}K_\nu\left(4\pi e^{\frac{-i\pi}{4}}\sqrt{\frac{inz}{q}}\right)\right\}\mathrm{d}z\\
&+2\sum_{n=1}^\infty\sigma_{-\nu,\bar{\chi}}(n)\ n^{\nu/2}\int_{\alpha\gamma_1\beta} f(z)(-iz)^{-\frac{\nu}{2}}\left\{e^{\frac{i\pi\nu}{4}}K_\nu\left(4\pi e^{\frac{i\pi}{4}}\sqrt{\frac{-inz}{q}}\right)\right.\\
&\left.+e^{\frac{-i\pi\nu}{4}}K_\nu\left(4\pi e^{\frac{-i\pi}{4}}\sqrt{\frac{-inz}{q}}\right)\right\}\mathrm{d}z.\end{aligned}$$

Simplifying we get

$$\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)}\sum_{\alpha<j<\beta}\bar{\sigma}_{-\nu,\chi}(j)f(j)=\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)}L(1-\nu,\chi)\int_\alpha^\beta\frac{f(t)}{t^\nu}\mathrm{d}t$$

$$+2\sum_{n=1}^{\infty}\sigma_{-\nu,\bar{\chi}}(n)\ n^{\nu/2}\int_{\alpha\gamma_2\beta} f(z)z^{-\frac{\nu}{2}}\left\{K_\nu\left(4\pi i\sqrt{\frac{nz}{q}}\right)+e^{\frac{-i\pi\nu}{2}}K_\nu\left(4\pi\sqrt{\frac{nz}{q}}\right)\right\}\mathrm{d}z$$
$$+2\sum_{n=1}^{\infty}\sigma_{-\nu,\bar{\chi}}(n)\ n^{\nu/2}\int_{\alpha\gamma_1\beta} f(z)z^{-\frac{\nu}{2}}\left\{e^{\frac{i\pi\nu}{2}}K_\nu\left(4\pi\sqrt{\frac{nz}{q}}\right)+K_\nu\left(-4\pi i\sqrt{\frac{nz}{q}}\right)\right\}\mathrm{d}z.$$

Employing the residue theorem again, this time for each of the integrals inside the two sums, and simplifying, we obtain

$$\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)}\sum_{\alpha<j<\beta}\bar{\sigma}_{-\nu,\chi}(j)f(j)=\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)}L(1-\nu,\chi)\int_\alpha^\beta\frac{f(t)}{t^\nu}\mathrm{d}t+2\sum_{n=1}^{\infty}\sigma_{-\nu,\bar{\chi}}(n)\ n^{\nu/2}$$
$$\times\int_\alpha^\beta f(t)t^{-\frac{\nu}{2}}\left\{K_\nu\left(4\pi i\sqrt{\frac{nt}{q}}\right)+K_\nu\left(-4\pi i\sqrt{\frac{nt}{q}}\right)\right.$$
$$\left.+2\cos\left(\frac{\pi\nu}{2}\right)K_\nu\left(4\pi\sqrt{\frac{nt}{q}}\right)\right\}\mathrm{d}t. \tag{5.17}$$

Hereby [7, p. 848, equation (7.15)], we have

$$K_\nu(ix)+K_\nu(-ix)=-\pi\left(J_\nu(x)\sin(\frac{\pi\nu}{2})+Y_\nu(x)\cos(\frac{\pi\nu}{2})\right), \tag{5.18}$$

where J_ν and Y_ν are the Bessel functions defined in (1.1) and (1.2), respectively. Now, we replace x by $4\pi\sqrt{nt/q}$ in (5.18) and substitute in (5.17), we get (5.2).

Next, we demonstrate that Theorem 5.2.1 is equivalent to Theorem 5.2.2.

