Construction of some new diffeomorphisms using a combinatorial approach: The approximation by conjugation method

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Dedicated to my family

Certificate

This is to certify that the thesis entitled **"Construction of some new diffeomorphisms using a combinatorial approach: The approximation by conjugation method"** being submitted by **"Ms. Divya Khurana"** to the **Indraprastha Institute of Information Technology Delhi**, for the award of the Degree of **Doctor of Philosophy** is a record of the original bonafide research work carried out by her under our supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

Ergodic theory often involves dealing with transformations of measure spaces that preserve specific measures, such as volume or probability measures. These transformations establish a framework for studying the behavior of dynamical systems.

Combinatorial methods are pivotal in ergodic theory, providing valuable techniques for understanding the behavior of dynamical systems. They are often utilized in the study of measure-preserving transformations, aiding in characterizing invariant measures, exploring ergodic decomposition, and establishing connections between different dynamical systems. Moreover, numerous scenarios exist where we can effectively characterize a measure-preserving transformation and its long-term behaviour by approaching it as the limit of specific finite objects, such as periodic processes.

This thesis discusses combinatorial constructions within Ergodic Theory, employing the "Approximation by Conjugation" method. This technique facilitates the construction of maps with specific topological and measure-theoretic characteristics on the manifolds that support a non-trivial circle action. We provide examples of volume-preserving diffeomorphisms with zero topological entropy. These examples exhibit intricate ergodic properties in both smooth and analytic categories.

Specifically, we present an example of a smooth diffeomorphism with an invariant measure, which is a generic but non-ergodic volume measure. This diffeomorphism satisfies various other ergodic and topological dynamic properties on the 2-torus. In ergodic theory, generic points are essential for understanding the statistical and dynamical behavior of systems, as their orbits cover most of the phase space according to the invariant measure. We examine the distinctions between generic and non-generic points in smooth dynamical systems, focusing on establishing bounds for the sizes of their respective sets. We construct an explicit collection of sets that encompass all generic points within the system and focus on determining bounds for their Hausdorff dimension, leading to more insightful and compelling conclusions.

Additionally, we explore the ergodic properties of diffeomorphisms and examine their differential maps with respect to a smooth measure in the projectivization of the tangent bundle.

We investigate examples of diffeomorphisms with complex ergodic properties within the projectivized tangent bundle for the smooth and analytic case. We construct a smooth diffeomorphism whose differential is weakly mixing with respect to a smooth measure in the projectivization of the tangent bundle. In the case of the 2-torus, we also obtain the analytic counterpart of such a diffeomorphism by utilizing an analytic approximation technique. We present an example of an analytic diffeomorphism whose projectivized derivative extension exhibits weak mixing with respect to a smooth measure.

All of these constructions are based on the original and quantitative version of the "Approximation by Conjugation" method. This approach utilizes highly explicit well-defined conjugation maps, explicit partial partitions of the space, specific approximation strategies, and geometric and combinatorial criteria to meet specific requirements.

Contents

List of Symbols

Symbol	**Meaning**
$\mathbb{N}$	*The set of Natural numbers defined as* $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$	*The set of Integers defined as* $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{R}$	*The set of Real numbers*
$\lvert A \rvert$	*Cardinality* of the set A
$\mathbb{T}^1$	*Circle or* S^1, *defined as quotient space* $\mathbb{R}/\mathbb{Z}$
$\mathbb{T}^2$	*Flat Torus* $\equiv \mathbb{T}^1 \times \mathbb{T}^1$
$\dim_H V$	The *Hausdorff dimension* of a set V
$\mathrm{C}^k(M, N)$	*The set of* C^k *functions defined from* M *to* N
$\mathrm{Diff}^k(M)$	*The set of* k *times differentiable diffeomorphism defined from* M *to* M
$\mathrm{Diff}^\infty(M)$	*The set of smooth diffeomorphism defined on* M
$\mathrm{Diff}^\infty(M, \mu)$	*The set of smooth measure preserving diffeomorphism defined on* M
$\mathrm{Ham}^\infty(M, \mu)$	*The set of smooth hamiltonian diffeomorphism defined on* M

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1
Introduction

A dynamical system serves as a mathematical framework that describes the evolution of a system over time. Typically represented as a function T defined on a phase space or set X, it is denoted as (X, T). This subject branches into several areas, each with its own distinct mathematical framework. Two of the major branches are measure-theoretic dynamical systems and topological dynamical systems. Measure-theoretic dynamical systems focus on studying dynamical systems through the lens of measure theory. In this framework, the evolution of a system is often depicted by measure-preserving transformations. In contrast, topological dynamical systems focus on the qualitative aspects of dynamical systems, emphasizing the role of topological properties over measures. Despite their differences, both approaches offer valuable insights into the behavior of complex dynamical systems across various mathematical contexts.

In this thesis, we focus on discrete dynamical systems where $T : X \to X$ is a map, and we analyze its behavior under repeated iteration. This iterative process is denoted by $\Phi = \{T^t \mid t \in G\}$, where G represents the set of time steps or group elements. More generally, one can study the action of a group G on the space X and examine the behavior of orbits in both a measure-theoretic and topological context.

Ergodic theory explores the dynamics of group actions G on measure spaces. It focuses on the statistical properties of dynamical systems, such as the long-term behavior of orbits $T^t(x)$ for $t \in G$, and their distribution across the space X. This includes studying the existence and properties of invariant measures – measures that remain unchanged under the action of T. The origins of ergodic theory can be traced back to Poincaré's recurrence theorem in classical mechanics and Boltzmann's ergodic hypothesis in statistical mechanics.

The mathematical development of ergodic theory gained momentum in the 1930s through the seminal work of J. Von Neumann, G.D. Birkhoff, who introduced fundamental ergodic theorems and numerous results. Their work establishes a connection between microscopic particle behaviour and macroscopic system properties observed in statistical mechanics. Ergodic theorems demonstrate that, under certain conditions, the time average of a system's observable quantities (such as energy, momentum, etc.) along the trajectory of a typical point in phase space converges to the ensemble average of those quantities with respect to an invariant measure. In the context of the Boltzmann Hypothesis, which asserts that the long-term behavior of a system is equivalent to phase averages, the ergodic theorems provide a rigorous mathematical foundation. They show that in systems where ergodicity holds, the long-term time average behavior indeed matches the ensemble average behavior predicted by statistical mechanics. This connection provides a deeper understanding of the relationship between dynamical systems theory and statistical mechanics. For a detailed survey, refer to [39].

Subsequently, research in ergodic theory expanded in various directions, exploring additional aspects such as the geometrical and statistical properties of smooth dynamical systems. Smooth dynamical systems describe the continuous evolution of a system using smooth functions or mappings. They are often represented by a set of differential equations or a flow on a smooth manifold, where smooth vector fields govern the dynamics. Specifically, the existence of geodesic flows on compact Riemannian manifolds of negative curvature holds great significance for ergodic theory and statistical mechanics. E. Hopf addressed this topic in his work [30] on the ergodicity of the geodesic flow on negatively curved surfaces.

During the 1950s, A. Kolmogorov, V. Arnold, and J. Moser developed a new study of the perturbation of integrable Hamiltonian systems, known as perturbative KAM theory. Their work associates the terms "stability" and "instability" of perturbative systems with the Diophantine and Liouvillean behaviour of the linear model, respectively. Specifically, the translation and linear flow on a finite-dimensional torus are discussed in [17, Section 1]. The perturbation of linear and nonlinear models with the Liouvillean factor often reveals unexpected quantitative effects, such as the emergence of ergodic properties after a sufficiently long period of perturbation. Their work claims the existence of ergodic diffeomorphisms near elliptic fixed points, while also addressing obstacles to achieving ergodicity in Hamiltonian systems. Furthermore, investigating the existence of such systems exhibiting specific types of dynamic behavior is of interest, and a constructive approach is adopted.

The foundations of this work date back to the 1970s when D.V. Anosov and A.B. Katok presented the first example of a weakly mixing area-preserving smooth diffeomorphism on the disc $\mathbb{D}^2$ in [1]. They introduced an even more general technique known as the 'Approximation by conjugation' method, alternatively referred to as the 'Anosov-Katok method'. This method is applicable to any compact connected smooth manifold M admitting an effective circle action $\mathbb{T}^1$. This method remains one of the powerful tools for constructing measure-preserving systems with intricate dynamics in both smooth and analytic categories.

Their research work also addresses a major question in ergodic theory known as the 'Smooth Realization Problem,' originally posed by J. Von Neumann in 1932, as detailed in their foundational paper [41]. This problem involves determining whether the abstract concepts of ergodic theory can be given smooth counterparts. Specifically, it explores whether there exists a smooth diffeomorphism T

that preserves a smooth measure μ-which corresponds to the volume element on a compact smooth manifold M and is isomorphic to a measure-preserving system (X, f, ν). Despite extensive efforts, the problem has largely remained unsolved, with smooth realization being feasible only for limited classes of systems. Few restrictions are known that prevent any space (X, f, ν) from being realized as a smooth dynamical system. However, it is necessary that such diffeomorphisms possess finite entropy, as pointed out by A.G. Kushnirenko [37]. In lower dimensions, additional restrictions apply: any circle diffeomorphism with an invariant smooth measure is conjugate to a rotation, and according to Pesin's theory [42], any weakly mixing surface diffeomorphism with positive metric entropy is Bernoulli. Nevertheless, the overall constraints on the problem remain unidentified. For further information, one can refer to the survey articles [17] and [15].

In the context of smooth realization, when considering an abstract space $(X, f,$ $\nu)$ and seeking the existence of a smooth diffeomorphism T that preserves a smooth measure μ on a compact smooth manifold M. The goal is for (M, T, μ) to be metrically isomorphic to (X, f, ν). The problem is specifically referred to as the *non-standard smooth realization* of the space (X, f, ν) if M and X are not diffeomorphic.

Utilizing the aforementioned method, [1, Sections 4] presented the first instance of a non-standard smooth realization for an irrational circle rotation via a smooth ergodic diffeomorphism preserving the measure on a compact smooth manifold. Additionally, they achieved similar results for certain ergodic translations on the finite-dimensional torus, as noted in [1, Sections 6]. Moreover, in [20], B. Fayad, M. Saprikiyna, and A. Windsor extended the non-standard smooth realization result to include all Liouville irrational rotations of the circle. They even achieved this for uniquely ergodic diffeomorphisms by tailoring the method and ensuring control over all orbits, in contrast to the original method, which only controlled almost every orbit. This modified method, providing control over all orbits, is often referred to as the topological version of the AbC method and may introduce additional aspects to the theory. Subsequently, in [6], M. Benhenda extended the non-standard smooth realization of certain ergodic translations to include vectors with one arbitrary Liouvillian coordinate on the finite-dimensional torus. Afterwards, S. Banerjee expanded the non-standard smooth realization problem by addressing the real analytic realization of certain irrational rotations as real analytic diffeomorphisms [2]. It is important to note that requiring the diffeomorphism to be real-analytic in the described technique introduces additional challenges, as discussed in [17, Section 7].

This method has gained substantial attention, particularly due to the significant body of work by M. Foreman and B. Weiss [23, 24]. Their research established anticlassification theorems for smooth diffeomorphisms. The problem of anticlassification or classification of measure-preserving transformations up to an appropriate equivalence relation belongs to the "isomorphism problem," as originally proposed by J. Von Neumann in the 1930s. This was discussed in the same foundational paper. Despite concerted efforts, the problem remained intractable, with only a few classifications of special classes of transformations based on their finite entropy or spectrum being solved.

In the influential paper by M. Foreman, D.J. Rudolph, and B. Weiss [22], it was demonstrated that solving the isomorphism problem is not feasible due to the complexity of the structures. They showed that determining the measure isomorphism between ergodic transformations is inaccessible. Hence, the class of ergodic measure-preserving transformations cannot be classified by the measure isomorphism relation. Subsequently, in a series of papers, M. Foreman and B. Weiss extended the anticlassification result to a more restricted class of transformations within the smooth category. They introduced the notion of coding a specific category of Anosov-Katok transformations into a symbolic representation. This representation is defined as uniform circular systems. They established the anticlassification of smooth ergodic measure-preserving systems up to the measure isomorphism relation. Their approach was later modified to obtain an analogous result for the analytic category by S. Banerjee and P. Kunde (see [3]). Additionally, numerous other anticlassification results have been established using the method. These include results for a weak class of isomorphism relations, such as those for the Kakutani equivalence relation [26]. The method has also been applied to a more general class of Anosov-Katok transformations, including results for weak mixing diffeomorphisms [36].

This method remains one of the few available for exploring the potential existence of diffeomorphisms within the smooth or real-analytic categories. It aids in identifying those diffeomorphisms that satisfy a prescribed set of ergodic or topological properties. Transformations or diffeomorphisms with extreme ergodic and topological properties often hold significant importance within their respective contexts. For example, the existence of minimal but not uniquely ergodic diffeomorphisms in the smooth category has been explored in [44]. Additionally, recent studies have revealed emerging connections, such as linking number-theoretic properties like the rotation number at the boundary to the dynamic behavior of a diffeomorphism, as discussed in [19].

D.V. Anosov and A.B. Katok, as cited in [1], provided examples of smooth measure-preserving diffeomorphisms that demonstrate weak mixing within the space

$$\mathcal{A}(M) = \overline{\{h \circ S_t \circ h^{-1} : t \in \mathbb{T}^1, h \in \mathrm{Diff}^\infty(M, \mu)\}}^{C^\infty},$$

for any compact connected manifold that admits a non-trivial action $\mathcal{S} = \{S_t\}_{t \in \mathbb{T}^1}$. Subsequently, B. Fayad and M. Saprykina, in [19], produced smooth weakly mixing diffeomorphisms within the more restricted space

$$\mathcal{A}_\alpha(M) = \overline{\{h \circ S_\alpha \circ h^{-1} : h \in \mathrm{Diff}^\infty(M, \mu)\}}^{C^\infty},$$

for any Liouville number α. A Liouville number α is defined such that for every n, there exist positive integers p and $q > 1$ where $0 < |\alpha - \frac{p}{q}| < \frac{1}{q^n}$. Moreover, they demonstrated that for the disc $\mathbb{D}^2$ or the annulus $\mathbb{A} = \mathbb{T}^1 \times [0, 1]$, limiting diffeomorphisms that exhibit weak mixing properties have a rotation number S_α that corresponds to a Liouville number α when restricted to the boundary. Diophantine numbers are irrational numbers that are not considered Liouville numbers. On the other hand, diffeomorphisms of $\mathbb{D}^2$ or $\mathbb{A}$ with a Diophantine rotation number S_α for Diophantine α face challenges in exhibiting even ergodicity because of an invariant positive measure set present at the boundary, as stated in 'Herman's last geometric theorem' in [21].

Both of the constructions detailed in [1] and [19] utilize the approximation by conjugation method: A diffeomorphism is derived as the limit of sequences $f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$ where $\alpha_{n+1} \in \mathbb{Q}$ and $H_n = h_1 \circ \ldots \circ h_n$ where h_n is a measure preserving diffeomorphism satisfying $S_{\alpha_n} \circ h_n = h_n \circ S_{\alpha_n}$. Here, the conjugation map h_n transfers a finite version of the specific property to f_n, which we eventually need to achieve for the limiting diffeomorphism. The choice of $\alpha_{n+1} \in \mathbb{Q}$ is close enough to α_n to allow the convergence of f_n in the appropriate topology, and the sequence of α_n converges to some Liouville number $\alpha \in \mathbb{Q}^c$. For the diffeomorphism f_n to converge in the space $\mathcal{A}_\alpha(M)$, for any α, it requires the construction of more explicit conjugation maps h_n and very precise norm estimates. This is generally more difficult compared to convergence in the space $\mathcal{A}(M)$. In this thesis, we also apply the quantitative version of this method to construct a diffeomorphism with a specific Liouvillean rotation number. This involves ensuring that the sequence α_n converges to α, and the resulting transformation $f \in \mathcal{A}_\alpha(\mathbb{T}^2)$ satisfies the specified ergodic property. This process is discussed in detail in [19].

This approach has been effectively applied in numerous instances to construct a variety of diffeomorphisms for smooth manifolds of dimension ($m \geq 2$), that support a non-trivial circle action. These constructions exhibit particular ergodic attributes in both the smooth and analytic categories. Notable examples include weak-mixing diffeomorphisms (refer to [1, Section 6], [28], or [19]), uniquely ergodic diffeomorphisms (see [20], [18]), minimal but non-uniquely ergodic diffeomorphisms (see [44], [4]), and uniformly rigid and weakly mixing diffeomorphisms (see [32]), among others. These topics are explored in depth in [17] and a recent chapter in the book [15, Chapter 10].

1.1 Main Results

1.1.1 Existence of Generic and Non-Generic points

In this thesis, we apply the aforementioned method to construct a variety of diffeomorphisms exhibiting specific ergodic properties in both smooth and analytic categories. First, we present an example of a smooth area-preserving diffeomorphism with an invariant measure on the 2-torus. This measure is generic but non-ergodic, while still satisfying other topological and ergodic properties. For a probability preserving dynamical system $(M, \mathcal{B}, \mu, T)$, we define the set of μ-*generic points*

$$L_\mu = \left\{ x \in M : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} f(T^i x) = \int_X f d\mu \ \forall\, f \in C_c(M) \right\},$$

where $C_c(M)$ is the set of all compactly supported real valued continuous functions. Generic points form a powerful tool in ergodic theory in quantifying the difference between two invariant measures. The measure μ is called a *generic measure* if $L_\mu \neq \emptyset$. The celebrated Birkhoff ergodic theorem asserts that $\mu(L_\mu) = 1$ for any μ-ergodic transformation. But in general, the non-ergodic measure may or may not have generic points at all.

There has been a considerable amount of interest in the existence of generic measures, particularly in the realm of interval-exchange transformations. J. Chaika and H. Masur [12] showed that there exists a minimal non-uniquely ergodic interval exchange transformation on 6 intervals with 2 ergodic measures, which also has a non-ergodic generic measure. Later, V. Cyr and B. Kra [14] found a criterion for establishing upper bounds on the number of distinct non atomic generic measures for subshifts based on complexity, and as a consequence, they showed that for $k > 2$, a minimal exchange of k intervals has at most $k-2$ generic measures. On the other hand, K. Gelfert and D. Kwietniak [25] gave an example of a topologically mixing subshift that can have exactly two ergodic measures, none of whose convex combination is generic.

However, to the best of our knowledge, there have been no discussions on differentiable or even continuous mappings on any manifold with such characteristics. This raises the question within contexts such as smooth dynamical systems, real analytic systems, or symplectic systems: Is it feasible to create a non-ergodic generic measure in a smooth dynamical system? Given the flexibility of the Anosov-Katok method, this inquiry can also be addressed within a framework, along with incorporating additional topological and ergodic properties.

In general, a uniquely ergodic measure preserving transformation on a compact metric space is minimal on the support of the measure. However, the converse is not true. A. Markov produced the first counterexample. Further, A. Windsor, in [44], constructed a minimal measure preserving diffeomorphism in $\mathcal{A}_\alpha(M)$ with exactly a finite number of ergodic measures. Afterwards, S. Banerjee and P. Kunde, in [4], produced a similar result for the real analytic category on $\mathbb{T}^2$.

We present several results in this thesis that investigate the existence of non-ergodic generic measures within this framework. Our constructions will be smooth and, in some cases, even real analytic. We further extend these results to produce additional examples with diverse measure-theoretical and topological dynamical properties, as detailed in Chapter 3.

Theorem 1.1. *For any natural number r, and any Liouvillian number α, there exists a minimal $T \in \mathcal{A}_\alpha(\mathbb{T}^2)$ such that the Lebesgue measure is a generic measure for T, and there exists r absolutely continuous w.r.t. to Lebesgue measures $\mu_1, \mu_2, \ldots, \mu_r$ such that T is weakly mixing w.r.t. each of these measures.*

In fact, the approximation by conjugation method on $\mathbb{T}^2$ offers enough flexibility to repeat the construction using a block-slide type of maps ([4], Theorem 5) and achieve the result within the analytic set-up. Moreover, we can also potentially employ the analytic approximation technique, as discussed in Chapter 7, to produce volume-preserving diffeomorphism with the non-ergodic generic measure on the 2-torus.

Theorem 1.2. *For any natural number r, there exists a minimal real-analytic $T \in \text{Diff}^{\,\omega}(\mathbb{T}^2, \mu)$ constructed using the approximation by conjugation method, such that the Lebesgue measure is a generic measure for T, and there exists r absolutely continuous w.r.t. to Lebesgue measures $\mu_1, \mu_2, \ldots, \mu_r$ such that T is weakly mixing w.r.t. each of these measures.*

Another principal aim of this thesis is to investigate the set of μ-generic and μ-non-generic points associated with a particular smooth dynamical system. Points that do not exhibit generic behavior with respect to the invariant measure μ-that is, whose empirical measures fail to converge to μ are referred to as μ-non-generic points. We establish rigorous bounds on the size of these sets. Rather than quantifying their measure-theoretic size via Lebesgue measure, which may be trivial, our analysis centers on determining their Hausdorff dimension. This geometric perspective yields more refined and meaningful results, as elaborated in Chapter 4 and Chapter 5.

Theorem 1.3. *For any* $1 < s < 2$*, there exists a smooth diffeomorphism* $T \in \mathrm{Diff}^{\infty}(\mathbb{T}^2, \mu)$*, constructed via the approximation by conjugation method, such that a non-empty set* $\mathrm{B} \subset \mathbb{T}^2$ *consisting of* μ*-generic points of* T *satisfies*

$$\dim_H(\mathrm{B}) \leq s, \quad \textit{and} \quad \mu(\mathrm{B}) = 0.$$

To facilitate this construction, we explicitly introduce certain narrow sets of small measure-most notably, horizontal strips of small width-within a carefully designed framework on the 2-Torus. The images of these sets under the selected conjugation maps exhibit a fractal-like structure, which plays a crucial role in establishing nontrivial upper bounds for the sets of generic points considered in our analysis. We construct a highly explicit scheme involving distinct, purpose-specific domains on the 2-Torus, accompanied by a deliberate and structured choice of conjugation maps. This construction is supported by appropriate combinatorial and geometric methods, which together enable the formation of the desired example.

Moreover, we demonstrate the existence of a nontrivial set of generic points possessing a positive lower bound on Hausdorff dimension. This is achieved by introducing a Cantor-type fractal set and its generalized variants into our framework, which allows us to obtain the following result:

Theorem 1.4. *There exists a smooth diffeomorphism* $T \in \mathrm{Diff}^{\infty}(\mathbb{T}^2, \mu)$*, constructed via the approximation by conjugation method, such that a set* $\mathrm{B} \subset \mathbb{T}^2$ *consisting of* μ*-generic points of* T *satisfies*

$$\dim_H(\mathrm{B}) \geq \frac{\log 2}{\log 3}, \quad \textit{and} \quad \mu(\mathrm{B}) = 0.$$

Theorem 1.5. *For any $1 < \gamma < 2$, there exists a smooth diffeomorphism $T \in \mathrm{Diff}^\infty(\mathbb{T}^2, \mu)$, constructed via the approximation by conjugation method, such that a set $\mathrm{B} \subset \mathbb{T}^2$ consisting of μ-generic points of T satisfies*

$$\dim_H(\mathrm{B}) \geq \gamma - 1, \quad \textit{and} \quad \mu(\mathrm{B}) = 0.$$

The application of this result also challenges the validity of the variational-type formula presented in [9, Theorem 2.3.1] for smooth dynamical systems. In that theorem, the author introduced a variational-type formula for the full shift on a two-symbol alphabet (Ω, σ). However, within our specific framework, it becomes clear that this theorem's claim of a variational-type formula does not hold. For instance, if $f : \mathbb{T}^2 \to \mathbb{R}$ is any continuous function, and $\gamma = \int f d\mu$ where $1 < \gamma < 2$ and μ is the standard Lebesgue measure, then the Hausdorff dimension of $E_f(\gamma)$ is greater than zero (see Theorem 1.5) where $E_f(\gamma) = \{x \in \mathbb{T}^2 : \lim_{N \longrightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = \gamma\}$. However, according to [16, Theorem 1] and [9, Theorem 2.3.1], this dimension should be zero, as both topological entropy and all measure-theoretic entropy, in our context, are consistently zero.

In any probability dynamical system $(M, \mathcal{B}, \mu, T)$, the points that are not μ-generic points are referred to as μ-non-generic points. While Birkhoff's Ergodic Theorem guarantees that the set of non-generic points has μ measure zero when T is ergodic, the Hausdorff dimension of this exceptional set can exhibit rich and nontrivial behaviour. In particular, its size from the perspective of dimension theory may significantly differ from its measure-theoretic triviality. By implementing a slight modification to the combinatorial structure in our construction of Theorem 1.3 and 1.5, we can extend our approach to establish analogous results for the set of non-generic points in the case of a smooth ergodic diffeomorphism.

Moreover, the diffeomorphisms produced in Theorems 1.3, 1.4 and 1.5 could be made minimal by employing the same combinatorial construction outlined in Chapter 3, as used in proving Theorem 1.1.

1.1.2 Weak mixing behavior for the projectivized derivative extension

In this thesis, our objective is to explore the ergodic properties of diffeomorphisms. We also examine their differential maps with respect to a smooth measure in the projectivization of the tangent bundle. We investigate examples of diffeomorphisms with complex ergodic properties within the projectivized tangent bundle, employing the Anosov-Katok method for both smooth and analytic categories.

The derivative extension of a smooth map, $f : M \to M$, on an m-dimensional manifold M is a transformation defined on the tangent bundle $\mathrm{T}M$, denoted as $(f, \mathrm{d}f)$. The projective tangent bundle $\mathbb{P}\mathrm{T}M$ is obtained by identifying the tangent space of a point $p \in M$, $T_p(M)$, with $\mathbb{R}^m$, and considering its projective space $\mathbb{P}\mathbb{R}^m$. It is important to note that a diffeomorphism preserving a measurable Riemannian metric induces an invariant measure for the projectivized derivative extension, which is absolutely continuous on the fibers. Such a diffeomorphism, which preserves a measurable Riemannian metric and an absolutely continuous probability measure, is called an IM diffeomorphism. IM diffeomorphisms and IM group actions are discussed in detail in [28, section 3].

In [28], R. Gunesch and A.B. Katok constructed a weakly mixing volume preserving smooth diffeomorphism that preserves a measurable Riemannian metric. This work extends the research of D.V. Anosov and A.B. Katok [1]. For every Liouville number α, such diffeomorphisms are dense in the restricted space $\mathcal{A}_\alpha(M)$ in the C^∞ topology as proved in [29]. In [34], P. Kunde realized a similar construction in the analytic category. He constructed a real analytic weakly mixing diffeomorphism $f \in \mathrm{Diff}^\omega_\rho(\mathbb{T}^m, \mu)$ that preserves a measurable Riemannian metric.

It is a natural question to investigate the ergodic properties of the projectivized derivative extension with respect to the measure induced by the measurable Riemannian metric. In fact, all the constructions described in the previous paragraph, in [28] and [29], are as non-ergodic as possible. Their projectivized derivative extensions are isomorphic to the direct product of the diffeomorphism in the base with the trivial action in the fibers, resulting in each ergodic component intersecting almost every fiber at a single point. In [35], P. Kunde realized the other extreme possibility by constructing smooth diffeomorphisms whose differentials are ergodic with respect to such a smooth measure in the projectivization of the tangent bundle. He constructed an ergodic IM diffeomorphism in [35], and a weak mixing IM diffeomorphism in the restricted space $\mathcal{A}_\alpha(M)$ in [33], whose differentials are also ergodic with respect to such a measure.

In this thesis, we study the further ergodic properties of the projectivized derivative extension, and we focus on weak mixing behavior. We recall that a measure-preserving system $(X, \mathcal{B}, \mu, T)$ is said to be weakly mixing if there is no nonconstant function $h \in L^2(X, \mu)$ such that $h(Tx) = \lambda \cdot h(x)$ for some $\lambda \in \mathbb{C}$. Equivalently, $(X, \mathcal{B}, \mu, T)$ is weakly mixing iff there exists a sequence $\{m_n\} \in \mathbb{N}$ such that for any pair $A, B \in \mathcal{B} : |\mu(B \cap f^{-m_n}(A)) - \mu(B)\mu(A)| \longrightarrow 0$. We explore this aspect in both the smooth and analytic settings. Hereby, we solve Problems 6.2 and 10.7 in [15]. Our approach bases upon a new weak mixing criterion specifically tailored for the projectivized derivative extension in both the smooth and analytic settings. It requires the construction of appropriate smooth maps on the projectivized tangent bundle that yield a suitable distribution of partition elements. We refer to Chapter 6 for an overview of our construction. Here, we present the smooth version of our main result.

Theorem 1.6. *Let M be a smooth compact and connected manifold of dimension 2 with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}, S_{t+1} = S_t$, preserving a smooth volume μ. If $\alpha \in \mathbb{R}$ is Liouville, there exists a measure-preserving weak mixing diffeomorphism $f \in \mathcal{A}_\alpha(M)$, whose projectivized derivative extension $(f, \mathrm{d}f)$ is a weak mixing with respect to the measure $\bar{\mu}$ in the projectivized tangent bundle $\mathbb{P}\mathrm{T}M$ which is absolutely continuous in the fibers.*

The Anosov-Katok constructions are renowned for their flexibility. In this thesis, we have discussed several instances exploring intricate dynamics within the smooth category. However, challenges emerge in the real-analytic category, as emphasized in [17, Section 7.1] and [4, Section 6.3]: Maps with notably large derivatives in the real domain or their inverses tend to display singularities in a small complex neighbourhood. For a real analytic family $S_t,\ 0 \le t \le t_0$, with $S_0 = id$, the family $h \circ S_t \circ h^{-1}$ is expected to display singularities very proximate to the real domain for any $t > 0$. Consequently, the domain of analyticity for maps in the form of $f_n = H_n \circ S_t \circ H_n^{-1}$ diminishes progressively during each construction step. This leads to a limiting diffeomorphism that is not analytic. Hence, there is a need to identify specific nature conjugation maps that can be explicitly inverted, ensuring both the map and its inverse maintain analyticity across a broad complex domain.

M. Saprykina first implemented the Anosov-Katok method in the real-analytic category in [43] and constructed a uniquely ergodic real-analytic diffeomorphism on the 2-torus. Afterwards, B. Fayad and M. Saprykina, in [19], successfully constructed an example of volume-preserving weakly mixing real-analytic diffeomorphism on the 2-torus. B. Fayad and A. Katok, in [18], produced similar results for the odd-dimensional sphere, utilizing explicit analytic conjugation maps. The constructions mentioned above used highly explicit analytic conjugation maps in an explicit form. In contrast, a more general analytic approximation technique that utilizes block-slide-type maps on the finite-dimensional torus is developed by S. Banerjee in [2]. This technique offers sufficient flexibility to replicate the construction process and obtain numerous analytic counterparts of Anosov-Katok constructions, as demonstrated in [4] and [34]. However, this analytic approximation technique cannot produce a specific distribution for the elements in the space and tangent direction. Consequently, it hinders the achievement of ergodicity for the projectivized derivative extension in the analytic category, thus requiring alternative approximation techniques.

In this article, we have achieved an even stronger property than ergodicity-the weak mixing property-for the projectivized derivative extension in the analytic category, specifically for the torus case. We adapt an analytical approximation method, as presented in [7, Theorem 1.8]. In this work, the author proposed a particular approach for approximating an analytic transformation within the infinite annulus $[0, 1] \times \mathbb{R}$. This is achieved by accurately approximating it using a smooth transformation, given specific conditions. This technique is built on the generators of groups of Hamiltonian maps and their flows, employing specific conditions. This distinctive approach plays a crucial role and involves adapting the approximation technique for the torus case. This requires exercising sufficient control over the proximity of two functions and establishing the convergence of the Anosov-Katok method while maintaining the desired dynamical properties, as outlined in Chapter 7, within our setup.

Theorem 1.7. *There exists a real analytic diffeomorphism $f \in \mathrm{Diff}^{\omega}(\mathbb{T}^2, \mu)$, whose projectivized derivative extension $(f, \mathrm{d}f)$ is a weak mixing with respect to the measure $\bar{\mu}$ in the projectivized tangent bundle $\mathbb{P}T\mathbb{T}^2$. Additionally the complexification of this map f is entire.*

These constructions, in Theorem 1.6 and 1.7, represent the only known instances of volume-preserving diffeomorphisms whose differentials are weakly mixing with respect to a smooth measure in the projectivization of the tangent bundle.

Regarding the successful implementation of analytic AbC schemes, we mention the recent breakthrough work [8] by P. Berger. Based on a so-called *Anosov-Katok principle* it enables the realization, by analytic symplectomorphisms on annulus, sphere or disc, of dynamical properties which are C^0-realizable by the AbC method. However, the current results only demonstrate that an Anosov-Katok principle holds for transitivity. Therefore, this approach does not directly allow the realization of weakly mixing analytic diffeomorphisms, let alone properties for the projectivized derivative extension.

1.2 The structure of the thesis

In Chapter 2, we outline some basic results from ergodic theory and measure theory and present the preliminary lemmas necessary to prove our findings in subsequent chapters. We discuss both the smooth and analytic topologies on the space of diffeomorphisms and provide a detailed description of the Approximation by Conjugation method.

In Chapter 3, we present an example of a smooth diffeomorphism with an invariant measure, which is a generic but non-ergodic volume measure. This diffeomorphism satisfies various topological and ergodic properties on the 2-Torus. To explore these properties, we use the Approximation by Conjugation method. To achieve these properties effectively, we establish a highly explicit setup. We define distinct targeted domains on the 2-Torus, each serving specific purposes, and carefully select conjugation maps to match the intended behavior. Furthermore, we explicitly discuss the criteria for minimality, weak mixing, and the existence of μ-generic points within our framework.

In Chapter 4 and 5, we delve into the analysis of generic and non-generic points within the smooth category, aiming to establish bounds on the size of the sets they form. Our focus is on determining bounds for their Hausdorff dimension. We apply the Approximation by Conjugation method to construct such a smooth diffeomorphism, ensuring that the set containing all generic points has a nontrivial Hausdorff dimension. This approach leads to more insightful and interesting results. To target sets containing such points, we introduce fractal sets, specifically various types of Cantor sets, into our explicit construction. We establish a highly detailed setup with distinct targeted domains on the 2-Torus, each serving specific purposes. This includes the explicit selection of conjugation maps. Additionally, we extend our analysis to non-generic points, particularly focusing on the case of an ergodic Lebesgue measure on the 2-Torus. In particular, we provide proofs for Theorems 1.3, 1.4, and 1.5.

In Chapter 6, we delve into the ergodic properties of the projectivized derivative extension, with a particular focus on weak mixing behavior. This exploration is conducted within the smooth category using the Approximation by Conjugation method. We examine the structure of a derivative extension map corresponding to a smooth map over the projectivized tangent bundle. We explicitly induce an invariant smooth measure over this bundle. We present an example of a derivative extension map that exhibits weak mixing with respect to such a smooth measure. Additionally, we introduce a weak mixing criterion specifically tailored for the projectivized derivative extension in the smooth setting. To demonstrate the weak mixing property, we provide a suitable distribution of elements under the explicit choice of conjugation maps that satisfy the criterion's requirements. In particular, we prove Theorem 1.6.

In Chapter 7, we extend the analogous result from Chapter 6 to the analytic category for the torus case using the same method. We present an example of an analytic diffeomorphism whose projectivized derivative extension exhibits weak mixing with respect to a smooth measure. For the torus case, we explore an analytic approximation technique that allows us to closely approximate an analytic diffeomorphism with a smooth diffeomorphism. Additionally, we extend the weak mixing criterion for the projectivized derivative extension to the analytic setting. To demonstrate the weak mixing property of the projectivized derivative extension, we provide a suitable distribution of elements. This is achieved through the explicit choice of conjugation maps and approximation techniques that meet the necessary requirements of the criterion. In particular, we prove Theorem 1.7.

In Chapter 8, we provide a brief summary of the key findings of this study. In addition, we explore some interesting open research questions in this direction.

2

Preliminaries

In this chapter, we will first present some basic definitions, techniques, and results from Ergodic Theory that are necessary to prove our results in subsequent chapters.

2.1 Basics of ergodic theory

This section explains some basic definitions and standard techniques that will be used throughout the thesis; refer to [12] and [19]. Consider (X, d) be a compact metric space, $\mathcal{B}$ is a σ algebra, μ is a measure and $T : X \longrightarrow X$ is a measure preserving transformation *(mpt) i.e.* $\mu(T^{-1}(A)) = \mu(A)\ \forall A \in \mathcal{B}$.

Definition 2.1 (Ergodic)**.** A *mpt* $(X, \mathcal{B}, \mu, T)$ is called *ergodic* if every invariant set $E \in \mathcal{B}$ satisfies $\mu(E) = 0$ *or* $\mu(X \backslash E) = 0$. We say μ is ergodic measure.

Definition 2.2 (Generic point)**.** A point $x \in X$ is a *generic point* for μ if for every continuous compactly supported $\phi : X \to \mathbb{R}$, we have $\frac{1}{N}\sum_{i=0}^{N-1}\phi(T^i x) \longrightarrow \int_X \phi d\mu$. A measure is called *generic measure* if it has a generic point. It follows from the Birkhoff Ergodic Theorem that if the system is ergodic, then almost every point is a μ-generic point.

Definition 2.3 (Minimal)**.** Given a continuous map $T : X \longrightarrow X$ on a topological space X, the map T is called *minimal* if for each $x \in X$, the orbit $\{T^i(x)\}_{i\in\mathbb{N}}$ is dense in X. In the context of a metric space, equivalently, a map T is minimal if for every $x \in X$, any $\delta > 0$, and any δ-ball B_δ, there exists an $i \in \mathbb{N}$ such that $T^i(x) \in B_\delta$.

Definition 2.4 (Weak mixing)**.** A measure preserving diffeomorphism $T : X \longrightarrow X$ is said to be *weakly mixing* on the space $(X, \mathcal{B}, \mu, T)$ if there exists a sequence $\{m_n\} \in \mathbb{N}$ such that for any pair $A, B \in \mathcal{B} : |\mu(B \cap f^{-m_n}(A)) - \mu(B)\mu(A)| \longrightarrow 0$.

Definition 2.5 (Partial partition)**.** A collection of disjoint sets η_n on $\mathbb{T}^2$ is called a *partial decomposition or partial partition* of $\mathbb{T}^2$. A sequence of partial decompositions η_n converges to the decomposition into points (notation: $\eta_n \to \varepsilon$) if, for any measurable set A, there exists a measurable set A_n for any n, which is a union of elements of η_n, such that $\lim_{n\to\infty}\mu(A \triangle A_n) = 0$ (here, $\triangle$ denotes the symmetric difference).

2.1.1 A primer of Hausdorff Dimension

The Hausdorff dimension is a measure of the “fractal" or “irregular" dimension of a set.

Definition 2.6. For a compact set $A \subset \mathbb{R}^n$, the Hausdorff dimension is defined as follows: Let $d \geq 0$ be a real number, and consider covering A with a collection of open balls of radius r_i for $i = 1, 2, \ldots$. The “Hausdorff d-dimensional measure" of A is defined as:

$$\mathcal{H}^d_\delta(A) = \inf\left\{\sum_{i=1}^{\infty}(\mathrm{diam} B_{r_i})^d \ : \ A \subseteq \cup_{i=0}^{\infty} B_{r_i},\ \mathrm{diam} B_{r_i} \leq \delta\right\},$$

where the infimum is taken over all possible coverings of A with balls B_{r_i} of diameter at most δ. The Hausdorff dimension of A is then given by:

$$\dim_H(A) = \inf\{d \geq 0 \ : \ \mathcal{H}^d(A) = 0\}.$$

where $\mathcal{H}^d(A) = \lim_{\delta \to 0} \mathcal{H}^d_\delta(A)$. In simpler terms, $\dim_H(A)$ is the largest value of d for which the Hausdorff d-dimensional measure of A is positive and finite.

Lemma 2.1. *For a compact set $A \subset \mathbb{R}^n$ and a diffeomorphism $\phi : A \to \mathbb{R}^n$, the Hausdorff dimension is preserved under the action of ϕ. In particular, if $\mathcal{H}^d(A) = 0$, then $\mathcal{H}^d(\phi(A)) = 0$, and if $\mathcal{H}^d(A) > 0$, then $\mathcal{H}^d(\phi(A)) > 0$.*

Proof. We start by showing that if $\mathcal{H}^d(A) = 0$, then $\mathcal{H}^d(\phi(A)) = 0$. Suppose $\mathcal{H}^d(A) = 0$, and let $\epsilon > 0$ be given. Since $\mathcal{H}^d(A) = 0$, there exists a covering of A with balls B_i such that $\sum_i \operatorname{diam}(B_i)^d < \epsilon$. Since ϕ is a diffeomorphism, it is bi-Lipschitz, meaning that there exists a constant $C > 0$ such that $\operatorname{diam}(\phi(B_i)) \leq C \cdot \operatorname{diam}(B_i)$ for all B_i. Thus, we have:

$$\sum_i \operatorname{diam}(\phi(B_i))^d \leq C^d \cdot \sum_i \operatorname{diam}(B_i)^d < C^d \cdot \epsilon.$$

As ϵ can be arbitrarily small, it follows that $\mathcal{H}^d(\phi(A)) = 0$. Next, we prove that if $\mathcal{H}^d(A) > 0$, then $\mathcal{H}^d(\phi(A)) > 0$. Suppose $\mathcal{H}^d(A) > 0$. Then, for any $\epsilon > 0$, there exists a covering of A with balls B_i such that

$$\sum_i \operatorname{diam}(B_i)^d < \mathcal{H}^d(A) + \epsilon.$$

Applying the same bi-Lipschitz property of ϕ as above, we obtain:

$$\sum_i \operatorname{diam}(\phi(B_i))^d \leq C^d \cdot \sum_i \operatorname{diam}(B_i)^d < C^d \cdot (\mathcal{H}^d(A) + \epsilon).$$

Since ϵ can be arbitrarily small, we conclude that $\mathcal{H}^d(\phi(A)) > 0$. Combining both cases, we have shown that the Hausdorff dimension of $\phi(A)$ is the same as the Hausdorff dimension of A, i.e., $\dim_H(\phi(A)) = \dim_H(A)$. Thus, the Hausdorff dimension is preserved under the action of a diffeomorphism. □

2.1.2 A primer on symplectic dynamics

A symplectic form on a smooth manifold is a closed non-degenerate differential 2-form. A smooth manifold equipped with a symplectic form is called a symplectic manifold; refer to [38].

Definition 2.7 (Symplectomorphism)**.** A *symplectomorphism* $f : (M, \omega) \longrightarrow (M, \omega)$ is a smooth diffeomorphism satisfying $f^*\omega = \omega$, where f^* is the pullback of f. We denote the set of all symplectic diffeomorphsims of M by $\mathrm{Symp}(M, \omega)$.

Definition 2.8 (Hamlitonian Diffeomorphism)**.** A symplectomorphism $\phi \in \mathrm{Symp}(M, \omega)$ is said to be a *hamiltonian diffeomorphism* if there exists a hamiltonian isotopy h_t such that $\phi = h_1$. We say that h_t is a hamiltonian isotopy if there exists a smooth family of functions $H_t : M \to \mathbb{R}$ such that $i_{(h_t)}\omega = \mathrm{d}H_t$. In other words, if there exist a smooth isotopy $H : [0, 1] \times M \to M$ which can be associated to the time dependent vector field X_{H_t} defined by $i_{(X_{H_t})}(\omega) = \mathrm{d}H_t$.

Let's denote $\mathrm{Ham}^\infty(M, \omega)$ be the space of all smooth Hamiltonian diffeomorphism of M and $\mathrm{Ham}^\omega(M, \omega)$ be space of all entire Hamiltonian diffeomorphism of M. For smooth volume form μ, $\mathrm{Ham}^\infty(M, \mu) \subseteq \mathrm{Diff}^\infty(M, \mu)$.

2.2 Smooth topology

Smooth diffeomorphisms of $\mathbb{T}^2$ homotopic to the identity have a lift of type

$$\widetilde{F}(x_1, x_2) = (x_1 + f_1(x_1, x_2), x_2 + f_2(x_1, x_2)),$$

where $f_i : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are $\mathbb{Z}^2$ periodic for $i = 1, 2$. We can use the smooth topology of $\mathbb{R}^2$ defined by the norms as follows: For any continuous function $f : (0, 1)^2 \longrightarrow \mathbb{R}$, the norm is given by

$$||f||_0 := \sup_{z \in (0,1)^2} |f(z)|.$$

The partial derivative of a function is denoted as follows: for $(a_1, a_2) \in \mathbb{N}^2$, where $|a| = a_1 + a_2$,

$$\mathrm{D}_a := \frac{\partial^a}{\partial_r^{a_2} \partial_\theta^{a_1}}.$$

Consider $\mathrm{Diff}^k(\mathbb{T}^2)$ be the space of k-differentiable diffeomorphisms of the torus. For any $F, G \in \mathrm{Diff}^k(\mathbb{T}^2)$, denote their lifts by $\widetilde{F}$ and $\widetilde{G}$. For mappings $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, let F_i represent the i-th coordinate function, and denote

$$|||F|||_k := \max\{\|\mathrm{D}_a F_i\|_0, \|\mathrm{D}_a (F)_i^{-1}\|_0 \mid i = 1, 2,\ 0 \leq |a| \leq k\}.$$

Definition 2.9. For any two diffeomorphisms $F, G \in \mathrm{Diff}^k(\mathbb{T}^2)$, define the distance between them as

$$\tilde{d}_0(F, G) = \max_{i=1,2}\{\inf_{p\in\mathbb{Z}} ||(\widetilde{F}_i - \widetilde{G}_i) + p||_0\},$$

$$\tilde{d}_k(F, G) = \max\{\tilde{d}_0(F, G), ||\mathrm{D}_a(\widetilde{F}_i - \widetilde{G}_i)||_0|\ i = 1, 2,\ 1 \leq |a| \leq k\}.$$

We will employ a metric that measures the distance between diffeomorphisms and their inverses:

$$d_k(F, G) = \max\{\tilde{d}_k(F, G), \tilde{d}_k(F^{-1}, G^{-1})\}.$$

In the smooth topology on M, a sequence of diffeomorphisms in $\mathrm{Diff}^\infty(M)$ is considered convergent if it converges in $\mathrm{Diff}^k(M)$ for all k. The space $\mathrm{Diff}^\infty(M)$ is equipped with the metric

$$d_\infty(F, G) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(F, G)}{1 + d_k(F, G)}.$$

It is a compact metric space, and the Baire theorem holds for any of its closed subspaces.

2.3 Analytic topology

Real analytic diffeomorphisms of $\mathbb{T}^2$ homotopic to the identity have a lift of type

$$F(x_1, x_2) = (x_1 + f_1(x_1, x_2), x_2 + f_2(x_1, x_2)),$$

where the function $f_i : \mathbb{R}^2 \to \mathbb{R}$ are real analytic and $\mathbb{Z}^2$ periodic for $i = 1, 2$. Any real-analytic $\mathbb{Z}^2$-periodic function defined on $\mathbb{R}^2$ can be extended as a holomorphic (complex analytic) function from some complex neighbourhood of $\mathbb{R}^2$ in $\mathbb{C}^2$. For any fixed $\rho > 0$, we define the complex domain Ω_ρ as follows:

$$\Omega_\rho = \{(z_1, z_2) \in \mathbb{C}^2 \ : \ |\Im z_i| < \rho, \ i = 1, 2\}. \tag{2.10}$$

Now, for a function f defined on this set, we define the norm as

$$|f|_\rho = \sup_{z \in \Omega_\rho} |f(z)|.$$

Denote the space $C^\omega_\rho(\mathbb{T}^2)$ as the space of all $\mathbb{Z}^2$-periodic real-analytic functions on $\mathbb{R}^2$ that extend to holomorphic functions on Ω_ρ, with $|f|_\rho < \infty$. Let $\mathrm{Diff}^\omega_\rho(\mathbb{T}^2, \mu)$ be the set of all measure-preserving real-analytic diffeomorphisms of $\mathbb{T}^2$ homotopic to the identity, whose lift F to $\mathbb{R}^2$ extends to a holomorphic functions on Ω_ρ.

Definition 2.11. For two functions $f, g \in \mathrm{Diff}^\omega_\rho(\mathbb{T}^2, \mu)$, we define the norm $|f|_\rho$ and metric $d^\omega_\rho(f, g)$[1] as follows:

$$|f|_\rho = \max_{i=1,2} |f_i|_\rho,$$
$$d^\omega_\rho(f, g) = \max_{i=1,2}\{\inf_{p \in \mathbb{Z}} |f_i - g_i + p|_\rho\}.$$

Moreover, for a diffeomorphism T with lift $\widetilde{T}(x_1, x_2) = (T_1(x_1, x_2), T_2(x_1, x_2))$ we define

$$|DT|_\rho = \max\left\{ \left| \frac{\partial T_i}{\partial x_i} \right|_\rho \ : \ i, j = 1, 2 \right\}.$$

[1]Here, we denote the metric d_r for C^r and d^ω_ρ for C^ω_ρ topology.

Additionally, define $\mathrm{Ham}^{\omega}_{\rho}(\mathbb{T}^2,\mu)$ as a subset of $\mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^2,\mu)$, containing all the real analytic Hamiltonian diffeomorphisms of $\mathbb{T}^2$. Next, we denote $\mathrm{Diff}^{\omega}_{\infty}(\mathbb{T}^2,\mu) \subseteq \mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^2,\mu)$ for any $\rho > 0$, be the space containing all the real-analytic measure-preserving diffeomorphism of $\mathbb{T}^2$ whose lift F to $\mathbb{R}^2$ extends to an entire function on $\mathbb{C}^2$. Analogously, define $\mathrm{Ham}^{\omega}_{\infty}(\mathbb{T}^2,\mu)$ as a subset of $\mathrm{Diff}^{\omega}_{\infty}(\mathbb{T}^2,\mu)$, containing all the real analytic Hamiltonian diffeomorphisms of $\mathbb{T}^2$.

2.4 Preliminary Lemmas

Lemma 2.2. *Let $F, G \in \mathrm{Diff}^{\infty}(\mathbb{T}^2)$. For $k \in \mathbb{N}$, the norm estimates of the composition $F \circ G$ satisfy*

$$|||F \circ G|||_k \leq C|||F|||_k^k \cdot |||G|||_k^k, \tag{2.12}$$

where C is constant only depending on k.

Remark 2.3. The above can be deduced using the corollary of the Faa di Bruno formula; similar proof has been done in [[32], Lemma 4.1].

Lemma 2.4. *For any $\varepsilon > 0$, there is a smooth Lebesgue measure preserving diffeomorphism $\varphi = \varphi(\varepsilon)$ of $[0,1]^2$, equal to identity outside $[\varepsilon, 1-\varepsilon]^2$ and rotating the square $[2\varepsilon, 1-2\varepsilon]^2$ by $\pi/2$ in the clockwise direction.*

Proof. The proof directly follows from [[19], Lemma 5.3]. □

Any permutation map of a matrix of rectangles over a square can be well approximated by a smooth measure preserving transformation has appeared independently in [1], [40], and [24].

Theorem 2.5. *[[24], Theorem 34] Let n horizontal lines and m vertical lines divide $[0,1]\times[0,1]$ into an array of $m \cdot n$ equal size rectangles $\{R_i\}_{i=1}^{mn}$. Let σ be a permutation of the rectangles and $\varepsilon > 0$. Then there is a C^∞, invertible, measure-preserving transformation ϕ of $[0,1]\times[0,1]$ that is the identity of a neighborhood of the boundary of $[0,1]\times[0,1]$ such that for a set L of Lebesgue measure at least $1-\varepsilon$ such that : If $x \in L \cap R_i$, then $\phi(x) \in \sigma(R_i)$ for all i.*

This assertion builds upon Lemma 2.4, which establishes the existence of a C^∞ measure-preserving map that rotates the disc of radius $R - \delta$ inside $[0,1] \times [0,1]$ by an angle of $\pi/2$. This map is identically equal to zero in an arbitrarily small neighborhood of the disc of radius R and acts as an identity on the boundary of $[0,1] \times [0,1]$. Hence, any permutation σ can be represented as a composition of a transposition(rotation). Therefore, the smooth maps can closely approximate each transposition by choosing a small enough δ in the above lemma. Similar results have been used in [24], [19], and [20].

Lemma 2.6. *Any smooth area-preserving diffeomorphism $f : [0,1]^2 \to [0,1]^2$ that acts as identity in a neighborhood of the boundary of $[0,1]^2$ is a Hamiltonian diffeomorphism on the torus $\mathbb{T}^2$.*

Proof. Since the diffeomorphism $f : [0,1]^2 \to [0,1]^2$ acts as the identity close to the boundary of $[0,1]^2$, it can be extended to a diffeomorphism $\tilde{f}$ on $\mathbb{T}^2$ by identifying opposite edges of $[0,1]^2$. Note that any closed non-degenerate 2-form on a simply connected domain of $\mathbb{R}^n$, or a contractible symplectic manifold, is exact due to Poincaré's lemma [11]. Additionally, the exactness of the symplectic form on a contractible manifold implies the existence of a Hamiltonian vector field associated with any closed 2-form.

Furthermore, Moser's theorem guarantees a continuous family of area preserving diffeomorphisms $F_t : \mathbb{T}^2 \to \mathbb{T}^2$ such that $F_1 = \tilde{f}$ and F_0 represents the identity map on $\mathbb{T}^2$, implying that $\tilde{f}$ is isotopic to the identity through area-preserving diffeomorphisms on the torus.

Since $[0,1]^2$ is a contractible domain, the diffeomorphism f within $[0,1]^2$, when extended to the torus as $\tilde{f}$, is indeed a Hamiltonian diffeomorphism on the torus, following from Poincaré's lemma and Moser's theorem. $\square$

2.5 Approximation by Conjugation Method

Here, we outline the scheme of the approximation by conjugation method introduced in [1], which allows the construction of smooth area-preserving diffeomorphisms with specific ergodic properties. We present the following method specifically for the case of $\mathbb{T}^2$, but it is applicable to any compact connected smooth manifold M with a non-trivial circle action. Let us denote by R_t the measure-preserving smooth circle action $\mathbb{T}^1$ on the torus $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ defined by translation t in the first coordinate : $R_t(\theta, r) = (\theta + t, r)$. The required map f is constructed as the limit of a sequence of periodic measure preserving diffeomorphism f_n in the smooth topology. The sequence of f_n is defined iteratively as

$$f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}, \tag{2.13}$$

where $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} \in \mathbb{Q}\backslash\mathbb{Z}$ and $H_n \in \text{Diff}^\infty(\mathbb{T}^2)$. The diffeomorphism H_n is constructed successively as $H_n = h_1 \circ \ldots \circ h_n$, where h_n is an area preserving diffeomorphism of $\mathbb{T}^2$ that satisfies

$$h_n \circ R_{\alpha_n} = R_{\alpha_n} \circ h_n. \tag{2.14}$$

The rationals α_{n+1} is chosen close enough to α_n to ensure the closeness between f_n and f_{n-1} in the C^∞ or C^ω topology. Given α_{n+1}, H_n, at the $n+1$ stage of this iterative process, we construct h_{n+1} such that f_{n+1} satisfy a finite version of the specific property we eventually need to achieve for the limiting diffeomorphism. The explicit construction of h_{n+1} has been done in Chapter 3, 4, 5, 6, and 7 which serves our purpose. Then we construct α_{n+2} enough close to α_{n+1} by choosing some parameters $k_{n+1} \in \mathbb{N}$ and $l_{n+1} \in \mathbb{N}$ to be large enough such that it satisfies the certain condition and guarantees the convergence of iterative sequence f_{n+1} in the smooth or real analytic topologies, respectively. The limit obtained from this iterative sequence is the required smooth or real analytic diffeomorphism with the specific ergodic properties, $f_{n+1} \longrightarrow f \in \text{Diff}^\infty(\mathbb{T}^2, \mu)$.

Additionaly, we employ the quantitative version of this method to construct the diffeomorphism with a prescribed Liouvillean rotation number α. This involves ensuring that the sequence α_n converges to α and the limiting transformation $f \in \mathcal{A}_\alpha(\mathbb{T}^2)$, as discussed in [19].

3

A Non-Ergodic Generic Measure

3.1 Introduction

In this chapter, we present an example of a smooth diffeomorphism with an invariant measure, which is a generic but non-ergodic volume measure. This diffeomorphism satisfies various topological and ergodic properties on the 2-Torus. We explore an example of a measure-preserving smooth diffeomorphism that possesses a non-ergodic Lebesgue measure μ but exhibits μ-generic points on $\mathbb{T}^2$. Subsequently, we delve into the existence of such diffeomorphisms, with exactly r invariant measures which are weak mixing, and possess the minimality property. To explore these properties, we utilize the Approximation by Conjugation method. To achieve these properties effectively, we establish a highly explicit setup with distinct targeted domains on the 2-Torus that serve specific purposes, along with the explicit selection of conjugation maps. Additionally, we explicitly discuss the criteria for minimality, weak mixing, and the existence of μ-generic points within our setup. In particular, we provide a proof for Theorem 1.1.

3.2 Outline of the proof

To prove Theorem 1.1, we utilize the approximation by conjugation method as outlined in Chapter 2. We decompose the torus $\mathbb{T}^2$ into three distinct parts, each serving specific purposes. Firstly, we divide $\mathbb{T}^2$ into r disjoint sets denoted by N^t, where $t = 0, \ldots, r-1$. Each set naturally supports an absolutely continuous Lebesgue measure μ_t, obtained by normalizing the Lebesgue measure μ.

Secondly, we introduce two additional parts within $\mathbb{T}^2$, specifically designed to achieve other two dynamic properties explicitly along with explicit chosen conjugation maps. These parts are deliberately chosen to be measure-theoretically insignificant, ensuring that their measures approach zero, as outlined in Section 3.3.

The convergence of the approximation by conjugation scheme in the smooth topology is done under very precise norm estimates in Section 3.4. With appropriate geometrical and combinatorial criteria, as explained in the Section 3.5, we obtain the limit diffeomorphism T, obtained from scheme, which is minimal and exhibits r distinct weak mixing measures on $\mathbb{T}^2$, along with the Lebesgue measure μ as a generic measure.

3.3 Construction of the conjugacies

Let us denote S_u, a measure preserving circle action $\mathbb{T}^1$ on the torus $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ defined as a translation u in the first coordinate : $S_u(x_1, x_2) = (x_1 + u, x_2)$. We consider the following conjugacies for the Approximation by conjugation method, for any $0 < \sigma < \frac{1}{2}$, on the torus as

$$T_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1} \text{ where } H_n = h_1 \circ \ldots \circ h_n, \tag{3.1}$$

$$h_n = g_n \circ \phi_n \circ P_n \circ Q_n, \tag{3.2}$$

$$g_n(x, y) = (x + \lfloor n q_n^{\sigma} \rfloor y, y), \tag{3.3}$$

where the sequence $\alpha_{n+1} = p_{n+1}/q_{n+1}$ converging to α (a Liouville number), and the diffeomorphisms ϕ_n, P_n and Q_n commute with S_{α_n}, are constructed in section 3.3.2 below.

3.3.1 Explicit set-up

This subsequent section introduces a couple of fundamental domains on which our explicit construction of conjugation maps exhibits various ergodic properties. First, define the following subsets of $\mathbb{T}^2$, for $t = 0, \ldots, r-1$:

$$N^t = \mathbb{T}^1 \times \left[\frac{t}{r}, \frac{t+1}{r}\right), \tag{3.4}$$

and denote μ_t be a measure on N^t defined as normalized Lebesgue measure μ to N^t, i.e. $\mu_t(A) = \frac{\mu(A\cap N^t)}{\mu(N^t)}$ for measurable set $A \in \mathcal{B}(\mathbb{T}^2)$. Considering the following fundamental domain of N^t for $t \in \{0, \ldots, r-1\}$ as

1. The fundamental domain: $D_n^t = \left[0, \frac{1}{q_n}\right] \times \left[\frac{t}{r}, \frac{t+1}{r}\right)$.
2. Split the D_n^t into two halves:
$$D_n^{t,1} = \left[0, \frac{1}{2q_n}\right) \times \left[\frac{t}{r}, \frac{t+1}{r}\right) \quad \text{and} \quad D_n^{t,2} = \left[\frac{1}{2q_n}, \frac{1}{q_n}\right) \times \left[\frac{t}{r}, \frac{t+1}{r}\right).$$
3. $D_{n,j}^t$, the shift of fundamental domain: $D_{n,j}^t = S_{j/q_n}(D_n^t)$, and so $D_{n,j}^{t,i} = S_{j/q_n}(D_n^{t,i})$.

3.3.1.1 Construction of the conjugacies

The aim is to construct the conjugation map h_n, which allows the limiting diffeomorphism T, defined by (3.1), to have $r \in \mathbb{N}$ weak mixing measures and have the Lebesgue measure as a generic measure, and be a minimal map. Here, we proceed with the construction of conjugation map ϕ_n in the following three steps and combining all together; we define the smooth measure-preserving diffeomorphism

$\phi_n : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ as

$$\phi_n = \phi_n^g \circ \phi_n^m \circ \phi_n^w. \tag{3.5}$$

The notations ϕ_n^g, ϕ_n^m, and ϕ_n^w are used to represent the diffeomorphisms constructed in the following steps to attain specific ergodic properties, namely genericity, minimality, and weak mixing, respectively.

Step-1:- Define the map $\phi_n^w : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ to achieve r weak mixing invariant measures on $\mathbb{T}^2$:

$$\phi_n^w(x) = \begin{cases} \phi_{n,0}(x) & \text{if } x \in N^0 \\ \phi_{n,1}(x) & \text{if } x \in N^1 \\ \vdots & \vdots \\ \phi_{n,r-1}(x) & \text{if } x \in N^{r-1} \end{cases} \tag{3.6}$$

where $\phi_{n,t}$ is a smooth measure-preserving diffeomorphism defined on $\mathbb{T}^2$ for $t = 0, 1, \dots, r-1$ as described in the following paragraph. Consider a map $\phi_{n,t} : \left[0, \frac{1}{q_n}\right] \times \left[\frac{t}{r}, \frac{t+1}{r}\right) \longrightarrow \left[0, \frac{1}{q_n}\right] \times \left[\frac{t}{r}, \frac{t+1}{r}\right)$:

$$\phi_{n,t} = \begin{cases} C_{n,t}^{-1} \circ \varphi^{-1}(\varepsilon_n^{(1)}) \circ C_{n,t} & \text{on } D_n^{t,1} \\ Id & \text{otherwise.} \end{cases} \tag{3.7}$$

Here, $C_{n,t} : \left[0, \frac{1}{q_n}\right] \times \left[\frac{t}{r}, \frac{t+1}{r}\right) \to [0,1] \times [0,1]$ is defined as $C_{n,t}(x,y) = (q_n x, ry - t)$, and φ is a measure-preserving diffeomorphism, defined as in Lemma 2.4, for $\epsilon_n^{(1)} \in (0,1)$. We can extend this map $\phi_{n,t}$ $\frac{1}{q_n}$-equivariantly to the whole N^t, as done in [19].

Step-2:- Here, we construct a smooth measure-preserving diffeomorphism $\phi_n^g : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ differently to ensure the existence of a generic point. Consider a map $\phi_n^g : \left[0, \frac{1}{q_n}\right] \times \mathbb{T}^1 \longrightarrow \left[0, \frac{1}{q_n}\right] \times \mathbb{T}^1$ defined as

$$\phi_n^g = \tilde{C}_n^{-1} \circ \varphi^{-1}(\varepsilon_n^{(3)}) \circ \varphi(\varepsilon_n^{(2)}) \circ \tilde{C}_n,$$

where $\tilde{C}_n(x,y) = (q_n x, y)$ and φ is a smooth measure-preserving diffeomorphism defined in Lemma 2.4 with the specific choice of $\varepsilon_n^{(2)}, \varepsilon_n^{(3)} \in (0,1)$ satisfying the condition $\varepsilon_n^{(2)} < \varepsilon_n^{(3)}$ [1]. As in the above step, we extend the ϕ_n^g equivariantly on $\mathbb{T}^2$. For $i = 0, \dots, q_n - 1$, we will use the notation

$$B_{n,i} = \left[\frac{i}{q_n} + \frac{2\varepsilon_n^{(2)}}{q_n}, \frac{i+1}{q_n} - \frac{2\varepsilon_n^{(2)}}{q_n}\right] \times [2\varepsilon_n^{(2)}, \varepsilon_n^{(3)}], \text{ and}$$

$$Y_{n,i} = \left[\frac{i+1}{q_n} - \frac{\varepsilon_n^{(3)}}{q_n}, \frac{i+1}{q_n} - \frac{2\varepsilon_n^{(2)}}{q_n}\right] \times [2\varepsilon_n^{(2)}, 1 - 2\varepsilon_n^{(2)}].$$

Remark 3.1. This scheme is called the "double rotation effect", as $\varphi^{-1}(\varepsilon_n^{(3)}) \circ \varphi(\varepsilon_n^{(2)})$ first rotate the whole square with the error $\varepsilon_n^{(2)}$, i.e. rotate inside the square $[2\varepsilon_n^{(2)}, 1-2\varepsilon_n^{(2)}]^2$, by $\frac{\pi}{2}$ in the clockwise direction and act as an identity outside the square $[\varepsilon_n^{(2)}, 1-\varepsilon_n^{(2)}]^2$ (see Lemma 2.4). Similarly, we rotate the whole square with the error $\varepsilon_n^{(3)}$, i.e. $[2\varepsilon_n^{(3)}, 1-2\varepsilon_n^{(3)}]^2$, in the anticlockwise direction. Note that with the specific choice of $\varepsilon_n^{(2)}$ and $\varepsilon_n^{(3)}$, and the condition $\varepsilon_n^{(2)} < \varepsilon_n^{(3)}$. The map ϕ_n^g satisfies the following properties:

1. ϕ_n^g rotates the region $B_{n,i}$ by $\pi/2$, which maps the region $B_{n,i}$ into $Y_{n,i}$, i.e. $\phi_n^g(B_{n,i}) = Y_{n,i}$.

2. ϕ_n^g acts as an identity on the region $\Sigma_1 \cup \Sigma_2$, where
   - $\Sigma_1 = \bigcup_{i=0}^{q_n-1} \left(\left[\frac{i}{q_n}, \frac{i}{q_n} + \frac{\varepsilon_n^{(2)}}{q_n}\right] \cup \left[\frac{i+1}{q_n} - \frac{\varepsilon_n^{(2)}}{q_n}, \frac{i+1}{q_n}\right]\right) \times \left([0, \varepsilon_n^{(2)}] \cup [1-\varepsilon_n^{(2)}, 1]\right)$
   - $\Sigma_2 = \bigcup_{i=0}^{q_n-1} \left[\frac{i}{q_n} + \frac{2\varepsilon_n^{(3)}}{q_n}, \frac{i+1}{q_n} - \frac{2\varepsilon_n^{(3)}}{q_n}\right] \times [2\varepsilon_n^{(3)}, 1-2\varepsilon_n^{(3)}]$.

Remark 3.2. The region $\mathbb{E}_n^g \subset \mathbb{T}^2 \backslash ((\cup_{i=0}^{q_n-1} B_{n,i} \cup Y_{n,i}) \cup (\Sigma_1 \cup \Sigma_2))$, say as Error zone, comes from the smoothing of the map ϕ_n^g.

Step 3:- In the same spirit, we define $R_n = \left[0, \frac{\varepsilon_n^{(2)}}{q_n}\right] \times \mathbb{T}^1$ and the map ϕ_n^m :

[1] $\varepsilon_n^{(1)}, \varepsilon_n^{(2)}, \varepsilon_n^{(3)}$ and $\varepsilon_n^{(4)}$ are just notations of different parameters, not related by any power of $\varepsilon_n^{(1)}$.

$\left[0, \frac{1}{q_n}\right] \times \mathbb{T}^1 \longrightarrow \left[0, \frac{1}{q_n}\right] \times \mathbb{T}^1$ differently to achieve minimality as

$$\phi_n^m = \begin{cases} \hat{C}_n^{-1} \circ \varphi(\varepsilon_n^{(4)}) \circ \hat{C}_n & \text{on } R_n \\ Id & \text{otherwise,} \end{cases} \tag{3.8}$$

where $\hat{C}_n(x,y) = \left(\frac{q_n}{\varepsilon_n^{(2)}}x, y\right)$ and $\varepsilon_n^{(4)} \in (0,1)$ satisfying that $\varepsilon_n^{(4)} < \varepsilon_n^{(2)}$. We extend the map ϕ_n^m equivariantly on $\mathbb{T}^2$ such that it acts as an identity outside the region $R_{n,i} = \left[\frac{i}{q_n}, \frac{i}{q_n} + \frac{\varepsilon_n^{(2)}}{q_n}\right] \times \mathbb{T}^1$ (defined as the shift of domain: $S_{\frac{i}{q_n}}(R_n) = R_{n,i},\ \forall\ i \in \{0, 1, \ldots, q_n - 1\}$).

Remark 3.3. With specific choice of $\varepsilon_n^{(4)}$, the map ϕ_n^m rotates the region

$$\left[\frac{i}{q_n} + \frac{2\varepsilon_n^{(4)}}{q_n}, \frac{i}{q_n} + \frac{\varepsilon_n^{(2)}}{q_n} - \frac{2\varepsilon_n^{(4)}}{q_n}\right] \times [2\varepsilon_n^{(4)}, 1 - 2\varepsilon_n^{(4)}],$$

inside $R_{n,i}$, by $\pi/2$ and acts as an identity outside the region $R_{n,i}$. The region

$$\begin{aligned}\mathbb{E}_n^m = \bigcup_{i=0}^{q_n-1} \Bigg(& \left[\frac{i}{q_n} + \frac{\varepsilon_n^{(4)}}{q_n}, \frac{i}{q_n} + \frac{2\varepsilon_n^{(4)}}{q_n}\right] \bigcup \left[\frac{i}{q_n} + \frac{\varepsilon_n^{(2)}}{q_n} - \frac{2\varepsilon_n^{(4)}}{q_n}, \frac{i}{q_n} + \frac{\varepsilon_n^{(2)}}{q_n} - \frac{\varepsilon_n^{(4)}}{q_n}\right] \Bigg) \\ & \times \left([\varepsilon_n^{(4)}, 2\varepsilon_n^{(4)}] \cup [1 - 2\varepsilon_n^{(4)}, 1 - \varepsilon_n^{(4)}]\right), \end{aligned} \tag{3.9}$$

the error zone comes from the smoothing of the map ϕ_n^m.

3.3.1.2 Choice of parameters and norm estimates

A suitable choice of parameters to achieve the desired combinatorics is as follows: $\varepsilon_n^{(1)} = \frac{1}{3nr}$, $\varepsilon_n^{(2)} = \frac{\varepsilon_n^{(1)}}{2^{n+3}}$, $\varepsilon_n^{(3)} = \frac{\varepsilon_n^{(1)}}{2^{n+1}}$, and $\varepsilon_n^{(4)} = \frac{1}{2^n s_n q_n}$,

Lemma 3.4. *The diffeomorphism ϕ_n constructed above satisfy: for all $k \in \mathbb{N}$, $|||\phi_n|||_k \leq c_k(n,k) q_n^{2k^3+k}$ where $c_k(n,k)$ is independent of q_n.*

Proof. For any $a \in \mathbb{N}^2$ with $|a| = k$, we have $\|(D_a \phi_n^m)_j\|_0 \leq c_m q_n^k$, and similarly, $\|(D_a (\phi_n^m)^{-1})_j\|_0 \leq c_m q_n^k$, for $j = 1, 2$. Hence $|||\phi_n^m|||_k \leq c_m(n,k) q_n^k$, where c_m is a constant and independent of q_n. Analogously, we have $|||\phi_n^g|||_k \leq c_g(n,k) q_n^k$ and

$|||\phi_{n,i}|||_k \leq c_i(n,k) q_n^k$ for $i \in \{0,\ldots,r-1\}$ where c_g and c_i are constants independent of q_n. With triangle inequality on the norm, we have $|||\phi_n^w|||_k \leq c_w(n,k)\cdot r \cdot q_n^k$. Using the above estimate and Lemma (2.2), we have

$$
\begin{aligned}
|||\phi_n|||_k &\leq c_k(n,k)|||\phi_n^g|||_k^k \cdot |||\phi_n^m \circ \phi_n^w|||_k^k \\
&\leq c_k(n,k)|||\phi_n^g|||_k^k \cdot |||\phi_n^m|||_k^{k^2} \cdot |||\phi_n^w|||_k^{k^2} \\
&\leq c_k(n,k)\cdot q_n^{2k^3+k},
\end{aligned}
$$

where $c_k(n,k)$ is a constant independent of q_n. □

3.3.2 The conjugation map h_n

The final conjugacy map $h_n : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ is defined as a composition of the following maps as

$$h_n = g_n \circ \phi_n \circ P_n \circ Q_n, \tag{3.10}$$

where the map $P_n : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ is smooth measure-preserving diffeomorphism defined by $P_n(x,y) = (x, y + \kappa_n(x))$ with a smooth map $\kappa_n : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$. For our specific situation, we choose $\tilde{\kappa}_n : \left[0, \frac{1}{q_n}\right] \longrightarrow \mathbb{T}^1$ as follows, and then extend it $\frac{1}{q_n}$-periodically on the whole $\mathbb{T}^1$,

$$
\tilde{\kappa}_n(x) = \begin{cases} \frac{2q_n}{n^2\varepsilon_n^{(2)}}x & \text{if } x \in [0, \frac{\varepsilon_n^{(2)}}{2q_n}] \\ -\frac{2q_n}{n^2\varepsilon_n^{(2)}}x + \frac{2}{n^2} & \text{if } x \in [\frac{\varepsilon_n^{(2)}}{2q_n}, \frac{\varepsilon_n^{(2)}}{q_n}] \\ 0 & \text{if } x \in [\frac{\varepsilon_n^{(2)}}{q_n}, \frac{1}{q_n}]. \end{cases} \tag{3.11}
$$

Let κ_n be the smooth approximation of $\tilde{\kappa}_n$ on $[0,1]$ by convolving it with a mollifier refer to [31]. Let ρ be the standard mollifier on $\mathbb{R}$, and set

$$
\rho(x) = \begin{cases} c \exp \frac{1}{|x|^2-1} & \text{if } |x| < 1 \\ 0 & \text{otherwise,} \end{cases}
$$

where c is numerical constant to ensures normalization i.e. $\int_{\mathbb{R}} \rho(x) = 1$. Then, $\kappa_n(x) = \lim_{\delta\to 0} \delta^{-1} \int_{\mathbb{T}^1} \rho\left(\frac{x-y}{\delta}\right) \tilde{\kappa}_n(y)dy$.

Analogously, the map $Q_n : \mathbb{T}^2 \to \mathbb{T}^2$ is a smooth measure-preserving diffeomorphism defined by $Q_n(x,y) = (x, y + \kappa_n'(x))$, where $\kappa_n' : \mathbb{T}^1 \to \mathbb{T}^1$ is a smooth function. The function κ_n' serves as a smooth approximation of $\overline{\kappa}_n$ on $[0,1]$, where $\overline{\kappa}_n$ is initially defined on $\left[0, \frac{1}{q_n}\right] \to \mathbb{T}^1$ and then extended periodically with period $\frac{1}{q_n}$ over the entire circle $\mathbb{T}^1$,

$$\overline{\kappa}_n(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{\varepsilon_n^{(2)}}{q_n}] \\ 2\varepsilon_n^{(2)} + \frac{\varepsilon_n^{(3)} - 2\varepsilon_n^{(2)}}{2} & \text{if } x \in [\frac{\varepsilon_n^{(2)}}{q_n}, \frac{1}{q_n} - \frac{\varepsilon_n^{(2)}}{q_n}] \\ 0 & \text{if } x \in [\frac{1}{q_n} - \frac{\varepsilon_n^{(2)}}{q_n}, \frac{1}{q_n}]. \end{cases} \tag{3.12}$$

Remark 3.5. The map $\kappa_n(x) = q_n x$ on $\mathbb{T}^1$ is considered in [20] to control almost all the orbits of space.

Remark 3.6. For minimality, the orbit of every point has to be dense. The map ϕ_n^m takes care of all the points inside $\mathbb{T}^2$ except for the points whose whole orbit gets trapped inside the Error zone $\mathbb{E}_n^m$ (where we do not have any control) of ϕ_n^m. The map P_n acts as the vertical translation ensuring that such an orbit enters the minimality zone, and no whole orbit of a point gets trapped inside the Error zone. Additionally, note that P_n acts as an identity outside the region $\bigcup_{i=0}^{q_n-1} R_{n,i}$ and the map Q_n acts as an identity on the region $\bigcup_{i=0}^{q_n-1} R_{n,i}$.

Remark 3.7. For a generic point, the map Q_n acts as a vertical translation on the entire $S_{\alpha_{n+1}}$-orbit of the point $(0,0)$, causing almost the entire orbit to remain confined within the region $B_{n,j}$. Moreover, this region is mapped to $Y_{n,j}$ under ϕ_n^g, which distributes across the entire $\mathbb{T}^1$, thereby ensuring the genericity of the point.

Remark 3.8. Also note that $\|D^k \kappa_n\|_0 \leq \max_{x\in[-1,1]} |\tilde{\kappa}_n| \cdot \|D^k \rho\|_0 < (2k\sqrt{18})^{2k} k! \cdot q_n^k$.

3.4 Convergence

There are some standard results on the closeness between the maps constructed as the conjugation of translations on the torus. The following two lemmas are identical to [20, Lemma 3, 4] with minor to no modifications. Hence, the proofs have been included in the Appendix for reference.

Lemma 3.9. *Let $k \in \mathbb{N}$. For all $\alpha, \beta \in \mathbb{R}$ and all $h \in \mathrm{Diff}^\infty(\mathbb{T}^2)$, we have the estimate*

$$d_k(hS_\alpha h^{-1}, hS_\beta h^{-1}) \leq C_k \max\{|||h|||_{k+1}, |||h^{-1}|||_{k+1}\}|\alpha - \beta|,$$

where C_k is a constant that depends only on k.

Lemma 3.10. *For any $\epsilon > 0$, let k_n be a sequence of natural numbers satisfying $\sum_{n=1}^{\infty} \frac{1}{k_n} < \epsilon$. Suppose for any Liouville α, there exists a sequence of rationals $\{\alpha_n\}$ that satisfy:*

$$|\alpha - \alpha_n| < \frac{1}{2^{n+1} k_n C_{k_n} q_n |||H_n|||_{k_{n+1}}^{k_{n+1}}} \tag{3.13}$$

where C_{k_n} is the same constant as in Lemma 3.9. Then the sequence of diffeomorphisms $T_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$ converges to $T \in \mathrm{Diff}^\infty(\mathbb{T}^2, \mu)$ in the C^∞ topology. Moreover, for any $m \leq q_{n+1}$, we have

$$d_0(T^m, T_n^m) \leq \frac{1}{2^{n+1}}. \tag{3.14}$$

Lemma 3.11. *For any $k \in \mathbb{N}$, the conjugation diffeomorphisms, defined by (3.5) and (3.10), satisfies the following norm estimates as*

1. *$|||h_n|||_k \leq c_k(n,k) \cdot q_n^{2k^4+2k^2}$, where $c_k(n,k)$ is constant independent of q_n.*
2. *$|||H_n|||_k \leq \hat{c}_k(n,k) \cdot q_n^{2k^5+2k^3+k}$, where $\hat{c}_k(n,k)$ is constant independent of q_n.*
3. *For α Liouville, there exists a sequence of rational $\{\alpha_n\}$ satisfying (3.13).*

Proof. The map h_n is defined by

$$h_n(x,y) = g_n \circ \phi_n \circ P_n \circ Q_n(x,y)$$

$$= ([\phi_n(x, y + \kappa_n(x)) + \kappa_n'(x)]_1 + \lfloor nq_n{}^\sigma \rfloor [\phi_n(x, y + \kappa_n(x) + \kappa_n'(x))]_2,$$
$$[\phi_n(x, y + \kappa_n(x) + \kappa_n'(x))]_2).$$

By Lemma 2.2 and Remark 3.8, we have estimate:

$$\begin{aligned} |||h_n|||_k &\leq 2 \cdot (nq_n)^{k-1} \cdot |||\phi_n|||_k^k \cdot |||\kappa_n|||_k^k \\ &\leq c_k(n,k) \cdot q_n^{2k^4+2k^2}. \end{aligned}$$

Similarly, $|||H_n|||_k = |||H_{n-1} \circ h_n|||_k \leq |||H_{n-1}|||_k^k \cdot |||h_n|||_k^k$. Since the kth order derivative of H_{n-1} is independent of q_n, we can conclude $|||H_n|||_k \leq \hat{c}_k(n,k) \cdot q_n^{2k^5+2k^3+k}$. For α being a Liouville, we can choose a sequence of rationals $\alpha_n = \frac{p_n}{q_n}$ (p_n, q_n are coprime) that satisfy the following property:

$$\begin{aligned} |\alpha - \alpha_n| &\leq \frac{1}{2^{n+1} k_n C_{k_n} q_n^{2(k_n+1)^5 + 2(k_n+1)^3 + (k_n+1)+1}} \\ &\leq \frac{1}{2^{n+1} k_n C_{k_n} q_n |||H_n|||_{k_{n+1}}^{k_{n+1}}}. \end{aligned}$$

□

Remark 3.12. Finally, we have proven the estimate on the norms of the conjugation map H_n, as shown in [19]. Additionally, the existence of rationals satisfying (3.13) guarantees the convergence of the sequence T_n to $T \in \text{Diff}^\infty(\mathbb{T}^2, \mu)$, as stated in Lemma 3.10. The limiting diffeomorphism T is minimal and has r weak mixing invariant measures, and the Lebesgue measure as a generic measure has been proven in the next section 3.5.4.

3.5 Weak mixing, Minimal and Generic points

In order to establish Theorem 1.1, we require a few preliminary results. Here we state a few definitions and the criterion for weak mixing described in [19] for $\mathbb{T}^2$.

3.5.1 A Fubini criterion for weak mixing

Recall the notion of (γ,δ,ϵ)-distribution of a horizontal interval in the vertical direction.

Definition 3.15. ((γ,δ,ϵ)- distribution):- A diffeomorphism $\Phi : M \longrightarrow M$, (γ,δ,ϵ) distributes a horizontal interval $I \in \eta$, where η is the partial decomposition of M (or $\phi(I)$ is (γ,δ,ϵ)- distributed on M), if

- $J = \pi_y(\Phi(I))$ is an interval with $1-\delta \leq \lambda(J) \leq 1$, where π_y is the projection map onto the y coordinate,
- $\Phi(S) \subseteq K_{c,\gamma} = [c, c+\gamma] \times J$ for some c (i.e. $\Phi(S)$ is almost vertical),
- for any interval $\tilde{J} \subseteq J$ we have: $\left|\frac{\lambda(I\cap\Phi^{-1}(\mathbb{T}\times\tilde{J}))}{\lambda(I)} - \frac{\lambda(\tilde{J})}{\lambda(J)}\right| \leq \epsilon\frac{\lambda(\tilde{J})}{\lambda(J)}$.

Proposition 3.13 ([19], Proposition 3.9)**.** Assume $T_n = H_n \circ S_{\alpha_n+1} \circ H_n^{-1}$ is the sequence of diffeomorphism constructed by (3.1), (3.3) and (3.10) such that all n, $\|DH_{n-1}\|_0 < \ln q_n$ holds. Suppose $\lim_{n\to\infty} T_n = T$ exists. If there exists a sequence of natural numbers $\{\mathfrak{m}_n\}$ such that $d_o(f^{\mathfrak{m}_n}, f_n^{\mathfrak{m}_n}) < \frac{1}{2^n}$, and a sequence of standard partial decomposition $\{\eta_n\}_{n\in\mathbb{N}}$ of M into horizontal intervals of length less than $\frac{1}{q_n}$ satisfying

1. $\eta_n \to \varepsilon$,
2. for $I_n \in \eta_n$, the diffeomorphism $\Phi_n = \phi_n \circ S^{\mathfrak{m}_n}_{\alpha_n+1} \circ \phi_n^{-1}$ is $(\frac{1}{nq_n^\sigma}, \frac{1}{n}, \frac{1}{n})$ uniformly distribute the interval I_n.

Then limiting diffeomorphism T is weak mixing.

3.5.2 Proof for weak mixing

The specific scheme that we describe here builds on the construction in [19]. First, we consider a subset, $\mathbb{E}_n^w \subset \mathbb{T}^2$ as follows:

$$\left(\bigcup_{k=0}^{2q_n-1}\left[\frac{k}{2q_n} - \frac{2\varepsilon_n^{(1)}}{q_n}, \frac{k}{2q_n} + \frac{2\varepsilon_n^{(1)}}{q_n}\right] \times \mathbb{T}^1\right) \bigcup \left(\bigcup_{t=0}^{r-1} \mathbb{T}^1 \times \left[\frac{t}{r} - 2\varepsilon_n^{(1)}, \frac{t}{r} + 2\varepsilon_n^{(1)}\right]\right). \tag{3.16}$$

3.5.2.1 Action of ϕ_n

Consider the interval, $I_{n,j} \subseteq D_{n,j}^{t,1}$ for some fixed t and j of the form $I_{n,j} = I_{n,j}^0 \times \{s\}$ where $s \in \left[\frac{t}{r}, \frac{t+1}{r}\right]$, and

$$I_{n,j}^0 = \left[\frac{j}{q_n} + \frac{2}{3nq_nr}, \frac{j}{q_n} + \frac{1}{2q_n} - \frac{2}{3nq_nr}\right]. \tag{3.17}$$

From our construction of ϕ_n, the image of $I_{n,j}$ under both ϕ_n and ϕ_n^{-1} is an interval of type $\{\theta\} \times \left[\frac{t}{r} + \frac{2}{3nr}, \frac{t+1}{r} - \frac{2}{3nr}\right]$ for some $\theta \in I_{n,j}^0$.

3.5.2.2 Choice of $\mathfrak{m}_n$- mixing sequence

Consider $\mathfrak{m}_n = \min\left\{m \leq q_{n+1} \mid \inf_{k\in\mathbb{Z}} \left|m\frac{q_np_{n+1}}{q_{n+1}} - \frac{1}{2} + k\right| \leq \frac{q_n}{q_{n+1}}\right\}$ and

$$\mathfrak{a}_n = \left(\mathfrak{m}_n\alpha_{n+1} - \frac{1}{2q_n} \mod \frac{1}{q_n}\right),$$

as defined in Fayad's paper for the torus case and with the growth assumption, $q_{n+1} > 10n^2q_n$ would result here:

$$|\mathfrak{a}_n| \leq \frac{1}{q_{n+1}} \leq \frac{1}{10n^2q_n}.$$

Further, if we define a precise domain as $\overline{D}_{n,j}^{t,1} = I_{n,j}^0 \times \left[\frac{t}{r}, \frac{t+1}{r}\right] \subset D_{n,j}^{t,1}$ for some $j \in \mathbb{Z}$, then we would have $S_{\alpha_{n+1}}^{\mathfrak{m}_n}(\overline{D}_{n,j}^{t,1}) \subset D_{n,j'}^{t,2}$ for some $j' \in \mathbb{Z}$.