Theorem 5.2.1 ⇒ Theorem 5.2.2 It is sufficient to prove the theorem for $\theta=h/q$, where q is prime and $0<h<q$. Employing (2.25), we consider

$$\sum_{\alpha<j<\beta}\sum_{d/j}d^{-\nu}\cos\left(\frac{2\pi jh}{dq}\right)f(j)$$
$$=\sum_{\alpha<j<\beta}j^{-\nu}f(j)\sum_{d/j}d^{\nu}\cos\left(\frac{2\pi dh}{q}\right)$$
$$=\sum_{\alpha<j<\beta}j^{-\nu}f(j)\left(\sum_{\substack{d|j\\ q|d}}d^{\nu}+\sum_{\substack{d|j\\ q\nmid d}}d^{\nu}\cos\left(\frac{2\pi dh}{q}\right)\right)$$
$$=\sum_{\frac{\alpha}{q}<m<\frac{\beta}{q}}m^{-\nu}f(qm)\sum_{d|m}d^{\nu}+\sum_{\alpha<j<\beta}j^{-\nu}f(j)\sum_{\substack{d|j\\ q\nmid d}}\frac{d^{\nu}}{\phi(q)}\sum_{\chi\ even}\chi(d)\chi(h)\tau(\bar{\chi})$$

$$
\begin{aligned}
&= \sum_{\frac{\alpha}{q}<m<\frac{\beta}{q}} m^{-\nu} f(qm) \sum_{d|m} d^{\nu} - \sum_{\alpha<j<\beta} j^{-\nu} f(j) \sum_{\substack{d|j\\ q\nmid d}} \frac{d^{\nu}}{\phi(q)} \chi_0(d) \\
&\quad + \sum_{\alpha<j<\beta} j^{-\nu} f(j) \sum_{\substack{d|j\\ q\nmid d}} \frac{d^{\nu}}{\phi(q)} \sum_{\substack{\chi\neq\chi_0\\ \chi even}} \chi(d)\chi(h)\tau(\bar{\chi}) \\
&= \sum_{\frac{\alpha}{q}<m<\frac{\beta}{q}} m^{-\nu} f(qm) \sum_{d|m} d^{\nu} - \sum_{\alpha<j<\beta} j^{-\nu} f(j) \frac{1}{\phi(q)} \left\{ \sum_{d/j} d^{\nu} - \sum_{\substack{d|j\\ q/d}} d^{\nu} \right\} \\
&\quad + \sum_{\alpha<j<\beta} j^{-\nu} f(j) \sum_{\substack{d|j\\ q\nmid d}} \frac{d^{\nu}}{\phi(q)} \sum_{\substack{\chi\neq\chi_0\\ \chi even}} \chi(d)\chi(h)\tau(\bar{\chi}) \\
&= \frac{q}{\phi(q)} \sum_{\frac{\alpha}{q}<m<\frac{\beta}{q}} m^{-\nu} f(qm)\sigma_{\nu}(m) - \frac{1}{\phi(q)} \sum_{\alpha<j<\beta} j^{-\nu} f(j)\sigma_{\nu}(j) \\
&\quad + \frac{1}{\phi(q)} \sum_{\substack{\chi\neq\chi_0\\ \chi even}} \chi(h)\tau(\bar{\chi}) \sum_{\alpha<j<\beta} j^{-\nu} f(j)\sigma_{\nu,\chi}(j) \\
&= \frac{q}{\phi(q)} \sum_{\frac{\alpha}{q}<m<\frac{\beta}{q}} \sigma_{-\nu}(m)\ f(qm) - \frac{1}{\phi(q)} \sum_{\alpha<j<\beta} \sigma_{-\nu}(j)\ f(j) \\
&\quad + \frac{1}{\phi(q)} \sum_{\substack{\chi\neq\chi_0\\ \chi even}} \chi(h)\tau(\bar{\chi}) \sum_{\alpha<j<\beta} \bar{\sigma}_{-\nu,\chi}(j) f(j),
\end{aligned}
\tag{5.19}
$$

in the last step, we used $\sigma_{\nu}(m) = m^{\nu}\sigma_{-\nu}(m)$ and $\bar{\sigma}_{-\nu,\chi}(m) = m^{-\nu}\sigma_{\nu,\chi}(m)$. Now, we first evaluate the first two sums on the right-hand side of (5.19). Applying Proposition 1.4.2 with $f(x) = f(qx)$, then simplifying, we obtain