3.5.2.3 Choice of decomposition η_n^t

For fixed $t \in \{0, 1, \ldots, r-1\}$, we consider the partial decomposition η_n^t of the set N^t, outside $\mathbb{E}_n^w$, which consists of two types of horizontal intervals: $I_{n,j} = I_{n,j}^0 \times \{s\} \subset D_{n,j}^{t,1}$ and $\overline{I}_{n,j} = \overline{I}_{n,j}^0 \times \{s'\} \subset D_{n,j}^{t,2}$ where $s, s' \in \left[\frac{t}{r}, \frac{t+1}{r}\right]$, and $I_{n,j}^0$ by (3.17), and

$$\overline{I}_{n,j}^0 = \left[\frac{j}{q_n} + \frac{1}{2q_n} - \frac{2}{3nq_nr} - \mathfrak{a}_n, \frac{j+1}{q_n} - \frac{2}{3nq_nr} - \mathfrak{a}_n\right]. \tag{3.18}$$

Note that for any element $I_n \in \eta_n^t$, we have $\pi_y(\phi_n(I_n)) \subset \left[\frac{t}{r}, \frac{t+1}{r}\right]$. Since the length of intervals goes to zero and $\sum_{I_n \in \eta_n} \lambda(I_n) \leq 1 - \lambda(\mathbb{E}_n^w) \leq 1 - \frac{4}{n} \to 1$, that implies $\eta_n^t \to 0$ as $n \to \infty$.

Lemma 3.14. *For any $t \in \{0, 1, ..., r-1\}$. The map $\Phi_n = \phi_n \circ P_n \circ Q_n \circ S_{\alpha_{n+1}}^{\mathfrak{m}_n} \circ Q_n^{-1} \circ P_n^{-1} \circ \phi_n^{-1}$ transform the elements of the partial decomposition, i.e. $I_{n,j} = I_{n,j}^0 \times \{s\} \in \eta_n^t$, into vertical interval of the form $\{\theta\} \times [\frac{t}{r} + \frac{2}{3nr}, \frac{t+1}{r} - \frac{2}{3nr}]$, for some $\theta \in I_{n,j}^0$ (see Figure 3.1).*

Proof. The map Q_n acts as a vertical translation by a factor of $\mathfrak{b}_n$ on the region $[\frac{\varepsilon_n^{(2)}}{q_n}, \frac{1}{q_n} - \frac{\varepsilon_n^{(2)}}{q_n}] \times \mathbb{T}^1$, where $\mathfrak{b}_n = \frac{\varepsilon_n^{(3)} + 2\varepsilon_n^{(2)}}{2}$ as defined in section 3.3.1.2. From our construction of $\phi_n \circ P_n \circ Q_n$, an interval $I_{n,j} = I_{n,j}^0 \times \{s\} \subset D_{n,j}^{t,1}$ where $s \in \left[\frac{t}{r} + \frac{2}{3nr}, \frac{t+1}{r} - \frac{2}{3nr}\right]$, we obtain

$$Q_n^{-1} \circ P_n^{-1} \circ \phi_n^{-1}(I_{n,j}) = \{\theta\} \times \left[\frac{t}{r} + \frac{2}{3nr} - \mathfrak{b}_n, \frac{t+1}{r} - \frac{2}{3nr} - \mathfrak{b}_\mathfrak{n}\right]$$

for some $\theta \in I_{n,j}^0$.
With the specific choice of sequence $\mathfrak{m}_n$ and the condition mentioned in section 3.5.2.2, we get

$$S_{\alpha_{n+1}}^{\mathfrak{m}_n} \circ Q_n^{-1} \circ P_n^{-1} \circ \phi_n^{-1}(I_{n,j}) = \{\theta'\} \times \left[\frac{t}{r} + \frac{2}{3nr} - \mathfrak{b}_n, \frac{t+1}{r} - \frac{2}{3nr} - \mathfrak{b}_n\right] \subset D_{n,j'}^{t,2},$$

for some $\theta' \in \mathbb{T}$ and $j' \in \mathbb{Z}$.

Since κ_n acts as an identity on $[\frac{\varepsilon_n^{(2)}}{q_n}, \frac{1}{q_n}]$, and the fact that ϕ_n acts as an identity on $D_{n,j'}^{t,2}$ concludes the Claim. Similarly, for the interval $\overline{I}_{n,j} = \overline{I}_{n,j}^0 \times \{s\} \subset D_{n,j}^{t,2}$, we deduced that

$$\phi_n \circ P_n \circ Q_n \circ S_{\alpha_{n+1}}^{\mathfrak{m}_n} \circ Q_n^{-1} \circ P_n^{-1} \circ \phi_n^{-1}(\overline{I}_{n,j}) = \{\theta'\} \times \left[\frac{t}{r} + \frac{1}{3nr}, \frac{t+1}{r} - \frac{1}{3nr}\right] \subset D_{n,j'}^{t,1},$$

for some $j' \in \mathbb{Z}$ and $\theta' \in \mathbb{T}$. □

Refer to Figure 3.1, for an illustration of action ϕ_n and Φ_n on the fundamental domains inside the $\mathbb{T}^2$. The orange region, $B_{n,j}$, is transformed into the green

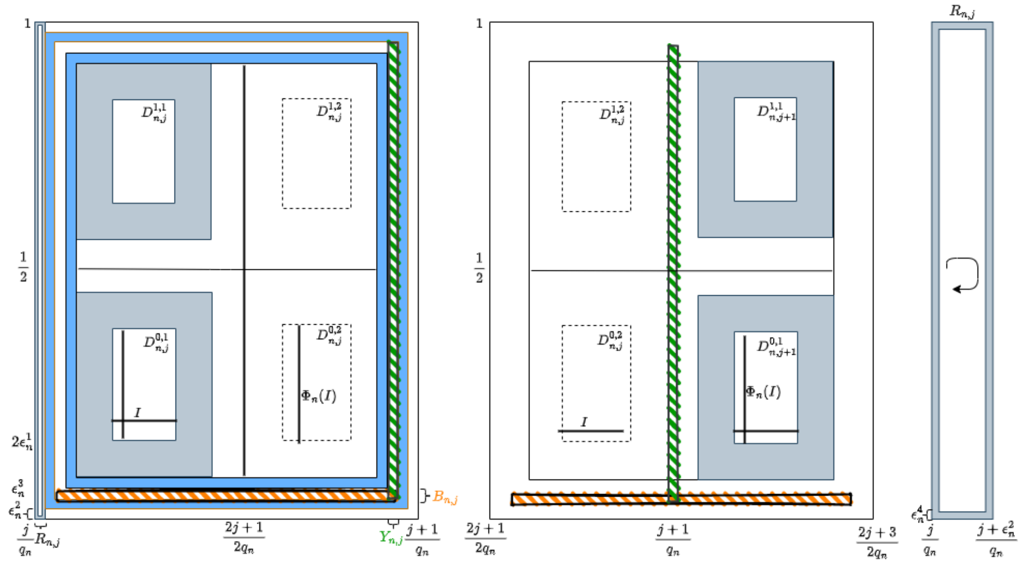

FIGURE 3.1: An example of action ϕ_n and Φ_n on the fundamental domains inside the $\mathbb{T}^2$ for $r = 2$.

region, $Y_{n,j}$, under the action of ϕ_n. In (a), the horizontal line I lying inside $D^{0,1}_{n,j}$ is transformed into vertical by ϕ_n^{-1} and then transferred to the right $D^{0,2}_{n,j}$ under the action of Φ_n. Whereas in (b), the horizontal line I lying inside $D^{0,2}_{n,j}$ is transferred to $D^{0,1}_{n,j+1}$ first and then transformed into vertical by ϕ_n under the action of Φ_n. The same action of Φ_n will be followed inside regions $D^{1,1}_{n,j}$ and $D^{1,2}_{n,j}$ in both (a) and (b) respectively. In (c), the region inside $R_{n,j}$ is being rotated by the map ϕ_n by $\pi/2$. The blue and grey shaded regions represent the error region for ϕ_n.

3.5.3 A criterion for minimality

The main idea used here to achieve minimality first appeared in [44] and was subsequently reinforced by its appearance in [17]. This section aims to deduce a criterion for minimality for our explicit construction, particularly to understand the action of ϕ_n on the region $R_{n,i}$ explained in step 3, section 3.3.1.1. Here, we define the following partition of the set $R_{n,i}$, excluding the set $\mathbb{E}^m_n$, for any natural number s_n satisfying $s_n > n^2 q_n^2$.

$$\begin{aligned}A_{i,k}^n &:= \left[\frac{i}{q_n} + \frac{2\varepsilon_n^{(4)}}{q_n} + \frac{k(\varepsilon_n^{(2)} - 4\varepsilon_n^{(4)})}{s_n q_n}, \frac{i}{q_n} - \frac{2\varepsilon_n^{(4)}}{q_n} + \frac{(k+1)(\varepsilon_n^{(2)} - 4\varepsilon_n^{(4)})}{s_n q_n}\right) \\ &\quad \times \left[2\varepsilon_n^{(4)}, 1 - 2\varepsilon_n^{(4)}\right], \\ B_{i,k}^n &:= \left[\frac{i}{q_n} + \frac{2\varepsilon_n^{(4)}}{q_n}, \frac{i}{q_n} + \frac{\varepsilon_n^{(2)}}{q_n} - \frac{2\varepsilon_n^{(4)}}{q_n}\right) \\ &\quad \times \left[2\varepsilon_n^{(4)} + \frac{k(1-4\varepsilon_n^{(4)})}{s_n}, 2\varepsilon_n^{(4)} + \frac{(k+1)(1-4\varepsilon_n^{(4)})}{s_n}\right].\end{aligned}$$

Let us denote the family of these subsets by $\mathcal{A}_n = \{A_{i,k}^n, \quad i = 0, \dots, q_n - 1, \; k = 0, \dots, s_n - 1\}$ and $\mathcal{B}_n = \{B_{i,k}^n, \quad i = 0, \dots, q_n - 1, \; k = 0, \dots, s_n - 1\}$.

Remark 3.15. Note that under the transformation ϕ_n, the elements of $\mathcal{A}_n$ map to the elements of $\mathcal{B}_n$. In particular, by (3.8), we get $\phi_n^m(A_{i,k}^n) = B_{i,k}^n$ for all i, k as defined above. Since $R_{n,i}$ lies inside Σ_1 and the maps ϕ_n^w, ϕ_n^g act as an identity on Σ_1. Therefore $\phi_n(A_{i,k}^n) = B_{i,k}^n$.

Lemma 3.16. *Let $x \in \mathbb{T}^2$ and $q_{n+1} > s_n q_n^2$ be arbitrary, the orbit $\{S_{\alpha_{n+1}}^{k'}(x)\}_{k'=0}^{q_{n+1}-1}$ intersects every set $P_n^{-1}(A_{i_1,i_2}^n)$.*

Proof. Let fix $x = (x_1, x_2) \in \mathbb{T}^2$, $i_1 \in \{0, \dots, q_n - 1\}$ and $i_2 \in \{0, \dots, s_n - 1\}$. For $z \in \left(0, \frac{\varepsilon_n^{(2)}}{q_n}\right)$, the map P_n acts as the vertical translation on $\mathbb{T}^2$ by a factor $b \cdot z$ and with the choice of κ_n function, $b = \frac{2q_n}{n^2\varepsilon_n^{(2)}}$ (see (3.12)). Note that the map P_n^{-1} induces a net vertical translation on the set A_{i_1,i_2}^n by approximately $b \cdot \gamma$, where $\gamma = \frac{(\epsilon_n^{(2)} - 4\epsilon_n^{(4)})}{s_n q_n}$. Since $[2\varepsilon_n^{(4)}, 1 - 2\varepsilon_n^{(4)}] \subseteq \pi_y(A_{i_1,i_2}^n)$, and $4\varepsilon_n^{(4)} < b \cdot \gamma = \frac{(\varepsilon_n^{(2)} - 4\varepsilon_n^{(4)})}{n^2(\varepsilon_n^{(2)})s_n}$, it satisfy $\pi_y(P_n^{-1}(A_{i_1,i_2}^n)) = \mathbb{T}^1$.

Since $\{k'\alpha_{n+1}\}_{k'=0,1,\dots,q_{n+1}-1}$ is equidistributed on $\mathbb{T}^1$ and $S_{\alpha_{n+1}}$ acts as horizontal translation on $\mathbb{T}^2$, therefore there exists $k' \in \{0, 1, \dots, q_{n+1} - 1\}$ such that $S_{\alpha_{n+1}}^{k'}(x) \in P_n^{-1}(A_{i_1,i_2}^n)$, in other words, there exists $k' \in \{0, 1, \dots, q_{n+1} - 1\}$ such that $x_1 + k'\alpha_{n+1} \in \pi_x(P_n^{-1}(A_{i_1,i_2}^n))$ and $x_2 \in \pi_y(P_n^{-1}(A_{i_1,i_2}^n))$. $\square$

Proposition 3.17. *1. For every $z \in \mathbb{T}^2$, the iterates of z under the map $\phi_n \circ P_n \circ Q_n \circ S_{\alpha_{n+1}}^{k'} \circ H_n^{-1}$, i.e. $\{\phi_n \circ P_n \circ Q_n \circ S_{\alpha_{n+1}}^{k'} \circ H_n^{-1}(z)\}_{k'=0}^{q_{n+1}-1}$ meets every* *set of the form* $\left[\frac{i}{q_n}, \frac{i+1}{q_n}\right] \times \left[\frac{j}{s_n}, \frac{j+1}{s_n}\right]$*, where* $s_n \in \mathbb{N}$ *satisfying* $s_n > n^2 q_n^2$*.*

2. *Suppose the sequence of diffeomorphism* $T_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$ *converges to* $T \in \text{Diff}^\infty(\mathbb{T}^2, \mu)$ *in the* C^∞ *topology and satisfies the proximity condition,* $d_0(T_n^{k'}, T^{k'}) < \frac{1}{2^n}$ $\forall$ $k' = 0, \dots, q_{n+1} - 1$*, then the limiting diffeomorphism* T *is minimal.*

Proof. Let $x \in \mathbb{T}^2$ and $i \in \{0, 1, \dots, q_n - 1\}$ and $j \in \{0, 1, \dots, s_n - 1\}$ be arbitrary. Note that if q_{n+1} is chosen such that $q_{n+1} > s_n q_n^2$ and by above lemma, there exists $k' \in \{0, 1, \dots, q_{n+1} - 1\}$ such that $S_{\alpha_{n+1}}^{k'}(x) \in P_n^{-1}(A_{i,j}^n)$. The map Q_n acts as identity on this region $P_n^{-1}(A_{i,j}^n)$. Thus under the conjugation map, we have

$$\begin{aligned} \phi_n \circ P_n \circ Q_n \circ S_{\alpha_{n+1}}^{k'}(x) &\in \phi_n(A_{i,j}^n) = B_{i,j}^n; \\ B_{i,j}^n &\subset \left[\frac{i}{q_n}, \frac{i+1}{q_n}\right] \times \left[\frac{j}{s_n}, \frac{j+1}{s_n}\right]. \end{aligned} \tag{3.19}$$

It shows that for $x = H_n^{-1}(z)$, the orbit $\{\phi_n \circ P_n \circ Q_n \circ S_{\alpha_{n+1}}^{k'} \circ H_n^{-1}(z)\}_{k'=0,1,\dots,q_{n+1}-1}$ meets every set of type $\left[\frac{i}{q_n}, \frac{i+1}{q_n}\right] \times \left[\frac{j}{s_n}, \frac{j+1}{s_n}\right]$. Also, record that the collection of such sets $\left[\frac{i}{q_n}, \frac{i+1}{q_n}\right] \times \left[\frac{j}{s_n}, \frac{j+1}{s_n}\right]$ for $0 \le i < q_n,\ 0 \le j < s_n$ covers the whole space $\mathbb{T}^2$ and

$$diam\left(H_{n-1} \circ g_n\left(\left[\frac{i}{q_n}, \frac{i+1}{q_n}\right] \times \left[\frac{j}{s_n}, \frac{j+1}{s_n}\right]\right)\right) \le \|DH_{n-1}\|_0 \cdot \|Dg_n\|_0 \cdot \frac{2}{s_n},$$

which goes to 0 as $n \to \infty$ (using $\|DH_{n-1}\|_0 \cdot \|Dg_n\|_0 < q_n^2$ and $s_n > n^2 q_n^2$). Hence, for $\varepsilon > 0$ and $y \in \mathbb{T}^2$ there is $n_1 \in \mathbb{N}$: there exists a set

$$H_{n-1} \circ g_n\left(\left[\frac{i}{q_n}, \frac{i+1}{q_n}\right] \times \left[\frac{j}{s_n}, \frac{j+1}{s_n}\right]\right) \subset B_{\frac{\varepsilon}{2}}(y)\ \forall\ n > n_1.$$

For $H_n = H_{n-1} \circ g_n \circ \phi_n \circ P_n \circ Q_n$, we utilize the convergence condition for the sequence of diffeomorphisms T_n, where $d_0(T_n^{k'}, T^{k'}) < \frac{1}{2^n}$ holds for any $k' \in \mathbb{N}$. Hence, we can conclude that for arbitrary $x, y \in \mathbb{T}^2$ and $\varepsilon > 0$, there exists $n_2 \in \mathbb{N}$ such that $d_0(T_n^{k'}, T^{k'}) < \frac{\varepsilon}{2}$ $\forall$ $k' = 0, \dots, q_n - 1,\ n > n_2$. Assuming $n > \max\{n_1, n_2\}$, there is a set $H_{n-1} \circ g_n\left(\left[\frac{i}{q_n}, \frac{i+1}{q_n}\right] \times \left[\frac{j}{s_n}, \frac{j+1}{s_n}\right]\right) \subset B_{\frac{\varepsilon}{2}}(y)$ and

$T_n^{k'}(x) \in H_{n-1} \circ g_n \left(\left[\frac{i}{q_n}, \frac{i+1}{q_n} \right] \times \left[\frac{j}{s_n}, \frac{j+1}{s_n} \right] \right) \subset B_{\frac{\varepsilon}{2}}(y)$ for some $k' < q_{n+1}$.
With the triangle inequality, we have

$$\begin{aligned} d_0(T^{k'}(x), y) &\leq d_0(T^{k'}(x), T_n^{k'}(x)) + d_0(T_n^{k'}(x), y) \\ &\leq d_0(T^{k'}, T_n^{k'}) + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

i.e. $T^{k'}(x) \in B_\varepsilon(y)$ and which implies T is minimal. $\square$

3.5.4 A Generic Measure

The following results allow us to show the existence of generic points residing inside the region $\mathcal{G}_n = \cup_{i=0}^{q_n-1} B_{n,i}$. We denote $\mathcal{Y}_n = \cup_{i=0}^{q_n-1} Y_{n,i}$ (defined in Section 3.3.1.1, step 2) and $\mathcal{D}_n = \mathbb{T}^2$. First, let us introduce partitions of the sets $\mathcal{G}_n$, $\mathcal{Y}_n$, and $\mathcal{D}_n$ for any sequence of natural numbers $\{s_n\}_{n\in\mathbb{N}}$, where each $s_n > q_n$. These partitions are defined using the families of subsets $G_{i,j}^n$, $Y_{i,j}^n$, and $\Delta_{i,j}^n$, with indices satisfying $0 \leq i < q_n$ and $0 \leq j < s_n$, as follows:

$$\begin{aligned} G_{i,j}^n &:= \left[\frac{i}{q_n} + \frac{2\varepsilon_n^{(2)}}{q_n} + \frac{j(1-4\varepsilon_n^{(2)})}{s_n q_n}, \frac{i}{q_n} + \frac{2\varepsilon_n^{(2)}}{q_n} + \frac{(j+1)(1-4\varepsilon_n^{(2)})}{s_n q_n} \right] \times \left[2\varepsilon_n^{(2)}, \varepsilon_n^{(3)} \right]; \\ Y_{i,j}^n &:= \left[\frac{i}{q_n} + \frac{1-\varepsilon_n^{(3)}}{q_n}, \frac{i}{q_n} + \frac{1-2\varepsilon_n^{(2)}}{q_n} \right] \\ &\quad \times \left[2\varepsilon_n^{(2)} + \frac{j(1-4\varepsilon_n^{(2)})}{s_n}, 2\varepsilon_n^{(2)} + \frac{(j+1)(1-4\varepsilon_n^{(2)})}{s_n} \right]; \\ \Delta_{i,j}^n &:= \left[\frac{i}{q_n}, \frac{i+1}{q_n} \right] \times \left[\frac{j}{s_n}, \frac{j+1}{s_n} \right]. \end{aligned} \tag{3.20}$$

Remark 3.18. For $x \in \mathbb{T}^1 \times (2\varepsilon_n^{(2)}, \varepsilon_n^{(3)})$, since the sequence $\{m\alpha_{n+1}\}_{m\in\mathbb{N}}$ equidistributed over $\mathbb{T}^1$, the orbit of x (say $\mathcal{O}^x$) under the $S_{\alpha_{n+1}}$ equidistributed among the element of $\mathcal{G}_n$. There are at most $(4\varepsilon_n^{(2)} \frac{q_{n+1}}{q_n})$ exceptional points that are trapped inside the error region $\mathbb{E}_n^g$ (see Remark 3.2). Therefore, any element $G_{i,j}^n \in \mathcal{G}_n$ captures at least $\left(1 - 4\varepsilon_n^{(2)}\right) \frac{q_{n+1}}{s_n q_n}$ points of the orbit $\mathcal{O}^x$.

Remark 3.19. Note that under the transformation ϕ_n, the elements of $\mathcal{G}_n$ map to the elements of $\mathcal{Y}_n$. In particular, $\phi_n^g(G_{i,j}^n) = Y_{i,s_n-j}^n$ and conversely, $\phi_n^g(Y_{i,j}^n) = G_{i,s_n-j}^n$ for all i, j. By construction, the maps ϕ_n^w, ϕ_n^m and P_n act as an identity on the set $\mathcal{G}_n$.

Proposition 3.20. *For $\epsilon > 0$ and $r \in \mathbb{N}$, consider a $(\frac{\sqrt{2}}{q_n}, \epsilon)$-uniformly continuous function $\psi : \mathbb{T}^2 \longrightarrow \mathbb{R}$, i.e. $\psi(B_{\frac{\sqrt{2}}{q_n}}(x)) \subset B_\epsilon(\psi(x))$. The point $x \in \mathbb{T}^1 \times \{0\}$ satisfy the following estimate:*

$$\left| \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \psi(\phi_n \circ P_n \circ Q_n \circ S_{\alpha_{n+1}}^i(x)) - \int_{\mathbb{T}^2} \psi d\mu \right| \leq 4\epsilon + \frac{2}{nr} \|\psi\|_0.$$

Proof. Fix $x \in \mathbb{T}^1 \times \{0\}$. Since the orbit of x under the $Q_n \circ S_{\alpha_{n+1}}$ is being almost trapped inside the elements of $\mathcal{G}_n$, therefore there exists a $i_0 \in \mathbb{N}$ such that $Q_n \circ S_{\alpha_{n+1}}^{i_0}(x) \in G_{i,s_n-j}^n$ for some $i, j \in \mathbb{N}$. Under the action of ϕ_n and by Remark 3.19 and equation (3.5), we have $\phi_n \circ P_n \circ Q_n \circ S_{\alpha_{n+1}}^{i_o}(x) \in Y_{i,j}^n \subset \Delta_{i,j}^n$. Therefore for any $y \in \Delta_{i,j}^n$, we have

$$d(\phi_n \circ P_n \circ Q_n \circ S_{\alpha_{n+1}}^{i_o}(x), y) \leq diam(\phi_n \circ P_n \circ Q_n \circ S_{\alpha_{n+1}}^{i_o}(x), y) \leq \frac{\sqrt{2}}{q_n}.$$

Using the hypothesis on ψ, we have $|\psi(\phi_n \circ P_n \circ Q_n \circ S_{\alpha_{n+1}}^{i_o}(x)) - \psi(y)| < 2\epsilon$. Take the average for all $y \in \Delta_{i,j}^n$ in the above equation, we get

$$|\psi(\phi_n \circ P_n \circ Q_n \circ S_{\alpha_{n+1}}^{i_o}(x)) - \frac{1}{\mu(\Delta_{i,j}^n)} \int_{\Delta_{i,j}^n} \psi(y) d\mu| < 2\epsilon.$$

Let us denote $J_\Delta = \{k \in 0, 1, \ldots, q_{n+1} - 1 \ : \phi_n \circ P_n \circ Q_n \circ S_{\alpha_{n+1}}^k(x) \in \Delta\}$ for all $\Delta \in \mathcal{D}_n$. By Remark 3.18, we have $|J_\Delta| > \left(1 - \frac{2}{nr}\right) \frac{q_{n+1}}{s_n q_n}$ (use $4\varepsilon_n^{(2)} < \frac{2}{nr}$). Now using the count on $|J_\Delta|$ and triangle inequality in the above equation, we get

$$\begin{aligned}\left| \frac{1}{q_{n+1}} \sum_{i \in J_\Delta} \psi(\phi_n \circ P_n \circ Q_n \circ S_{\alpha_{n+1}}^i(x)) - \int_{\Delta_{i,j}^n} \psi d\mu \right| \\ < 2\epsilon\mu(\Delta_{i,j}^n) + \frac{2}{nr}(\|\psi\|_0 + 2\epsilon)\mu(\Delta_{i,j}^n)\end{aligned}$$

$$< \left(4\epsilon + \frac{2}{nr}\|\psi\|_0\right)\mu(\Delta^n_{i,j}).$$

Since the last inequality holds for arbitrary $\Delta \in \mathcal{D}_n$, therefore, we conclude

$$\begin{aligned}
&\left|\frac{1}{q_{n+1}}\sum_{i=0}^{q_{n+1}-1}\psi(\phi_n \circ P_n \circ Q_n \circ S^i_{\alpha_{n+1}}(x)) - \int_{\mathbb{T}^2}\psi d\mu\right| \\
&\leq \left|\sum_{\Delta\in\mathcal{D}_n}\left(\frac{1}{q_{n+1}}\sum_{i\in J_\Delta}\psi(\phi_n \circ P_n \circ Q_n \circ S^i_{\alpha_{n+1}}(x) - \int_\Delta \rho d\mu\right)\right| + \frac{q_n}{q_{n+1}}||\psi||_0 \\
&\leq 4\epsilon + \frac{2}{nr}\|\psi\|_0.
\end{aligned}$$

□

3.5.5 Existence of invariant measures ξ^t

For the existence of the T-invariant measure, the proof follows the same line of reasoning as in [4, Section 5.4]. Here, we construct the T-invariant measures using the normalized measure μ_t of the Lebesgue measure on the sets N^t, as defined in Section 3.4. First, we define the sequence of measures $\xi^n_t := (H_n)_*\mu_t$ and prove their T_n-invariance: $(T_n)_*\xi^n_t = (T_n)_*(H_n)_*\mu_t = (T_n \circ H_n)_*\mu_t = (H_n \circ S_{\alpha_{n+1}})_*\mu_t = (H_n)_*(S_{\alpha_{n+1}})_*\mu_t = \xi^n_t$.

Next, we estimate $\mu(H_{n+1}(N^t)\triangle H_n(N^t))$. To do so, we analyze which parts of the set N^t are mapped back to N^t under the transformation $h_{n+1} = g_{n+1} \circ \phi_{n+1}$. The measure difference arises from the regions where h_{n+1} is specifically constructed to ensure minimality and genericity. The map g_{n+1} acts as a horizontal translation, and the subset $\left(\left[\frac{k}{q_n} + \frac{2\varepsilon^{(3)}_n}{q_n}, \frac{k+1}{q_n} - \frac{2\varepsilon^{(3)}_n}{q_n}\right] \times [2\varepsilon^{(3)}_n, 1 - 2\varepsilon^{(3)}_n]\right) \bigcap N^t$, for $k = 0, \ldots, q_n - 1$, represents the portion that is mapped back to N^t under ϕ_{n+1}. Thus, combining these estimates, we obtain

$$\mu(H_{n+1}(N^t)\triangle H_n(N^t)) \leq 8\varepsilon^{(3)}_n \leq \frac{1}{2^n}.$$

Following the proof of [4, Section 4], the above estimate is sufficient to conclude the existence of a limit $B_t := \lim_{n\to\infty} H_n(N^t)$, and the weak convergence of the measures ξ_t^n to a measure ξ_t, given by $\xi_t(A) = \frac{\mu(A\cap B_t)}{\mu(B_t)}$ for any measurable set $A \subset \mathbb{T}$. The measures ξ_t are T-invariant; for proof, refer to [4, Section 4].

Proof of Theorem 1.1. We will construct a minimal map $T \in \mathrm{Diff}^\infty(\mathbb{T}^2, \mu)$, obtained by (3.5), (3.1), (3.3), and (3.10), for any Liouville α satisfying (3.13). The map will have r distinct weak mixing invariant measures ξ_t and the Lebesgue measure μ as a generic measure.

Let us fix a countable set of Lipshitz functions $\Psi = \{\psi_i\}_{i\in\mathbb{N}}$, which is dense in $C^0(\mathbb{T}^2, \mathbb{R})$. Let L_n be a uniform Lipshitz constant for $\psi_1, \psi_2, \ldots, \psi_n$. Choose $q_{n+1} = l_n k_n q_n^2$ large enough by selecting l_n arbitrarily large enough such that it satisfies: $l_n > n^2 \cdot \max_{1\le i\le n} L_i \cdot \|DH_{n-1}\|_{n-1} \cdot \|Dg_n\|_0$. This assumption implies that $\psi_1 H_{n-1} g_n, \psi_2 H_{n-1} g_n, \ldots, \psi_n H_{n-1} g_n$ are $(\frac{\sqrt{2}}{q_n}, \frac{2}{nr})$- uniformly continuous.
Claim 1: *The point $x = (0,0)$ is a generic point for the Lebesgue measure μ.*
Using the fact that ϕ_n, P_n, and Q_n are measure-preserving and act as the identity on the point $x = (0,0)$, and that the maps g_n act as horizontal translations on $\mathbb{T}^2$ with $g_n(0,0) = (0,0)$, it follows that $h_n(x) = x$ for all n, and therefore $H_n^{-1}(x) = x$. Applying Proposition 3.20 with $\epsilon = \frac{2}{nr}$ for $1 \le k \le n$ and for $x \in \mathbb{T}^1 \times \{0\}$, we get:

$$\left| \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \psi_k(H_n S^i_{\alpha_{n+1}} x) - \int_{\mathbb{T}^2} \psi_k H_n d\mu \right| < \frac{2}{nr}\|\psi_k\|_0 + \frac{8}{nr}.$$

By applying relation (3.1), it follows that for each $\psi_k \in \Psi$:

$$\left| \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \psi_k(T_n^i x) - \int_{\mathbb{T}^2} \psi_k d\mu \right| < \frac{2}{nr}\|\psi_k\|_0 + \frac{8}{nr}.$$

Further, using the triangle inequality and the uniform approximation bound, i.e. $d_0(T_n^i, T^i) \le \frac{1}{2^n}$ for all $i \in \mathbb{N}$, we obtain:

$$\left| \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \psi_k(T^i x) - \int_{\mathbb{T}^2} \psi_k d\mu \right| < \frac{2}{nr}\|\psi_k\|_0 + \frac{8}{nr} + \frac{1}{2^n}. \tag{3.21}$$

For any large $N \in \mathbb{N}$, suppose it lies in $q_j < N < q_{j+1}$ for some j. There exists an integer $m \in \mathbb{N}$ such that $N = mq_{j-1} + r$, where $0 < r < q_{j-1}$,

$$\frac{1}{N}\sum_{i=0}^{N-1}\psi_k(T_n^i x) = \frac{mq_{j-1}}{N}\left\{\frac{1}{mq_{j-1}}\sum_{i=0}^{mq_{j-1}-1}\psi_k(T_n^i x)\right\} + \frac{1}{N}\sum_{i=mq_{j-1}}^{N-1}\psi_k(T_n^i x).$$

We can write

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{i=0}^{N-1}\psi_k(T^i x) - \int_{\mathbb{T}^2}\psi_k d\mu\right| \\
&\leq \left|\frac{mq_{j-1}}{N}\left\{\frac{1}{mq_{j-1}}\sum_{i=0}^{mq_{j-1}-1}\psi_k(T^i x)\right\} - \int_{\mathbb{T}^2}\psi_k d\mu\right| + \left|\frac{1}{N}\sum_{i=mq_{j-1}}^{N-1}\psi_k(T^i x)\right| \\
&\leq \frac{mq_{j-1}}{N}\left|\frac{1}{mq_{j-1}}\sum_{i=0}^{mq_{j-1}-1}\psi_k(T^i x) - \frac{N}{mq_{j-1}}\int_{\mathbb{T}^2}\psi_k d\mu\right| + \frac{r}{N}\|\psi_k\|_0 \\
&\leq \frac{mq_{j-1}}{N}\left|\frac{1}{mq_{j-1}}\sum_{i=0}^{mq_{j-1}-1}\psi_k(T^i x) - \int_{\mathbb{T}^2}\psi_k d\mu\right| + \left|1 - \frac{N}{mq_{j-1}}\right|\left|\int_{\mathbb{T}^2}\psi_k d\mu\right| + \frac{r}{N}\|\psi_k\|_0 \\
&\leq \frac{mq_{j-1}}{N}\left(\frac{2}{(j-2)r}\|\psi_k\|_0 + \frac{8}{(j-2)r} + \frac{1}{2^{j-2}}\right) + \left|1 - \frac{N}{mq_{j-1}}\right|\|\psi_k\|_0 + \frac{r}{N}\|\psi_k\|_0.
\end{aligned}$$

As $N \to \infty$, we have $j \to \infty$, $\frac{mq_{j-1}}{N} \to 1$ and $\frac{r}{N} \to 0$. Thus,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1}\psi_k(T^i x) \to \int_{\mathbb{T}^2}\psi_k d\mu,$$

and x is a generic point for μ.

In order to prove that the map T is weakly mixing with respect to the invariant measure ξ_t, for $t = 0, \ldots, r-1$, defined in Subsection 3.5.5, we will apply Proposition 3.13. For this purpose, consider the sequence $(\mathfrak{m}_n)$ and the decomposition η_n^t described in Sections (3.5.2.2)-(3.5.2.3). It suffices to show that $\eta_n^t \to \varepsilon$, and that the diffeomorphism $\Phi_n(I_n) = \phi_n \circ P_n \circ Q_n \circ S_{\alpha_{n+1}}^{\mathfrak{m}_n} \circ Q_n^{-1} \circ P_n^{-1} \circ \phi_n^{-1}(I_n)$ is $(0, 2/3q_n, 0)$-distributed for any $I_n \in \eta_n$.

Clearly, $\eta_n \to \varepsilon$, since η_n consists of all intervals of each length less than $1/q_n$. By Lemma 3.14, for any $I_n \in \eta_n^t$, $J = \pi_y(\Phi_n(I_n)) = \left[\frac{t}{r} + \frac{2}{3nr}, \frac{t+1}{r} - \frac{2}{3nr}\right]$, and

$\Phi_n(I_n)$ is a vertical interval. Hence, we take $\delta = 2/3n$ and $\gamma = 0$. Finally, since the restriction of $\Phi_n(I_n)$ is an affine map, it verifies the condition for $\varepsilon = 0$. Therefore, the map T is weak mixing with respect to the measures ξ_t for $(t = 0, \ldots, r-1)$. For a more detailed proof, one can refer to [19]. The map T is minimal and has been proved in Proposition 3.17, which completes the proof. $\square$

Remark 3.21. The measure $\mu = \frac{1}{r}(\mu_0 + \mu_1 + \ldots + \mu_{r-1})$ is a nonergodic Lebesgue measure but a generic measure on $\mathbb{T}^2$.

4

The Set of Generic Points with Non-Trivial Hausdorff Upper Bound

4.1 Introduction

In this chapter, we delve into the analysis of generic and non-generic points within the smooth setting, aiming to establish bounds on the size of the sets they form. Instead of quantifying the set of generic points using the Lebesgue measure, we focus on determining bounds on their Hausdorff dimension, which leads to more insightful and interesting results. We explore various examples of measure-preserving smooth diffeomorphisms, each associated with a set containing all the generic points with zero measure and exhibiting non-trivial bounds on their Hausdorff dimension. To construct such sets and investigate their properties, we employ the Approximation by Conjugation method.

To facilitate this construction, we explicitly introduce certain narrow sets of small measure, most notably, horizontal strips of small width within a carefully designed framework on the 2-Torus. We establish a highly explicit setup with distinct targeted domains on the 2-Torus, each serving specific purposes, along with the explicit selection of conjugation maps. Additionally, we extend our analysis to capture all non-generic points of such a system as well. In particular, we provide proofs for Theorems 1.3.

4.2 Outline of the proof

To prove Theorem 1.3, we construct a $T \in \mathrm{Diff}^\infty(\mathbb{T}^2, \mu)$ using the approximation by conjugation method, as discussed in Chapter 3. However, we will modify the combinatorics in the setup to achieve the desired result. We introduce two parts within $\mathbb{T}^2$, G_n and NG_n, specifically designed to achieve the two dynamic properties explicitly, along with explicitly chosen conjugation maps h_n at each induction step. We define the combinatorics such that the limiting set B consists of all the generic points of the system, and the limiting set NB contains all the non-generic points, as outlined in Section 4.3 utilizing the explicit conjugation maps h_n and partial partitions G_n and NG_n. Finally, the convergence of the scheme in the smooth topology is achieved under very precise norm estimates in Section 4.4.

4.3 Construction of the Generic sets

4.3.1 Explicit set-up

We begin by introducing a decomposition of the two-dimensional torus $\mathbb{T}^2$ into a pair of disjoint measurable sets: $\mathbb{T}^2 = \mathrm{G}_n \cup \mathrm{NG}_n$. This decomposition depends on a parameter $m > 0$, determined by a real number $s \in (1, 2)$ via the relation $m = \frac{1}{s-1} - 1$. Given a sequence $\{q_n\}_{n\in\mathbb{N}}$ of positive integers and the associated scaling parameter $\lambda_{n,m} := \frac{1}{q_n^m}$, we define the horizontal strip

$$\mathrm{G}_n := \mathbb{T}^1 \times I^{n,m}, \quad \text{where} \quad I^{n,m} := \left[\frac{1}{2} - \lambda_{n,m}, \frac{1}{2} + \lambda_{n,m}\right], \tag{4.1}$$

and set $\mathrm{NG}_n := \mathbb{T}^2 \setminus \mathrm{G}_n$.

On the set G_n, we consider a smooth conjugation map $\bar{\phi}_n$ whose action is confined to the strip $\mathcal{I}^{n,m} := \mathbb{T}^1 \times I^{n,m}$. The map $\bar{\phi}_n$ is constructed so as to redistribute this thin strip vertically across the torus in a periodic fashion. This spreading ensures the presence of generic points whose orbits become uniformly distributed with respect to the Lebesgue measure on $\mathbb{T}^2$. However, the Lebesgue measure of this set vanishes, as the strip width $2\lambda_{n,m}$ tends to zero with increasing n. In contrast, on the complement NG_n, the map $\bar{\phi}_n$ acts on the two complementary regions $\mathcal{J}^0 := \mathbb{T}^1 \times \left[0, \frac{1}{2} - \lambda_{n,m}\right)$, $\mathcal{J}^1 := \mathbb{T}^1 \times \left[\frac{1}{2} + \lambda_{n,m}, 1\right)$, and redistributes them in such a way that their vertical support remains confined within the intervals $\mathbb{T}^1 \times \left(\frac{t}{2}, \frac{t+1}{2}\right)$, for $t = 0$ or $t = 1$ respectively. As a consequence, points in NG_n exhibit non-generic behavior, as their orbits fail to become uniformly distributed over $\mathbb{T}^2$.

Finally, let $\{q_n\}_{n\in\mathbb{N}}$ and $\{s_n\}_{n\in\mathbb{N}}$ be sequences of natural numbers satisfying the growth conditions: $s_n \geq q_n$ and $q_n > 3^n$ for all $n \in \mathbb{N}$. Under these assumptions, we introduce the following partition of $\mathbb{T}^2$, which will play a central role in the subsequent construction:

$$\mathrm{G}_n := \left\{\mathcal{I}_{i_1}^{n,m} = \left[\frac{i_1}{s_n q_n}, \frac{i_1+1}{s_n q_n}\right) \times I^{n,m} \ : 0 \leq i_1 < s_n q_n\right\};$$

$$\mathrm{NG}_n := \left\{\begin{aligned} &\mathcal{J}_{i_1}^{n,1} = \left[\tfrac{i_1}{s_n q_n}, \tfrac{i_1+1}{s_n q_n}\right) \times \left[0, \tfrac{1}{2} - \lambda_{n,m}\right) : \ 0 \leq i_1 < s_n q_n; \\ &\mathcal{J}_{i_1}^{n,1} = \left[\tfrac{i_1}{s_n q_n}, \tfrac{i_1+1}{s_n q_n}\right) \times \left[\tfrac{1}{2} + \lambda_{n,m}, 1\right) : 0 \leq i_1 < s_n q_n \end{aligned}\right\};$$

$$\mathrm{V}_n := \left\{\mathcal{V}_{i_1,i_2}^{n,m} = \left[\tfrac{i_1}{q_n}, \tfrac{i_1}{q_n} + \tfrac{2\lambda_{n,m}}{q_n}\right) \times \left[\tfrac{i_2}{s_n}, \tfrac{i_2+1}{s_n}\right) \ : \ 0 \leq i_1 < q_n, 0 \leq i_2 < s_n; \right\};$$

$$\mathrm{W}_n := \left\{\begin{aligned} &\mathcal{W}_{i_1,i_2}^{n,t} = \left[\tfrac{i_1}{q_n} + \tfrac{2\lambda_{n,m}}{q_n}, \tfrac{i_1+1}{q_n}\right) \times \left[\tfrac{t}{2} + \tfrac{i_2}{2s_n}, \tfrac{t}{2} + \tfrac{i_2+1}{2s_n}\right) : \ t = 0, 1; \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq i_1 < q_n, 0 \leq i_2 < s_n. \end{aligned}\right.$$

4.3.2 The conjugation map h_n

Now we define the following permutation maps $\widetilde{\phi}_n : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ of the above partition $\mathrm{G}_n \cup \mathrm{NG}_n$ which maps to the elements of partition $\mathrm{V}_n \cup \mathrm{W}_n$ [1]. Consider the map $\widetilde{\phi}_n : \left[\frac{i}{q_n}, \frac{i+1}{q_n}\right) \times \mathbb{T}^1 \longrightarrow \left[\frac{i}{q_n}, \frac{i+1}{q_n}\right) \times \mathbb{T}^1$ as following and extend it to the whole $\mathbb{T}^2$ as $\frac{1}{q_n}$-equivariantly.

$$\widetilde{\phi}_n(\mathcal{I}^{n,m}_{i_1}) = \mathcal{V}^{n,m}_{j_1,j_2} \quad \text{where} \quad j_1 = \left\lfloor \frac{i_1}{s_n} \right\rfloor, j_2 = i_1 \mod s_n, \tag{4.2}$$

$$\widetilde{\phi}_n(\mathcal{J}^{n,t}_{i'_1}) = \mathcal{W}^{n,t}_{j'_1,j'_2} \quad \text{where} \quad j'_1 = \left\lfloor \frac{i'_1}{s_n} \right\rfloor, j'_2 = t.s_n + i'_1 \mod s_n, \ t = 0, 1. \tag{4.3}$$

The map $\widetilde{\phi}_n$ effectively rearranges elements of equal size on $\mathbb{T}^2$, and its visualization can be further understood by observing the corresponding rectangles, as shown below: For the case $n \geq 1$,

$$\widetilde{\phi}_n\left(\left[\frac{i}{q_n}, \frac{i+1}{q_n}\right) \times \left[\frac{1}{2} - \lambda_{n,m}, \frac{1}{2} + \lambda_{n,m}\right)\right) = \left[\frac{i}{q_n}, \frac{i}{q_n} + \frac{2\lambda_{n,m}}{q_n}\right) \times \mathbb{T}^1; \tag{4.4}$$

$$\widetilde{\phi}_n\left(\left[\frac{i}{q_n}, \frac{i+1}{q_n}\right) \times \left[0, \frac{1}{2} - \lambda_{n,m}\right)\right) = \left[\frac{i}{q_n} + \frac{2\lambda_{n,m}}{q_n}, \frac{i+1}{q_n}\right) \times \left[0, \frac{1}{2}\right); \tag{4.5}$$

$$\widetilde{\phi}_n\left(\left[\frac{i}{q_n}, \frac{i+1}{q_n}\right) \times \left[\frac{1}{2} + \lambda_{n,m}, 1\right)\right) = \left[\frac{i}{q_n} + \frac{2\lambda_{n,m}}{q_n}, \frac{i+1}{q_n}\right) \times \left[\frac{1}{2}, 1\right). \tag{4.6}$$

We extend the map $\widetilde{\phi}_n : \left[0, \frac{1}{q_n}\right) \times \mathbb{T}^1 \to \left[0, \frac{1}{q_n}\right) \times \mathbb{T}^1$, $\frac{1}{q_n}$-equivariantly to define a piecewise transformation on the entire torus $\mathbb{T}^2$, preserving the same combinatorial structure across horizontal strips. This extended map can then be approximated by a smooth measure-preserving diffeomorphism that agrees with the combinatorics of $\widetilde{\phi}_n$ up to a small error set $E_n \subset \mathbb{T}^2$, as described in Subsection 4.4.

Such an approximation is ensured by Theorem 2.5, which states that any piecewise permutation map on the torus can be approximated arbitrarily well by a smooth diffeomorphism that coincides with the permutation map outside a small exceptional set E_n, and acts as the identity near the boundary of the unit square

[1] We refer to the map $\tilde{\phi}_n$ as a permutation map because it rearranges the elements of the partition.

$[0,1]\times[0,1]$. We denote by $\overline{\phi}_n$ the resulting smooth measure-preserving diffeomorphism approximating the action of $\widetilde{\phi}_n$ on $\mathbb{T}^2$.
Here we define our final conjugation diffeomorphism as

$$h_n = \overline{\phi}_n, \tag{4.7}$$

where, $\overline{\phi}_n$ denotes a smooth approximation of the map $\widetilde{\phi}_n$. By construction, the map $\overline{\phi}_n$ commutes with the rotation S_{α_n}; that is, $\overline{\phi}_n \circ S_{\alpha_n} = S_{\alpha_n} \circ \overline{\phi}_n$.

4.4 Convergence and Estimates

To exclude regions where we do not have sufficient control over the combinatorics, we consider a subset $E_n = E_{v,n} \cup E_{h,n}$, where $E_{v,n}$ and $E_{h,n}$ denote the vertical and horizontal error sets, respectively. Each of these is further decomposed as follows: $E_{v,n} = E^{(1)}_{v,n} \cup E^{(2)}_{v,n}$ and $E_{h,n} = E^{(1)}_{h,n} \cup E^{(2)}_{h,n}$, where the sets $E^{(i)}_{v,n}$ and $E^{(i)}_{h,n}$ for $i = 1, 2$, are subsets of the 2-torus $\mathbb{T}^2$ defined by:

$$\begin{aligned}
E^{(1)}_{v,n} &= \bigcup_{i=0}^{s_nq_n-1} \left[\frac{i}{s_nq_n} - \frac{\epsilon'_n}{2s_nq_n}, \frac{i}{s_nq_n} + \frac{\epsilon'_n}{2s_nq_n}\right] \times \mathbb{T}^1;\\
E^{(2)}_{v,n} &= \bigcup_{i=0}^{q_n-1} \left[\frac{i}{q_n} + \frac{2\lambda_{n,m}}{q_n} - \frac{\epsilon'_n}{2q_n}, \frac{i}{q_n} + \frac{2\lambda_{n,m}}{q_n} + \frac{\epsilon'_n}{2q_n}\right] \times \mathbb{T}^1;\\
E^{(1)}_{h,n} &= \mathbb{T}^1 \times \left[\frac{1}{2} - \lambda_{n,m} - \frac{\epsilon'_n}{2}, \frac{1}{2} - \lambda_{n,m} + \frac{\epsilon'_n}{2}\right] \cup \left[\frac{1}{2} + \lambda_{n,m} - \frac{\epsilon'_n}{2}, \frac{1}{2} + \lambda_{n,m} + \frac{\epsilon'_n}{2}\right];\\
E^{(2)}_{h,n} &= \mathbb{T}^1 \times \bigcup_{j=1}^{2s_n-1} \left[\frac{j}{2s_n} - \frac{\epsilon'_n}{4s_n}, \frac{j}{2s_n} + \frac{\epsilon'_n}{4s_n}\right].
\end{aligned}$$

We choose $\epsilon'_n := \left(\frac{1}{2^n s_n^2 q_n \|DH_{n-1}\|_1^2}\right)^{2^n}$, so that the measure of the exceptional set satisfies $\mu(E_n) = 10 s_n \epsilon'_n < \frac{1}{e^{3^n}}$. We define the complementary sets

$$F_n := \mathbb{T}^2 \setminus E_n \quad \text{and} \quad F^{(1)}_{h,n} := \mathbb{T}^2 \setminus E^{(1)}_{h,n}, \tag{4.8}$$

such that $F_n \subseteq F^{(1)}_{h,n}$, and the measure of F_n satisfies $\mu(F_n) > 1 - \frac{1}{e^{3n}}$.
To analyze the distribution of generic and non-generic points, we now introduce the following families of subsets of the torus. These will be referred to as the "trapping generic zones" and "trapping non-generic zones," respectively. For each $i_1 = 0, \dots, q_n s_n - 1$, we define

$$\mathcal{X}^{n,m}_{i_1} := \mathcal{I}^{n,m}_{i_1} \cap (\mathbb{T}^1 \setminus E^{(1)}_{h,n} \cup E^{(1)}_{v,n}) \quad \text{and} \quad \mathcal{Y}^{n,k}_{i_1} := \mathcal{J}^{n,k}_{i_1} \cap F_n, \quad k = 0, 1. \tag{4.9}$$

Remark 4.1. For any $x \in \mathcal{I}^{n,m}_{i_1} \cap F^{(1)}_{h,n}$, after removing the error set $E^{(1)}_{v,n}$, the number of iterates of the orbit $S^i_{\alpha_{n+1}}(x)$ that lie in each set $\mathcal{X}^{n,m}_{i_1}$ is at least $\left(1 - \frac{2}{n^2}\right) \frac{q_{n+1}}{s_n q_n}$. Similarly, for any $x \in \mathcal{J}^{n,k}_{i_1} \cap F_n$, the number of iterates of the orbit $S^i_{\alpha_{n+1}}(x)$ that lie in each set $\mathcal{Y}^{n,k}_{i_1}$ is at least $\left(1 - \frac{2}{n^2}\right) \frac{q_{n+1}}{s_n q_n}$.

Remark 4.2. Recall that the image of $\mathcal{X}^{n,m}_{i_1}$, under the conjugation map h_n, contained inside $\mathcal{V}^{n,m}_{\lfloor \frac{i_1}{s_n} \rfloor, i_1 \mod s_n}$ and conversely, $\mathcal{V}^{n,m}_{i_1,i_2}$ is uniquely mapped onto $\mathcal{X}^{n,m}_{i_1.s_n}$. Thus, the number of iterates $k \in \{0, 1, \dots, q_{n+1} - 1\}$ such that $h_n \circ S^k_{\alpha_{n+1}}(x) \in \mathcal{V}^{n,m}_{i_1,i_2}$ for $x \in \mathcal{X}^{n,m}_{i_1}$ is at least $\left(1 - \frac{2}{n^2}\right) \frac{q_{n+1}}{s_n q_n}$.

Remark 4.3. Note that under the action of h_n, every element from $\mathrm{NG}_n \cap F^{(1)}_{h,n}$ transforms as follows:

$$h_n \left(\bigcup_{i_1=0}^{s_n q_n - 1} \mathcal{Y}^{n,k}_{i_1} \right) = \bigcup_{i_1=0}^{s_n q_n - 1} \overline{\phi}_n(\mathcal{J}^{n,k}_{i_1} \cap F_n) \subseteq \mathbb{T}^1 \times \left[\frac{k}{2}, \frac{k+1}{2} \right), \ k = 0, 1.$$

4.4.1 Generic points estimates

Proposition 4.4. *For $\epsilon > 0$, consider a $(\frac{\sqrt{2}}{q_n}, \epsilon)$-uniformly continuous function $\psi : \mathbb{T}^2 \longrightarrow \mathbb{R}$, i.e. $\psi(B_{\frac{\sqrt{2}}{q_n}}(x)) \subset B_\epsilon(\psi(x))$. Then, for any $x \in \mathrm{G}_n \cap F^{(1)}_{h,n}$, we have the following estimate:*

$$\left| \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \psi(h_n \circ S^i_{\alpha_{n+1}}(x)) - \int_{\mathbb{T}^2} \psi d\mu \right| \leq 4\epsilon + \frac{2}{n^2} \|\psi\|_0. \tag{4.10}$$

Proof. For any $x \in \mathrm{G}_n \cap F^{(1)}_{h,n}$ and $\Delta^n_{i_1,i_2} \in \Delta^n_{i,j}$ (see equation (3.20)). Precisely, $x \in \mathbb{T}^1 \times I^{n,m}$. Since the orbit of x under the $S^k_{\alpha_{n+1}}$ is almost trapped by the domains $\{\mathcal{X}_{t_1}^{n,m}\}$, therefore there exists a $i_0 \in \mathbb{N}$ such that $S_{\alpha_{n+1}}^{i_0}(x) \in \mathcal{X}_{i_1.s_n+i_2}^{n,m}$. With the action of h_n and by Remark 4.2, we have $h_n \circ S_{\alpha_{n+1}}^{i_o}(x) \in \mathcal{V}_{i_1,i_2}^{n,m} \subset \Delta_{i_1,i_2}^n$. Therefore for any $y \in \Delta_{i,j}^n$, we conclude

$$d(h_n \circ S_{\alpha_{n+1}}^{i_o}(x), y) \leq diam(h_n \circ S_{\alpha_{n+1}}^{i_o}(x), y) \leq \sqrt{2}/q_n.$$

Now using the hypothesis on ψ, we have $|\psi(h_n \circ S_{\alpha_{n+1}}^{i_o}(x)) - \psi(y)| < 2\epsilon$. Take the average for all $y \in \Delta_{i,j}^n$ in the last equation, we get

$$|\psi(h_n \circ S_{\alpha_{n+1}}^{i_o}(x)) - \frac{1}{\mu(\Delta_{i,j}^n)} \int_{\Delta_{i,j}^n} \psi(y) d\mu| < 2\epsilon.$$

Let us denote $J_\Delta = \{k \in 0, 1, \ldots, q_{n+1} - 1 \ : h_n \circ S_{\alpha_{n+1}}^k(x) \in \Delta\}$ for all $\Delta \in \mathcal{D}_n$, where $\mathcal{D}_n$ defined by (3.20). Using the count estimate described in Remark 4.2 and triangle inequality in the last equation, we have

$$\left| \frac{1}{q_{n+1}} \sum_{i \in J_\Delta} \psi(h_n \circ S_{\alpha_{n+1}}^i(x)) - \int_{\Delta_{i,j}^n} \psi d\mu \right| < (4\epsilon + \frac{2}{n^2}\|\psi\|_0)\mu(\Delta_{i,j}^n).$$

Further, we follow the analogous estimation as done in Proposition 3.20, and we have the estimate (4.10) as required. □

4.4.2 Non-Generic points estimates

We begin by partitioning the torus $\mathbb{T}^2$ into two regions:

$$\mathrm{D}^1 := \mathbb{T}^1 \times \left[0, \tfrac{1}{2}\right), \quad \mathrm{D}^2 := \mathbb{T}^1 \times \left[\tfrac{1}{2}, 1\right).$$

Our goal is to study how often the composition $H_n := h_1 \circ h_2 \circ \cdots \circ h_n$, evaluated along the orbit $S_{\alpha_{n+1}}^i(x)$, lands in the region D^1.

Proposition 4.5. *For any $x \in \mathrm{NG}_n \cap F_{h,n}^{(1)} \cap \mathrm{D}^1, \lambda_{n,m} = \frac{1}{q_n^m}$ for $m > 0$, and $N \geq q_{n+1}$. The set $J_N^{(n)}(x) \subset \{0, 1, \ldots, N-1\}$, be defined by*

$$J_N^{(n)}(x) := \left\{i \in \{0, 1, \ldots, N-1\} : h_1 \circ h_2 \circ \cdots \circ h_n(S_{\alpha_{n+1}}^i x) \in \mathrm{D}^1\right\}.$$

Then, the following lower bound holds:

$$|J_N^{(n)}(x)| \geq \left(1 - \frac{1}{e^{3^2}}\right) \left(\prod_{j=2}^{n-1} \left(1 - (1-2\lambda_{j,m}) \cdot \lambda_{j-1,m} - \frac{1}{e^{3^j}} - \frac{1}{e^{3(j-1)}}\right)\right)$$
$$\left(1 - \frac{2}{n^2}\right) \left(1 - \lambda_{n,m} - \frac{1}{2 \cdot e^{3(n-1)}}\right) N.$$

Moreover, for all sufficiently large n, there exists a constant $c > 0.6$ such that $|J_N^{(n)}(x)| \geq cN$.

Proof. To estimate a lower bound for $|J_N^{(n)}(x)|$, we construct nested subsets $A_j \subset \{0, 1, \ldots, N-1\}$ as follows: At level $j = n$, define

$$A_n := \left\{i \in \{0, 1, \ldots, N-1\} : h_n(S^i_{\alpha_{n+1}} x) \in \mathrm{NG}_{n-1} \cap F_{n-1} \cap \mathrm{D}^1\right\}.$$

For $j = n-1, \ldots, 2$, define inductively

$$A_j := \left\{i \in A_{j+1} : h_j \circ h_{j+1} \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x) \in \mathrm{NG}_{j-1} \cap F_{j-1} \cap \mathrm{D}^1\right\};$$
$$A_1 := \left\{i \in A_2 : h_1 \circ h_2 \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x) \in \mathrm{D}^1\right\}.$$

Here, $F_{j-1} = \mathbb{T}^2 \setminus E_{j-1}$, where E_{j-1} denotes the error set at the jth stage (see subsection 4.4). Suppose that at each level $j = n, n-1, \ldots, 2, 1$, the proportion of iterates satisfying the required condition (as defined by the sets A_j) is denoted by $\alpha_j \in [0, 1]$. Specifically, we define: $\alpha_n = \frac{|A_n|}{N}, \alpha_{n-1} = \frac{|A_{n-1}|}{|A_n|}, \ldots, \alpha_1 = \frac{|A_1|}{|A_2|}$. Using the fact established in Remark 4.3, we know that if $i \in A_j$, meaning $h_j \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x) \in \mathrm{NG}_{j-1} \cap F_{j-1} \cap \mathrm{D}^1$, then it follows that $h_{j-1} \circ h_j \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x) \in \mathrm{D}^1$. In particular, if $i \in A_1$, then all necessary conditions are satisfied to ensure that the full composition $h_1 \circ h_2 \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x)$ lies in D^1. Therefore, we conclude that $A_1 \subseteq J_N^{(n)}(x)$, and consequently, $|J_N^{(n)}(x)| \geq |A_1|$. Now, note the recursive structure:

$$|A_1| = \alpha_1 |A_2| = \alpha_1 \alpha_2 |A_3| = \cdots = \left(\prod_{j=1}^{n} \alpha_j\right) (N).$$

Thus, $|J_N^{(n)}(x)| \geq |A_1| = \prod_{j=1}^{n} \alpha_j \cdot N$.

Note that for $j=n$ and any $x \in \mathrm{NG}_n \cap F^{(1)}_{h,n} \cap \mathrm{D}^1$. Since $\mathrm{NG}_n = \bigcup_{i_1=0}^{s_n q_n - 1} \mathcal{J}^{n,k}_{i_1}$, it follows that $x \in \bigcup_{i_1=0}^{s_n q_n - 1} \mathcal{J}^{n,0}_{i_1} \cap \mathrm{D}^1$ for some indices i_1. Then the number of iterates in the orbit $\{S^i(x)\}_{i=0}^{N-1}$ that lie in the set $\bigcup_{i_1=0}^{s_n q_n - 1} \mathcal{Y}^{n,0}_{i_1} := (\bigcup_{i_1=0}^{s_n q_n - 1} \mathcal{J}^{n,0}_{i_1} \cap F_n) \cap \mathrm{D}^1$ is at least $\left(1-\frac{2}{n^2}\right)N$ and $h_n(\bigcup_{i_1=0}^{s_n q_n - 1} \mathcal{Y}^{n,0}_{i_1}) \subseteq \mathrm{D}^1$, (see Remark 4.3). Thus, the fraction of iterates in the image orbit $\{h_n(S^i_{\alpha_{n+1}}(x))\}_{i=0}^{N-1}$ that lie in $\mathrm{NG}_{n-1} \cap F_{n-1} \cap \mathrm{D}^1$ is $\left(1-\lambda_{n,m}-\frac{1}{2\cdot e^{3(n-1)}}\right)$. Combining these two observations, we conclude that

$$\alpha_n \geq \left(1-\frac{2}{n^2}\right)\left(1-\lambda_{n,m}-\frac{1}{2\cdot e^{3(n-1)}}\right).$$

Let $j=n-1$. For each $i \in A_n$, we have $h_n(S^i_{\alpha_{n+1}}x) \in \mathrm{NG}_{n-1} \cap F_{n-1} \cap \mathrm{D}^1$, which implies that $h_{n-1} \circ h_n(S^i_{\alpha_{n+1}}x) \in \mathrm{D}^1$. In particular, the image set $\left\{h_{n-1} \circ h_n(S^i_{\alpha_{n+1}}x)\right\}_{i \in A_n} \subseteq \bigcup_{i=0}^{q_{n-1}-1}\left[\frac{i}{q_{n-1}}+\frac{2\lambda_{n-1,m}}{q_{n-1}}, \frac{i+1}{q_{n-1}}\right] \times \left[0, \frac{1}{2}\right)$.

Furthermore, each point in this image set lies either in $\mathrm{D}^1 \cap \mathrm{NG}_{n-2}$ or $\mathrm{D}^1 \cap \mathrm{G}_{n-2}$. We now estimate the proportion of points that lie in the "good" set, i.e., $\mathrm{NG}_{n-2} \cap F_{n-2} \cap \mathrm{D}^1$.

First, observe that the proportion of the set $\mathrm{D}^1 \cap \mathrm{G}_{n-2}$ within the region $\bigcup_{i=0}^{q_{n-1}-1}\left[\frac{i}{q_{n-1}}+\frac{\lambda_{n-1,m}}{q_{n-1}}, \frac{i+1}{q_{n-1}}\right] \times \left[0, \frac{1}{2}\right)$ is at most $(1-2\lambda_{n-1,m}) \cdot \lambda_{n-2,m}$. Next, consider the error set E_{n-2} which occupies some complete $\frac{1}{q_{n-1}}$-fundamental domains and contributes at most $\frac{1}{e^{3^{n-2}}}$ to the total proportion. Additionally, for each such $\frac{1}{q_{n-1}}$-fundamental domain, the horizontal and vertical boundary error strips contribute at most $\frac{4\epsilon_{n-1}}{q_{n-1}}$. Hence, we can remove the entire error set E_{n-1}, which contributes at most $\frac{1}{e^{3^{n-1}}}$.

Combining all the above, the proportion of the set $\mathrm{NG}_{n-2} \cap F_{n-2} \cap \mathrm{D}^1$ that intersects the orbit points $\left\{h_{n-1} \circ h_n(S^i_{\alpha_{n+1}}x)\right\}_{i \in A_n}$ is accounted after removing the contributions from $\mathrm{D}^1 \cap \mathrm{G}_{n-2}$, E_{n-2}, and E_{n-1}. Thus, we conclude that the fraction of orbit points $\left\{h_{n-1} \circ h_n(S^i_{\alpha_{n+1}}x)\right\}_{i \in A_n}$ that lie in the set $\mathrm{NG}_{n-2} \cap F_{n-2} \cap \mathrm{D}^1$ is at least

$$\alpha_{n-1} \geq 1-(1-2\lambda_{n-1,m}) \cdot \lambda_{n-2,m} - \frac{1}{e^{3^{n-2}}} - \frac{1}{e^{3^{n-1}}}.$$

Analogously, for any $2 \leq j \leq n-2$, consider the set A_{j+1}. For each $i \in A_{j+1}$, we have $h_j \circ h_{j+1} \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x) \in \bigcup_{i=0}^{q_j-1} \left[\frac{i}{q_j} + \frac{\lambda_{j,m}}{q_j}, \frac{i+1}{q_j}\right] \times \left(0, \frac{1}{2}\right)$. Furthermore, if $h_j \circ h_{j+1} \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x) \in \mathrm{NG}_{j-1} \cap F_{j-1} \cap \mathrm{D}^1$, then it follows that $i \in A_j$. To estimate the fraction of such indices, observe that the image set $\left\{h_j \circ h_{j+1} \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x)\right\}_{i \in A_{j+1}}$ is contained in the union of $\frac{1}{q_j}$-fundamental domains at stage j, each of which intersects either $\mathrm{D}^1 \cap \mathrm{NG}_{j-1}$ or $\mathrm{D}^1 \cap \mathrm{G}_{j-1}$. The proportion of points falling into the "bad" region $\mathrm{D}^1 \cap \mathrm{G}_{j-1}$ is at most $(1 - 2\lambda_{j,m}) \cdot \lambda_{j-1,m}$. Additionally, we account for the error sets from stages j and $j-1$, which contribute at most $\frac{1}{e^{3^{j-1}}}$ and $\frac{1}{e^{3^j}}$, respectively, to the total error proportion. Therefore, after removing these contributions, the proportion of points in the orbit set $\left\{h_j \circ h_{j+1} \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x)\right\}_{i \in A_{j+1}}$ that lie in the set $\mathrm{NG}_{j-1} \cap F_{j-1} \cap \mathrm{D}^1$ is at least

$$\alpha_j \geq 1 - (1 - 2\lambda_{j,m}) \cdot \lambda_{j-1,m} - \frac{1}{e^{3^{j-1}}} - \frac{1}{e^{3^j}}.$$

For $j = 1$, observe that for each $i \in A_2$, the image $h_2 \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x)$ lies in the set $NG_1 \cap F_1 \cap \mathrm{D}^1$. This implies that all such indices $i \in A_2$ also belong to A_1, except for those that fall into the error set E_2. Therefore, after removing the contribution from the error set E_2, the proportion of elements in the set $\left\{h_1 \circ h_2 \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x)\right\}_{i \in A_2}$ that remain within D^1 is at least

$$\alpha_1 \geq \left(1 - \frac{1}{e^{3^2}}\right).$$

Therefore, we obtain the following lower bound as

$$|J_N^{(n)}(x)| \geq \left(1 - \frac{1}{e^{3^2}}\right) \left(\prod_{j=2}^{n-1} \left(1 - (1 - 2\lambda_{j,m}) \cdot \lambda_{j-1,m} - \frac{1}{e^{3^j}} - \frac{1}{e^{3(j-1)}}\right)\right)$$
$$\left(1 - \frac{2}{n^2}\right) \left(1 - \lambda_{n,m} - \frac{1}{2 \cdot e^{3(n-1)}}\right) N.$$

For sufficiently large values of n and putting $\lambda_{j,m} = \frac{1}{q_j^m}$ and $q_j \geq 3^j$ for $m > 0$, we estimate each component of the inequality separately.

- Since $e^9 > 8000$, we have $1 - \frac{1}{e^9} > 0.9998$.

- For the second product term, define $\beta_j := (1-2\lambda_{j,m})\cdot\lambda_{j-1,m} = \left(1-\frac{2}{q_j^m}\right)\frac{1}{q_{j-1}^m} < \frac{1}{q_{j-1}^m} < \frac{1}{3^{(j-1)m}}$. Including the exponentially small terms $\frac{1}{e^{3j}}$ and $\frac{1}{e^{3j-1}}$, the total sum is $\sum_{j=2}^{\infty}\epsilon_j < \sum_{j=2}^{\infty}\left(\frac{1}{3^{(j-1)m}}+\frac{1}{e^{3j}}+\frac{1}{e^{3j-1}}\right) < \infty$. More precisely, this sum satisfies $\sum_{j=2}^{\infty}\epsilon_j < 0.3$ for all $m > 0$. Using the inequality for convergent infinite products,

$$\prod_{j=2}^{\infty}(1-\epsilon_j) \geq \exp(-1\sum_{j=2}^{\infty}\epsilon_j) > \exp(-0.3) \geq 0.7.$$

- For large n, we estimate the remaining terms: $1-\frac{2}{n^2} > 0.98$, and $1-\lambda_{n,m}-\frac{1}{2e^{3(n-1)}} = 1-\frac{1}{q_n^m}-\frac{1}{2e^{3(n-1)}} > 0.973$.

Multiplying all the bounds, we obtain: $|J_N^{(n)}(x)| \geq (0.9998)(0.7)(0.98)(0.973)N > 0.667N$. Hence, the constant $c > 0.6$ exists for all sufficiently large n.

$\square$

Lemma 4.6. *The sequence of diffeomorphisms $T_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$, such that $H_n = h_1 \circ h_2 \ldots \circ h_n$ and h_n defined by (4.7) and α_{n+1} converges to a Liouvillian number, converges to $T \in \mathrm{Diff}^\infty(\mathbb{T}^2,\mu)$ in the C^∞ topology. Moreover, for any $m \leq q_{n+1}$, we have*

$$d_0(T^m, T_n^m) \leq \frac{1}{2^{n+1}}. \tag{4.11}$$

4.4.3 Estimate of Hausdorff dimension

Proposition 4.7. *Let $1 \leq s < 2$, and $\lambda_{n,m} = \frac{1}{q_n^m}$, where $m = \frac{1}{s-1} - 1$. Then the set $\mathrm{B} := \liminf_{n\to\infty} H_n(\mathrm{G}_n \cap F_{h,n}^{(1)})$ and $\mathrm{E} := \limsup_{n\to\infty} H_n(E_{h,n}^{(1)})$, where $F_{h,n}^{(1)}$ and $E_{h,n}^{(1)}$) defined by (4.8), satisfies*

$$\dim_H(\mathrm{B}) \leq s \quad \textit{and} \quad \dim_H(\mathrm{E}) \leq s.$$

Proof. Define the set $\mathrm{G}_n^{(1)} = \mathrm{G}_n \cap F_{h,n}^{(1)}$, we aim to show that $\dim_H(\liminf_{n\to\infty} H_n(\mathrm{G}_n^{(1)})) \leq s$. For each $n \in \mathbb{N}$, define $\mathrm{B}_n = \cap_{k=n}^{\infty} H_k(\mathrm{G}_k^{(1)})$ and $\mathrm{B}_n^{(m)} = \cap_{k=n}^{m} H_k(\mathrm{G}_k^{(1)})$. Then clearly,

$$\mathrm{B} = \liminf_{n\to\infty} H_n(\mathrm{G}_n^{(1)}) = \bigcup_{n=1}^{\infty} \mathrm{B}_n,$$

and $\mathrm{B}_n \subset \mathrm{B}_n^{(m)}$ for all $m \geq 0$. We estimate the Hausdorff s-measure of $\mathrm{B}_n^{(n+3)}$, which has the form

$$\mathrm{B}_n^{(n+3)} = H_n\left(\mathrm{G}_n^{(1)} \cap h_{n+1}\left(\mathrm{G}_{n+1}^{(1)} \cap h_{n+2}\left(\mathrm{G}_{n+2}^{(1)} \cap h_{n+3}(\mathrm{G}_{n+3}^{(1)})\right)\right)\right).$$

The set $\mathrm{G}_{n+3}^{(1)}$ is the union of horizontal strips given by $\mathrm{G}_{n+3}^{(1)} = \bigcup_{i=0}^{s_{n+3}q_{n+3}-1} \mathcal{X}_i^{n+3,m}$ (see equation 4.9). Under the map h_{n+3}, we have

$$h_{n+3}(\mathrm{G}_{n+3}^{(1)}) = \bigcup_{i=0}^{q_{n+3}-1}\left(\left[\frac{i}{q_{n+3}}, \frac{i}{q_{n+3}} + 2\frac{\lambda_{n+3,m}}{q_{n+3}}\right] \times \mathbb{T}^1\right) \cap F_{n+3}.$$

Intersecting with G_{n+2}, which consists of a horizontal strip of height $2\lambda_{n+2,m}$ centered at $y = \frac{1}{2}$, yields rectangles of width $2\lambda_{n+2,m}$ and length $2\lambda_{n+3,m}/q_{n+3}$. Next, applying the map $h_{n+2} = \bar{\phi}_{n+2}$ to the rectangles contained in $\mathrm{G}_{n+2}^{(1)}$, scales it horizontally and vertically by a factor of q_{n+2}. As a result, we obtain rectangles of dimensions $\frac{\lambda_{n+2,m}}{q_{n+2}} \times \frac{\lambda_{n+3,m}q_{n+2}}{q_{n+3}}$, and each $\frac{1}{q_{n+2}}$-fundamental domain contains $\frac{q_{n+3}}{q_{n+2}}$ such rectangles.

Continuing, intersecting with $\mathrm{G}_{n+1}^{(1)}$ and applying h_{n+1} further transforms these into rectangles of size $\frac{\lambda_{n+3,m}q_{n+2}}{q_{n+3}q_{n+1}} \times \frac{\lambda_{n+2,m}q_{n+1}}{q_{n+2}}$. Finally, within each $\frac{1}{q_{n+1}}$-fundamental domain, the number of such rectangles intersected with the sets in $\mathrm{G}_n^{(1)}$ is bounded above by $\left(\frac{\lambda_{n,m}q_{n+2}}{q_{n+1}}\right) \cdot \left(\frac{\lambda_{n+1,m}q_{n+3}}{q_{n+2}}\right)$.

Combining altogether, the full intersection,

$$\mathrm{G}_n^{(1)} \cap h_{n+1}\left(\mathrm{G}_{n+1}^{(1)} \cap h_{n+2}\left(\mathrm{G}_{n+2}^{(1)} \cap h_{n+3}(\mathrm{G}_{n+3}^{(1)})\right)\right)$$

is covered by rectangles, say $\mathcal{U}_j$'s of form $\frac{\lambda_{n+3,m}q_{n+2}}{q_{n+3}q_{n+1}} \times \frac{\lambda_{n+3,m}q_{n+2}}{q_{n+3}q_{n+1}}$. The total number of such rectangles is at most

$$q_{n+1}\left(\frac{\lambda_{n+2,m}q_{n+1}}{q_{n+2}}\frac{q_{n+3}q_{n+1}}{\lambda_{n+3,m}q_{n+2}}\right)\left(\frac{\lambda_{n,m}q_{n+2}}{q_{n+1}}\right)\left(\frac{\lambda_{n+1,m}q_{n+3}}{q_{n+2}}\right).$$

Further, substituting $\lambda_{k,m} = q_k^{-m}$, we can compute

$$\sum_j \operatorname{diam}(\mathcal{U}_j)^s \leq \frac{q_{n+3}^2}{\lambda_{n+3,m}} \frac{\lambda_{n+3,m}^s}{q_{n+3}^s} = \frac{q_{n+3}^{2-s}}{q_{n+3}^{m(s-1)}} \leq \frac{1}{(q_{n+3})^{s-2+m(s-1)}}. \tag{4.12}$$

Further, using the above estimate and the fact that $\operatorname{diam}(\mathcal{U}_j) \to 0$ as $n \to \infty$, we observe that since H_n is a smooth diffeomorphism on $\mathrm{B}_n^{(n+3)}$, it preserves the s-dimensional Hausdorff measure (see Lemma 2.1). That is, $\mathcal{H}^s(H_n^{-1}(\mathrm{B}_n^{(n+3)})) = \mathcal{H}^s(\mathrm{B}_n^{(n+3)})$. Thus, we can compute an upper bound for the s-dimensional Hausdorff measure of the following set as

$$\begin{aligned}
\mathcal{H}^s(\liminf_{n\to\infty} H_n(\mathrm{G}_n^{(1)})) &\leq \mathcal{H}^s(\cup_{n=1}^\infty \mathrm{B}_n) \\
&\leq \sum_{n=1}^\infty \mathcal{H}^s(\mathrm{B}_n) \\
&\leq \sum_{n=1}^\infty \mathcal{H}^s(\mathrm{B}_n^{(n+3)}) \\
&\leq \sum_{n=1}^\infty \mathcal{H}^s(H_n^{-1}\mathrm{B}_n^{(n+3)}) \leq \sum_{n=1}^\infty \frac{1}{(q_{n+3})^{s-2+m(s-1)}} < \infty.
\end{aligned}$$

Thus, $\mathcal{H}^s(\liminf_{n\to\infty} H_n(\mathrm{G}_n^{(1)})) < \infty$, which implies $\dim_H(\liminf_{n\to\infty} H_n(\mathrm{G}_n^{(1)})) \leq s$.

Claim 1: *Prove that* $\dim_H(\mathrm{E}) \leq s$.
By the definition of the limsup, we have

$$\mathrm{E} := \limsup_{n\to\infty} H_n(E_{h,n}^{(1)}) = \bigcap_{m=0}^\infty \bigcup_{n\geq m} H_n(E_{h,n}^{(1)}) \subseteq \bigcup_{n\geq m} H_n(E_{h,n}^{(1)}), \quad \text{for all } m \in \mathbb{N}.$$

Using the inclusion $h_n(E_{h,n}^{(1)}) \subset E_n$ and the identity $H_n = H_{n-1} \circ h_n$, we obtain $H_n(E_{h,n}^{(1)}) \subset H_{n-1}(E_n)$, and hence,

$$\mathrm{E} \subseteq \bigcup_{n\geq m} H_{n-1}(E_n). \tag{4.13}$$

We now estimate the Hausdorff s-measure of the union $\bigcup_{n\geq m} H_{n-1}(E_n)$. For each

n, the set E_n is defined in subsection 4.4 as $E_n := E_{h,n} \cup E_{v,n}$, where:

- $E_{h,n}$ consists of $2s_n + 2$ horizontal strips, each of dimensions $1 \times \epsilon'_n$,
- $E_{v,n}$ consists of $s_n q_n + q_n$ vertical strips, each of dimensions $\epsilon'_n \times 1$.

Each strip (horizontal or vertical) can be covered by a collection of squares $\{U_{n,j}\}$ of side length ϵ'_n. Hence, the total number of such squares required to cover E_n is at most $N_n = \frac{s_n q_n + q_n}{\epsilon'_n} + \frac{2s_n+2}{\epsilon'_n} \leq \frac{4s_n q_n}{\epsilon'_n}$.

Since H_{n-1} is a smooth map defined on a compact domain, its image $H_{n-1}(E_n)$ can be covered by at most N_n sets $\mathcal{U}_{n,j}$, each of diameter at most $\operatorname{diam}(\mathcal{U}_{n,j}) \leq \|DH_{n-1}\|_1 \cdot 2\sqrt{2}\epsilon'_n$, where $\|DH_{n-1}\|_1$ denotes the operator norm of the derivative on the domain. Therefore, the total contribution of $H_{n-1}(E_n)$ to the s-dimensional Hausdorff measure is bounded above by

$$\begin{aligned}\sum_{j=1}^{N_n} \operatorname{diam}(\mathcal{U}_{n,j})^s &\leq \frac{4s_n q_n}{\epsilon'_n} \cdot \left(2\sqrt{2}\epsilon'_n \|DH_{n-1}\|_1\right)^s \\ &\leq 16 s_n q_n (\epsilon'_n)^{s-1} \|DH_{n-1}\|_1^s.\end{aligned}$$

In a similar manner to the previous estimate, we analyze the contribution from the next level of the error set, namely the image $H_n(E_{n+1})$. Each strip in E_{n+1} can be covered by a collection of squares $\mathcal{U}_{n+1,j}$ of side length $2\epsilon'_{n+1}$. Under the smooth map H_n, the image of each square has diameter at most $\operatorname{diam}(\mathcal{U}_{n+1,j}) \leq 2\sqrt{2}\epsilon'_{n+1}\|DH_n\|_1$. The number of such covering squares required to cover E_{n+1} is at most $\frac{4s_{n+1}q_{n+1}}{\epsilon'_{n+1}}$. Consequently, the total contribution to the s-dimensional Hausdorff measure from the image $H_n(E_{n+1})$ is bounded above by $\sum_j \operatorname{diam}(\mathcal{U}_{n+1,j})^s \leq 16 s_{n+1} q_{n+1} (\epsilon'_{n+1})^{s-1} \|DH_n\|_1^s$.

Continuing in this manner and combining all contributions from the sets $H_{j-1}(E_j)$, we obtain an upper bound on the total Hausdorff s-measure of the union:

$$\mathcal{H}^s\left(\bigcup_{n\geq m} H_{n-1}(E_n)\right) \leq \sum_{n\geq m} 16 s_n q_n \|DH_{n-1}\|_1^s (\epsilon'_n)^{s-1}.$$

With specific choice $\epsilon'_n := \left(\frac{1}{s_n q_n \|DH_{n-1}\|_1^2 2^n}\right)^{2^n} < 1$. There exists a sufficiently large integer n such that $\frac{1}{2^n} \leq s-1$ and consequently, we obtain the inequality $(\epsilon'_n)^{s-1} \leq (\epsilon'_n)^{1/2^n}$. Using this estimate, the total Hausdorff s-measure becomes

$$\mathcal{H}^s\left(\bigcup_{n\geq m} H_{n-1}(E_n)\right) \leq \sum_{n\geq m} 16 s_n q_n \|DH_{n-1}\|_1^s (\epsilon'_n)^{1/2^n}.$$

$$\leq \sum_{n\geq m} 16 s_n q_n \|DH_{n-1}\|_1^s \cdot \frac{1}{s_n q_n \|DH_{n-1}\|_1^2 2^n} \leq \sum_{n\geq m} \frac{16}{2^n} < \infty.$$

Hence, by equation 4.13, we conclude that the Hausdorff s-measure of the error set $\mathcal{H}^s(\mathrm{E})$ is finite, which implies that $\dim_H(\mathrm{E}) \leq s$. □

4.5 The set of generic and non-generic points

Proof of Theorem 1.3. Let us fix a countable set of Lipshitz functions denoted by $\Psi = \{\psi_i\}_{i\in\mathbb{N}}$, which is dense in $C^0(\mathbb{T}^2, \mathbb{R})$. Let L_n to be a uniform Lipshitz constant for $\psi_1, \ldots, \psi_n$. Choose $q_{n+1} = l_n k_n s_n q_n$ large enough by choosing l_n enuogh arbitrary large such that it satisfies:

$$l_n > n^2 \cdot ||DH_{n-1}||_{n-1} \cdot \max_{0\leq i\leq n} L_n. \tag{4.14}$$

The latter assumption guarantees the convergence of sequences of diffeomorphism $\{T_n\}$ and implies that $\psi_1 H_{n-1}, \psi_2 H_{n-1}, \ldots, \psi_n H_{n-1}$ are $(\frac{\sqrt{2}}{q_n}, \frac{1}{n^2})$-uniformly continuous.

Claim 1: *For any fixed $s \in (1,2)$, we choose $m = \frac{1}{s-1} - 1$, and the set $\mathrm{B} = \liminf_{n\to\infty} H_n(\mathrm{G}_n \cap F^{(1)}_{h,n})$ consists of generic points where G_n defined by (4.1) and $F^{(1)}_{h,n}$ by (4.8).*

Let $y \in \mathrm{B}$, i.e. $y \in H_n(\mathrm{G}_n \cap F^{(1)}_{h,n})$ $\forall$ n except for finitely many n. Say, $x_n = H_n^{-1}(y) \in \mathrm{G}_n \cap F^{(1)}_{h,n}$. Apply Propostition 4.4 with $\varepsilon = \frac{1}{n^2}$, $1 \leq k \leq n$, and for $x_n \in \mathrm{G}_n \cap F^{(1)}_{h,n}$ (see (4.1)),

$$\left| \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \psi_k(H_n S^i_{\alpha_{n+1}} x_n) - \int_{\mathbb{T}^2} \psi_k H_n d\mu \right| < \frac{2}{n^2}||\psi_k||_0 + \frac{4}{n^2}.$$

Using the fact that H_n is an area-preserving smooth diffeomorphism and $H_n(x_n) = y$ with the convergence estimate (4.6) in the last equation, we find that for every $\psi_k \in \Psi$,

$$\left| \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \psi_k(T^i y) - \int_{\mathbb{T}^2} \psi_k d\mu \right| < \frac{2}{n^2}||\psi_k||_0 + \frac{4}{n^2} + \frac{1}{2^{n+1}}.$$

Following the argument in Claim 1 of the proof of Theorem 1.1, the triangle inequality, together with the subsequent averaging criterion that allows us to transfer the limit over the sequence $\{q_{n+1}\}$ to a limit over all $N \in \mathbb{N}$, ensures that the point y is generic for the measure μ in the sense of Definition 2.2. That is, for every test function $\psi_k \in C^\infty(\mathbb{T}^2)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \psi_k(T^i y) = \int_{\mathbb{T}^2} \psi_k \, d\mu.$$

Claim 2: *The set of generic points is non-empty and has Hausdorff dimension at most s.*

Note that the point $(0, \frac{1}{2})$ is a generic point. Therefore, $(0, \frac{1}{2}) \in \mathrm{B} \neq \emptyset$. It is sufficient to show that the set B satisfies $\dim_H(\mathrm{B}) \leq s$, and that the horizontal error set-a small exceptional set-satisfies $\dim_H \left(\limsup_{n\to\infty} H_n(E^{(1)}_{h,n}) \right) \leq s$. The dimension estimates for both of these sets are established in Proposition 4.7. Hence, the claim follows.

Claim 3: *Every point inside the set* $\mathrm{NB} = \limsup\limits_{n\to\infty} H_n(\mathrm{NG}_n \cap F^{(1)}_{h,n})$ *is a non-generic point.*

It is enough to show that for each $y \in \mathrm{NB}$, there exists a $\delta' > 0$ and $\psi \in C(\mathbb{T}^2, \mathbb{R})$ such that

$$\left| \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \psi(T^i(y)) - \int_{\mathbb{T}^2} \psi d\mu \right| \geq \delta'.$$

For any point $y \in \mathrm{NB}$, there exists an increasing sequence $\{n_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ such that $y \in H_{n_j}(\mathrm{NG}_{n_j} \cap F^{(1)}_{h,n_j})$ for all j. Define $x_{n_j} := H^{-1}_{n_j}(y)$, so that $x_{n_j} \in \mathrm{NG}_{n_j} \cap F^{(1)}_{h,n_j}$.