$$
\begin{aligned}
\frac{q}{\phi(q)} \sum_{\frac{\alpha}{q}<m<\frac{\beta}{q}} \sigma_{-\nu}(m) f(qm) &= \frac{1}{\phi(q)} \int_{\alpha}^{\beta} f(t) \left\{ q^{\nu}\zeta(1-\nu)\ t^{-\nu} + \zeta(\nu+1)\ \right\} dt \\
+ \frac{2\pi q^{\frac{\nu}{2}}}{\phi(q)} \sum_{n=1}^{\infty} \sigma_{-\nu}(n) n^{\nu/2} &\int_{\alpha}^{\beta} f(t)(t)^{-\frac{\nu}{2}} \left\{ \left(\frac{2}{\pi} K_{\nu}(4\pi\sqrt{\frac{nt}{q}}) - Y_{\nu}(4\pi\sqrt{\frac{nt}{q}}) \right) \cos\left(\frac{\pi\nu}{2}\right) \right. \\
&\left. - J_{\nu}(4\pi\sqrt{\frac{nt}{q}}) \sin\left(\frac{\pi\nu}{2}\right) \right\} dt.
\end{aligned}
\tag{5.20}
$$

By using Proposition 1.4.2, we get the second sum of the right-hand side of (5.19).

Finally, we evaluate the third sum of the right-hand side of (5.19). Now we multiply both sides of identity (5.2) in Theorem 5.2.1 by $\chi(h)\tau(\bar{\chi})/\phi(q)$, then sum on χ, where χ is a non-principal even primitive character χ modulo q and obtain

$$\begin{aligned}
&\frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\tau(\bar{\chi})\sum_{\alpha<j<\beta}\bar{\sigma}_{-\nu,\chi}(j)f(j)\\
&=\frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\tau(\bar{\chi})L(1-\nu,\chi)\int_\alpha^\beta\frac{f(t)}{t^\nu}\mathrm{d}t\\
&+\frac{2\pi}{q^{1-\frac{\nu}{2}}\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\tau(\bar{\chi})\tau(\chi)\sum_{n=1}^\infty\sigma_{-\nu,\bar{\chi}}(n)n^{\nu/2}\int_\alpha^\beta f(t)(t)^{-\frac{\nu}{2}}\\
&\times\left\{\left(\frac{2}{\pi}K_\nu\left(4\pi\sqrt{\frac{nt}{q}}\right)-Y_\nu\left(4\pi\sqrt{\frac{nt}{q}}\right)\right)\cos\left(\frac{\pi\nu}{2}\right)-J_\nu\left(4\pi\sqrt{\frac{nt}{q}}\right)\sin\left(\frac{\pi\nu}{2}\right)\right\}dt\\
&=\frac{\Gamma(\nu)\cos(\frac{\pi\nu}{2})}{(2\pi)^\nu}\frac{2q^\nu}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)L(\nu,\bar{\chi})\int_\alpha^\beta f(t)\ t^{-\nu}dt\\
&+\frac{2\pi q^{\frac{\nu}{2}}}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\sum_{n=1}^\infty\sigma_{-\nu,\bar{\chi}}(n)n^{\nu/2}\int_\alpha^\beta f(t)(t)^{-\frac{\nu}{2}}\left\{\left(\frac{2}{\pi}K_\nu\left(4\pi\sqrt{\frac{nt}{q}}\right)\right.\right.\\
&\qquad\left.\left.-Y_\nu\left(4\pi\sqrt{\frac{nt}{q}}\right)\right)\cos\left(\frac{\pi\nu}{2}\right)-J_\nu\left(4\pi\sqrt{\frac{nt}{q}}\right)\sin\left(\frac{\pi\nu}{2}\right)\right\}dt, \qquad (5.21)
\end{aligned}$$

where in the last line, we used the functional equation of L-function (2.13). By utilising (2.9), (2.13), (2.3), (2.21) and (2.23), we arrive at

$$\begin{aligned}
&\frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)L(\nu,\bar{\chi})\\
&=\frac{1}{2q^\nu}\{\zeta(\nu,h/q)+\zeta(\nu,1-h/q)\}-\frac{1}{\phi(q)}(1-q^{-\nu})\zeta(\nu). \qquad (5.22)
\end{aligned}$$

Using (5.22) and functional equation of Riemann zeta- function (2.9), we obtain the first sum in the right-hand side of (5.21) equals

$$\frac{\Gamma(\nu)\cos(\frac{\pi\nu}{2})}{(2\pi)^\nu}\{\zeta(\nu,h/q)+\zeta(\nu,1-h/q)\}\int_\alpha^\beta f(t)\ t^{-\nu}dt$$

$$-\frac{1}{\phi(q)}(q^{\nu}-1)\zeta(1-\nu)\int_{\alpha}^{\beta} f(t)\ t^{-\nu}dt \tag{5.23}$$

Now, we define for $y \in \mathbb{R}$,

$$W_{\nu}\left(y\ \right)=\left(\frac{2}{\pi}K_{\nu}\left(y\right)-Y_{\nu}\left(y\right)\right)\cos\left(\frac{\pi\nu}{2}\right)-J_{\nu}\left(y\right)\sin\left(\frac{\pi\nu}{2}\right). \tag{5.24}$$

Now we consider

$$\begin{aligned}
&\frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\sum_{n=1}^{\infty}\sigma_{-\nu,\bar{\chi}}(n)n^{\nu/2}W_{\nu}\left(4\pi\sqrt{\frac{nt}{q}}\ \right)\\
&=\frac{1}{\phi(q)}\sum_{n=1}^{\infty}n^{\nu/2}W_{\nu}\left(4\pi\sqrt{\frac{nt}{q}}\ \right)\sum_{d/n}d^{-\nu}\sum_{\substack{\chi\neq\chi_0\\ \chi even}}\chi(h)\bar{\chi}(d)\\
&=\frac{1}{\phi(q)}\sum_{n=1}^{\infty}n^{\nu/2}W_{\nu}\left(4\pi\sqrt{\frac{nt}{q}}\ \right)\sum_{d/n}d^{-\nu}\left\{\sum_{\chi even}\chi(h)\bar{\chi}(d)-\chi_0(d)\right\}\\
&=\frac{1}{\phi(q)}\sum_{m=1}^{\infty}m^{\nu/2}\sum_{d=1}^{\infty}d^{-\nu/2}W_{\nu}\left(4\pi\sqrt{\frac{mdt}{q}}\ \right)\sum_{\chi even}\chi(h)\bar{\chi}(d)\\
&\quad-\frac{1}{\phi(q)}\sum_{n=1}^{\infty}n^{\nu/2}W_{\nu}\left(4\pi\sqrt{\frac{nt}{q}}\ \right)\left(\sigma_{-\nu}(n)-q^{-\nu}\sigma_{-\nu}(\frac{n}{q})\right)\\
&=\frac{1}{2}\sum_{m=1}^{\infty}m^{\nu/2}\sum_{\substack{d=1\\ d\equiv\pm h(q)}}^{\infty}\frac{W_{\nu}\left(4\pi\sqrt{\frac{mdt}{q}}\ \right)}{d^{\nu/2}}\\
&\qquad-\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)n^{\nu/2}W_{\nu}\left(4\pi\sqrt{\frac{nt}{q}}\ \right)\\
&\qquad+\frac{q^{-\nu}}{\phi(q)}\sum_{n=1}^{\infty}\sigma_{-\nu}\left(\frac{n}{q}\right)n^{\nu/2}W_{\nu}\left(4\pi\sqrt{\frac{nt}{q}}\ \right)\\
&=\frac{1}{2}\sum_{m=1}^{\infty}m^{\nu/2}\sum_{r=0}^{\infty}\left\{\frac{W_{\nu}(4\pi\sqrt{m(rq+h)\frac{t}{q}})}{(rq+h)^{\nu/2}}+\frac{W_{\nu}(4\pi\sqrt{m(rq+q-h)\frac{t}{q}})}{(rq+h)^{\nu/2}}\right\}\\
&\quad-\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)n^{\nu/2}W_{\nu}\left(4\pi\sqrt{\frac{nt}{q}}\ \right)+\frac{q^{-\nu/2}}{\phi(q)}\sum_{r=1}^{\infty}\sigma_{-\nu}(r)r^{\nu/2}W_{\nu}(4\pi\sqrt{rt})
\end{aligned}$$