By the construction of $\mathrm{NG}_{n_j} \cap F^{(1)}_{h,n_j}$, it follows that there exists a fixed $k \in \{0,1\}$ such that $x_{n_j} \in \bigcup_{i_1=0}^{s_n q_n-1} \mathcal{J}^{n,k}_{i_1} \cap F^{(1)}_{h,n_j} \subset \mathrm{NG}_{n_j} \cap F^{(1)}_{h,n_j}$. Moreover, for fixed $k=0$ or $k=1$, the family of intervals $\bigcup_{i_1=0}^{s_n q_n-1} \mathcal{J}^{n,k}_{i_1}$ lies entirely within either D^1 or D^2. Without loss of generality, we may assume that $\bigcup_{i_1=0}^{s_n q_n-1} \mathcal{J}^{n,k}_{i_1} \subset \mathrm{D}^1$.
Now, for any $x \in \mathrm{NG}_n \cap F^{(1)}_{h,n_j} \cap \mathrm{D}^1$, define the set of iterates:

$$J_N^{(n)}(x) := \left\{ i \in \{0,1,\dots,N-1\} : h_1 \circ h_2 \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x) \in \mathrm{D}^1 \right\}.$$

This set represents the indices for which the orbit remains inside D^1 and further, applying Proposition 4.5, we have the following estimate as:

$$\begin{aligned} |J_N^{(n)}(x)| \geq \left(1 - \frac{1}{e^{3^2}}\right) &\left(\prod_{j=2}^{n-1} \left(1 - (1-2\lambda_{j,m}) \cdot \lambda_{j-1,m} - \frac{1}{e^{3^j}} - \frac{1}{e^{3(j-1)}}\right)\right) \\ &\left(1 - \frac{2}{n^2}\right)\left(1 - \lambda_{n,m} - \frac{1}{2 \cdot e^{3(n-1)}}\right) N. \end{aligned}$$

As $n \to \infty$, we have $\frac{|J_N^{(n)}(x)|}{N} \geq 0.6$.
Further, consider a continuous function $\psi : \mathbb{T}^2 \to \mathbb{R}$ satisfying the following as:

$$\psi(x) = \begin{cases} 1, & x \in \mathrm{D}^1 \\ 0, & x \in \mathrm{D}^2, \end{cases} \tag{4.15}$$

and where $\int_{\mathbb{T}^2} \psi d\mu = \frac{1}{2}$. Since $T^i_{n_j}(y) = H_{n_j} \circ S^i_{\alpha_{n_j+1}}(x_{n_j}) \in \mathrm{D}^1$ for all $i \in J_N^{(n_j)}(x)$, it follows that $\psi(T^i_{n_j}(y)) = 1$ for all such i. From Proposition 4.5, there exists a $k' \in \mathbb{N}$ such that for all $n_j \geq k'$, the following holds as:

$$\begin{aligned} \left| \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \psi(T^i_{n_j}(y)) - \int_{\mathbb{T}^2} \psi \, d\mu \right| &= \left| \lim_{N\to\infty} \frac{1}{N} \sum_{i \in J_N^{(n_j)}(x)} \psi(T^i_{n_j}(y)) - \int_{\mathbb{T}^2} \psi \, d\mu \right| \\ &\geq |0.6 - 0.5| := \delta. \end{aligned}$$

With the convergence estimate from Lemma 4.6, we have $|\psi(T^i_{n_j}) - \psi(T^i)| \leq \frac{\|\psi\|_0}{2^{n_j}}$ for all $i \in \mathbb{N}$, and large enough $n_j \in \mathbb{N}$. Combining this with the above estimate, we obtain

$$\left| \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \psi(T^i(y)) - \int_{\mathbb{T}^2} \psi d\mu \right| \geq \delta - \frac{\|\psi\|_0}{2^{n_j}} = \delta'.$$

This implies that y is a non-generic point. Since the argument holds for any $y \in \mathrm{NB}$ therefore every point in NB is a non-generic point. $\square$

5

The Set of Generic Points with Non-Trivial Hausdorff Lower Bound

5.1 Introduction

In this chapter, we delve into the analysis of generic and non-generic points within the smooth setting, aiming to establish bounds on the size of the sets they form. Instead of quantifying the set of generic points using the Lebesgue measure, we focus on determining bounds on their Hausdorff dimension, which leads to more insightful and interesting results. We explore various examples of measure-preserving smooth diffeomorphisms, each associated with a set containing all the generic points with zero measure and exhibiting non-trivial bounds on their Hausdorff dimension. To construct such sets and investigate their properties, we employ the Approximation by Conjugation method. In targeting sets containing such points, we introduce fractal sets, particularly various types of Cantor sets, into our explicit construction. We establish a highly explicit setup with distinct targeted domains on the 2-Torus, each serving specific purposes, along with the explicit selection of conjugation maps. In particular, we provide proofs for Theorem 1.4 and 1.5.

5.2 Preliminaries

5.2.1 The middle third Cantor set

Consider the middle third Cantor set $C \subset [0,1]$, obtained by removing the open middle third interval and then repeating the same process with each remaining interval. After completing the n stage of removing middle intervals from $[0,1]$, we have 2^n closed intervals enumerated as $I_l^n,\ l = 0,1,...,2^n - 1$ and have 2^{n-1} removed open interval denoted as $J_l^n,\ l = 0,1,...,2^{n-1}-1$. Precisely, the interval I_l^n is of the form $\left[\frac{3k}{3^n}, \frac{3k+1}{3^n}\right]$ or $\left[\frac{3k+2}{3^n}, \frac{3k+3}{3^n}\right]$, and interval J_l^n of the form $\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$, for $k = 0,1,\ldots,3^{n-1}-1$. The explicit closed form of the Cantor set is defined as

$$C = \bigcap_{n\geq 1} \bigcup_{l=0}^{2^n-1} I_l^n = [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{l=0}^{2^{n-1}-1} J_l^n. \tag{5.1}$$

5.2.2 The Cantor set associated with a sequence

For any sequence $\lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ such that $\sum \lambda_k = K$, there exists a Cantor set C_λ associated with it, defined on the interval $I_{0,\lambda} = [0,K]$ and also known as generalised Cantor set. It is constructed in a similar way to the middle third Cantor set and has the same topological and measure properties. Precisely, it is a compact, perfect, totally disconnected subset of the real line and has measure zero.

Let $\lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ be a sequence such that $\sum \lambda_k = K$. The set C_λ is obtained by removing open intervals whose lengths correspond to the terms of the sequence λ. There is a unique way to choose these intervals such that the subsequent steps of the construction can be carried out properly. This process generalizes the standard

Cantor set construction, but it is specifically designed to yield a set of Lebesgue measure zero. However, there also exist generalizations of the Cantor set with positive Lebesgue measure.

In the first step, an open interval $J^1_{0,\lambda}$ of length λ_1 is removed from $I_{0,\lambda}$, obtaining two closed intervals $I^1_{0,\lambda}, I^1_{1,\lambda}$. In the second step, we remove an open interval of length λ_2 and λ_3 from $I^1_{0,\lambda}$ and $I^1_{1,\lambda}$, respectively. After k complete steps, we have 2^k closed intervals denoted as $\{I^k_{l,\lambda}\}_{l=0}^{2^k-1}$ and 2^{k-1} removed open intervals denoted as $\{J^k_{l,\lambda}\}_{l=0}^{2^{k-1}-1}$ of length equal to the previously used terms of the sequence. And continue in this way, removing an open interval $J^{k+1}_{l,\lambda}$ of length λ_{2^k+l} from interval $I^k_{l,\lambda}$ we have $I^{k+1}_{2l,\lambda}$ and $I^{k+1}_{2l+1,\lambda}$. Since $\sum_k \lambda_k = K$, the location of each interval $J^k_{l,\lambda}$ to be removed is determined uniquely, and the Cantor set C_λ is well defined as

$$C_\lambda = \bigcap_{n\geq 1} \bigcup_{l=0}^{2^n-1} I^n_{l,\lambda} = [0, K] \setminus \bigcup_{n=1}^{\infty} \bigcup_{l=0}^{2^{n-1}-1} J^n_{l,\lambda}. \tag{5.2}$$

There is a unique way of choosing this such that the subsequent steps of the construction can be carried out. This is a generalization of the standard construction but one constructed to give Lebesgue measure 0. There are generalization of the Cantor set with positive Lebesgue measure.

Remark 5.1. Since the length of the interval $I_{0,\lambda}$ equals the sum of the lengths of all the intervals removed in the construction, there is a unique way of doing this construction. The length of each of the remaining intervals at step k should be exactly the sum of the lengths of all the gaps that will be removed from it later in the construction. So, the two remaining intervals of step 1 will have lengths:

$$|I^1_{0,\lambda}| = \sum_{n=1}^{\infty} \sum_{j=0}^{2^{n-1}-1} \lambda_{2^n+j}, \qquad |I^1_{1,\lambda}| = \sum_{n=1}^{\infty} \sum_{j=2^{n-1}}^{2^n-1} \lambda_{2^n+j}.$$

In particular, for any $k \in \mathbb{N}$ and $l = 0, 1, ..., 2^k - 1$, we have

$$|I^k_{l,\lambda}| = \sum_{n=k}^{\infty} \sum_{j=l2^{n-k}}^{(l+1)2^{n-1}-1} \lambda_{2^n+j}.$$

Remark 5.2. Clearly, given a sequence $\sum_{k\in\mathbb{N}} \lambda_k = K$ can be normalize to get $\lambda' = \{\lambda'_k\} = \{\frac{\lambda_k}{K}\}$ so that $\sum_{k\in\mathbb{N}} \lambda'_k = 1$ and we get $C_{\lambda'} \subset [0,1]$.

Remark 5.3. In the subsequent sections we will consider $\sum_{k=1}^{\infty} \frac{1}{k^p}$, which is finite for $p > 1$. Normalizing as above we get a sequence

$$\lambda(p) = \left\{ \frac{1}{\zeta(p)k^p} \right\},$$

where $\zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p}$. It can be shown that its Hausdorff dimension is

$$0 < \dim_H(C_\lambda(p)) = \frac{1}{p} < 1, \tag{5.3}$$

as described in more detail in [10]. In our explicit construction, we choose $p = \frac{1}{\gamma-1} > 1$ for $1 < \gamma < 2$.

5.3 Outline of the proof

To prove Theorem 1.4 and Theorem 1.5, we construct a $T \in \mathrm{Diff}^\infty(\mathbb{T}^2, \mu)$ using the approximation by conjugation method, as discussed in Chapter 3. However, we will modify the combinatorics in the setup to achieve the desired result. We introduce two parts within $\mathbb{T}^2$, G_n and NG_n, specifically designed to achieve the two dynamic properties explicitly, along with explicitly chosen conjugation maps h_n at each induction step. We define the combinatorics such that the limiting set B consists of all the generic points of the system, and the limiting set NB contains all the non-generic points, as outlined in Section 5.4 utilizing the explicit conjugation maps h_n and partial partitions G_n and NG_n. Finally, the convergence of the scheme in the smooth topology is achieved under very precise norm estimates in Section 5.5.

5.4 Construction of the Generic sets

5.4.1 Explicit set-up

Consider the following collection of disjoint subsets of $\mathbb{T}^2 : \mathbb{T}^2 = (\mathrm{G}_n \cup \mathrm{NG}_n)$ such that

$$\mathrm{G}_n = \mathbb{T}^1 \times \bigcup_{l=0}^{2^n-1} I_l^n \quad \text{and} \quad \mathrm{NG}_n = \mathbb{T}^1 \times \bigcup_{k=0}^{n-1} \bigcup_{l=0}^{2^{k-1}-1} J_l^k, \tag{5.4}$$

where I_l^n and J_l^n are intervals of $[0,1]$ as defined in section 5.2.1. We split the interval J_0^1 into two halves as $J_0^1 = \hat{J}_0^1 \cup \hat{J}_1^1$, where $\hat{J}_0^1 = \left(\frac{1}{3}, \frac{1}{2}\right)$ and $\hat{J}_1^1 = \left(\frac{1}{2}, \frac{2}{3}\right)$.

Analogous to the construction in the previous Chapter 4, we define a smooth conjugation map $\bar{\phi}_n$ supported on the set G_n, acting within the strip $\mathcal{I}_l^n := \mathbb{T}^1 \times I_l^n$. This map redistributes the strip vertically across the torus in a periodic manner, producing generic points whose orbits become uniformly distributed with respect to the Lebesgue measure. However, since the strip width $\frac{1}{3^n} \to 0$ as $n \to \infty$, the total Lebesgue measure of this set vanishes.

On the complementary set NG_n, the map $\bar{\phi}_n$ acts on regions $\mathcal{J}_l^k := \mathbb{T}^1 \times J_l^k$, spreading them into vertical bands supported within the intervals $\mathbb{T}^1 \times \left(\frac{t}{2}, \frac{t+1}{2}\right)$, for $t = 0$ or 1. As a result, points in NG_n exhibit non-generic behavior, since their orbits fail to equidistribute on $\mathbb{T}^2$.

Furthermore, we present the following partition of $\mathbb{T}^2$ for any sequence of natural numbers $\{q_n\}_{n\in\mathbb{N}}$ and $\{s_n\}_{n\in\mathbb{N}}$, with the condition that each $s_n > q_n$, as outlined below:

$$\mathrm{G}_n := \left\{ \mathcal{I}_{i_1,i_2}^n = \left[\frac{i_1}{s_n q_n}, \frac{i_1+1}{s_n q_n}\right) \times I_{i_2}^n \ : 0 \le i_1 < s_n q_n, 0 \le i_2 < 2^n \right\};$$

$$\mathrm{NG}_n := \left\{ \begin{array}{l} \mathcal{J}_{i_1,i_2}^{n,k} = \left[\frac{i_1}{s_n q_n}, \frac{i_1+1}{s_n q_n}\right) \times J_{i_2}^k \ : \ 2 \le k \le n, \ 0 \le i_2 < 2^{n-1}; \\ \mathcal{J}_{i_1,i_2'}^{n,1} = \left[\frac{i_1}{s_n q_n}, \frac{i_1+1}{s_n q_n}\right) \times \hat{J}_{i_2'}^1 \ : \ 0 \le i_1 < s_n q_n, \ i_2' = 0, 1 \end{array} \right\};$$

$$\mathrm{V}_n := \left\{ \begin{array}{l} \mathcal{V}_{i_1,i_2,i_3}^n = \left[\frac{i_1}{q_n} + \frac{i_2}{3^n q_n}, \frac{i_1}{q_n} + \frac{i_2+1}{3^n q_n}\right) \times \left[\frac{i_3}{s_n}, \frac{i_3+1}{s_n}\right) \ : \ 0 \le i_1 < q_n, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le i_2 < 2^n, \ 0 \le i_3 < s_n \end{array} \right\};$$

$$\mathrm{W}_n := \left\{ \begin{array}{l} \mathcal{W}^{n,k}_{i_1,i_2} = \left[\frac{i_1}{q_n} + \frac{2^k}{3^k q_n} + \frac{2i_2}{3^k q_n}, \frac{i_1}{q_n} + \frac{2^k}{3^k q_n} + \frac{2(i_2+1)}{3^k q_n}\right) \times \left[\frac{t}{2s_n}, \frac{t+1}{2s_n}\right) : \\ 0 \leq i_1 < q_n, 2 \leq k \leq n, 0 \leq i_2 < 2^{k-1}, t = \begin{cases} 0, & \text{if } 0 \leq i_2 \leq 2^{k-2}-1 \\ 1, & \text{if } 2^{k-2} \leq i_2 < 2^{k-1}. \end{cases} \end{array} \right\}$$

$$\mathrm{W}^1_n := \left\{ \begin{array}{r} \mathcal{W}^{n,1}_{i_1,i_2'} = \left[\frac{i_1}{q_n} + \frac{2}{3q_n}, \frac{i_1+1}{q_n}\right) \times \left[\frac{1}{2} + \frac{i_2'}{2s_n}, \frac{1}{2} + \frac{i_2'+1}{2s_n}\right) : 0 \leq i_1 < q_n, \\ 0 \leq i_2' < s_n. \end{array} \right\}.$$

5.4.2 The conjugation map h_n

Now we define the following permutation maps $\widetilde{\phi}_n : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ of the above partition $\mathrm{G}_n \cup \mathrm{NG}_n$ which maps to the elements of partition $\mathrm{V}_n \cup \mathrm{W}_n \cup \mathrm{W}^1_n$ [1]. Consider the map $\widetilde{\phi}_n : \left[\frac{i}{q_n}, \frac{i+1}{q_n}\right) \times \mathbb{T}^1 \longrightarrow \left[\frac{i}{q_n}, \frac{i+1}{q_n}\right) \times \mathbb{T}^1$ as following and extend it to the whole $\mathbb{T}^2$ as $\frac{1}{q_n}$-equivariantly.

$$\widetilde{\phi}_n(\mathcal{I}^n_{i_1,i_2}) = \mathcal{V}^n_{j_1,j_2,j_3} \quad \text{where} \quad j_1 = \left\lfloor \frac{i_1}{s_n} \right\rfloor,\ j_2 = i_2,\ j_3 = i_1 \mod s_n, \tag{5.5}$$

$$\widetilde{\phi}_n(\mathcal{J}^{n,k}_{i_1',i_2'}) = \mathcal{W}^{n,k}_{j_1',j_2'} \quad \text{where} \quad j_1' = \left\lfloor \frac{i_1'}{s_n} \right\rfloor,$$

$$j_2' = \begin{cases} i_2'.s_n + i_1' \mod s_n & \text{for } 2 \leq k \leq n \\ i_1' \mod s_n & \text{for } k = 1\ \&\ i_2' = 0 \\ s_n + i_1' \mod s_n & \text{for } k = 1\ \&\ i_2' = 1. \end{cases} \tag{5.6}$$

The map $\widetilde{\phi}_n$ effectively rearranges elements of equal size on $\mathbb{T}^2$, and its visualization can be further understood by observing the corresponding rectangles, as shown below: For the case $n = 1$,

$$\widetilde{\phi}_n\left(\left[\frac{i}{q_n}, \frac{i+1}{q_n}\right) \times \left(\frac{1}{3}, \frac{1}{2}\right)\right) = \left[\frac{i}{q_n} + \frac{2}{3q_n}, \frac{i}{q_n} + \frac{2}{3q_n} + \frac{1}{3q_n}\right) \times \left(0, \frac{1}{2}\right);$$

$$\widetilde{\phi}_n\left(\left[\frac{i}{q_n}, \frac{i+1}{q_n}\right) \times \left(\frac{1}{2}, \frac{2}{3}\right)\right) = \left[\frac{i}{q_n} + \frac{2}{3q_n}, \frac{i}{q_n} + \frac{2}{3q_n} + \frac{1}{3q_n}\right) \times \left(\frac{1}{2}, 1\right).$$

[1] We refer to the map $\tilde{\phi}_n$ as a permutation map because it rearranges the elements of the partition.

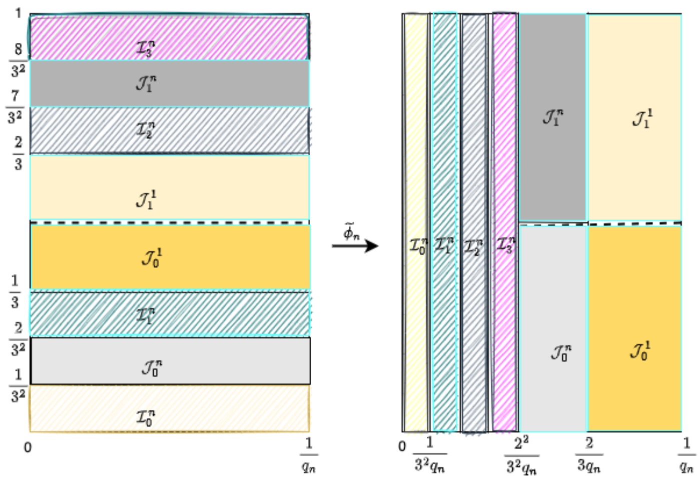

FIGURE 5.1: An example of action $\widetilde{\phi}_n$ on the elements of $\mathrm{G}_n \cup \mathrm{NG}_n$ for $n = 2$.

For the case $n \geq 2$,

$$\widetilde{\phi}_n\left(\left[\frac{i}{q_n}, \frac{i+1}{q_n}\right) \times I_l^n\right) = \left[\frac{i}{q_n} + \frac{l}{3^n q_n}, \frac{i}{q_n} + \frac{l+1}{3^n q_n}\right) \times \mathbb{T}^1, \ 0 \leq l \leq 2^n - 1;$$

$$\widetilde{\phi}_n\left(\left[\frac{i}{q_n}, \frac{i+1}{q_n}\right) \times J_l^k\right) = \left[\frac{i}{q_n} + \frac{2^k}{3^k q_n} + \frac{2l}{3^k q_n}, \frac{i}{q_n} + \frac{2^k}{3^k q_n} + \frac{2^k}{3^k q_n} + \frac{2(l+1)}{3^k q_n}\right)$$
$$\times \left(\frac{t}{2}, \frac{t+1}{2}\right), \ \forall \ 2 \leq k \leq n, t = \begin{cases} 0, \ \text{if } 0 \leq l \leq 2^{k-2} - 1 \\ 1, \ \text{if } 2^{k-2} \leq l \leq 2^{k-1}. \end{cases}$$

Here we define our final conjugation diffeomorphism as

$$h_n = \overline{\phi}_n, \tag{5.7}$$

where, $\overline{\phi}_n$ denotes the smooth approximation of the map $\widetilde{\phi}_n$ on $\mathbb{T}^2$. Note that, by construction, the map h_n satisfies the commutation relation $h_n \circ S_{\alpha_n} = S_{\alpha_n} \circ h_n$.

5.5 Convergence and Estimates

To exclude the region where we don't have control over the combinatorics, we consider a subset $E_n = \cup_{i=1}^3 E_{v,n}^{(i)} \cup_{j=1}^2 E_{h,n}^{(j)}$ of $\mathbb{T}^2$ defined as:

$$
\begin{aligned}
E_{v,n}^{(1)} &= \bigcup_{i_1=0}^{s_nq_n-1} \left(\frac{i_1}{s_nq_n}, \frac{i_1}{s_nq_n} + \epsilon_n \right) \times \mathbb{T}^1; \\
E_{v,n}^{(2)} &= \bigcup_{i_1=0}^{q_n-1} \bigcup_{i_2=0}^{2^n-1} \left(\frac{i_1}{q_n} + \frac{i_2}{3^nq_n}, \frac{i_1}{q_n} + \frac{i_2}{3^nq_n} + \epsilon_n \right) \times \mathbb{T}^1; \\
E_{v,n}^{(3)} &= \bigcup_{i_1=0}^{q_n-1} \bigcup_{i_2=0}^{2^{n-1}-1} \left(\frac{i_1}{q_n} + \frac{2^n}{3^nq_n} + \frac{2i_2}{3^nq_n}, \frac{i_1}{q_n} + \frac{2^n}{3^nq_n} + \frac{2i_2}{3^nq_n} + \epsilon_n \right) \times \mathbb{T}^1; \\
E_{h,n}^{(1)} &= \bigcup_{k=0}^{3^{n-1}-1} \mathbb{T}^1 \times \left(\left(\frac{3k+1}{3^n}, \frac{3k+1}{3^n} + \epsilon_n \right) \bigcup \left(\frac{3k+2}{3^n} - \epsilon_n, \frac{3k+2}{3^n} \right) \right) \qquad (5.8) \\
&\quad \bigcup \mathbb{T}^1 \times \left(\left(\frac{1}{2}, \frac{1}{2} + \epsilon_n \right) \bigcup \left(\frac{2}{3} - \epsilon_n, \frac{2}{3} \right) \right); \\
E_{h,n}^{(2)} &= \bigcup_{i_2=0}^{2s_n-1} \mathbb{T}^1 \times \left(\frac{i_2}{2s_n}, \frac{i_2}{2s_n} + \epsilon_n \right).
\end{aligned}
$$

We choose $\epsilon_n \le \left(\frac{1}{3^n e^{3^n} s_n q_n^3 \|DH_{n-1}\|_1^2} \right)^{2^n}$, so that the measure of the exceptional set satisfies $\mu(E_n) = 10 s_n q_n \epsilon_n < \frac{1}{e^{3^n}}$. We define the complementary sets

$$
F_n := \mathbb{T}^2 \setminus E_n \quad \text{and} \quad F_{h,n}^{(1)} := \mathbb{T}^2 \setminus E_{h,n}^{(1)}, \qquad (5.9)
$$

such that $F_n \subseteq F_{h,n}^{(1)}$, and the measure of F_n satisfies $\mu(F_n) > 1 - \frac{1}{e^{3^n}}$. To analyze the distribution of generic and non-generic points, we now introduce the following families of subsets of the torus. These will be referred to as the "trapping generic zones" and "trapping non-generic zones," respectively. For each $i_1 = 0, \dots, q_n s_n - 1$, we define

$$
\mathcal{Y}_{i_1,i_2}^{n,k} := \mathcal{J}_{i_1}^{n,k} \cap F_n, \quad k = 0, 1. \qquad (5.10)
$$

Remark 5.4. Recall that the image of $\mathcal{I}_{i_1,i_2}^n$, under the conjugation map h_n, contained inside $\mathcal{V}^n_{\lfloor \frac{i_1}{s_n} \rfloor, i_2, i_1 \mod s_n}$ and conversely, $\mathcal{V}^n_{i_1,i_2,i_3}$ is uniquely mapped onto

$\mathcal{I}^n_{i_1.s_n+i_3,i_2}$. Thus, the number of iterates $k \in \{0, 1, \dots, q_{n+1} - 1\}$ such that $h_n \circ S^k_{\alpha_{n+1}}(x) \in \mathcal{V}_{i_1,i_2,i_3}$ for $x \in \mathbb{T}^1 \times I^n_{t_1}$ is at least $\left(1 - \frac{2}{n^2}\right) \frac{q_{n+1}}{s_n q_n}$.

Remark 5.5. Note that under the action of h_n, every element from NG_n transforms as follows (for $i_2 = 0, 1, \dots, 2^{n-1} - 1$):

$$h_n \left(\bigcup_{i_1=0}^{s_n q_n - 1} \mathcal{Y}^{n,k}_{i_1,i_2} \right) = \bigcup_{i_1=0}^{s_n q_n - 1} \overline{\phi}_n(\mathcal{J}^{n,k}_{i_1,i_2} \cap F_n) \subseteq \mathbb{T}^1 \times \left[\frac{t}{2}, \frac{t+1}{2}\right), \quad 2 \leq k \leq n,$$

$$t = \begin{cases} 0, & \text{for } 0 \leq i_2 \leq 2^{k-2} - 1 \\ 1, & \text{for } 2^{k-2} \leq i_2 \leq 2^{k-1} - 1. \end{cases}$$

$$h_n \left(\bigcup_{i_1=0}^{s_n q_n - 1} \mathcal{Y}^{n,1}_{i_1,t} \right) = \bigcup_{i_1=0}^{s_n q_n - 1} \overline{\phi}_n(\mathcal{J}^{n,1}_{i_1,t} \cap F_n) \subseteq \mathbb{T}^1 \times \left[\frac{t}{2}, \frac{t+1}{2}\right), \ t = 0, 1.$$

5.5.1 Generic points estimates

Proposition 5.6. *For $\varepsilon > 0$, consider a $(\frac{\sqrt{2}}{q_n}, \varepsilon)$-uniformly continuous function $\psi : \mathbb{T}^2 \longrightarrow \mathbb{R}$, i.e. $\psi(B_{\frac{\sqrt{2}}{q_n}}(x)) \subset B_\varepsilon(\psi(x))$. Then, for any $x \in \mathrm{G}_n$, we have the following estimate:*

$$\left| \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \psi(h_n \circ S^i_{\alpha_{n+1}}(x)) - \int_{\mathbb{T}^2} \psi d\mu \right| \leq 4\varepsilon + \frac{2}{n^2} \|\psi\|_0. \tag{5.11}$$

Proof. For any $x \in \mathrm{G}_n$ and $\Delta^n_{i_1,i_2} \in \Delta^n_{i,j}$ (see (3.20)). Precisely, $x \in \mathbb{T}^1 \times I^n_l$ for some l. Since the orbit of x under the $S^k_{\alpha_{n+1}}$ is almost trapped by the domains $\{\mathcal{I}^n_{t_1,t_2}\}$, therefore there exists a $i_0 \in \mathbb{N}$ such that $S^{i_0}_{\alpha_{n+1}}(x) \in \mathcal{I}^n_{i_1.s_n+i_2,l}$. With the action of h_n and Remark 5.4, we have $h_n \circ S^{i_o}_{\alpha_{n+1}}(x) \in \mathcal{V}^n_{i_1,l,i_2} \subset \Delta^n_{i_1,i_2}$. Therefore for any $y \in \Delta^n_{i,j}$, we conclude

$$d(h_n \circ S^{i_o}_{\alpha_{n+1}}(x), y) \leq diam(h_n \circ S^{i_o}_{\alpha_{n+1}}(x), y) \leq \sqrt{2}/q_n.$$

Now using the hypothesis on ψ, we have $|\psi(h_n \circ S^{i_o}_{\alpha_{n+1}}(x)) - \psi(y)| < 2\varepsilon$. Take the average for all $y \in \Delta^n_{i,j}$ in the last equation, we get

$$|\psi(h_n \circ S^{i_o}_{\alpha_{n+1}}(x)) - \frac{1}{\mu(\Delta^n_{i,j})} \int_{\Delta^n_{i,j}} \psi(y) d\mu| < 2\varepsilon.$$

Let us denote $J_\Delta = \{k \in 0, 1, \dots, q_{n+1} - 1 \ : h_n \circ S^k_{\alpha_{n+1}}(x) \in \Delta\}$ for all $\Delta \in \mathcal{D}_n$, where $\mathcal{D}_n$ defined by (3.20). Using the count estimate described in Remark 5.4 and triangle inequality in the last equation, we have

$$\left| \frac{1}{q_{n+1}} \sum_{i \in J_\Delta} \psi(h_n \circ S^i_{\alpha_{n+1}}(x)) - \int_{\Delta^n_{i,j}} \psi d\mu \right| < (4\varepsilon + \frac{2}{n^2} \|\psi\|_0) \mu(\Delta^n_{i,j}).$$

Further, we follow the analogous estimation as done in Proposition 3.20, and we have the estimate (5.11) as required. □

5.5.2 Non-Generic points estimates

We begin by partitioning the torus $\mathbb{T}^2$ into two regions:

$$\mathrm{D}^1 := \mathbb{T}^1 \times \left[0, \tfrac{1}{2}\right), \quad \mathrm{D}^2 := \mathbb{T}^1 \times \left[\tfrac{1}{2}, 1\right).$$

Our goal is to study how often the composition $H_n := h_1 \circ h_2 \circ \cdots \circ h_n$, evaluated along the orbit $S^i_{\alpha_{n+1}}(x)$, lands in the region D^1.

Proposition 5.7. *For any $x \in \mathrm{NG}_n \cap F^{(1)}_{h,n} \cap \mathrm{D}^1$ and $N \geq q_{n+1}$, where $F^{(1)}_{h,n}$ defined by equation (5.9). We define $J^{(n)}_N(x) \subset \{0, 1, \dots, N-1\}$ as*

$$J^{(n)}_N(x) := \left\{ i \in \{0, 1, \dots, N-1\} : h_1 \circ h_2 \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x) \in \mathrm{D}^1 \right\}.$$

Then, the following lower bound holds:

$$|J^{(n)}_N(x)| \geq \left(1 - \frac{1}{e^{3^2}}\right) \left(\prod_{j=2}^{n-1} \left(1 - \left(1 - \frac{2^j}{3^j}\right) \frac{2^{j-2}}{3^{j-1}} - \frac{1}{e^{3^j}} - \frac{1}{e^{3(j-1)}}\right)\right) \left(1 - \frac{2}{n^2}\right) \left(1 - \frac{2^{n-1}}{3^n} - \frac{1}{2 \cdot e^{3(n-1)}}\right) N.$$

Moreover, for all sufficiently large n, there exists a constant $c > 0.6$ such that $|J_N^{(n)}(x)| \geq cN$.

Proof. The proof follows the same structure as in Proposition 4.5. To estimate a lower bound for $|J_N^{(n)}(x)|$, we define a sequence of nested subsets $A_j \subset \{0, 1, \ldots, N-1\}$ and corresponding proportions α_j, as described in that proposition.

At level $j = n$, consider any point $x \in \mathrm{NG}_n \cap F_{h,n}^{(1)} \cap \mathrm{D}^1$. Since $\mathrm{NG}_n = \bigcup_{k=0}^{n-1} \bigcup_{l=0}^{2^{k-1}-1} \mathcal{J}_l^k$, it follows that $x \in \mathcal{J}_l^k \cap \mathrm{D}^1$ for some l, k. Define $\mathcal{Y}_l^k := \mathcal{J}_l^k \cap F_n \cap \mathrm{D}^1$. By Remark 5.5, the orbit $\{S^i(x)\}_{i=0}^{N-1}$ visits $\mathcal{Y}_l^k$ at least $\left(1 - \frac{2}{n^2}\right) N$ times. Since $h_n(\mathcal{Y}_l^k) \subset \mathrm{D}^1$, the image orbit $\left\{h_n(S^i_{\alpha_{n+1}}(x))\right\}_{i=0}^{N-1}$ spends a proportion of at least $\alpha_n \geq \left(1 - \frac{2}{n^2}\right)\left(1 - \frac{2^{n-1}}{3^n} - \frac{1}{2e^{3^{n-1}}}\right)$ in the set $\mathrm{NG}_{n-1} \cap F_{n-1} \cap \mathrm{D}^1$.

Analogously, for any $2 \leq j \leq n-1$, consider the set A_{j+1}. For each $i \in A_{j+1}$, we have $h_j \circ h_{j+1} \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x) \in \cup_{i=0}^{q_j - 1} \left[\frac{i}{q_j} + \frac{2^j}{3^j q_j}, \frac{i+1}{q_j}\right] \times \left(0, \frac{1}{2}\right)$. Furthermore, if $h_j \circ h_{j+1} \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x) \in \mathrm{NG}_{j-1} \cap F_{j-1} \cap \mathrm{D}^1$, then it follows that $i \in A_j$. To estimate the fraction of such indices, observe that the image set $\left\{h_j \circ h_{j+1} \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x)\right\}_{i \in A_{j+1}}$ is contained in the union of $\frac{1}{q_j}$-fundamental domains at stage j, each of which intersects either $\mathrm{D}^1 \cap \mathrm{NG}_{j-1}$ or $\mathrm{D}^1 \cap \mathrm{G}_{j-1}$. The proportion of points falling into the "bad" region $\mathrm{D}^1 \cap \mathrm{G}_{j-1}$ is at most $\left(1 - \frac{2^j}{3^j}\right) \cdot \frac{2^{j-2}}{3^{j-1}}$. Additionally, we account for the error sets from stages j and $j-1$, which contribute at most $\frac{1}{e^{3^{j-1}}}$ and $\frac{1}{e^{3^j}}$, respectively, to the total error proportion. Therefore, after removing these contributions, the proportion of points in the orbit set $\left\{h_j \circ h_{j+1} \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x)\right\}_{i \in A_{j+1}}$ that lie in the set $\mathrm{NG}_{j-1} \cap F_{j-1} \cap \mathrm{D}^1$ is at least

$$\alpha_j \geq 1 - \left(1 - \frac{2^j}{3^j}\right) \cdot \frac{2^{j-2}}{3^{j-1}} - \frac{1}{e^{3^{j-1}}} - \frac{1}{e^{3^j}}.$$

At level $j = 1$, for each $i \in A_2$, the image $h_2 \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x)$ lies in $\mathrm{NG}_1 \cap F_1 \cap \mathrm{D}^1$, except for a negligible contribution from E_2. Thus, $\alpha_1 \geq \left(1 - \frac{1}{e^{3^2}}\right)$. Combining all levels, we obtain:

$$|J_N^{(n)}(x)| \geq \left(1 - \frac{1}{e^{3^2}}\right) \left(\prod_{j=2}^{n-1} \left(1 - \left(1 - \frac{2^j}{3^j}\right) \frac{2^{j-2}}{3^{j-1}} - \frac{1}{e^{3^j}} - \frac{1}{e^{3^{j-1}}}\right)\right)$$

$$\cdot \left(1 - \frac{2}{n^2}\right)\left(1 - \frac{2^{n-1}}{3^n} - \frac{1}{2e^{3^{n-1}}}\right) N.$$

For sufficiently large n, each factor is close to 1, and the product is bounded below by 0.6, yielding: $|J_N^{(n)}(x)| \geq 0.6N$. A detailed derivation of this estimate is provided in Proposition 4.5. $\square$

Lemma 5.8. *The sequence of diffeomorphisms $T_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$, such that $H_n = h_1 \circ h_2 \ldots \circ h_n$ and h_n defined by (5.7) and α_{n+1} converges to a Liouvillian number, converges to $T \in \mathrm{Diff}^\infty(\mathbb{T}^2, \mu)$ in the C^∞ topology. Moreover, for any $m \leq q_{n+1}$, we have*

$$d_0(T^m, T_n^m) \leq \frac{1}{2^{n+1}}. \tag{5.12}$$

5.6 The set of generic and non-generic points

Proof of Theorem 1.4. Let us fix a countable set of Lipshitz functions denoted by $\Psi = \{\psi_i\}_{i\in\mathbb{N}}$, which is dense in $C^0(\mathbb{T}^2, \mathbb{R})$. Let L_n to be a uniform Lipshitz constant for $\psi_1, \ldots, \psi_n$. Choose $q_{n+1} = l_n k_n q_n^2$ large enough by choosing l_n enuogh arbitrary large such that it satisfies:

$$l_n > n^2 \cdot ||DH_{n-1}||_{n-1} \cdot \max_{0\leq i\leq n} L_n. \tag{5.13}$$

The latter assumption guarantees the convergence of sequences of diffeomorphism $\{T_n\}$ and implies that $\psi_1 H_{n-1}, \psi_2 H_{n-1}, \ldots, \psi_n H_{n-1}$ are $(\frac{\sqrt{2}}{q_n}, \frac{1}{n^2})$-uniformly continuous.

Claim 1: *Every point inside the set $\mathrm{B} = \liminf_{n\to\infty} \mathrm{B}_n$ is a generic point, where $\mathrm{B}_n = H_n(\mathrm{G}_n)$.*

Let $y \in \mathrm{B}$, i.e. $y \in \mathrm{B}_n$ $\forall n$ except for finitely many n. Say, $x_n = H_n^{-1}(y) \in \mathrm{G}_n$. Apply Propostition 5.6 with $\varepsilon = \frac{1}{n^2}$, $1 \leq k \leq n$, and for $x_n \in \mathrm{G}_n$ (see (5.4)),

$$\left| \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \psi_k(H_n S^i_{\alpha_{n+1}} x_n) - \int_{\mathbb{T}^2} \psi_k H_n d\mu \right| < \frac{2}{n^2} ||\psi_k||_0 + \frac{4}{n^2}.$$

Using the fact that H_n is an area-preserving smooth diffeomorphism and $H_n(x_n) = y$ with the convergence estimate (5.12) in the last equation, we find that for every $\psi_k \in \Psi$,

$$\left| \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \psi_k(T^i y) - \int_{\mathbb{T}^2} \psi_k d\mu \right| < \frac{2}{n^2} ||\psi_k||_0 + \frac{4}{n^2} + \frac{1}{2^{n+1}}.$$

Following the steps of Claim 1 in the proof of Theorem 1.1, the triangle inequality together with the subsequent criterion taking the average q_{n+1} to N, which ensures that y is a generic point for μ in the sense of Definition 2.2, such that

$$\lim_{N \longrightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} \psi_k(T^i y) \longrightarrow \int_{\mathbb{T}^2} \psi_k d\mu.$$

Since $y \in \mathrm{B}$ was chosen arbitrarily, therefore every point $y \in \mathrm{B}$ is a generic point.
Claim 2: *The set of generic points has Hausdorff dimension at least* $\dim_H(C) = \frac{\log 2}{\log 3}$.
By construction, the map H_n acts as the identity near the boundary of $\mathbb{T}^2$, which implies that the set $\{0\} \times C \subseteq \mathrm{B}$. Therefore, we obtain the estimate $\dim_H(C) \leq \dim_H(\mathrm{B})$. Furthermore, a computation similar to the one carried out in Proposition 4.7 shows that $\dim_H \left(\limsup_{n\to\infty} H_n(E^{(1)}_{h,n}) \right) \geq \frac{\log 2}{\log 3}$, which completes the proof of the claim.
Claim 3: *Every point inside the set* $\mathrm{NB} = \limsup_{n\to\infty} H_n(\mathrm{NG}_n \cap F^{(1)}_{h,n})$ *is a non-generic point.*
It is enough to show that for each $y \in \mathrm{NB}$, there exists a $\delta' > 0$ and $\psi \in C(\mathbb{T}^2, \mathbb{R})$ such that

$$\left| \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \psi(T^i(y)) - \int_{\mathbb{T}^2} \psi d\mu \right| \geq \delta'.$$

For any $y \in \mathrm{NB}$, there exists an increasing sequence $\{n_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ such that $y \in H_{n_j}(\mathrm{NG}_{n_j} \cap F^{(1)}_{h,n_j})$. Define $x_{n_j} = H^{-1}_{n_j}(y)$. Since $x_{n_j} \in \mathrm{NG}_{n_j} \cap F^{(1)}_{h,n_j}$, it follows that there exist $l, k \in \mathbb{N}$ such that $x_{n_j} \in \mathcal{J}^k_l \cap F^{(1)}_{h,n_j} \subset \mathrm{NG}_{n_j} \cap F^{(1)}_{h,n_j}$ (since

$\mathrm{NG} = \bigsqcup_k \mathrm{NG}_k = \bigsqcup_k \bigsqcup_l \mathcal{J}_l^k)$. The sets $\mathcal{J}_l^k$ either lie in D^1 or D^2; assume without loss of generality that $\mathcal{J}_l^k \subset \mathrm{D}^1$.
Now, for any $x \in \mathrm{NG}_n \cap F_{h,n}^{(1)} \cap \mathrm{D}^1$, define the set of iterates:

$$J_N^{(n)}(x) := \left\{ i \in \{0, 1, \dots, N-1\} : h_1 \circ h_2 \circ \cdots \circ h_n(S^i_{\alpha_{n+1}} x) \in \mathrm{D}^1 \right\}.$$

This set represents the indices for which the orbit remains inside D^1 and further, applying Proposition 5.7, we have the following estimate as:

$$\begin{aligned} |J_N^{(n)}(x)| \geq \left(1 - \frac{1}{e^{3^2}}\right) \left(\prod_{j=2}^{n-1} \left(1 - \left(1 - \frac{2^j}{3^j}\right) \frac{2^{j-2}}{3^{j-1}} - \frac{1}{e^{3^j}} - \frac{1}{e^{3(j-1)}}\right)\right) \\ \left(1 - \frac{2}{n^2}\right) \left(1 - \frac{2^{n-1}}{3^n} - \frac{1}{2 \cdot e^{3(n-1)}}\right) N. \end{aligned}$$

As $n \to \infty$, we have $\frac{|J_N^{(n)}(x)|}{N} \geq 0.6$.
Further, consider a continuous function $\psi : \mathbb{T}^2 \to \mathbb{R}$ satisfying the following as:

$$\psi(x) = \begin{cases} 1, & x \in \mathrm{D}^1 \\ 0, & x \in \mathrm{D}^2, \end{cases} \tag{5.14}$$

and where $\int_{\mathbb{T}^2} \psi d\mu = \frac{1}{2}$. Since $T^i_{n_j}(y) = H_{n_j} \circ S^i_{\alpha_{n_j+1}}(x_{n_j}) \in \mathrm{D}^1$ for all $i \in J_N^{(n_j)}(x)$, it follows that $\psi(T^i_{n_j}(y)) = 1$ for all such i. From Proposition 5.7, there exists a $k' \in \mathbb{N}$ such that for all $n_j \geq k'$, the following holds as:

$$\begin{aligned} \left| \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \psi(T^i_{n_j}(y)) - \int_{\mathbb{T}^2} \psi \, d\mu \right| &= \left| \lim_{N\to\infty} \frac{1}{N} \sum_{i \in J_N^{(n_j)}(x)} \psi(T^i_{n_j}(y)) - \int_{\mathbb{T}^2} \psi \, d\mu \right| \\ &\geq |0.6 - 0.5| := \delta. \end{aligned}$$

With the convergence estimate from Lemma 5.8, we have $|\psi(T^i_{n_j}) - \psi(T^i)| \leq \frac{\|\psi\|_0}{2^{n_j}}$ for all $i \in \mathbb{N}$, and large enough $n_j \in \mathbb{N}$. Combining this with the above estimate, we obtain

$$\left| \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \psi(T^i(y)) - \int_{\mathbb{T}^2} \psi d\mu \right| \geq \delta - \frac{\|\psi\|_0}{2^{n_j}} = \delta'.$$

This implies that y is a non-generic point. Since the argument holds for any $y \in \mathrm{NB}$ therefore every point in NB is a non-generic point. $\square$

5.6.1 Proof of Theorem 1.5

Here, we construct a couple of sets containing all the generic points for the interesting values of their Hausdorff dimension. The sets can be constructed in a similar manner to the set G constructed in the last subsection (see (5.4)). Therefore we will only mention the respective changes that need to be made.