$$= \frac{1}{2q^{\frac{\nu}{2}}} \sum_{m=1}^{\infty} m^{\nu/2} \sum_{r=0}^{\infty} \left\{ \frac{W_\nu(4\pi\sqrt{m(r+h/q)t})}{(r+h/q)^{\nu/2}} + \frac{W_\nu(4\pi\sqrt{m(r+1-h/q)t})}{(r+1-h/q)^{\nu/2}} \right\}$$
$$- \frac{1}{\phi(q)} \sum_{n=1}^{\infty} \sigma_{-\nu}(n) n^{\nu/2} W_\nu\left(4\pi\sqrt{\frac{nt}{q}}\right) + \frac{q^{-\nu/2}}{\phi(q)} \sum_{r=1}^{\infty} \sigma_{-\nu}(r) r^{\nu/2} W_\nu(4\pi\sqrt{rt}). \tag{5.25}$$

Substituting (5.23) and (5.25) in (5.21), we obtain the third sum of the right-hand side of (5.19), and then combining (5.19) with Proposition 1.4.2 and (5.20), we get (5.3).

Theorem 5.2.2 $\Rightarrow$ Theorem 5.2.1 Let $\theta = h/q$, and χ be an even primitive non-principal character modulo q. Multiplying the identity (5.3) in Theorem 5.2.2 by $\bar{\chi}(h)/\tau(\bar{\chi})$, and then summing on h, $0 < h < q$, one can show that Theorem 5.2.2 implies Theorem 5.2.1.

□

Now, we see the proof of identities involving odd characters.

Proof of Theorem 5.2.5 and its equivalence with Theorem 5.2.6. The proof of Theorem 5.2.5 is similar to the proof of Theorem 5.2.1. Thus, we omit details. However, we establish the equivalence of Theorem 5.2.5 and 5.2.6.

Theorem 5.2.5 $\Rightarrow$ Theorem 5.2.6 It is sufficient to prove the theorem for $\theta = h/q$, where q is prime and $0 < h < q$. Now we multiply both sides of identity (5.6) in Theorem 5.2.5 by $\chi(h)\tau(\bar{\chi})/i\phi(q)$, then take the sum on odd primitive character χ modulo q. The left-hand side of (5.2.5) becomes

$$\frac{1}{i\phi(q)} \sum_{\chi\ odd} \chi(h)\tau(\bar{\chi}) \sum_{\alpha<j<\beta} \sigma_{-\nu,\chi}(j) f(j) = \frac{1}{i\phi(q)} \sum_{\alpha<j<\beta} \sum_{d|j} d^{-\nu} \sum_{\chi\ odd} \chi(d)\chi(h)\tau(\bar{\chi}) f(j)$$
$$= \sum_{\alpha<j<\beta} \sum_{d|j} d^{-\nu} \sin\left(\frac{2\pi dh}{q}\right) f(j), \tag{5.26}$$

where we have used the identity (2.24). The right hand side of (5.2.5) becomes

$$\frac{1}{i\phi(q)} \sum_{\chi\ odd} \chi(h)\tau(\bar{\chi}) \sum_{\alpha<j<\beta} \sigma_{-\nu,\chi}(j) f(j)$$

$$= \frac{1}{i\phi(q)} \sum_{\chi\ odd} \chi(h)\tau(\bar{\chi})L(1+\nu,\chi) \int_{\alpha}^{\beta} f(t)\mathrm{d}t$$

$$- \frac{2\pi}{\phi(q)q^{\frac{\nu}{2}}} \sum_{\chi\ odd} \chi(h) \sum_{n=1}^{\infty} \bar{\sigma}_{-\nu,\bar{\chi}}(n)\ n^{\nu/2} \int_{\alpha}^{\beta} f(t)t^{-\frac{\nu}{2}} \left\{ \left(\frac{2}{\pi} K_{\nu}\left(4\pi\sqrt{\frac{nt}{q}}\right) + Y_{\nu}\left(4\pi\sqrt{\frac{nt}{q}}\right)\right) \sin\left(\frac{\pi\nu}{2}\right) - J_{\nu}\left(4\pi\sqrt{\frac{nt}{q}}\right)\cos\left(\frac{\pi\nu}{2}\right)\right\} dt. \tag{5.27}$$

Utilising (2.13), (2.3), (2.21) and (2.22), we get

$$\frac{1}{i\phi(q)} \sum_{\chi\ odd} \chi(h)\tau(\bar{\chi})L(1+\nu,\chi) = -(2\pi)^{\nu}\Gamma(-\nu)\sin\left(\frac{\pi\nu}{2}\right)\left\{\zeta(-\nu,\frac{h}{q}) - \zeta(-\nu,1-\frac{h}{q})\right\}. \tag{5.28}$$

We define for $y \in \mathbb{R}$,

$$Z_{\nu}\left(y\ \right) = \left(\frac{2}{\pi}K_{\nu}\left(y\right) + Y_{\nu}\left(y\right)\right)\sin\left(\frac{\pi\nu}{2}\right) - J_{\nu}\left(y\right)\cos\left(\frac{\pi\nu}{2}\right). \tag{5.29}$$

Next, we consider

$$\begin{aligned}
&\frac{1}{\phi(q)} \sum_{\chi\ odd} \chi(h) \sum_{n=1}^{\infty} \bar{\sigma}_{-\nu,\bar{\chi}}(n)\ n^{\nu/2} Z_{\nu}\left(4\pi\sqrt{\frac{nt}{q}}\right) \\
&= \frac{1}{\phi(q)} \sum_{n=1}^{\infty} n^{\nu/2} Z_{\nu}\left(4\pi\sqrt{\frac{nt}{q}}\right) \sum_{d|n} d^{-\nu} \sum_{\chi\ odd} \chi(h)\bar{\chi}\left(\frac{n}{d}\right) \\
&= \frac{1}{2}\sum_{d=1}^{\infty} d^{-\nu} \left\{ \sum_{\substack{r=1\\ r\equiv h(q)}}^{\infty} (dr)^{\nu/2} Z_{\nu}\left(4\pi\sqrt{\frac{drt}{q}}\right) - \sum_{\substack{r=1\\ r\equiv -h(q)}}^{\infty} (dr)^{\nu/2} Z_{\nu}\left(4\pi\sqrt{\frac{drt}{q}}\right)\right\} \\
&= \frac{q^{\frac{\nu}{2}}}{2} \sum_{d=1}^{\infty} d^{-\frac{\nu}{2}} \sum_{m=0}^{\infty} \left\{ \left(m+\frac{h}{q}\right)^{\frac{\nu}{2}} Z_{\nu}\left(4\pi\sqrt{d\left(m+\frac{h}{q}\right)t}\right) \right. \\
&\qquad \left. - \left(m+1-\frac{h}{q}\right)^{\frac{\nu}{2}} Z_{\nu}\left(4\pi\sqrt{d\left(m+1-\frac{h}{q}\right)t}\right)\right\}.
\end{aligned} \tag{5.30}$$

Employing (5.28), (5.29) (5.30), we deduce the expression for (5.27). Now, equating the resulting expression with (5.26), we get (5.7).

Theorem 5.2.6 ⇒ Theorem 5.2.5 Let $\theta = h/q$, and let χ be an odd primitive character modulo q. Multiplying the identity (5.7) in Theorem 5.2.6 by $\bar{\chi}(h)/\tau(\bar{\chi})$, and then summing on h, $0 < h < q$, the left-hand side of the identity (5.7) becomes

$$\begin{aligned} &\frac{1}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\sum_{\alpha<j<\beta} f(j)\sum_{d|j} d^{-\nu}\sin\left(2\pi dh/q\right)\\ &=\frac{1}{2i\tau(\bar{\chi})}\sum_{\alpha<j<\beta} f(j)\sum_{d|j} d^{-\nu}\sum_{h=1}^{q-1}\bar{\chi}(h)\left(e^{2\pi idh/q}-e^{-2\pi idh/q}\right)\\ &=\frac{1}{2i}\sum_{\alpha<j<\beta} f(j)\sum_{d|j} d^{-\nu}\left(\chi(d)-\chi(-d)\right)\\ &=i^{-1}\sum_{\alpha<j<\beta}\sigma_{-\nu,\chi}(j)f(j), \end{aligned} \tag{5.31}$$

where in the penultimate step, we used (2.26). Employing (2.3) and (2.21), we can observe that

$$\frac{1}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\left(\zeta(-\nu,h/q)-\zeta(-\nu,1-h/q)\right) = -2q^{-\nu-1}\tau(\chi)L(-\nu,\bar{\chi}). \tag{5.32}$$