For any $1 < \gamma < 2$, and consider a Cantor set C_λ associated with the sequence $\lambda = \{\lambda_k\}_{k\in\mathbb{N}}$, where $\lambda_k = \frac{1}{c_0}(\frac{1}{k})^{\frac{1}{\gamma-1}}$, the constant $c_0 = \sum_{k\in\mathbb{N}} \lambda_k$, explained in section 5.2.2. At first, just replace the Cantor set C with C_λ, I_l^n with $I_{l,\lambda}^n$, and J_l^n with $J_{l,\lambda}^n$ in equation (5.4) and subsection (5.4.2) to get following collection of disjoint subsets of $\mathbb{T}^2 : \mathbb{T}^2 = (\mathrm{G}_{n,\lambda} \cup \mathrm{NG}_{n,\lambda})$ where

$$\mathrm{G}_{n,\lambda} = \mathbb{T}^1 \times \bigcup_{l=0}^{2^n-1} I_{l,\lambda}^n \quad \text{and} \quad \mathrm{NG}_{n,\lambda} = \mathbb{T}^1 \times \bigcup_{k=0}^{n-1} \bigcup_{l=0}^{2^{k-1}-1} J_{l,\lambda}^k, \tag{5.15}$$

where $I_{l,\lambda}^n$ and $J_{l,\lambda}^n$ are intervals of $[0,1]$ as defined in section 5.2.2. We split the interval $J_{0,\lambda}^1$ into two halves as $J_{0,\lambda}^1 = \hat{J}_{0,\lambda}^1 \cup \hat{J}_{1,\lambda}^1$.

Consider the following permutation map $\widetilde{\phi}_{n,\lambda} : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$, which follows the same combinatorics as $\widetilde{\phi}_n$ from section 5.4.1,

$$\widetilde{\phi}_{n,\lambda}\left(\left[\frac{i}{q_n},\frac{i+1}{q_n}\right)\times I^n_{l,\lambda}\right)=\left[\frac{i}{q_n}+\sum_{k=0}^{l-1}\frac{|I^n_{k,\lambda}|}{q_n},\frac{i}{q_n}+\sum_{k=0}^{l}\frac{|I^n_{k,\lambda}|}{q_n}\right)\times\mathbb{T}^1,\quad 0\leq l<2^n$$

$$\begin{aligned}\widetilde{\phi}_{n,\lambda}\left(\left[\frac{i}{q_n},\frac{i+1}{q_n}\right)\times J^k_{l,\lambda}\right)=&\left[\frac{i}{q_n}+\sum_{l=0}^{2^k-1}\frac{|I^k_{l,\lambda}|}{q_n}+\sum_{j=0}^{l-1}\frac{2|J^k_{j,\lambda}|}{q_n},\frac{i}{q_n}+\sum_{l=0}^{2^k-1}\frac{|I^k_{l,\lambda}|}{q_n}\right.\\&\left.+\sum_{j=0}^{l}\frac{2|J^k_{j,\lambda}|}{q_n}\right)\times\left(\frac{t}{2},\frac{t+1}{2}\right),\ 0\leq l<2^{k-1},\ 2\leq k\leq n,\end{aligned}$$

where $t=\begin{cases}0, & \text{if } 0\leq l\leq 2^{k-2}-1\\ 1, & \text{if } 2^{k-2}\leq l\leq 2^{k-1}.\end{cases}$ The action of $\widetilde{\phi}_{n,\lambda}$ on the interval $J^1_{0,\lambda}=\hat{J}^1_{0,\lambda}\cup\hat{J}^1_{1,\lambda}$ as

$$\begin{aligned}\widetilde{\phi}_{n,\lambda}\left(\left[\frac{i}{q_n},\frac{i+1}{q_n}\right)\times\hat{J}^1_{l,\lambda}\right)=&\left[\frac{i}{q_n}+\sum_{l=0}^{1}\frac{|I^1_{l,\lambda}|}{q_n},\frac{i}{q_n}+\sum_{l=0}^{1}\frac{|I^1_{l,\lambda}|}{q_n}+\frac{2|\hat{J}^1_{l,\lambda}|}{q_n}\right)\\&\times\left(\frac{l}{2},\frac{l+1}{2}\right),\ \forall\ l=0,1.\end{aligned}$$

Then the final conjugation map $h_n:\mathbb{T}^2\longrightarrow\mathbb{T}^2$ can be described as

$$h_n=\overline{\phi}_{n,\lambda},$$

where $\overline{\phi}_{n,\lambda}$ is a smooth approximation of the map $\widetilde{\phi}_{n,\lambda}$.
In the same spirit as the construction in Subsection 5.5, we introduce an error set $E_{n,\lambda}\subset\mathbb{T}^2$ to designate regions where we lack sufficient control over the combinatorics. These regions are excluded from the construction to maintain the desired structural properties. Let each interval $J^n_{l,\lambda}\subset\mathbb{T}^1$ be denoted by its endpoints as $J^n_{l,\lambda}=(a^n_{l,\lambda},\,b^n_{l,\lambda})$. In particular, we define the *horizontal error set* $E_{h,n,\lambda}\subset E_{n,\lambda}$ by

$$E_{h,n,\lambda}=\bigcup_{l=0}^{2^{n-1}-1}\mathbb{T}^1\times\left((a^n_{l,\lambda},\,a^n_{l,\lambda}+\epsilon_n)\cup(b^n_{l,\lambda}-\epsilon_n,\,b^n_{l,\lambda})\right),$$

which removes narrow horizontal strips of width ϵ_n near the endpoints of each interval $J^n_{l,\lambda}$. The parameter ϵ_n is chosen appropriately to ensure that the total measure of the error set satisfies the decay condition $\mu(E_{n,\lambda}) < \frac{1}{e^{3n}}$. We denote the corresponding good sets by

$$F_{n,\lambda} = \mathbb{T}^2 \setminus E_{n,\lambda} \quad \text{and} \quad F^{(1)}_{n,\lambda} = \mathbb{T}^2 \setminus E_{h,n,\lambda},$$

which satisfy $\mu(F_{n,\lambda}) > 1 - \frac{1}{e^{3n}}$.
Analogously, we consider the specific domains as defined in equation (5.10). The following result is derived in the spirit of Lemma 5.8 and Proposition 5.6.

Proposition 5.9. *For $\varepsilon > 0$, consider a $(\frac{\sqrt{2}}{q_n}, \varepsilon)$-uniformly continuous function $\psi : \mathbb{T}^2 \longrightarrow \mathbb{R}$, i.e. $\psi(B_{\frac{\sqrt{2}}{q_n}}(x)) \subset B_\varepsilon(\psi(x))$. Then, for any $x \in \mathrm{G}_{n,\lambda}$, we have the following estimate:*

$$\left| \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \psi(h_n \circ S^i_{\alpha_{n+1}}(x)) - \int_{\mathbb{T}^2} \psi d\mu \right| \leq 4\varepsilon + \frac{2}{n^4} \|\psi\|_0. \tag{5.16}$$

Proof of Theorem 1.5. The proof will follow on the same line as the proof of Theorem 1.4. We start by choosing L_n to be uniform Lipshitz constant and $q_{n+1} = l_n q_n^2$ where l_n satisfying (5.13). Now it is enough to show that every point inside $\mathrm{B}_\lambda = \liminf_{n\to\infty} \mathrm{B}_{n,\lambda}$ where $\mathrm{B}_{n,\lambda} = H_n(\mathrm{G}_{n,\lambda})$ is a generic point, and its Hausdorff dimension is atleast $\alpha - 1$. The latter fact is followed by using Proposition 5.9 as done in Claim 2, and $\dim_H(C_\lambda) = \alpha - 1$.
In our specific case, the similar relations as mentioned in Remark 5.5 are satisfied, and which shows that every point inside the $\mathrm{NB}_\lambda = \limsup_{n\to\infty} H_n(\mathrm{NG}_{n,\lambda} \cap F^{(1)}_{n,\lambda})$ is a non-generic point. This completes the proof. $\square$

6

Smooth Weak Mixing Projectivized Derivative Extension

6.1 Introduction

In this chapter, we delve into the extreme ergodic properties of the projectivized derivative extension, with a particular focus on weak mixing. We explore this example for smooth category using the Approximation by Conjugation method. We examine the structure of a derivative extension map corresponding to a smooth map over the projectivized tangent bundle and explicitly induce an invariant smooth measure over it. We present an example of a derivative extension map that exhibits weak mixing with respect to such measure. In contrast to the approach in [35] and [33], we introduce a weak mixing criterion specifically tailored for the projectivized derivative extension in the smooth category. To demonstrate the weak mixing property of the projectivized derivative extension, we provide a suitable distribution of elements under the explicit choice of conjugation maps that meet the requirements of the criterion. In particular, we prove Theorem 1.6.

6.2 Preliminaries

6.2.1 Projectivized derivative Extension

We refer to [29, section 2.1] for useful definitions and notations. Subsequently, we introduce the concept of the invariant measure for the projectivized derivative extension.

We consider the derivative extension of a smooth diffeomorphism, $f : M \to M$, as a transformation defined on the tangent bundle TM, and denoted as $(f, \mathrm{d}f)$. For any point $p \in M$, we identify the tangent space T_pM with $\mathbb{R}^2$ in the case of a 2-dimensional manifold. Subsequently, we consider its projective space $\mathbb{PR}^2$, which is diffeomorphic to the circle $\mathbb{T}^1$ and introduce the notation $[a, b] \subset \mathbb{PR}^2$ to describe the allowed values for the spherical coordinate $\phi \in \mathbb{R}/\pi\mathbb{Z}$. This results in the formation of the projectivized tangent bundle denoted as $\mathbb{P}\mathrm{T}M$.

We consider the projectivized derivative extension of a diffeomorphism $f : M \to M$ on the projectivized tangent bundle $\mathbb{P}\mathrm{T}M$, denoted as $(f, \mathrm{d}f)$ again by abusing the notation. We employ the notation $c \times [a, b] \subset \mathbb{P}\mathrm{T}M$, with $c \subset M$, to represent the subset in $\mathbb{P}\mathrm{T}M$ comprising base points $x \in c$ and corresponding spherical coordinates $\phi \in [a, b]$.

6.2.1.1 Existence of invariant measure on the $\mathbb{P}\mathrm{T}M$:

The existence of an invariant measure is explored following the framework detailed in [13, chapter 5.1]. Initially, we consider the cotangent bundle TM^* along with the projection maps $\pi_1 : TM \to M$ and $\pi_2 : TM^* \to M$. Subsequently, we define the canonical 1-form ω on TM^* by $\omega|_\tau = \pi_2^*\tau$, where $\omega|_\tau$ denotes 1 form ω evaluated at $\tau \in TM^*$. Additionally, the canonical 2-form Ω on TM^* is defined as $\Omega = \mathrm{d}\omega$, exhibiting symplectic properties.

Let M be a Riemannian manifold, and $V : M \to \mathbb{R}$ be a function. We examine the Lagrangian $L : TM \to \mathbb{R}$ given by $L(\zeta) = |\zeta|^2 - V \circ \pi_1(\zeta)$, where $|\zeta|$ is calculated using the Riemannian metric. Associated with this Lagrangian, we define a bundle map $F_L : TM \to TM^*$ defined by $F_L(\zeta)(\eta) = \frac{d}{dt}(L(\zeta + t\eta))|_{t=0}$ for $p \in M$, and $\zeta, \eta \in T_pM$. Subsequently, we define $\Theta = F_L^*\Omega$ and $\nu = F_L^*\omega$.

In [13, Chapter 5.1], they examine the differential form $\nu \wedge \Theta$ on the unit tangent bundle SM, which is proven to be locally a product, up to a constant multiple, of the Riemannian volume on M with the Lebesgue 1-form on the unit tangent spheres of M with respect to the Riemannian metric. Particularly, for any $\nu \wedge \Theta$-integrable function g on SM, the "integration over the fibers", i.e. $\int_{SM} g\nu \wedge \Theta = c \int_M \mathrm{dVol}(p) \int_{S_pM} g|_{S_pM} \mathrm{d}\mu_p$. Here, Vol represents the volume form induced by the Riemannian metric, and μ_p is the standard Borel measure on the tangent sphere S_pM concerning the Riemannian metric.

Using a similar method, we derive an analogous formula for the invariant measurable Riemannian metric ω_∞ and for any integrable function on $\mathbb{P}\mathrm{T}M$. The associated measure will be represented by $\bar{\mu}$. Additionally, the measure derived from the measurable Riemannian metric ω_∞ in our explicit construction corresponds with the measure μ on M. Since ω_∞ is invariant under f, it follows that $\bar{\mu}$ is invariant under $(f, \mathrm{d}f)$.

6.3 Outline of the proof

In order to prove the Theorem 1.6, we utilize the Approximation by Conjugation method as outlined in Chapter 2. Similar to the constructions in [29], we introduce two sequences of partial partitions, η_n and ζ_n, for the torus, both converging to point decompositions. The partition η_n will satisfy the requirements to demonstrate the weak mixing property, and the other partition ζ_n consists of more refined partition elements of η_n. The conjugation map h_n acts as an isometry on the partition elements of ζ_n, allowing the construction of an invariant measurable Riemannian metric. Subsequently, the conjugation diffeomorphism $h_n = g_n \circ \phi_n$ is formulated. Compared to [29] and [35], modifications are made to g_n and ϕ_n

to establish the weak-mixing property of the projectivized derivative extension. Specifically, g_n introduces shear in the θ direction, acting as an isometry in the image ϕ_n' of any partition element $I \in \zeta_n$. Meanwhile, ϕ_n acts as an isometry on elements $I \in \zeta_n$ and simultaneously arranges elements of η_n to meet weak mixing criteria. The map ϕ_n is defined as $i_n \circ \tilde{\phi}_n$ in half of the fundamental domain of the torus and as an identity on the other half. Here, $\tilde{\phi}_n$ acts as an isometry and distributes the η_n partition elements as required. The map i_n is a composition of translation and rotation in different sections of domains in ζ_n, similar to [35], with a modified rotation domain according to our purpose, outlined in Section 6.4. The convergence of the approximation by conjugation scheme in the smooth topology and the existence of an invariant measure on $\mathbb{P}\mathrm{T}M$ are proved in Sections 6.8 and 6.9, respectively.

Compared to [29], we introduce a weak mixing criterion for the projectivized derivative extension, with a sequence of partial partition $\hat{\eta}_n$ for the space $\mathbb{P}\mathrm{T}M$ converges to point decomposition. This criterion is based upon the notion of $(\gamma, \delta, \varepsilon_1, \varepsilon_2)$ distribution of the map $(\Phi_n, \mathrm{d}\Phi_n) = (\phi_n, \mathrm{d}\phi_n) \circ (R_{\alpha_{n+1}}, dR_{\alpha_{n+1}})^{m_n} \circ (\phi_n, \mathrm{d}\phi_n)^{-1}$ with a specific choice of sequence $(m_n)_n \in \mathbb{N}$. This criterion is based on [29], but adapted for the derivative extension $(\Phi_n, \mathrm{d}\Phi_n)$ in Section 6.6. Finally, the application of the following criterion is carried out in Section 6.10 and 6.11.

6.4 Smooth AbC scheme

In our approximation by conjugation scheme in the smooth category we will inductively construct maps

$$f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}, \text{ where } H_n = h_1 \circ \ldots \circ h_n \tag{6.1}$$

with conjugation maps of the form

$$h_n = g_n \circ \phi_n, \tag{6.2}$$

where the area-preserving diffeomorphisms g_n and ϕ_n commute with R_{α_n} and will be explicitly constructed in Sections 6.4.1 and 6.7.2, respectively. We will choose the sequence of rationals $\alpha_n = \frac{p_n}{q_n}$ converging to a prescribed Liouville number α in such a way that α_{n+1} is sufficiently close to α_n in order to guarantee convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ to a limit diffeomorphism. In our explicit inductive construction process at step n, we have $H_{n-1} \in \mathrm{Diff}^\infty(\mathbb{T}^2, \mu)$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{T}^1$. Moreover, we are given some parameters $k_n \in \mathbb{N}$ and $l_n \in \mathbb{N}$ such that the following properties hold:

(P1) The parameter $k_n \geq n^5$ satisfies that for every subset $\hat{S} \subset \mathbb{P}\mathrm{T}M$ with $\mathrm{diam}(\hat{S}) < \frac{3}{k_n}$, we have

$$\mathrm{diam}\left((H_{n-1}, \mathrm{d}H_{n-1})(\hat{S})\right) \leq \frac{1}{n^2}. \tag{6.3}$$

(P2) The sequence $(k_n)_{n\in\mathbb{N}}$ increases rapidly enough to guarantee

$$\sum_{u=n}^{\infty} \frac{1}{k_u^5} \leq \frac{1}{4k_n^4} \quad \text{for every } n \in \mathbb{N}. \tag{6.4}$$

(P3) The parameter q_n is chosen to be large enough such that $q_n^{0.25} > 2k_n$. Note that this is possible since the parameter k_n is independent of q_n and depends only on the conjugation map H_{n-1}.

(P4) Growth condition on $l_n \geq 2 \cdot k_n^{10} \cdot q_n^2 \cdot \|\mathrm{d}H_{n-1}\|_0$, and $q_{n+1} > 2k_n^{12} q_n^2$.

6.4.1 The conjugation map g_n

We introduce the smooth map g_n as an approximation by suitably chosen step functions to the shear function $\tilde{g}_n(\theta, r) = (\theta + br, r)$, where $b = [nq_n^\sigma]$ and $0.25 < \sigma < 0.5$. The purpose of the map g_n is to introduce the appropriate kind of shear in the θ direction to achieve the weak mixing property on M. Additionally, the map g_n must act as an isometry on the image $\phi_n(I_n)$ of all partition elements $I_n \in \zeta_n$, where ζ_n is a specific partial partition of M defined in subsection 6.7.1.2

below. This property will allow us to show that f admits an invariant measurable Riemannian metric on the space, inducing an invariant measure on $\mathbb{P}\mathrm{T}M$ in our setup. This approach follows from [29].

Proposition 6.1. *Let* $a_n = k_n^5, \varepsilon_n = \frac{1}{2n^5 k_n^{10}}$, *and* $b_n = [nq_n^\sigma]$ *for* $0.25 < \sigma < 0.5$. *Then there exists* $g_n \in Ham^\infty(\mathbb{T}^2, \mu)$ *such that*

1. $g_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ g_n$.
2. *For any* $j \in \{0, \dots, a_n - 1\}$, g_n *acts on* $\mathbb{T}^1 \times [\frac{j+2\varepsilon_n}{a_n}, \frac{j+1-2\varepsilon_n}{a_n}]$ *as translation in the* θ*-direction by* $b_n \cdot \frac{j}{a_n}$.
3. $|||g_n|||_r \leq c_{n,k_n,r} \cdot q_n^r$, *where the constant* $c_{n,k_n,r}$ *is independent of* q_n.

Proof. Let $a, b \in \mathbb{N}$ and $\varepsilon > 0$ satisfying $\frac{1}{2\varepsilon} \in \mathbb{N}$. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function that is equal to 0 for $x \leq -1$ and takes the value 1 for $x \geq 1$. Consider the map $\tilde{\psi}_{a,b,\varepsilon} : [0,1] \to \mathbb{R}$ as

$$\tilde{\psi}_{a,b,\varepsilon}(x) = \frac{b}{a} \cdot \sum_{i=1}^{a-1} \rho\left(\frac{ax - i}{2\varepsilon}\right). \tag{6.5}$$

Observe that for any $0 \leq j \leq a-1$, we have $\tilde{\psi}_{a,b,\varepsilon}|_{[\frac{j+2\varepsilon}{a}, \frac{j+1-2\varepsilon}{a}]} = b \cdot \frac{j}{a} \mod 1$. Additionally, we can estimate $\|D^r \tilde{\psi}_{a,b,\varepsilon}\|_0 \leq \frac{b \cdot a^{r-1}}{\varepsilon^r} \cdot \|D^r \rho\|_0$.
We consider an antiderivative map $\varrho_{a,b,\varepsilon} : \mathbb{T} \to \mathbb{R}$ where $\frac{\mathrm{d}}{\mathrm{d}r} \varrho_{a,b,\varepsilon}(r) = \tilde{\psi}_{a,b,\varepsilon}(r)$. Furthermore, we can introduce a symplectic vector field

$$X_H = (\partial_r(H), -\partial_\theta(H)) = (\tilde{\psi}_{a,b,\varepsilon}, 0)$$

defined by the Hamiltonian $H \in C^\infty(\mathbb{T}^2, \mathbb{R})$ given by $H(\theta, r) = \varrho_{a,b,\varepsilon}(r)$. Finally, we obtain a Hamiltonian diffeomorphism $g_{a,b,\varepsilon} : \mathbb{T}^2 \to \mathbb{T}^2$ given by

$$g_{a,b,\varepsilon}(\theta, r) = (\theta + \tilde{\psi}_{a,b,\varepsilon}(r) \mod 1, r). \tag{6.6}$$

as the time-1 map of the Hamiltonian $H(\theta, r) = \varrho_{a,b,\varepsilon}(r)$.
In our explicit construction, we will use $g_n = g_{a_n,b_n,\varepsilon_n}$, where $a_n = k_n^5$, $\varepsilon_n = \frac{1}{2n^5 k_n^{10}}$

and $b_n = [nq_n^\sigma]$ for $0.25 < \sigma < 0.5$. Observe that $g_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ g_n$ and

$$|||g_n|||_r \leq \frac{b_n \cdot a_n^{r-1}}{\varepsilon_n^r} \cdot \|D^r\rho\|_0 \leq 2[nq_n^\sigma] \cdot k_n^{3(r-1)+2r} \cdot q_n^{r-1} \cdot \|D^r\rho\|_0 \leq c_{n,k_n,r} \cdot q_n^r,$$

where $c_{n,k_n,r}$ is a constant that depends only on n, k_n but is independent of q_n. $\square$

We define the "good domain" of g_n as

$$\mathcal{G}_n = \bigcup_{j\in\{0,\ldots,a_n-1\}} \mathbb{T}^1 \times \left[\frac{j+2\varepsilon_n}{a_n}, \frac{j+1-2\varepsilon_n}{a_n}\right]. \tag{6.7}$$

By the second part of Proposition 6.1 the map g_n acts as a translation on $\mathcal{G}_n$ and, hence, its differential map $\mathrm{d}_p g_n = \mathrm{id}$ for any base point $p \in \mathcal{G}_n$.

6.5 Criterion of an invariant Riemannian metric

We follow the criteria for the existence of an f-invariant measurable Riemannian metric, as deduced in [28, section 4.8] and [34, section 7]. Let ω_0 be the standard Riemannian metric for $\mathbb{T}^2$. The following definition will help to express closeness to being a local isometry for conjugation map h_n on the elements of some partial partition ζ_n.

Definition 6.8. We can define the *deviation from isometry* for a diffeomorphism f defined on a compact subset U of a smooth Riemannian manifold by

$$\mathrm{dev}_U(f) := \max_{v\in\mathrm{T}U, \|v\|=1} |\log \|\mathrm{d}f(v)\||.$$

Remark 6.2. Observe that $\mathrm{dev}_U(f) = 0$ if and only if f is a smooth isometry of U. Furthermore, we have

- $\mathrm{dev}_U(f) = \mathrm{dev}_{f(U)}\left(f^{-1}\right);$
- $\mathrm{dev}_U\left(\tilde{f} \circ f\right) \leq \mathrm{dev}_{f(U)}\left(\tilde{f}\right) + \mathrm{dev}_U(f).$

Proposition 6.3 (Criterion for the existence of a f-invariant measurable Riemannian metric)**.** *Let $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, defined by (6.1) and (6.2), such that $(f_n)_{n\in\mathbb{N}}$ converges to a limit diffeomorphism f in the* Diff^∞*-topology. Let $(\zeta_n)_{n\in\mathbb{N}}$ be a sequence of partial partitions whose elements cover a set of measure at least $1-\frac{1}{n^2}$ for every $n \in \mathbb{N}$. There exists a function $\kappa : \mathrm{Diff}^\infty(M,\mu) \to (0,1)$ such that for every decreasing sequence $(\mathfrak{d}_n)_{n\in\mathbb{N}}$ of positive reals satisfying $\sum_{k\geq n} \mathfrak{d}_k < \kappa(H_{n-1})$ for every $n \in \mathbb{N}$ the following holds: If for every $n \in \mathbb{N}$ and every partition element $I_n \in \zeta_n$ the conjugation map h_n satisfies $dev_{I_n}(h_n) \leq \frac{2\mathfrak{d}_n}{\|DH_{n-1}\|_0^2}$, then the limit diffeomorphism f admits an invariant measurable Riemannian metric.*

Proof. The proof follows along the lines of [34, section 7]. First, we put $\omega_n := (H_n^{-1})^*\omega_0$, and each ω_n is a smooth Riemannian metric as it is defined as the pullback of a smooth metric via a diffeomorphism. Since $R^*_{\alpha_{n+1}}\omega_0 = \omega_0$ the metric ω_n is f_n-invariant:

$$\begin{aligned} f_n^*\omega_n &= (H_n \circ R_{\alpha_{n+1} \circ H_n^{-1}})^*(H_n^{-1})^*\omega_0 = (H_n^{-1})^* R^*_{\alpha_{n+1}} H_n^*(H_n^{-1})^*\omega_0 \\ &= (H_n^{-1})^* R^*_{\alpha_{n+1}}\omega_0 = \omega_n. \end{aligned}$$

With our assumptions that $\mathrm{dev}_{I_n}(h_n) \leq \frac{\mathfrak{d}_n}{\|DH_{n-1}\|_0^2}$ for every $I_n \in \zeta_n$ and ζ_n covering a set of measure at least $1 - \frac{1}{n^2}$, we can compute as in [34, Lemma 7.2] that $\omega_\infty := \lim_{n\to\infty}\omega_n$ exists almost everywhere with respect to Lebesgue measure μ. To show that ω_∞ is a measurable Riemannian metric, we follow the proof of [34, Lemma 7.3]. This proof requires sufficient closeness of ω_∞ to $\omega_{n-1} = (H_{n-1}^{-1})^*\omega_0$ on a set of large measure. We express this requirement via the function κ. Finally, we apply [34, Lemma 7.5] to conclude that ω_∞ is f-invariant. □

6.6 Criterion for weak mixing of the derivative extension

Building on criteria for weak mixing for AbC diffeomorphisms on m-dimensional manifolds in [19] and [29], here we introduce such a criterion for weak mixing of the projectivized derivative extension.

Throughout this section we assume that the limiting smooth diffeomorphism $(f, \mathrm{d}f)$ (see 6.1) and the limiting analytic diffeomorphism $(\hat{f}, \mathrm{d}\hat{f})$ (see 7.1), admit an invariant measure $\bar{\mu}$. Strictly speaking we will be working with two different metrics for f and $\hat{f}$ and, hence, different measures for the smooth and analytic case, as the metrics will depend on the limiting diffeomorphims. To avoid unnecessary notational complexity we will use only one notation, namely $\bar{\mu}$.

6.6.1 $(\gamma, \delta, \varepsilon_1, \varepsilon_2)$ distribution

A key concept in our weak mixing criterion for the projectivized derivative extension is the notion of $(\gamma, \delta, \varepsilon_1, \varepsilon_2)$ distribution that we introduce in this subsection. It generalizes the notion of $(\gamma, \delta, \varepsilon)$ distribution from [19] that plays a central role in criteria for weak mixing for AbC diffeomorphisms in [19] and [29].

To express the $(\gamma, \delta, \varepsilon_1, \varepsilon_2)$ distribution property, we consider a partial partition, denoted as $\hat{\eta}_n$, of the space $\mathbb{P}\mathrm{T}M$ of a particular form. We let k_n be the number defined by condition **(P1)** and we let $\tilde{\eta}_n$ be a partial decomposition of M satisfying the following properties:

(D1) The conjugation map $h_n = g_n \circ \phi_n$ acts as a composition of translations and rotations, thus acting as an isometry, on the elements of the partial partition $\tilde{\eta}_n$.

(D2) Each partition element $\tilde{I}_n \in \tilde{\eta}_n$ is of the form $\tilde{I}_n = \bigcup_{l=0}^{k_n-1} \tilde{I}_{n,l}$, where each $\tilde{I}_{n,l}$ is a union of squares with a side length smaller than $\frac{1}{2k_n^5 q_n}$, lying in the good domain $\mathcal{G}_n$ of the map g_n (see (6.7)).

(D3) Each set $\tilde{I}_{n,l}$ has the same measure of at least $\frac{1}{2k_n^6 q_n}\left(1 - \frac{16}{k_n^5}\right)$. The union of elements from $\tilde{\eta}_n$ covers a set with a measure of at least $1 - \frac{25}{k_n^5}$ in M.

(D4) For any $\tilde{I}_{n,l}$, we have $\mu(\tilde{I}_{n,l} \triangle [\cup_{\tilde{I}^i_{n,s} \in \bar{\Lambda}_{\tilde{I}_{n,l}}} h_n(\tilde{I}^i_{n,s})]) \leq \frac{41}{k_n^5} \mu(\tilde{I}_{n,l})$, where

$$\bar{\Lambda}_{\tilde{I}_{n,l}} = \{\tilde{I}^i_{n,s} \; : \; h_n(\tilde{I}^i_{n,s}) \cap \tilde{I}_{n,l} \neq \emptyset, \; \tilde{I}^i_n = \sqcup_{s=0}^{k_n-1} \tilde{I}^i_{n,s} \in \tilde{\eta}_n\}.$$

(D5) Each $\tilde{I}_{n,l}$ is covered by elements of $\tilde{\eta}_{n+1}$ by measure of at least $\left(1-\frac{25}{k_{n+1}^5}\right)\mu(\tilde{I}_{n,l})$ and, in particular,

$$\mu\left(\tilde{I}_{n,l}\triangle[\cup_{\tilde{I}^i_{n+1,l'}\in\Lambda_{\tilde{I}_{n,l}}}\tilde{I}^i_{n+1,l'}]\right)\leq\frac{25}{k_{n+1}^5}\mu(\tilde{I}_{n,l}),$$

where

$$\Lambda_{\tilde{I}_{n,l}}=\{\tilde{I}^i_{n+1,l'}\ :\ \tilde{I}^i_{n+1,l'}\cap\tilde{I}_{n,l}\neq\emptyset,\ \tilde{I}^i_{n+1}=\sqcup_{l'=0}^{k_{n+1}-1}\tilde{I}^i_{n+1,l'}\in\tilde{\eta}_{n+1}\}.$$

Then we let $\hat{\eta}_n$ be a partial partition of the space $\mathbb{P}\mathrm{T}M$ of the following form:

$$\hat{\eta}_n=\left\{\tilde{I}_n\times T_j\ :\ \tilde{I}_n\in\tilde{\eta}_n,\ j=0,\dots,k_n-1\right\},\tag{6.9}$$

where $T_j=\left[\frac{j}{k_n},\frac{j+1}{k_n}\right]$, that is, elements $\hat{I}_{n,j}\in\hat{\eta}_n$ are of the form

$$\hat{I}_{n,j}=\tilde{I}_n\times T_j=\bigcup_{l=0}^{k_n-1}\tilde{I}_{n,l,j},\quad\text{where}\quad\tilde{I}_{n,l,j}:=\tilde{I}_{n,l}\times T_j,$$

with $\tilde{I}_n=\bigcup_{l=0}^{k_n-1}\tilde{I}_{n,l}\in\tilde{\eta}_n$ and $j\in\{0,\dots,k_n-1\}$.

Remark 6.4. For our explicit constructions in Section 6.7, properties **(D1)**, **(D2)**, **(D3)**, **(D4)**, and **(D5)** are verified in Remark 6.28, Remark 6.15, Remark 6.33, and Remark 6.16 respectively.

Definition 6.10. A diffeomorphism $(\Phi_n,\mathrm{d}\Phi_n):\mathbb{P}\mathrm{T}M\longrightarrow\mathbb{P}\mathrm{T}M$ is $(\gamma,\delta,\varepsilon_1,\varepsilon_2)$-distributing $\hat{I}_{n,j}=\tilde{I}_n\times T_j\in\hat{\eta}_n$, where $\tilde{I}_n=\cup_{l=0}^{k_n-1}\tilde{I}_{n,l}\in\tilde{\eta}_n$, if the following properties hold

1. for any $l\in\{0,\dots,k_n-1\}$, we have $J_l\subseteq\pi_r(\Phi_n(\tilde{I}_{n,l}))$ with $1-\delta\leq\lambda(J_l)\leq1$ where π_r is the projection onto the r-axis;
2. for any $l\in\{0,\dots,k_n-1\}$, we have $\Phi_n(\tilde{I}_{n,l})\subseteq K_{c_l,\gamma}:=[c_l,c_l+\gamma]\times[0,1]$ for some constant $c_l\in\mathbb{T}^1$;

3. $\Phi_n : M \to M$ is ε_1-distributing $\tilde{I}_n \in \tilde{\eta}_n$, that is, for $l \in \{0, \ldots, k_n - 1\}$ and for any $\tilde{J} \subseteq J_l$,

$$\left| \frac{\mu(\tilde{I}_{n,l} \cap \Phi_n^{-1}(\mathbb{T} \times \tilde{J}))}{\mu(\tilde{I}_{n,l})} - \frac{\lambda(\tilde{J})}{\lambda(J_l)} \right| \leq \varepsilon_1 \frac{\lambda(\tilde{J})}{\lambda(J_l)};$$

4. $(\Phi_n, \mathrm{d}\Phi_n)$ is ε_2-distributing $\hat{I}_{n,j} \in \hat{\eta}_n$ in the tangent direction in the following sense: for all $j, k \in \{0, 1, \ldots, k_n - 1\}$, there exists a unique $l \in \{0, 1, \ldots, k_n - 1\}$ such that $k \equiv l + j \mod k_n$, and for $\tilde{J} \subseteq J_l$ and $T_k = \left[\frac{k}{k_n}, \frac{k+1}{k_n}\right]$ we have

$$\left| \frac{\mu\left(\pi_M\left(\tilde{I}_{n,l} \times T_j \cap (\Phi_n, \mathrm{d}\Phi_n)^{-1}(\mathbb{T} \times \tilde{J} \times T_k)\right)\right)}{\mu(\tilde{I}_{n,l})} - \frac{\lambda(\tilde{J})}{\lambda(J_l)} \right| \leq \varepsilon_2 \frac{\lambda(\tilde{J})}{\lambda(J_l)},$$

where π_M denotes the projection onto the manifold M.

We will rewrite the fourth condition of the above definition as follows:

$$\left| \mu\left(\pi_M\left(\tilde{I}_{n,l,j} \cap (\Phi_n, \mathrm{d}\Phi_n)^{-1}(\mathbb{T} \times \tilde{J} \times T_k)\right)\right) \lambda(J_l) - \mu(\tilde{I}_{n,l})\lambda(\tilde{J}) \right| \leq \varepsilon_2 \mu(\tilde{I}_{n,l})\lambda(\tilde{J}).$$

In particular, the distribution in the tangent direction implies that the product set of the lth segment $\tilde{I}_{n,l}$ of a partition element $\tilde{I}_n \in \tilde{\eta}_n$ on the manifold with the jth tangent element T_j is mapped under $(\Phi_n, \mathrm{d}\Phi_n)$ to the kth tangent element T_k, where $k \equiv l + j \mod k_n$.

Key Idea of proof: For $\tilde{I}_n = \cup_{l=0}^{k_n-1} \tilde{I}_{n,l} \in \tilde{\eta}_n$ and $l, j \in \{0, \ldots, k_n - 1\}$, we consider $\Gamma_{n,l,j} = (H_{n-1}, \mathrm{d}H_{n-1})(g_n, dg_n)(\tilde{I}_{n,l} \times T_j)$. We approximate its measure by using the "good domain" of H_{n-1}, defined as $\widetilde{G}_n$ in (6.14), where the measure $\bar{\mu}$ is computed with the length in the tangent direction and it follows

$$|\bar{\mu}(\Gamma_{n,l,j}) - \frac{1}{k_n}\mu(\tilde{I}_{n,l})| \to 0 \quad \text{as} \quad n \to \infty.$$

In fact, it subsequently follows in Lemma 6.6 for $\Gamma_{n,j} = \cup_{l=0}^{k_n-1} \Gamma_{n,l,j}$ that

$$|\bar{\mu}(\Gamma_{n,j}) - \mu(\tilde{I}_{n,l})| \to 0 \quad \text{as} \quad n \to \infty. \tag{6.11}$$

Analogously, we obtain in Lemma 6.6 for any $l, j \in \{0, \ldots, k_n - 1\}$ that $|\bar{\mu}(\Gamma_{n,l,j} \cap (f_n, \mathrm{d}f_n)^{-m_n}(C_{n,k})) - \frac{1}{k_n}\mu(\pi_M(\tilde{I}_{n,l} \times T_j \cap (g_n \circ \Phi_n, \mathrm{d}g_n \circ \Phi_n)^{-1}(\tilde{S}_{n,k}))|$ tends to 0, where $C_{n,k} = (H_{n-1}, \mathrm{d}H_{n-1})(\tilde{S}_{n,k})$ and $\{\tilde{S}_{n,k}\}$ is a collection of small cubes inside $\mathbb{P}\mathrm{T}M$ that covers almost the entire space.
Together with the distribution property of $(\Phi_n, \mathrm{d}\Phi_n)$ and $(g_n, \mathrm{d}g_n)$ proved in the Lemma 6.8, this yields the estimate

$$|\bar{\mu}(\Gamma_{n,l,j} \cap (f_n, \mathrm{d}f_n)^{-m_n}(C_{n,k})) - \mu(\tilde{I}_{n,l})\bar{\mu}(C_{n,k})| \to 0. \tag{6.12}$$

Finally using equations 6.11 and 6.12, we obtain that for all $j, k \in \{0, \ldots, k_n - 1\}$, the following estimate holds:

$$|\bar{\mu}(\Gamma_{n,j} \cap (f_n, \mathrm{d}f_n)^{-m_n}(C_{n,k})) - \bar{\mu}(\Gamma_{n,j})\bar{\mu}(C_{n,k})| \to 0, \tag{6.13}$$

which is the final step required for the criterion of weak mixing in Proposition 6.11.

6.6.2 Preliminary lemmas for weak mixing

Using the partial partition $\tilde{\eta}_n$, we define a "good domain" for H_{n-1}, where the composition of conjugation maps $h_{n+s}^{-1} \circ \ldots \circ h_{n+1}^{-1} \circ h_n^{-1}$, $s \geq 0$, acts as an isometry. This will be useful for approximating the $\bar{\mu}$ measure of specific sets (see Lemma 6.6). We denote $G_n = \cup_{\tilde{I}_n \in \tilde{\eta}_n} h_n(\tilde{I}_n)$, and define

$$\widetilde{G}_n = G_n \cap \bigcap_{j=1}^{\infty} h_n \circ h_{n+1} \circ \ldots \circ h_{n+j-1}(G_{n+j}), \tag{6.14}$$

where $\tilde{I}_n = \cup_{l=0}^{k_n-1} \tilde{I}_{n,l} \in \tilde{\eta}_n$. Then, we define the elements $\breve{I}_{n,l}$ as $\breve{I}_{n,l} = \tilde{I}_{n,l} \cap \widetilde{G}_n$.

Note that each $\tilde{I}_{n,l}$ is almost covered by the elements $\tilde{I}_{n+1} \in \tilde{\eta}_{n+1}$ by at least measure of $\left(1 - \frac{25}{k_{n+1}^5}\right)\mu(\tilde{I}_{n,l})$, by **(D5)**. Specifically, we have

$$\mu(\tilde{I}_{n,l} \triangle \cup_{\tilde{I}^i_{n+1,l'} \in \Lambda_{\tilde{I}_{n,l}}} \tilde{I}^i_{n+1,l'}) \leq \frac{25}{k_{n+1}^5}\mu(\tilde{I}_{n,l}).$$

Moreover, property **(D4)** states that for each $\tilde{I}^i_{n+1,l'}$, we have

$$\mu(\tilde{I}^i_{n+1,l'} \triangle [\cup_{\tilde{I}^k_{n+1,s} \in \bar{\Lambda}_{\tilde{I}^i_{n+1,l'}}} h_{n+1}(\tilde{I}^k_{n+1,s})]) \leq \frac{41}{k^5_{n+1}} \mu(\tilde{I}^i_{n+1,l'}).$$

We denote $\mathcal{C}_{\tilde{I}_{n,l}} = \cup_{\tilde{I}^i_{n+1,l'} \in \Lambda_{\tilde{I}_{n,l}}} \cup_{\tilde{I}^k_{n+1,s} \in \bar{\Lambda}_{\tilde{I}^i_{n+1,l'}}} \tilde{I}^k_{n+1,s}$. Altogether, for each $\tilde{I}_{n,l}$, we have

$$\begin{aligned}
&\mu\Big(h_n(\tilde{I}_{n,l}) \triangle [\cup_{\tilde{I}^k_{n+1,s} \in \mathcal{C}_{\tilde{I}_{n,l}}} h_n \circ h_{n+1}(\tilde{I}^k_{n+1,s})] \Big) \\
&= \mu \left(\tilde{I}_{n,l} \triangle [\cup_{\tilde{I}^k_{n+1,s} \in \mathcal{C}_{\tilde{I}_{n,l}}} h_{n+1}(\tilde{I}^k_{n+1,s})] \right) \\
&\leq \mu \left(\cup_{\tilde{I}^i_{n+1,l'} \in \Lambda_{\tilde{I}_{n,l}}} \tilde{I}^i_{n+1,l'} \triangle [\cup_{\tilde{I}^k_{n+1,s} \in \mathcal{C}_{\tilde{I}_{n,l}}} h_{n+1}(\tilde{I}^k_{n+1,s})] \right) \\
&\quad + \mu \left(\cup_{\tilde{I}^i_{n+1,l'} \in \Lambda_{\tilde{I}_{n,l}}} \tilde{I}^i_{n+1,l'} \triangle \tilde{I}_{n,l} \right) \\
&\leq \mu \left(\cup_{\tilde{I}^i_{n+1,l'} \in \Lambda_{\tilde{I}_{n,l}}} \tilde{I}^i_{n+1,l'} \triangle [\cup_{\tilde{I}^k_{n+1,s} \in \mathcal{C}_{\tilde{I}_{n,l}}} h_{n+1}(\tilde{I}^k_{n+1,s})] \right) + \frac{25}{k^5_{n+1}} \mu(\tilde{I}_{n,l}) \\
&= \sum_{\tilde{I}^i_{n+1,l'} \in \Lambda_{\tilde{I}_{n,l}}} \mu \left(\tilde{I}^i_{n+1,l'} \triangle [\cup_{\tilde{I}^k_{n+1,s} \in \bar{\Lambda}_{\tilde{I}^i_{n+1,l'}}} h_{n+1}(\tilde{I}^k_{n+1,s})] \right) + \frac{25}{k^5_{n+1}} \mu(\tilde{I}_{n,l}) \\
&\leq \sum_{\tilde{I}^i_{n+1,l'} \in \Lambda_{\tilde{I}_{n,l}}} \frac{41}{k^5_{n+1}} \mu(\tilde{I}^i_{n+1,l'}) + \frac{25}{k^5_{n+1}} \mu(\tilde{I}_{n,l}) \\
&= \frac{41}{k^5_{n+1}} \mu(\cup_{\tilde{I}^i_{n+1,l'} \in \Lambda_{\tilde{I}_{n,l}}} \tilde{I}^i_{n+1,l'}) + \frac{25}{k^5_{n+1}} \mu(\tilde{I}_{n,l}) \\
&\leq \frac{41}{k^5_{n+1}} \left(\mu(\tilde{I}_{n,l}) + \frac{25}{k^5_{n+1}} \mu(\tilde{I}_{n,l}) \right) + \frac{25}{k^5_{n+1}} \mu(\tilde{I}_{n,l}) \\
&\leq \frac{70}{k^5_{n+1}} \mu(\tilde{I}_{n,l}).
\end{aligned} \tag{6.15}$$

Similarly, using properties **(D4)** and **(D5)**, we find that each $\tilde{I}^i_{n+1,l'}$ is covered by elements of $h_{n+2}(\tilde{I}^k_{n+2,s'})$, where $\tilde{I}^k_{n+2,s'} \in \mathcal{C}_{\tilde{I}^i_{n+1,l'}}$. This concludes the following estimation:

$$\mu \left(h_n \circ h_{n+1}(\tilde{I}^i_{n+1,l'}) \triangle \cup_{\tilde{I}^k_{n+2,s'} \in \mathcal{C}_{\tilde{I}^i_{n+1,l'}}} h_n \circ h_{n+1} \circ h_{n+2}(\tilde{I}^k_{n+2,s'}) \right)$$

$$\leq \frac{70}{k_{n+2}^5}\mu(h_{n+1}(\tilde{I}^i_{n+1,l'})). \tag{6.16}$$

Given any set $\tilde{I}_{n,l}$, we consider the following family of sets:

$$\begin{aligned}
A_1 &= h_n(\tilde{I}_{n,l}),\\
A_2 &= \bigcup_{\tilde{I}^i_{n+1,l'} \in \mathcal{C}_{\tilde{I}_{n,l}}} h_n \circ h_{n+1}(\tilde{I}^i_{n+1,l'}),\\
A_3 &= \bigcup_{\tilde{I}^i_{n+1,l'} \in \mathcal{C}_{\tilde{I}_{n,l}}} \bigcup_{\tilde{I}^k_{n+2,s} \in \mathcal{C}_{\tilde{I}^i_{n+1,l'}}} h_n \circ h_{n+1} \circ h_{n+2}(\tilde{I}^k_{n+2,s}),\\
&\vdots\\
A_{n_1} &= \bigcup_{\tilde{I}^i_{n+1,l'} \in \mathcal{C}_{\tilde{I}_{n,l}}} \bigcup_{\tilde{I}^k_{n+2,s} \in \mathcal{C}_{\tilde{I}^i_{n+1,l'}}} \cdots \bigcup_{\tilde{I}^\ell_{n+n_1-1,s''} \in \mathcal{C}_{\tilde{I}_{n+n_1-1,s'}}} h_n \circ \ldots \circ h_{n+n_1-1}(\tilde{I}^\ell_{n+n_1-1,s''}).
\end{aligned}$$

Analogous to equations (6.15) and (6.16), we have estimates for $n_1 = 2, 3, 4, \ldots$,

$$\mu(A_{n_1-1} \triangle A_{n_1}) \leq \frac{70}{k_{n+n_1-1}^5}\mu(A_{n_1-1}). \tag{6.17}$$

Using the triangle inequality on the aforementioned equation, we conclude that

$$\begin{aligned}
\mu(A_1 \triangle A_3) &\leq \mu(A_1 \triangle A_2) + \mu(A_2 \triangle A_3)\\
&\leq \mu(A_1 \triangle A_2) + \frac{70}{k_{n+2}^5}\mu(A_2)\\
&\leq \mu(A_1 \triangle A_2) + \frac{70}{k_{n+2}^5}(\mu(A_1) + \mu(A_1 \triangle A_2))\\
&\leq \left(1 + \frac{70}{k_{n+2}^5}\right)\mu(A_1 \triangle A_2) + \frac{70}{k_{n+2}^5}\mu(A_1)\\
&\leq \left(\left(1 + \frac{70}{k_{n+2}^5}\right)\frac{70}{k_{n+1}^5} + \frac{70}{k_{n+2}^5}\right)\mu(A_1).
\end{aligned}$$

Likewise,

$$\mu(A_1 \triangle A_4) \leq \mu(A_1 \triangle A_3) + \mu(A_3 \triangle A_4)$$

$$\leq \left(1+\frac{70}{k_{n+3}^5}\right)\mu(A_1\triangle A_3)+\frac{70}{k_{n+3}^5}\mu(A_1)$$
$$\leq \left(\left(1+\frac{70}{k_{n+3}^5}\right)\left(1+\frac{70}{k_{n+2}^5}\right)\frac{70}{k_{n+1}^5}+\left(1+\frac{70}{k_{n+3}^5}\right)\frac{70}{k_{n+2}^5}+\frac{70}{k_{n+3}^5}\right)\mu(A_1).$$

By applying recursion using equation (6.17), we obtain

$$\begin{aligned}\mu(A_1\triangle A_{n_1}) &\leq \left(1+\frac{70}{k_{n+n_1-1}^5}\right)\mu(A_1\triangle A_{n_1-1})+\frac{70}{k_{n+n_1-1}^5}\mu(A_1)\\ &\leq \left(\sum_{u=1}^{n_1-2}\prod_{j=u+1}^{n_1-1}\left(1+\frac{70}{k_{n+j}^5}\right)\frac{70}{k_{n+u}^5}+\frac{70}{k_{n+n_1-1}^5}\right)\mu(A_1). \end{aligned} \tag{6.18}$$

Following this, we apply mathematical induction to derive the approximation for the infinite intersection of these sets $\{A_n\}_{n\in\mathbb{N}}$. According to equation (6.18), it can be inferred that, for $n=3$:

$$\begin{aligned}\mu(A_1\cap A_2\cap A_3) &= \mu(A_1\cap A_2)-\mu(A_1\cap A_2\setminus A_3)\\ &\geq \mu(A_1\cap A_2)-\mu(A_2\setminus A_3)\\ &\geq \mu(A_1)-\mu(A_1\triangle A_2)-\mu(A_2\triangle A_3)\\ &\geq \mu(A_1)-\mu(A_1\triangle A_2)-\frac{70}{k_{n+2}^5}\mu(A_2)\\ &\geq \mu(A_1)-\mu(A_1\triangle A_2)-\frac{70}{k_{n+2}^5}(\mu(A_1)+\mu(A_1\triangle A_2))\\ &\geq \mu(A_1)-\left(1+\frac{70}{k_{n+2}^5}\right)\mu(A_1\triangle A_2)-\frac{70}{k_{n+2}^5}\mu(A_1)\\ &\geq \left(1-\left(1+\frac{70}{k_{n+2}^5}\right)\frac{70}{k_{n+1}^5}-\frac{70}{k_{n+2}^5}\right)\mu(A_1).\end{aligned}$$

Having started the induction process with $n_1=3$, we assume the following statement holds for $n=n_1$:

$$\mu(A_1\cap A_2\ldots\cap A_{n_1})\geq\left(1-\sum_{u=1}^{n_1-2}\prod_{j=u+1}^{n_1-1}\left(1+\frac{70}{k_{n+j}^5}\right)\frac{70}{k_{n+u}^5}-\frac{70}{k_{n+n_1-1}^5}\right)\mu(A_1).$$

We now show the result for $n = n_1 + 1$ with the above assumption:

$$\begin{aligned}
&\mu(A_1 \cap A_2 \ldots \cap A_{n_1} \cap A_{n_1+1}) \\
&= \mu(A_1 \cap A_2 \ldots \cap A_{n_1}) - \mu(A_1 \cap A_2 \ldots \cap A_{n_1} \setminus A_{n_1+1}) \\
&\geq \mu(A_1 \cap A_2 \ldots \cap A_{n_1}) - \mu(A_{n_1} \triangle A_{n_1+1}) \\
&\geq \left(1 - \sum_{u=1}^{n_1-2} \prod_{j=u+1}^{n_1-1} \left(1 + \frac{70}{k_{n+j}^5}\right) \frac{70}{k_{n+u}^5} - \frac{70}{k_{n+n_1-1}^5}\right) \mu(A_1) - \frac{70}{k_{n+n_1}^5} \mu(A_{n_1}) \\
&\geq \left(1 - \sum_{u=1}^{n_1-2} \prod_{j=u+1}^{n_1-1} \left(1 + \frac{70}{k_{n+j}^5}\right) \frac{70}{k_{n+u}^5} - \frac{70}{k_{n+n_1-1}^5}\right) \mu(A_1) - \frac{70}{k_{n+n_1}^5} \mu(A_1) \\
&\quad - \frac{70}{k_{n+n_1}^5} \left(\sum_{u=1}^{n_1-2} \prod_{j=u+1}^{n_1-1} \left(1 + \frac{70}{k_{n+j}^5}\right) \frac{70}{k_{n+u}^5} + \frac{70}{k_{n+n_1-1}^5}\right) \mu(A_1) \\
&\geq \left(1 - \left(1 + \frac{70}{k_{n+n_1}^5}\right) \left(\sum_{u=1}^{n_1-2} \prod_{j=u+1}^{n_1-1} \left(1 + \frac{70}{k_{n+j}^5}\right) \frac{70}{k_{n+u}^5}\right)\right) \mu(A_1) \\
&\quad \left(- \left(1 + \frac{70}{k_{n+n_1}^5}\right) \frac{70}{k_{n+n_1-1}^5} - \frac{70}{k_{n+n_1}^5}\right) \mu(A_1) \\
&\geq \left(1 - \sum_{u=1}^{n_1-2} \prod_{j=u+1}^{n_1} \left(1 + \frac{70}{k_{n+j}^5}\right) \frac{70}{k_{n+u}^5} - \left(1 + \frac{70}{k_{n+n_1}^5}\right) \frac{70}{k_{n+n_1-1}^5} - \frac{70}{k_{n+n_1}^5}\right) \mu(A_1) \\
&\geq \left(1 - \sum_{u=1}^{n_1-1} \prod_{j=u+1}^{n_1} \left(1 + \frac{70}{k_{n+j}^5}\right) \frac{70}{k_{n+u}^5} - \frac{70}{k_{n+n_1}^5}\right) \mu(A_1). \qquad (6.19)
\end{aligned}$$

By employing mathematical induction, the aforementioned estimates are valid for every natural number, leading to the following conclusion:

$$\begin{aligned}
\mu(\breve{I}_{n,l}) &= \mu\left(\tilde{I}_{n,l} \cap \widetilde{G}_n\right) \\
&\geq \left(1 - \sum_{u=0}^{\infty} \prod_{j=u+1}^{\infty} \left(1 + \frac{70}{k_{n+j}^5}\right) \frac{70}{k_{n+u}^5} - \frac{70}{k_{n+1}^5}\right) \mu(\tilde{I}_{n,l}) \\
&\geq \left(1 - 2 \cdot \sum_{u=0}^{\infty} \frac{70}{k_{n+u}^5}\right) \mu(\tilde{I}_{n,l}) \\
&\geq \left(1 - \frac{140}{4k_n^4}\right) \mu(\tilde{I}_{n,l}). \qquad (6.20)
\end{aligned}$$

We employ the fact that $\prod_{j=1}^{\infty}\left(1+\frac{70}{k_{n+j}^5}\right) \leq 2$, and under condition **(P2)**, the sequence $(k_j)_{j\in\mathbb{N}}$ increases rapidly enough to guarantee that $\sum_{u=0}^{\infty} \frac{140}{k_{n+u}^5} \leq \frac{140}{4k_n^4}$.

Remark 6.5. For any $S_n' \subseteq \mathbb{T}^2$, we define $\tilde{c}_{n,l} = \tilde{I}_{n,l} \cap \Phi_n^{-1} \circ g_n^{-1}(S_n')$ and $\breve{c}_{n,l} = \breve{I}_{n,l} \cap \Phi_n^{-1} \circ g_n^{-1}(S_n')$. Thus, we obtain $|\mu(\tilde{c}_{n,l}) - \mu(\breve{c}_{n,l})| \leq \mu(\tilde{I}_{n,l} \backslash \breve{I}_{n,l}) \leq \frac{35}{k_n^4}\mu(\tilde{I}_{n,l})$.

Lemma 6.6. *Denote* $\mathcal{A}_{n,l,j} = (H_{n-1}, \mathrm{d}H_{n-1}) \circ (g_n, \mathrm{d}g_n)(\tilde{c}_{n,l,j})$, *where* $\tilde{c}_{n,l,j} = \tilde{c}_{n,l} \times T_j$. *Then we have*

$$\left|\bar{\mu}(\mathcal{A}_{n,l,j}) - \frac{1}{k_n}\mu(\pi_M(\tilde{c}_{n,l,j}))\right| \leq \frac{35(k_n-1)}{k_n^5}\mu(\tilde{I}_{n,l}). \tag{6.21}$$

Moreover, for any $S_n \times T_k \subset \mathbb{P}\mathrm{T}M$ *and* $C_{n,k} = (H_{n-1}, \mathrm{d}H_{n-1})(\tilde{S}_n \times T_k)$, *where* $\tilde{S}_n = S_n \cap \widetilde{G}_n$, *we have*

$$\bar{\mu}(C_{n,k}) = \frac{1}{k_n}\mu(\tilde{S}_n). \tag{6.22}$$

Proof. Note that $\breve{c}_{n,l} = \tilde{c}_{n,l} \cap \widetilde{G}_n$. For any $x \in \breve{c}_{n,l}$, let $y = H_{n-1}(x)$. Then let $(y,v), (y,w)$ be defined as

$$(y,v) = (H_{n-1}(x), \mathrm{d}_x H_{n-1}(\bar{v})) \quad \text{and} \quad (y,w) = (H_{n-1}(x), \mathrm{d}_x H_{n-1}(\bar{w})),$$

with $x \in \breve{c}_{n,l} \subset \widetilde{G}_n$, $\bar{v}, \bar{w} \in T_j$. Using the definition of the f-invariant Riemannian metric ω_∞, which induces the measure $\bar{\mu}$ invariant under the map $(f, \mathrm{d}f)$, we conclude that

$$\begin{aligned}
\omega_\infty \restriction_y (v,w) &= \lim_{k\to\infty} (H_k^{-1})^* \omega_0 \restriction_y (v,w) \\
&= \lim_{k\to\infty} \omega_0 \restriction_{H_k^{-1}(y)} (\mathrm{d}_y H_k^{-1}(v), \mathrm{d}_y H_k^{-1}(w)) \\
&= \lim_{k\to\infty} \omega_0 \restriction_{h_k^{-1}\circ\ldots\circ h_n^{-1}(x)} (\mathrm{d}_x h_k^{-1} \circ \ldots \circ h_n^{-1}(\bar{v}), \mathrm{d}_x h_k^{-1} \circ \ldots \circ h_n^{-1}(\bar{w})).
\end{aligned}$$

Since $h_k^{-1} \circ \ldots \circ h_n^{-1}$ is an isometry with respect to ω_0 on $\widetilde{G}_n$ and ω_0 is independent from the base point, we conclude that $\omega_\infty \restriction_y (v,w) = \omega_0 \restriction_x (\bar{v}, \bar{w})$. Thus, for images under the map $(H_{n-1}, \mathrm{d}H_{n-1})$ of sets with base points $x \in \breve{c}_{n,l} \subseteq \widetilde{G}_n$, the measure

$\bar{\mu}$ acts as a uniform measure, preserving the length in the tangent direction. By utilizing the estimate in Remark 6.5 and noting that g_n^{-1} acts as translation on $\widetilde{G}_n$, we have

$$\frac{1}{k_n}\mu(g_n^{-1}(\breve{c}_{n,l})) \leq \bar{\mu}(\mathcal{A}_{n,l,j}) \leq \frac{1}{k_n}\mu(g_n^{-1}(\breve{c}_{n,l})) + \frac{35(k_n-1)}{k_n^5}\mu(g_n^{-1}(\tilde{I}_{n,l})).$$

Finally, using that g_n is measure preserving and estimate $\mu(\tilde{c}_{n,l}) \geq \mu(\breve{c}_{n,l}) \geq \mu(\tilde{c}_{n,l}) - \frac{35}{k_n^4}\mu(\tilde{I}_{n,l})$, we conclude that

$$\left|\bar{\mu}(\mathcal{A}_{n,l,j}) - \frac{1}{k_n}\mu(\pi_M(\tilde{c}_{n,l,j}))\right| \leq \frac{35(k_n-1)}{k_n^5}\mu(\tilde{I}_{n,l}).$$

Analogously to the above estimate, we conclude the other claim. □

Lemma 6.7. *Consider $g_n : \mathbb{T}^2 \to \mathbb{T}^2$ as defined in Proposition 6.1 with the same a_n and ε_n. Let K be an interval on r-axis of the form $[\frac{i+2\varepsilon_n}{a_n}, \frac{i+1-2\varepsilon_n}{a_n}]$ where $i \in \{0, ..., a_n - 1\}$. Denote $K_{c_l,\gamma} = [c_l, c_l + \gamma] \times K$ for some constant $c_l \in \mathbb{T}^1$ and γ. Let $L = [l_1, l_2]$ be an interval on the θ axis. If $b_n\lambda(K) > 2$, then for $Q = \pi_r(K_{c_l,\gamma} \cap g_n^{-1}(L \times K))$, it holds*

$$|\lambda(Q) - \lambda(K)\lambda(L)| \leq \frac{2}{b_n}\lambda(L) + \frac{2\gamma}{b_n} + \gamma\lambda(K) + \frac{b_n}{a_n}\lambda(K) + \frac{2}{a_n}, \tag{6.23}$$

where λ is the Lebesgue measure of $\mathbb{R}$.

Proof. The proof can be derived directly from [29, Lemma 6.4] by considering the dimension $m = 2$. □

Lemma 6.8. *Let $n > 4$, g_n be as in Proposition 6.1, and $\hat{I}_{n,j} = \cup_{l=0}^{k_n-1}\tilde{I}_{n,l,j} \in \hat{\eta}_n$, where $\hat{\eta}_n$ is a partial partition of $\mathbb{P}\mathrm{T}M$ as described by equation (6.9). Suppose the diffeomorphism $(\Phi_n, \mathrm{d}\Phi_n) : \mathbb{P}\mathrm{T}M \to \mathbb{P}\mathrm{T}M$ is $(\gamma, \delta, \varepsilon_1, \varepsilon_2)$-distributing $\hat{I}_{n,j} \in \hat{\eta}_n$ with $\gamma < \frac{1}{k_n^3 q_n}, \delta < \frac{50}{k_n^2}, \varepsilon_1, \varepsilon_2 < \frac{1}{n}$. Let S_n be a square of side length equal to $\frac{1}{k_n}$ lying in $\mathbb{T}^1 \times [\delta, 1-\delta]$. Furthermore, let $\tilde{S}_n = S_n \cap \widetilde{G}_n$ using the good domain $\widetilde{G}_n$ from (6.14). For $k \in \{0, \ldots, k_n - 1\}$, denote $\tilde{S}_{n,k} = \tilde{S}_n \times T_k$, where $T_k = \left[\frac{k}{k_n}, \frac{k+1}{k_n}\right]$.*

Then for every pair $j, k \in \{0, 1, \ldots, k_n - 1\}$, there exists $l \in \{0, 1, \ldots, k_n - 1\}$

such that $k \equiv (l+j) \mod k_n$ *and it holds:*

$$\begin{aligned}&\left|\mu\left(\pi_M\left(\tilde{I}_{n,l,j} \cap (\Phi_n, \mathrm{d}\Phi_n)^{-1} \circ (g_n, \mathrm{d}g_n)^{-1}(\tilde{S}_{n,k})\right)\right) - \mu(\tilde{I}_{n,l})\mu(\tilde{S}_n)\right| \\ &\leq \frac{68}{n}\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n).\end{aligned}$$

Proof. Denote $\tilde{S}_{n,k} = \tilde{S}_n \times T_k$ with $\tilde{S}_n = S_n \cap \widetilde{G}_n$, where S_n is a square of side length equal to $\frac{1}{k_n}$ lying in $\mathbb{T}^1 \times [\delta, 1-\delta]$. Let $\tilde{S}_\theta = \pi_\theta(\tilde{S}_n) \subseteq \pi_\theta(S_n) = [s_1, s_2]$, $\tilde{S}_r = \pi_r(\tilde{S}_n) \subseteq \pi_r(S_n)$, where π_θ and π_r are projections on θ and r axis, respectively. For $\tilde{I}_n = \cup_{l=0}^{k_n-1} \tilde{I}_{n,l} \in \tilde{\eta}_n$ and $l \in \{0, \dots, k_n - 1\}$, denote

$$J_l = \pi_r(\Phi_n(\tilde{I}_{n,l})).$$

Since g_n^{-1} acts as a translation on $\tilde{S}_n$, we have $\mathrm{d}_p g_n^{-1} \equiv \mathrm{id}$ for all base points $p \in \tilde{S}_n$. Thus, the following holds using $(\Phi_n, \mathrm{d}\Phi_n)$ being $(\gamma, \delta, \varepsilon_1, \varepsilon_2)$-distributing:

$$\Phi_n(\tilde{I}_{n,l}) \cap g_n^{-1}(\tilde{S}_n) \subseteq [c_l, c_l + \gamma] \times \tilde{S}_r \subseteq K_{c_l,\gamma}, \tag{6.24}$$

$$(g_n, \mathrm{d}g_n)^{-1}(\tilde{S}_n \times T_k) = g_n^{-1}(\tilde{S}_n) \times T_k. \tag{6.25}$$

Consider the following collection of sets, for $\tilde{S}_{n,\gamma} \subset \tilde{S}_n$ where $\tilde{S}_{n,\gamma} \subseteq [s_1 + \gamma, s_2 - \gamma] \times \tilde{S}_r$,

$$Q = \pi_r(K_{c_l,\gamma} \cap g_n^{-1}(\tilde{S}_n)) \quad \text{and} \quad Q_1 = \pi_r(K_{c_l,\gamma} \cap g_n^{-1}(\tilde{S}_{n,\gamma})).$$

Then we have the following containment relation using (6.24):

$$\Phi_n(\tilde{I}_{n,l}) \cap \mathbb{T}^1 \times Q_1 \subseteq \Phi_n(\tilde{I}_{n,l}) \cap g_n^{-1}(\tilde{S}_n) \subseteq \Phi_n(\tilde{I}_{n,l}) \cap \mathbb{T}^1 \times Q. \tag{6.26}$$

Subsequently, we use equation (6.25) to express the following: for any $k, j \in \{0, \dots, k_n - 1\}$, there exist a $l \in \{0, \dots, k_n - 1\}$ such that $k \equiv (l+j) \mod k_n$,

$$\begin{aligned}&(\Phi_n, \mathrm{d}\Phi_n)(\tilde{I}_{n,l,j}) \cap (g_n, \mathrm{d}g_n)^{-1}(\tilde{S}_n \times T_k) \\ &\qquad = (\Phi_n(\tilde{I}_{n,l}) \cap g_n^{-1}(\tilde{S}_n)) \times (\pi_v((\Phi_n, \mathrm{d}\Phi_n)(\tilde{I}_{n,l,j})) \cap T_k),\end{aligned} \tag{6.27}$$

where π_v represents the projection to the projective tangent vector, i.e. projection in the third coordinate. Thus, altogether with (6.26) and (6.27), we have the following inclusions

$$(\Phi_n, \mathrm{d}\Phi_n)(\tilde{I}_{n,l,j}) \cap (g_n, \mathrm{d}g_n)^{-1}(\tilde{S}_n \times T_k) \subseteq (\Phi_n, \mathrm{d}\Phi_n)(\tilde{I}_{n,l,j}) \cap (\mathbb{T}^1 \times Q \times T_k),$$
$$(\Phi_n, \mathrm{d}\Phi_n)(\tilde{I}_{n,l,j}) \cap (\mathbb{T}^1 \times Q_1 \times T_k) \subseteq (\Phi_n, \mathrm{d}\Phi_n)(\tilde{I}_{n,l,j}) \cap (g_n, \mathrm{d}g_n)^{-1}(\tilde{S}_n \times T_k).$$

Subsequently, we can conclude that

$$\begin{aligned}
&\left|\mu\left(\pi_M\left(\tilde{I}_{n,l,j} \cap (\Phi_n, \mathrm{d}\Phi_n)^{-1} \circ (g_n, \mathrm{d}g_n)^{-1}(\tilde{S}_{n,k})\right)\right)\lambda(J_l) - \mu(\tilde{I}_{n,l})\mu(\tilde{S}_n)\right| \\
&\leq \max\Big(\big|\mu\big(\pi_M\big(\tilde{I}_{n,l,j} \cap (\Phi_n, \mathrm{d}\Phi_n)^{-1}(\mathbb{T}^1 \times Q_1 \times T_k)\big)\big)\lambda(J_l) - \mu(\tilde{I}_{n,l})\mu(\tilde{S}_n)\big|, \\
&\quad \big|\mu\big(\pi_M\big(\tilde{I}_{n,l,j} \cap (\Phi_n, \mathrm{d}\Phi_n)^{-1}(\mathbb{T}^1 \times Q \times T_k)\big)\big)\lambda(J_l) - \mu(\tilde{I}_{n,l})\mu(\tilde{S}_n)\big|\Big).
\end{aligned} \tag{6.28}$$

Next we apply Lemma 6.7 for $S_n' = S_n \cap \mathcal{G}_n$, where $\mathcal{G}_n$ is defined by (6.7). In particular, $|\mu(S_n) - \mu(S_n')| \leq 4\varepsilon_n\mu(S_n)$. By our choices $\gamma < \frac{1}{k_n^3 q_n}, \delta < \frac{50}{k_n^2}, \varepsilon_1 = \varepsilon_2 < \frac{1}{n}, K = \pi_r(S_n'), L = \pi_\theta(S_n'), b_n = [nq_n^\sigma], a_n = k_n^5, \varepsilon_n \leq \frac{1}{2k_n^{10}}$ we have $\lambda(L) \leq \frac{1}{k_n}, \lambda(K) \leq \frac{1}{k_n}$, and for $n > 4$ using the condition **(P1)** and **(P3)**, we find that $\frac{1}{q_n^\sigma} < \frac{1}{2k_n}$, $k_n > n^2$, $\frac{b_n}{a_n} = \frac{[nq_n^\sigma]}{k_n^5} \leq \frac{1}{nk_n^2}$, and $b_n\lambda(K) > 2$. Altogether, we obtain

$$\begin{aligned}
&|\lambda(Q) - \mu(S_n')| \\
&\leq \left(\frac{2}{[nq_n^\sigma]}\lambda(S_\theta) + \frac{2}{[nq_n^\sigma]\cdot k_n^3 q_n} + \frac{1}{k_n^3 q_n}\lambda(S_r) + \frac{[nq_n^\sigma]}{2k_n^9 q_n}\lambda(S_r) + \frac{2}{2k_n^9 q_n}\right) \\
&\leq \left(\frac{4}{nq_n^\sigma}\lambda(S_\theta) + \frac{4}{nq_n^\sigma \cdot 2k_n^3 q_n} + \frac{1}{2k_n^3 q_n}\lambda(S_r) + \frac{1}{nk_n^2}\lambda(S_r) + \frac{2}{2k_n^3 q_n}\right) \\
&\leq \frac{13}{n}\mu(S_n).
\end{aligned}$$

Using the estimate $|\mu(S_n) - \mu(\tilde{S}_n)| \leq \frac{35}{k_n^4}\mu(S_n)$, we have

$$\begin{aligned}
|\lambda(Q) - \mu(\tilde{S}_n)| &\leq |\lambda(Q) - \mu(S_n')| + |\mu(S_n') - \mu(S_n)| + |\mu(S_n) - \mu(\tilde{S}_n)| \\
&\leq \frac{30}{n}\mu(\tilde{S}_n).
\end{aligned}$$

In particular, $-\frac{30}{n}\mu(\tilde{S}_n) \leq \lambda(Q) - \mu(\tilde{S}_n) \leq \frac{30}{n}\mu(\tilde{S}_n)$, i.e. $\lambda(Q) \leq (\frac{30}{n}+1)\mu(\tilde{S}_n) \leq 2\mu(\tilde{S}_n)$.
Similarly we estimate $\lambda(Q_1) \leq 2\mu(\tilde{S}_n)$ as well as $|\lambda(Q_1) - \mu(\tilde{S}_{n,\gamma})| \leq \frac{30}{n}\mu(\tilde{S}_n)$.
Since Q and Q_1 are finite unions of disjoint intervals contained within J_l, we utilize the $(\Phi_n, \mathrm{d}\Phi_n)$-$(\gamma, \delta, \varepsilon_1, \varepsilon_2)$ distribution property with $\tilde{J} = Q \subseteq J_l$, and we have

$$\begin{aligned}
&\left|\mu\left(\pi_M\left(\tilde{I}_{n,l,j} \cap (\Phi_n, \mathrm{d}\Phi_n)^{-1}(\mathbb{T}^1 \times Q \times T_k)\right)\right)\lambda(J_l) - \mu(\tilde{I}_{n,l})\mu(\tilde{S}_n)\right| \\
&\leq \left|\mu\left(\pi_M\left(\tilde{I}_{n,l,j} \cap (\Phi_n, \mathrm{d}\Phi_n)^{-1}(\mathbb{T}^1 \times Q \times T_k)\right)\right)\lambda(J_l) - \mu(\tilde{I}_{n,l})\lambda(Q)\right| \\
&\quad + \mu(\tilde{I}_{n,l})|\lambda(Q) - \mu(\tilde{S}_n)| \\
&\leq \left(\varepsilon_2 + \frac{30}{n}\right)\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n). \qquad (6.29)
\end{aligned}$$

Note that $\mu(\tilde{S}_{n,\gamma}) = \mu(\tilde{S}_n) - 2\gamma$, and $|\mu(\tilde{S}_{n,\gamma}) - \mu(\tilde{S}_n)| \leq \frac{2}{2k_n^3 q_n} \leq \frac{2}{n}\mu(\tilde{S}_n)$. Analogously we obtain,

$$\begin{aligned}
&\left|\mu\left(\pi_M\left(\tilde{I}_{n,l,j} \cap (\Phi_n, \mathrm{d}\Phi_n)^{-1}(\mathbb{T}^1 \times Q_1 \times T_k)\right)\right)\lambda(J_l) - \mu(\tilde{I}_{n,l})\mu(\tilde{S}_n)\right| \\
&\leq \frac{32}{n}\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n).
\end{aligned} \qquad (6.30)$$

Using estimates from (6.29) and (6.30) in (6.28), we have

$$\begin{aligned}
&\left|\mu\left(\pi_M\left(\tilde{I}_{n,l,j} \cap (\Phi_n, \mathrm{d}\Phi_n)^{-1} \circ (g_n, \mathrm{d}g_n)^{-1}(\tilde{S}_{n,k})\right)\right)\lambda(J_l) - \mu(\tilde{I}_{n,l})\mu(\tilde{S}_n)\right| \\
&\leq \frac{32}{n}\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n).
\end{aligned}$$

Finally, using the triangle inequality with $\lambda(J_l) \geq 1 - \frac{50}{k_n^2} \geq \frac{1}{2}$, and $\frac{1-\lambda(J_l)}{\lambda(J_l)} \leq \frac{100}{k_n^2} \leq \frac{4}{n}$ we conclude

$$\begin{aligned}
&\left|\mu\left(\pi_M\left(\tilde{I}_{n,l,j} \cap (\Phi_n, \mathrm{d}\Phi_n)^{-1} \circ (g_n, \mathrm{d}g_n)^{-1}(\tilde{S}_{n,k})\right)\right) - \mu(\tilde{I}_{n,l})\mu(\tilde{S}_n)\right| \\
&\leq \frac{1}{\lambda(J_l)}\left|\mu\left(\pi_M\left(\tilde{I}_{n,l,j} \cap (\Phi_n, \mathrm{d}\Phi_n)^{-1} \circ (g_n, \mathrm{d}g_n)^{-1}(\tilde{S}_{n,k})\right)\right)\lambda(J_l) - \mu(\tilde{I}_{n,l})\mu(\tilde{S}_n)\right| \\
&\quad + \frac{1-\lambda(J_l)}{\lambda(J_l)}\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{64}{n}\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n) + \frac{4}{n}\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n) \\
&\leq \frac{68}{n}\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n).
\end{aligned}
$$

This concludes the claim. □

Remark 6.9. The statement of the above lemma still holds for $\tilde{S}_n = S_n \cap \widetilde{G}_n$, where S_n is a square with a side length in the range $\frac{1}{4k_n}$ to $\frac{2}{k_n}$.

Lemma 6.10. *Let $(f, \mathrm{d}f) = \lim_{n\to\infty}(f_n, \mathrm{d}f_n)$ and $(m_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers satisfying $d_1(f_n^{m_n}, f^{m_n}) < \frac{1}{2^n}$ for all $n \in \mathbb{N}$. Furthermore, let $\hat{\nu}_n$ be a sequence of partial partitions satisfying $\hat{\nu}_n \to \epsilon$ and the following property: For every 3-dimensional cube $A \subseteq \mathbb{P}\mathrm{T}M$ and for every $\varepsilon \in (0, 1]$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, and every $\Gamma_n \in \hat{\nu}_n$, we have*

$$|\bar{\mu}(\Gamma_n \cap (f_n, \mathrm{d}f_n)^{-m_n}(A)) - \bar{\mu}(\Gamma_n)\bar{\mu}(A)| \leq \varepsilon\bar{\mu}(\Gamma_n)\bar{\mu}(A). \tag{6.31}$$

Then $(f, \mathrm{d}f)$ is weakly mixing on $\mathbb{P}\mathrm{T}M$.

Proof. The proof can be derived directly from [29, Lemma 6.1, 6.2] by considering the dimension $m = 3$. □

6.6.3 Criterion for smooth weakly mixing diffeomorphism

Here, we prove the criterion for weak mixing on $\mathbb{P}\mathrm{T}M$.

Proposition 6.11. *(Criterion for weak mixing in smooth setting) Let $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, defined by (6.1) and (6.2), such that $(f_n, \mathrm{d}f_n)$ converges to $(f, \mathrm{d}f)$ and satisfies $d_1(f_n^{m_n}, f^{m_n}) < \frac{1}{2^n}$ for every $n \in \mathbb{N}$. Additionally, we assume that the sequence $(H_n)_{n\in\mathbb{N}}$ satisfies condition* **(P1)**, *and the map g_n is as in Proposition 6.1. Consider a sequence of partial partitions $(\hat{\eta}_n)_{n\in\mathbb{N}}$, defined as in (6.9), such that $\hat{\eta}_n \to \epsilon$. Additionally, let the partial partition $\hat{\nu}_n$ be defined as*

$$\hat{\nu}_n = \left\{\Gamma_{n,j} = (H_{n-1}, \mathrm{d}H_{n-1}) \circ (g_n, \mathrm{d}g_n)(\hat{I}_{n,j}) \ : \quad \hat{I}_{n,j} \in \hat{\eta}_n\right\}, \tag{6.32}$$

and suppose that $\hat{\nu}_n \to \epsilon$. Suppose for a sequence $(m_n)_{n\in\mathbb{N}}$ and diffeomorphisms $\Phi_n := \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$ that $(\Phi_n, \mathrm{d}\Phi_n) : \mathbb{P}\mathrm{T}M \to \mathbb{P}\mathrm{T}M$ is $(\gamma, \delta, \varepsilon_1, \varepsilon_2)$-distributing all partition elements of $\hat{\eta}_n$ with $\gamma < \frac{1}{k_n^3 q_n}, \delta < \frac{50}{k_n^2}, \varepsilon_1, \varepsilon_2 < \frac{1}{n}$. Then $(f, \mathrm{d}f) = \lim_{n\to\infty}(f_n, \mathrm{d}f_n)$ is weakly mixing with respect to the measure $\overline{\mu}$ on $\mathbb{P}\mathrm{T}M$.

Proof. Fix $\epsilon > 0$ and $\hat{I}_{n,j} = \cup_{l=0}^{k_n-1}\tilde{I}_{n,l,j} \in \hat{\eta}_n$. Let S_n be a square of side length equal to $\frac{1}{k_n}$ contained in $\mathbb{T}^1 \times [\delta, 1-\delta]$. Using the good domain $\widetilde{G}_n$ from (6.14), we define $\tilde{S}_n = S_n \cap \widetilde{G}_n$. For $k \in \{0, \dots, k_n - 1\}$, let $\tilde{S}_{n,k} = \tilde{S}_n \times T_k$, where $T_k = \left[\frac{k}{k_n}, \frac{k+1}{k_n}\right]$. Consider $C_{n,k} = (H_{n-1}, \mathrm{d}H_{n-1})(\tilde{S}_{n,k})$ and $\Gamma_{n,l,j} = (H_{n-1}, \mathrm{d}H_{n-1}) \circ (g_n, \mathrm{d}g_n)(\tilde{I}_{n,l,j})$. We apply Lemma 6.6 and Lemma 6.8 to conclude that for any $j, k \in \{0, \dots, k_n-1\}$, there exists $l \in \{0, 1, \dots, k_n - 1\}$ satisfying $k \equiv l + j \mod k_n$, and we have

$$\begin{aligned}
&\bar{\mu}(\Gamma_{n,l,j} \cap (f_n, \mathrm{d}f_n)^{-m_n}(C_{n,k})) \\
\leq&\frac{1}{k_n}\mu(\pi_M(\tilde{I}_{n,l,j} \cap (\Phi_n, \mathrm{d}\Phi_n)^{-1} \circ (g_n, \mathrm{d}g_n)^{-1}(\tilde{S}_{n,k}))) + \frac{35(k_n-1)}{k_n^5}\mu(\tilde{I}_{n,l}) \\
\leq& \left(1 + \frac{68}{n}\right)\frac{1}{k_n}\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n) + \frac{35(k_n-1)}{k_n^5}\frac{1}{\bar{\mu}(C_{n,k})}\mu(\tilde{I}_{n,l})\bar{\mu}(C_{n,k}) \\
\leq& \left(1 + \frac{68}{n}\right)\frac{1}{k_n}\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n) + \frac{35(k_n-1)}{k_n^5}\frac{k_n}{\mu(\tilde{S}_n)}\mu(\tilde{I}_{n,l})\bar{\mu}(C_{n,k}) \\
\leq& \left(1 + \frac{68}{n}\right)\mu(\tilde{I}_{n,l})\bar{\mu}(C_{n,k}) + \frac{2}{n}\mu(\tilde{I}_{n,l})\bar{\mu}(C_{n,k}) \\
\leq& \left(1 + \frac{70}{n}\right)\mu(\tilde{I}_{n,l})\bar{\mu}(C_{n,k}).
\end{aligned}$$

Here, we used the inequality $\mu(\tilde{S}_n) \geq (1 - \frac{35}{k_n^4})\mu(S_n)$, which implies for $n > 4$, $\frac{1}{\mu(\tilde{S}_n)} \leq (1 - \frac{35}{k_n^4})^{-1}\frac{1}{\mu(S_n)} \leq \frac{2}{\mu(S_n)} \leq 2k_n^2$. Similarly, we conclude that

$$\bar{\mu}(\Gamma_{n,l,j} \cap (f_n, \mathrm{d}f_n)^{-m_n}(C_{n,k})) \geq \left(1 - \frac{70}{n}\right)\mu(\tilde{I}_{n,l})\bar{\mu}(C_{n,k}).$$

Altogether, we get

$$|\bar{\mu}(\Gamma_{n,l,j} \cap (f_n, \mathrm{d}f_n)^{-m_n}(C_{n,k})) - \mu(\tilde{I}_{n,l})\bar{\mu}(C_{n,k})| \leq \frac{70}{n}\mu(\tilde{I}_{n,l})\bar{\mu}(C_{n,k}). \tag{6.33}$$

Furthermore, consider $\Gamma_{n,j} = \bigcup_{l=0}^{k_n-1}\Gamma_{n,l,j}$. With $\bar{\mu}(\Gamma_{n,j}) = \sum_{l=0}^{k_n-1}\bar{\mu}(\Gamma_{n,l,j})$, and considering $\tilde{c}_{n,l}$ with $S'_n = \mathbb{T}^2$ in Lemma 6.6, we conclude that

$$|\bar{\mu}(\Gamma_{n,j}) - \mu(\tilde{I}_{n,l})| \leq \frac{35}{k_n^4}\mu(\tilde{I}_{n,l}). \tag{6.34}$$

Additionally, using the fact that for any $k, j \in \{0, \ldots, k_n - 1\}$ there exists a unique $l \in \{0, 1, \ldots, k_n - 1\}$ satisfying $k \equiv l + j \mod k_n$, we have

$$\Gamma_{n,j} \cap (f_n, \mathrm{d}f_n)^{-m_n}(C_{n,k}) = \Gamma_{n,l,j} \cap (f_n, \mathrm{d}f_n)^{-m_n}(C_{n,k}). \tag{6.35}$$

Since $C_{n,k} \subset \mathbb{P}\mathrm{T}M$, with $C_{n,k} = (H_{n-1}, \mathrm{d}H_{n-1})(\tilde{S}_{n,k})$ of $\mathrm{diam}(\tilde{S}_{n,k}) \leq \frac{3}{k_n}$, we can use condition **(P1)** and conclude that

$$\mathrm{diam}(C_{n,k}) \leq \mathrm{diam}((H_{n-1}, \mathrm{d}H_{n-1})(\tilde{S}_{n,k})) < \frac{1}{n^2}, \tag{6.36}$$

that is, $\mathrm{diam}(C_{n,k}) \to 0$ as $n \to \infty$.