Next, we consider

$$\begin{aligned} &\frac{1}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\sum_{d=1}^{\infty} d^{-\nu/2}\sum_{m=0}^{\infty}\Big\{(m+h/q)^{\frac{\nu}{2}}Z_\nu\left(4\pi\sqrt{d\left(m+h/q\right)t}\right)\\ &\qquad -\left(m+1-h/q\right)^{\frac{\nu}{2}}Z_\nu\left(4\pi\sqrt{d\left(m+1-h/q\right)t}\right)\Big\}\\ &=\frac{q^{-\nu/2}}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\sum_{d=1}^{\infty} d^{-\nu/2}\sum_{\substack{r=1\\ r\equiv h(q)}}^{\infty} r^{\nu/2}Z_\nu\left(4\pi\sqrt{\frac{drt}{q}}\right)\\ &-\frac{q^{-\nu/2}}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\sum_{d=1}^{\infty} d^{-\nu/2}\sum_{\substack{r=1\\ r\equiv -h(q)}}^{\infty} r^{\nu/2}Z_\nu\left(4\pi\sqrt{\frac{drt}{q}}\right)\\ &=\frac{2q^{-\nu/2}}{\tau(\bar{\chi})}\sum_{d=1}^{\infty}\sum_{r=1}^{\infty} d^{-\nu/2}r^{\nu/2}\bar{\chi}(r)Z_\nu\left(4\pi\sqrt{\frac{drt}{q}}\right) \end{aligned}$$

$$= -\frac{2\tau(\chi)}{q^{1+\nu/2}} \sum_{n=1}^{\infty} \bar{\sigma}_{-\nu,\bar{\chi}}(n) n^{\nu/2} Z_\nu \left(4\pi \sqrt{\frac{nt}{q}} \right), \tag{5.33}$$

where Z_ν is defined in (5.29). Substituting (5.31), (5.33) and utilising (5.32) in (5.7), one can get (5.6). □

Remark. The proofs of other remaining theorems will be similar, so we skip the proofs to avoid repetitions.

6
Conclusion and future work

It is well known that the Voronoi summation formula has served as the foundation for most attempts to find an upper bound for $\Delta(x)$ in the Dirichlet divisor problem. In light of this fact, in Chapter 5, we studied the Voronoi type summation formula for twisted sums of the divisor functions $\sigma_{z,\chi}(n), \bar{\sigma}_{z,\chi}(n)$ and $\sigma_{z,\chi_1,\chi_2}(n)$. In my future project, I would like to determine the truncated Voronoi summation formula for the number-theoretic error term $\delta_{k,\chi}(x)$ defined by

$$\delta_{k,\chi}(x) := \sum_{n\leq x} \sigma_{k,\chi}(n) - g_k(x), \tag{6.1}$$

where $g_k(x)$ denotes the main term. Next, I am interested in studying the higher power moments, i.e.,

$$\int_1^x \delta_{k,\chi}^j(t)\ dt \quad \text{and} \quad \sum_{n\leq x} \delta_{k,\chi}(n)^j, \quad \text{for} j \geq 1$$

where the error term $\delta_{k,\chi}(x)$ is defined in (6.1).

In [29], J. Furuya examined the connection between the discrete and continuous moments for the error term in the Dirichlet divisor problem. I would also like to find the analogous Furuya's results for the error term in (6.1). Finally, I intend to study sign change results for the error term $\delta_{k,\chi}(x)$ for real characters.

In 1916, Hardy [31] studied omega results for the divisor function,

$$\Delta(x) = \begin{cases} \Omega_+((x \log x)^{\frac{1}{4}} \log_2 x), \\ \Omega_-(x^{\frac{1}{4}}). \end{cases} \tag{6.2}$$

Here $\log_j$ denotes the jth iterated logarithm. After Hardy, number theorists tried to improve these results, but the best result is known due to K. Soundararajan's result [49]

$$\Delta(x) = \Omega\left((x \log x)^{1/4} (\log_2 x)^{(3/4)(2^{4/3}-1)} (\log_3 x)^{-5/8}\right).$$

In future, we plan to study omega results for the error term $\delta_{k,\chi}(x)$ in this direction.

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