Let $\{S_n^i \mid i \in \Lambda_n\}$ with a finite index set Λ_n be a collection of squares S_n^i of side length $\frac{1}{k_n}$ contained in $\mathbb{T}^1 \times [\delta, 1-\delta]$ such that $\mu(\cup_{i\in\Lambda_n} S_n^i) \geq 1 - 4\delta$. Let $\tilde{S}_n^i = S_n^i \cap \widetilde{G}_n$ and $\tilde{S}_{n,k}^i = \tilde{S}_n \times T_k$ with $k \in \{0, \ldots, k_n - 1\}$. Considering sets $C_{n,k}^i = (H_{n-1}, \mathrm{d}H_{n-1})(\tilde{S}_{n,k}^i)$ and index set $\tilde{\Lambda}_n = \{(i,k) \mid i \in \Lambda_n,\ k \in \{0, \ldots, k_n - 1\}\}$, we can conclude that

$$\begin{aligned}
\bar{\mu}(\cup_{(i,k)\in\tilde{\Lambda}_n} C_{n,k}^i) &= \sum_{(i,k)\in\tilde{\Lambda}_n} \bar{\mu}(C_{n,k}^i) = \sum_{i\in\Lambda_n}\sum_{k=0}^{k_n-1} \bar{\mu}(C_{n,k}^i) \\
&= \sum_{i\in\Lambda_n}\sum_{k=0}^{k_n-1} \frac{1}{k_n}\mu(\tilde{S}_n^i) \\
&\geq \sum_{i\in\Lambda_n}\left(1 - \frac{35}{k_n^4}\right)\mu(S_n^i) \\
&\geq \left(1 - \frac{35}{k_n^4}\right)(1 - 4\delta) \\
&\geq \left(1 - \frac{35}{k_n^4}\right)\left(1 - \frac{200}{k_n^2}\right) \\
&\geq \left(1 - \frac{235}{n^4}\right).
\end{aligned}$$

Thus, $\bar{\mu}\left(\cup_{(i,k)\in\tilde{\Lambda}_n} C^i_{n,k}\right) \to 1$ as $n \to \infty$. Therefore, any cube $A \subset \mathbb{P}TM$ can be approximated by a finite disjoint union of sets of the form $C_{n,k}$: For n sufficiently large there are sets $A_1 = \bigcup_{(i,k)\in\mathcal{C}^1_n} C^i_{n,k}$ and $A_2 = \bigcup_{(i,k)\in\mathcal{C}^2_n} C^i_{n,k}$ with finite sets $\mathcal{C}^1_n$ and $\mathcal{C}^2_n$ of indices such that $A_1 \subseteq A \subseteq A_2$ and $|\bar{\mu}(A) - \bar{\mu}(A_i)| \leq \frac{\epsilon}{3}\bar{\mu}(A)$ for $i = 1, 2$. Additionally, we choose n large enough such that $\frac{142}{n} < \frac{\epsilon}{3}$ holds. It follows, using estimates (6.33), (6.34), (6.35), and for $n > 4$, $\left(1 - \frac{35}{k_n^4}\right)^{-1} < 2$ that:

$$
\begin{aligned}
&\bar{\mu}(\Gamma_{n,j} \cap (f_n, \mathrm{d}f_n)^{-m_n}(A)) - \bar{\mu}(\Gamma_{n,j})\bar{\mu}(A) \\
\leq &\bar{\mu}(\Gamma_{n,j} \cap (f_n, \mathrm{d}f_n)^{-m_n}(A_2)) - \bar{\mu}(\Gamma_{n,j})\bar{\mu}(A_2) + \bar{\mu}(\Gamma_{n,j})(\bar{\mu}(A_2) - \bar{\mu}(A)) \\
\leq &\bar{\mu}(\cup_{l=0}^{k_n-1}\Gamma_{n,l,j} \cap (f_n, \mathrm{d}f_n)^{-m_n}(\cup_{(i,k)\in\mathcal{C}^2_n} C^i_{n,k})) - \bar{\mu}(\Gamma_{n,j})\bar{\mu}(\cup_{(i,k)\in\mathcal{C}^2_n} C^i_{n,k}) \\
&+ \frac{\epsilon}{3}\bar{\mu}(\Gamma_{n,j})\bar{\mu}(A) \\
\leq &\sum_{(i,k)\in\mathcal{C}^2_n} \left(\bar{\mu}(\Gamma_{n,l,j} \cap (f_n, \mathrm{d}f_n)^{-m_n}(C^i_{n,k})) - \mu(\tilde{I}_{n,l})\bar{\mu}(C^i_{n,k})\right) \\
&+ \sum_{(i,k)\in\mathcal{C}^2_n} \bar{\mu}(C^i_{n,k})(\mu(\tilde{I}_{n,l}) - \bar{\mu}(\Gamma_{n,j})) + \frac{\epsilon}{3}\bar{\mu}(\Gamma_{n,j})\bar{\mu}(A) + \frac{\epsilon}{3}\bar{\mu}(\Gamma_{n,j})\bar{\mu}(A) \\
\leq &\sum_{(i,k)\in\mathcal{C}^2_n} \left(\frac{70}{n} + \frac{35}{k_n^4}\right)\mu(\tilde{I}_{n,l})\bar{\mu}(C^i_{n,k}) + \frac{\epsilon}{3}\bar{\mu}(\Gamma_{n,j})\bar{\mu}(A) \\
\leq &\sum_{(i,k)\in\mathcal{C}^2_n} \frac{71}{n}\left(1 - \frac{35}{k_n^4}\right)^{-1} \bar{\mu}(\Gamma_{n,j})\bar{\mu}(C^i_{n,k}) + \frac{\epsilon}{3}\bar{\mu}(\Gamma_{n,j})\bar{\mu}(A) \\
\leq &\frac{142}{n}\bar{\mu}(\Gamma_{n,j})\bar{\mu}(A) + \frac{142}{n}\bar{\mu}(\Gamma_{n,j})(\bar{\mu}(A_2) - \bar{\mu}(A)) + \frac{\epsilon}{3}\bar{\mu}(\Gamma_{n,j})\bar{\mu}(A) \\
\leq &\frac{\epsilon}{3}\bar{\mu}(\Gamma_{n,j})\bar{\mu}(A) + \frac{\epsilon}{3}\bar{\mu}(\Gamma_{n,j})\bar{\mu}(A) + \frac{\epsilon}{3}\bar{\mu}(\Gamma_{n,j})\bar{\mu}(A) \\
\leq &\epsilon\bar{\mu}(\Gamma_{n,j})\bar{\mu}(A).
\end{aligned}
$$

Analogously, we estimate that

$$\bar{\mu}(\Gamma_{n,j} \cap (f_n, \mathrm{d}f_n)^{-m_n}(A)) - \bar{\mu}(\Gamma_{n,j})\bar{\mu}(A) > -\epsilon\bar{\mu}(\Gamma_{n,j})\bar{\mu}(A).$$

Both estimates together yield equation (6.31) required for Lemma 6.10. This implies that $(f, \mathrm{d}f)$ is weakly mixing with respect to $\bar{\mu}$ on the $\mathbb{P}TM$. □

6.7 Explicit constructions

6.7.1 Sequence of partial partitions

In this subsection, we define three sequences of partial partitions $\{\eta_n\}_{n\in\mathbb{N}}$, $\{\zeta_n\}_{n\in\mathbb{N}}$ and $\{\tilde{\eta}_n\}_{n\in\mathbb{N}}$ of the 2-torus. For each $n \in \mathbb{N}$, the partial partition ζ_n is a refined partition of the partial partition η_n, and the partition elements of $\tilde{\eta}_n$ are unions of the partition elements of ζ_n. We define these sequences of partial partitions using the sequences $\{k_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ of rapidly growing natural numbers satisfying (**P1**) – (**P4**).

6.7.1.1 Partial partition η_n

Consider the following collection of multidimensional intervals in following form:

$$\check{I}^{u_0,u_1,u_2}_{v_0} = \left[\frac{u_0}{2q_n} + \frac{u_1}{2k_nq_n} + \frac{u_2}{2k_n^6q_n} + \frac{1}{4n^5k_n^{16}q_n}, \frac{u_0}{2q_n} + \frac{u_1}{2k_nq_n} + \frac{u_2+1}{2k_n^6q_n} - \frac{1}{4n^5k_n^{16}q_n}\right] \times \left[\frac{v_0}{k_n^5} + \frac{1}{2n^5k_n^{15}}, \frac{v_0+1}{k_n^5} - \frac{1}{2n^5k_n^{15}}\right], \tag{6.37}$$

where $u_0 \in \{0,1,\dots,2q_n-1\}$, $u_1 \in \{0,1\dots,k_n-1\}$ and $v_0, u_2 \in \{1,\dots,k_n^5-2\}$. Denote by η_n the collection of such intervals $\check{I}^{u_0,u_1,u_2}_{v_0}$ of $\mathbb{T}^2$.

Remark 6.12. Note that η_n is a partial partition of $\mathbb{T}^2$. For every $n \in \mathbb{N}$, η_n consists of disjoint sets, covers a set of measure at least $\left(1-\frac{2}{k_n^5}\right)^4 \geq 1-\frac{8}{k_n^5}$ and for $I \in \eta_n$, $\operatorname{diam}(I) \leq \frac{\sqrt{2}}{k_n^5} \to 0$ as $n \to \infty$. Hence, $(\eta_n)_{n\in\mathbb{N}}$ converges to the decomposition into points.

6.7.1.2 Partial partition ζ_n

Consider the following collection of multidimensional intervals in the given form:

$$I^{u_0,u_1,u_2,u_3,u_4}_{v_0,v_1,v_2} = \left[\frac{u_0}{2q_n} + \frac{u_1}{2k_nq_n} + \frac{u_2}{2k_n^6q_n} + \frac{u_3}{2k_n^{11}q_n} + \frac{u_4}{2k_n^{16}q_n} + \frac{1}{4n^5k_n^{26}q_n}, \frac{u_0}{2q_n} + \frac{u_1}{2k_nq_n} + \frac{u_2}{2k_n^6q_n} + \frac{u_3}{2k_n^{11}q_n} + \frac{u_4+1}{2k_n^{16}q_n} - \frac{1}{4n^5k_n^{26}q_n}\right] \times$$

$$\left[\frac{v_0}{k_n^5}+\frac{v_1}{2k_n^{11}q_n}+\frac{v_2}{2k_n^{16}q_n}+\frac{1}{4n^5k_n^{26}q_n},\frac{v_0}{k_n^5}+\frac{v_1}{2k_n^{11}q_n}+\frac{v_2+1}{2k_n^{16}q_n}-\frac{1}{4n^5k_n^{26}q_n}\right], \quad (6.38)$$

where $u_0 \in \{0,1,\dots,2q_n-1\}$, $u_1 \in \{0,1\dots,k_n-1\}$, $v_0,v_2,u_2,u_3,u_4 \in \{1,\dots,k_n^5-2\}$, and $v_1 \in \{2q_n,\dots,2k_n^6q_n-2q_n-1\}$. Let ζ_n be the collection of such intervals $I_{v_0,v_1,v_2}^{u_0,u_1,u_2,u_3,u_4}$.

Remark 6.13. Note that ζ_n is a partial partition of $\mathbb{T}^2$. For every $n \in \mathbb{N}$, ζ_n consists of disjoint sets, covers a set of measure at least $\left(1-\frac{2}{k_n^5}\right)^8 \geq 1-\frac{16}{k_n^5}$ and for $I \in \zeta_n$, $\operatorname{diam}(I) \leq \frac{\sqrt{2}}{k_n^{15}q_n} \to 0$ as $n \to \infty$. Thus, $(\zeta_n)_{n\in\mathbb{N}}$ converges to the decomposition into points. We will use the sequence $(\zeta_n)_{n\in\mathbb{N}}$ in section 6.9 to verify that the assumptions of our criterion in Proposition 6.3 for the existence of a f-invariant measurable Riemannian metric are satisfied in our explicit constructions.

Remark 6.14. Note that ζ_n is a refined partition of η_n where each element $\check{I}_{v_0}^{u_0,u_1,u_2} \in \eta_n$ is covered by some union of elements $I_{v_0,v_1,v_2}^{u_0,u_1,u_2,u_3,u_4} \in \zeta_n$ with a measure of at least $\left(1-\frac{8}{k_n^5}\right)\mu(\check{I}_{v_0}^{u_0,u_1,u_2})$,

$$\bigcup_{v_1=2q_n}^{2k_n^6q_n-2q_n-1}\ \bigcup_{u_3,u_4,v_2=1}^{k_n^5-2} I_{v_0,v_1,v_2}^{u_0,u_1,u_2,u_3,u_4} \subseteq \check{I}_{v_0}^{u_0,u_1,u_2}. \quad (6.39)$$

6.7.1.3 Partial partition $\tilde{\eta}_n$

This collection will be useful in defining a partial partition of the space $\mathbb{P}\mathrm{T}M$ and referring to it as the "good domain" in our explicit construction. Denote $\tilde{\eta}_n$ be collection of the following elements for $u_0 \in \{0,\dots,2q_n-1\}$ and $v_0 \in \{1,\dots,k_n^5-2\}$ as

$$\tilde{I}_{v_0}^{u_0} = \bigcup_{u_1=0}^{k_n-1} \tilde{I}_{v_0}^{u_0,u_1};\ \text{where } \tilde{I}_{v_0}^{u_0,u_1} = \bigcup_{u_2=1}^{k_n^5-2}\ \bigcup_{v_1=2q_n}^{2k_n^6q_n-2q_n-1}\ \bigcup_{u_3,u_4,v_2=1}^{k_n^5-2} I_{v_0,v_1,v_2}^{u_0,u_1,u_2,u_3,u_4}. \quad (6.40)$$

Remark 6.15. Note that for each element $\tilde{I}_{v_0}^{u_0,u_1}$, we have $\frac{1}{2k_n^6q_n}\left(1-\frac{16}{k_n^5}\right) \leq \mu(\tilde{I}_{v_0}^{u_0,u_1}) \leq \frac{1}{2k_n^6q_n}$, and $\tilde{\eta}_n$ covers a set with a measure of at least $\left(1-\frac{25}{k_n^5}\right)$. This verify the partial partition $\tilde{\eta}_n$ satisfies the properties in **(D2)** and **(D3)**.

Furthermore, according to Remark 6.14, we have $\tilde{I}^{u_0,u_1}_{v_0} \subseteq \bigcup_{u_2=1}^{k_n^3-1} \check{I}^{u_0,u_1,u_2}_{v_0}$, and $\mu\left(\bigcup_{u_2=1}^{k_n^5-2} \check{I}^{u_0,u_1,u_2}_{v_0} \cap \tilde{I}^{u_0,u_1}_{v_0}\right) \geq \left(1-\frac{8}{k_n^5}\right)\mu\left(\bigcup_{u_2=1}^{k_n^5-2} \check{I}^{u_0,u_1,u_2}_{v_0}\right)$.

Remark 6.16. Furthermore, with $q_{n+1} > 2k_n^{10}q_n^2$ in **(P4)**, and using the definition of the sequence of partial partitions $\{\tilde{\eta}_n\}_{n\in\mathbb{N}}$, where each $\tilde{I}_{n+1} \in \tilde{\eta}_{n+1}$ is a refined partition element of $\tilde{I}_n \in \tilde{\eta}_n$, we can verify property **(D5)**.

6.7.2 Smooth conjugation maps ϕ_n

6.7.2.1 The conjugation map $\tilde{\phi}_n$

In [35, section 2.1] the smooth area-preserving diffeomorphism $\tilde{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2}$ on $\mathbb{T}^1\times[0,1]$ satisfying the subsequent properties is constructed and referred to as the "inner rotation of type A". With minor to no modifications, the proposition holds for the case of $[0,1]^2$.

Proposition 6.17. *Let $\varepsilon, \varepsilon_2 \in (0,\frac{1}{4})$ and $\lambda,\mu \in \mathbb{N}$. Then there is a smooth area-preserving diffeomorphism $\tilde{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2} : [0,1]^2 \to [0,1]^2$ such that*

1. *$\tilde{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2}$ coincides with the identity on $[0,1]^2\backslash[\varepsilon,1-\varepsilon]^2$;*
2. *$\tilde{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2}\left(x+\frac{l}{q_n},y\right) = \tilde{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2}(x,y)$ for all $(x,y) \in \left[\frac{l}{q_n},\frac{l+1}{q_n}\right) \times [0,1], l \in \mathbb{Z}$.*
3. *Let $t_2 \in \mathbb{Z}$, $\lceil 2\varepsilon\mu\rceil \leq t_2 \leq \mu - \lceil 2\varepsilon\mu\rceil - 1, |u_2| \leq \varepsilon_2$, and $u_1 \in (2\varepsilon,\frac{1}{2})$ be of the form $\frac{t_1}{\mu}$ with $t_1 \in \mathbb{Z}$. Then we have*

$$\tilde{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2}\left(\left[\frac{u_1}{\lambda},\frac{1-u_1}{\lambda}\right]\times\left[\frac{t_2+u_2}{\mu},\frac{t_2+1-u_2}{\mu}\right]\right) = \left[\frac{1}{\lambda}-\frac{t_2+1-u_2}{\lambda\mu},\frac{1}{\lambda}-\frac{t_2+u_2}{\lambda\mu}\right]\times[u_1,1-u_1];$$

4. *$\tilde{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2}$ acts as isometry on each cuboid*

$$\left[\frac{t_1+2\varepsilon_2}{\lambda\mu},\frac{t_1+1-2\varepsilon_2}{\lambda\mu}\right]\times\left[\frac{t_2+2\varepsilon_2}{\mu},\frac{t_2+1-2\varepsilon_2}{\mu}\right],$$

where $t_i \in \mathbb{Z}$, $\lceil 2\varepsilon\mu\rceil \leq t_i \leq \mu - \lceil 2\varepsilon\mu\rceil - 1$ for $i = 1,2$;

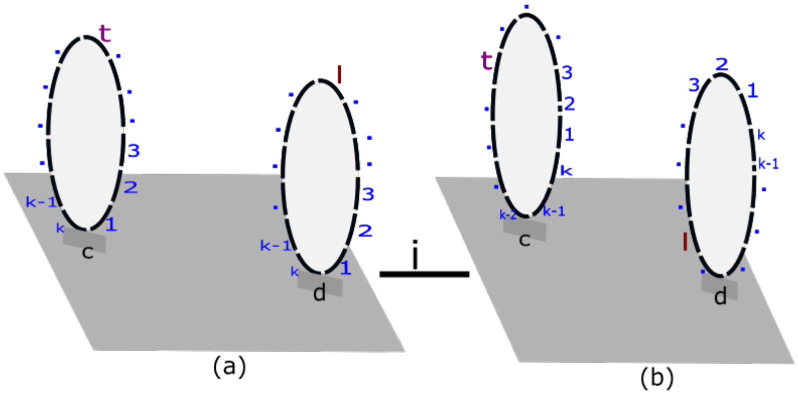

FIGURE 6.1: An example of the action i_n within the $\mathbb{T}^2$ transform on the fundamental domains is shown in (a) and (b).

5. $\left|\left|\left|\tilde{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2}\right|\right|\right|_r \leq c \cdot \lambda^r \mu^r$ *where c is a constant independent of λ and μ.*

In our explicit construction, we choose the parameters as follows to define $\tilde{\phi}_n : [0,1]^2 \to [0,1]^2$ by

$$\tilde{\phi}_n = \tilde{\phi}_{2k_n q_n, \frac{1}{2k_n^5}, k_n^5, \frac{1}{4n^5 k_n^{10}}}. \tag{6.41}$$

Remark 6.18. The action of the map $\tilde{\phi}_n$ on the elements of the partial partitions ζ_n and η_n can be described by

$$\tilde{\phi}_n(\check{I}^{u_0,u_1,u_2}_{v_0}) = \check{I}^{u_0,u_1,k_n^5-v_0-1}_{u_2}, \tag{6.42}$$

$$\tilde{\phi}_n(I^{u_0,u_1,u_2,u_3,u_4}_{v_0,v_1,v_2}) = I^{u_0,u_1,k_n^5-v_0-1,u_3,u_4}_{u_2,v_1,v_2}. \tag{6.43}$$

6.7.2.2 The conjugation map i_n

Within this subsection, we introduce what we refer to as the "inner rotations of type B". These rotations will play a crucial role in establishing the weak mixing property for the projectivized derivative extension. Specifically, we will use distinct rotation angles applied to specific regions, which allow us to analyze the distribution of elements in the tangent direction effectively.

Proposition 6.19. *Let* $b_n = 2k_nq_n, a_n = 2k_n^{11}q_n, c_n = k_n^5, \varepsilon_n = \frac{1}{2n^5k_n^{11}}$. *There is a smooth measure-preserving diffeomorphsim* $i_n : [0,1]^2 \to [0,1]^2$ *such that*

1. i_n *coincides with the identity on* $[0,1]^2 \backslash [\varepsilon_n, 1-\varepsilon_n]^2$.

2. *Each square of form* $\left[\frac{\mathrm{u}}{a_n}, \frac{\mathrm{u}+1}{a_n}\right] \times \left[\frac{\mathrm{v}}{a_n}, \frac{\mathrm{v}+1}{a_n}\right]$ *with* $\mathrm{u}, \mathrm{v} \in \mathbb{Z}$ *is mapped onto itself by* i_n *and* i_n *coincides with the identity on an* $\frac{\varepsilon_n}{a_n}$*- neighbourhood of its boundary.*

3. *For set* $\left[\frac{l}{b_n} + \frac{\mathrm{u}}{a_n} + \frac{s_1+2\varepsilon_n}{a_nc_n}, \frac{l}{b_n} + \frac{\mathrm{u}}{a_n} + \frac{s_1+1-2\varepsilon_n}{a_nc_n}\right] \times \left[\frac{\mathrm{v}}{a_n} + \frac{s_2+2\varepsilon_n}{a_nc_n}, \frac{\mathrm{v}}{a_n} + \frac{s_2+1-2\varepsilon_n}{a_nc_n}\right]$, *where* $s_1, s_2 \in \mathbb{Z}$, $1 \le s_1, s_2 \le c_n - 2$, $\mathrm{u} \in \{0, \dots, \frac{a_n}{b_n} - 1\}$, $\mathrm{v} \in \{0, \dots, a_n - 1\}$, i_n *is a composition of a translation and a rotation by* β_l, *where* $\beta_l \equiv \frac{s\pi}{k_n}$ *with* $s \equiv l \mod k_n$.

4. $i_n\left(x + \frac{p}{q_n}, y\right) = i_n(x,y)$ *for all* $(x,y) \in \left[\frac{p}{q_n}, \frac{p+1}{q_n}\right) \times [0,1]$, $p \in \mathbb{Z}$.

5. $|||i_n|||_r \le c'_{n,k_n,r} \cdot q_n^{r-1}$ *where the constant* $c'_{n,k_n,r}$ *is independent of* q_n.

Proof. Similar to [35, Lemma 2.3], there exist such a measure preserving diffeomorphism, $\psi_{c,\varepsilon,\beta} : [0,1]^2 \to [0,1]^2, c \in \mathbb{N}, \varepsilon \in (0, \frac{1}{5c}]$ and $\beta \in [0, \pi]$, is constructed with the aid of Moser's trick which satisfies following properties:

- $\psi_{c,\varepsilon,\beta}$ coincides with the identity on $[0,1]^2 \backslash [\varepsilon, 1-\varepsilon]^2$.
- On every square $\left[\frac{\mathrm{v}+2\varepsilon}{c}, \frac{\mathrm{v}+1-2\varepsilon}{c}\right] \times \left[\frac{\mathrm{k}+2\varepsilon}{c}, \frac{\mathrm{k}+1-2\varepsilon}{c}\right]$ with $1 \le \mathrm{v}, \mathrm{k} \le c-2$ the map $\psi_{c,\varepsilon,\beta}$ is equal to a composition of a translation and a rotation by arc β around the centre.

Define the map $\psi_{a,c,\varepsilon,\beta} : \left[0, \frac{1}{a}\right]^2 \to \left[0, \frac{1}{a}\right]^2$ by $\psi_{a,c,\varepsilon,\beta} = D_a^{-1} \circ \psi_{c,\varepsilon,\beta} \circ D_a$, where the dilation map $D_a : \left[0, \frac{1}{a}\right]^2 \to [0,1]^2$ defined by $D_a(\theta, r) = (a \cdot \theta, a \cdot r)$ for $a \in \mathbb{N}$. Since it coincide with the identity in a neighbourhood of the boundary, we can extend it to a smooth diffeomorphism on $[0,1]^2$ equivariantly by the description

$$\psi_{a,c,\varepsilon,\beta}\left(\theta + \frac{a_1}{a}, r + \frac{a_2}{a}\right) = \left(\frac{a_1}{a}, \frac{a_2}{a}\right) + \psi_{a,c,\varepsilon,\beta}(\theta, r),$$

for $a_1, a_2 \in \mathbb{N}$.

In our concrete construction, we define i_n by

$$i_n = \psi_{a_n,c_n,\varepsilon_n,\beta_l}, \tag{6.44}$$

where $a_n = 2k_n^{11}q_n, c_n = k_n^5, \varepsilon_n = \frac{1}{2n^5k_n^{10}}$ and the value of β_l varies with the function domain, specifically within the context of the given domain: On domain $\left[\frac{l}{b_n} + \frac{\mathrm{u}}{a_n}, \frac{l}{b_n} + \frac{\mathrm{u}+1}{a_n}\right] \times [0,1]$, for all $\mathrm{u} \in \{0,\ldots,\frac{a_n}{b_n} - 1\}$, choose the value of $\beta_l \equiv \frac{s\cdot\pi}{k_n}$ where $s \equiv l \mod k_n$. Moreover, the map $\psi_{c_n,\varepsilon_n,\beta_l}$ defines the rotation by the arcs and it depends on the parameter k_n but remains independent of q_n. Therefore, the norm of the map i_n defined by $i_n = D_{a_n}^{-1} \circ \psi_{c_n,\varepsilon_n,\beta_l} \circ D_{a_n}$ can be estimated as

$$|||i_n|||_r \leq a_n^{r-1} \cdot |||\psi_{c_n,\varepsilon_n,\beta_l}|||_r \leq c'_{n,k_n,r} q_n^{r-1},$$

where $c'_{n,k_n,r}$ is a constant depending upon n, k_n, and r, but independent of q_n. □

Remark 6.20. Compared to the analogous construction in [35, Proposition 2.2], our choice of the distinct rotation angles, denoted as β_l, has a different dependence on the specific domain. For illustration of the map i_n, we refer to Figure 6.1.

Remark 6.21. The action of the map i_n on the elements of the partial partitions ζ_n and η_n can be described as the composition of a rotation by an angle $\frac{u_1\cdot\pi}{k_n}$ and a translation, such that:

$$i_n(\check{I}_{v_0}^{u_0,u_1,u_2}) = \check{I}_{v_0}^{u_0,u_1,u_2}, \tag{6.45}$$

$$i_n(I_{v_0,v_1,v_2}^{u_0,u_1,u_2,u_3,u_4}) \subseteq \check{I}_{v_0}^{u_0,u_1,u_2}, \tag{6.46}$$

where $I_{v_0,v_1,v_2}^{u_0,u_1,u_2,u_3,u_4} \in \zeta_n$ and $\check{I}_{v_0}^{u_0,u_1,u_2} \in \eta_n$.

6.7.2.3 The conjugation map ϕ_n

We define the final conjugation map $\phi_n : [0,1]^2 \to [0,1]^2$, for $u_0 = 0, 1, \ldots, q_n - 1$:

$$\phi_n = \begin{cases} i_n \circ \tilde{\phi}_n & \text{on } \left[\frac{2u_0}{2q_n}, \frac{2u_0+1}{2q_n}\right] \times [0,1], \\ \mathrm{Id} & \text{on } \left[\frac{2u_0+1}{2q_n}, \frac{2u_0+2}{2q_n}\right] \times [0,1], \end{cases} \tag{6.47}$$

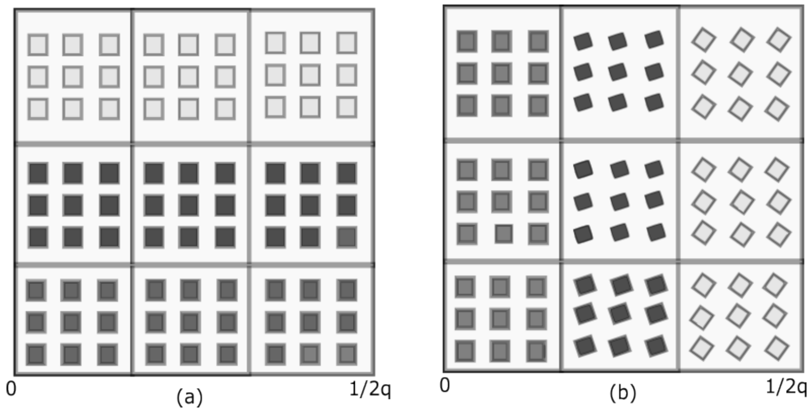

Figure 6.2: An example of the action of ϕ_n transforming from (a) to (b) on the fundamental domain within $\mathbb{T}^2$.

where the map $\tilde{\phi}_n$ and i_n is defined by (6.41) and (6.44) respectively.

Remark 6.22. Note that the map ϕ_n is a smooth area-preserving diffeomorphism on the 2-torus, given that both the maps $\tilde{\phi}_n$ and i_n act as identities in a neighborhood of the boundary of $[0,1]^2$. Consequently, ϕ_n is a well-defined Hamiltonian diffeomorphisms on the torus by Lemma 2.6. For illustration of the map ϕ_n, we refer to Figure 6.2.

6.8 Convergence of sequences $< f_n >_{n\in\mathbb{N}}$ in the smooth topology

The following two lemmas follow from Lemmas 3 and 4 in [20] with minor to no modifications; hence, we skip the proofs for brevity.

Lemma 6.23 ([20, Lemma 3])**.** *Let $r \in \mathbb{N}$. For all $\alpha, \beta \in \mathbb{R}$ and all $h \in \mathrm{Diff}^\infty(\mathbb{T}^2, \mu)$, we have the estimate*

$$d_r(hS_\alpha h^{-1}, hS_\beta h^{-1}) \leq C_r \max\{|||h|||_{r+1}, \left|\left|\left|h^{-1}\right|\right|\right|_{r+1}\}|\alpha - \beta|,$$

where C_r is a constant that depends only on r.

Lemma 6.24 ([20, Lemma 4]). *For any $\epsilon > 0$, let r_n be a sequence of natural numbers satisfying $\sum_{n=1}^{\infty} \frac{1}{r_n} < \epsilon$. Suppose for any Liouville number α that there exists a sequence of rationals $\{\alpha_n\}$ satisfying*

$$|\alpha - \alpha_n| < \frac{1}{2^{n+1} r_n \hat{C}_{r_n} q_n |||H_n|||_{r_{n+1}}^{r_{n+1}}}, \tag{6.48}$$

where $\hat{C}_{r_n}$ is the same constant as in Lemma 6.23. Then the sequence of diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converges to $f \in \mathrm{Diff}^\infty(\mathbb{T}^2, \mu)$ in the C^∞ topology. Moreover, for any $m \leq q_{n+1}$, we have

$$d_0(f^m, f_n^m) \leq \frac{1}{2^{n+1}}, \text{ and } \sup_{x \in \mathbb{P}\mathrm{T}M} d((f_n, \mathrm{d}f_n)^m(x), (f, \mathrm{d}f)^m(x)) \leq \frac{1}{2^{n+1}}. \tag{6.49}$$

Lemma 6.25. *For any $r_n \in \mathbb{N}$, the conjugation diffeomorphisms constructed in Sections 6.4 and 6.7.2 satisfy the following statements:*

1. $|||h_n|||_{r_n} \leq c_{n,k_n,r_n} \cdot q_n^{2r_n^3}$*, where c_{n,k_n,r_n} is constant independent of q_n.*
2. $|||H_n|||_{r_n} \leq \hat{c}_{n,k_n,r_n} \cdot q_n^{2r_n^4}$*, where $\hat{c}_{n,k_n,r_n}$ is constant independent of q_n.*
3. *For α Liouville, there exists a sequence of rational $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfying (6.48) and* **(P1)**-**(P4)**.

Proof. Using Lemma 2.2 and the norm estimates of conjugation maps ϕ_n and g_n, defined in Proposition 6.17, 6.19 and 6.1, we have estimate:

$$\begin{aligned}|||h_n|||_{r_n} &\leq |||g_n|||_{r_n}^{r_n} \cdot |||\phi_n|||_{r_n}^{r_n} \\ &\leq \tilde{c}_{n,k_n,r_n} \cdot q_n^{r_n^2} \cdot |||i_n|||_{r_n}^{r_n} \cdot \left|\left|\left|\tilde{\phi}_n\right|\right|\right|_{r_n}^{r_n} \leq c_{n,k_n,r_n} \cdot q_n^{2r_n^3}.\end{aligned}$$

Similarly, $|||H_n|||_{r_n} = |||H_{n-1} \circ h_n|||_{r_n} \leq |||H_{n-1}|||_{r_n}^{r_n} \cdot |||h_n|||_{r_n}^{r_n}$. Since all order derivatives of H_{n-1} are independent of q_n, we can conclude that $|||H_n|||_{r_n} \leq \hat{c}_{n,k_n,r_n} \cdot q_n^{2r_n^4}$. For α being a Liouville, we can choose a sequence of rationals $\alpha_n = \frac{p_n}{q_n}$ (with p_n, q_n relatively prime) that satisfy the following property, along with **(P1)** – **(P4)**:

$$|\alpha - \alpha_n| \leq \frac{1}{2^{n+1} \cdot r_n \cdot \hat{c}_{n,k_n,r_n} \cdot q_n^{2(r_n+1)^5+1}} \leq \frac{1}{2^{n+1} \cdot r_n \cdot \hat{C}_{r_n} \cdot q_n \cdot |||H_n|||_{r_n+1}^{r_n+1}},$$

where $\hat{C}_{r_n}$ is constant depends only on r_n. $\square$

Remark 6.26. Finally, we have proven the estimate on the norms of the conjugation map H_n, as shown in [19]. Additionally, the existence of rationals satisfying (6.48) guarantees the convergence of the sequence f_n to $f \in \mathrm{Diff}^\infty(\mathbb{T}^2, \mu)$, as stated in Lemma 6.24.

6.9 Existence of an invariant Riemannian metric

Next, we show that our conjugation maps h_n and $\hat{h}_n$ fulfill the requirements of Proposition 6.3.

Lemma 6.27. *Let $I_n \in \zeta_n$, defined as in section 6.7. The maps h_n defined by (6.2), satisfy the following for all $I_n \in \zeta_n$: $dev_{I_n}(h_n) = 0$;*

Proof. For the first statement, it is sufficient to show that h_n acts as an isometry with respect to ω_0 for all $I^{u_0,u_1,u_2,u_3,u_4}_{v_0,v_1,v_2} \in \zeta_n$. In our explicit construction, the map ϕ_n is defined by (6.47). If u_0 is odd, it acts as the identity and, hence, as an isometry. If u_0 is even, ϕ_n acts as $i_n \circ \tilde{\phi}_n$ on $I^{u_0,u_1,u_2,u_3,u_4}_{v_0,v_1,v_2}$. In Proposition 6.17, equation (6.41), and Remark 6.18, the mapping behavior of $\tilde{\phi}_n$ is explicitly discussed, and it acts as an isometry on $I^{u_0,u_1,u_2,u_3,u_4}_{v_0,v_1,v_2}$ as follows:

$$\tilde{\phi}_n(I^{u_0,u_1,u_2,u_3,u_4}_{v_0,v_1,v_2}) = I^{u_0,u_1,k_n^5-v_0-1,u_3,u_4}_{u_2,v_1,v_2}.$$

In Proposition 6.19, the second property defines the map i_n as a composition of a translation and some rotation on such sets $I^{u_0,u_1,k_n^5-v_0-1,u_3,u_4}_{u_2,v_1,v_2}$. In particular, $i_n(I^{u_0,u_1,k_n^5-v_0-1,u_3,u_4}_{u_2,v_1,v_2}) \subset \check{I}^{u_0,u_1,k_n^5-v_0-1}_{u_2}$. The map g_n, as defined in Proposition 6.1, acts as a translation on the image set $\phi_n(I^{u_0,u_1,u_2,u_3,u_4}_{v_0,v_1,v_2}) \subset \check{I}^{u_0,u_1,k_n^5-v_0-1}_{u_2}$. Altogether, we conclude the first statement. $\square$

Remark 6.28. Note that h_n acts as an isometry on the sets $I^{u_0,u_1,u_2,u_3,u_4}_{v_0,v_1,v_2}$, and the union of these elements defines $\tilde{\eta}_n$ as described in subsection 6.7.1.3. This construction verifies the property **(D1)**.

Remark 6.29. By Lemma 6.27 we can choose $\mathfrak{d}_n = 0$ for all $n \in \mathbb{N}$ when applying Proposition 6.3 for our sequence $(f_n)_{n\in\mathbb{N}}$ of C^∞ diffeomorphisms defined as in (6.1). Thus, the limit diffeomorphism $f = \lim_{n\to\infty} f_n$ admits an invariant measurable Riemannian metric.

6.10 Proof of weak mixing of the derivative extension

6.10.1 Choice of partial partition of $\mathbb{P}\mathrm{T}M$

Consider the 3-dimensional elements in $\mathbb{P}\mathrm{T}M$ for $\tilde{I}^{u_0}_{v_0} = \cup_{u_1=0}^{k_n-1} \tilde{I}^{u_0,u_1}_{v_0} \in \tilde{\eta}_n$ by (6.40), and $j \in \{0, \dots, k_n - 1\}$ as

$$\hat{I}_{u_0,v_0,j} = \tilde{I}^{u_0}_{v_0} \times T_j = \bigcup_{u_1=0}^{k_n-1} (\tilde{I}^{u_0,u_1}_{v_0} \times T_j), \quad \text{where} \quad T_j = \left[\frac{j}{k_n}, \frac{j+1}{k_n}\right].$$

Denote the following collection of elements to form a partial partition of $\mathbb{P}\mathrm{T}M$ by

$$\hat{\eta}_n := \left\{ \hat{I}_{u_0,v_0,j} \ : \ u_0 \in \{0, \dots, 2q_n - 1\}, v_0 \in \{1, \dots, k_n^5 - 2\}, j \in \{0, \dots, k_n - 1\} \right\}. \tag{6.50}$$

Remark 6.30. Note that $\hat{\eta}_n$ is a partial partition of $\mathbb{P}\mathrm{T}M$. For every $n \in \mathbb{N}$, $\hat{\eta}_n$ consists of disjoint sets, covers a set of measure at least $\left(1 - \frac{25}{k_n^5}\right)^2 \geq 1 - \frac{50}{k_n^5}$. For $I \in \hat{\eta}_n$, it is a finite union of elements of the form $I^{u_0,u_1,u_2,u_3,u_4}_{v_0,v_1,v_2} \times T_j$, and every element has $\operatorname{diam}(I^{u_0,u_1,u_2,u_3,u_4}_{v_0,v_1,v_2} \times T_j) \leq \frac{\sqrt{2}}{k_n} \to 0$ as $n \to \infty$. Thus, the decomposition $(\hat{\eta}_n)_{n\in\mathbb{N}}$ converges to a decomposition into points.

6.10.2 Choice of mixing sequence m_n

We consider

$$m_n = \min\left\{ m \leq q_{n+1} \mid \inf_{k\in\mathbb{Z}} \left| m \frac{q_n p_{n+1}}{q_{n+1}} - \frac{1}{2} + k \right| \leq \frac{q_n}{q_{n+1}} \right\}$$

and

$$\mathfrak{a}_n = \left(m_n \alpha_{n+1} - \frac{1}{2q_n} \mod \frac{1}{q_n} \right)$$

as defined in [19] for the torus case. Under the growth assumption $q_{n+1} > 2k_n^{12} q_n^2$ we obtain $|\mathfrak{a}_n| \leq \frac{1}{q_{n+1}} < \frac{1}{2k_n^{12} q_n^2}$.

Remark 6.31. Since the action of $R_{\alpha_{n+1}}^{m_{n+1}}$ acts as a translation in the θ direction by the factor $m_{n+1}\alpha_{n+1}$, its action is trivial in the tangent direction. Hence, the condition $\mathfrak{a}_n < \frac{1}{2k_n^{12} q_n^2}$ holds for the 'good domains' of the partial partitions η_n and $\tilde{\eta}_n$ such that

$$\mu(R_{\alpha_{n+1}}^{m_n}(\check{I}_{v_0}^{u_0,u_1,u_2}) \triangle \check{I}_{v_0}^{u_0+1,u_1,u_2}) < |\mathfrak{a}_n| \leq \frac{1}{2k_n^{12} q_n^2},$$
$$\mu(R_{\alpha_{n+1}}^{m_n}(\tilde{I}_{v_0}^{u_0,u_1}) \triangle \tilde{I}_{v_0}^{u_0+1,u_1}) < |\mathfrak{a}_n| \leq \frac{1}{2k_n^{12} q_n^2}.$$

6.10.3 Action of $(\Phi_n, \mathrm{d}\Phi_n)$

Note that $\tilde{I}_{v_0}^{u_0,u_1} \subseteq \bigcup_{u_2=1}^{k_n^5-2} \check{I}_{v_0}^{u_0,u_1,u_2}$ with $\check{I}_{v_0}^{u_0,u_1,u_2} \in \eta_n$ belongs to the "good domain" of the conjugation map ϕ_n by (6.39) and (6.40). Therefore, we can describe the mapping behaviour of the projectivized derivative extension $(\phi_n, \mathrm{d}\phi_n)$ on the elements $\hat{I}_{u_0,v_0,j} \in \hat{\eta}_n$ explicitly.

Lemma 6.32. *For any $\tilde{I}_{v_0}^{u_0,u_1} \times T_j \subseteq \mathbb{P}TM$, where u_0 is even and $\tilde{I}_{v_0}^{u_0,u_1}$ is defined by (6.40), the following holds:*

1. $\phi_n(\tilde{I}_{v_0}^{u_0,u_1}) \subseteq \bigcup_{u_2=1}^{k_n^5-2} \check{I}_{u_2}^{u_0,u_1,k_n^5-v_0-1}$, *and* $\phi_n^{-1}(\tilde{I}_{v_0}^{u_0,u_1}) \subseteq \bigcup_{u_2=1}^{k_n^5-2} \check{I}_{k_n^5-u_2-1}^{u_0,u_1,v_0}$.

2. $(\phi_n, \mathrm{d}\phi_n)(\tilde{I}_{v_0}^{u_0,u_1} \times T_j) \subseteq \bigcup_{u_2=1}^{k_n^5-2} \check{I}_{u_2}^{u_0,u_1,k_n^5-v_0-1} \times T_k$, *where* $k \equiv u_1 + j \mod k_n$. *Moreover,*

$$\mu\left(\pi_M\left((\phi_n, \mathrm{d}\phi_n)\big(\tilde{I}_{v_0}^{u_0,u_1} \times T_j\big) \triangle \left(\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{u_2}^{u_0,u_1,k_n^5-v_0-1} \times T_k \right)\right)\right) \leq \frac{8}{k_n^{11} q_n}. \tag{6.51}$$

3. $(\phi_n, \mathrm{d}\phi_n)^{-1}(\tilde{I}_{v_0}^{u_0,u_1} \times T_j) \subseteq \bigcup_{u_2=1}^{k_n^5-2} \check{I}_{k_n^5-u_2-1}^{u_0,u_1,v_0} \times T_{k'}$, *where* $k' \equiv -u_1 + j \mod k_n$.

Moreover,

$$\mu\left(\pi_M\left((\phi_n, \mathrm{d}\phi_n)^{-1}\left(\tilde{I}_{v_0}^{u_0,u_1} \times T_j\right) \triangle \left(\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{k_n^5-u_2-1}^{u_0,u_1,v_0} \times T_{k'}\right)\right)\right) \leq \frac{8}{k_n^{11} q_n}. \quad (6.52)$$

In particular, an element $\tilde{I}_{v_0}^{u_0,u_1}$ is mapped to a region $\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{u_2}^{u_0,u_1,k_n^5-v_0-1} \in \mathbb{T}^2$, which has almost full vertical length, and the jth tangent interval T_j is translated to the kth tangent interval T_k, where $k \equiv u_1 + j \mod k_n$, under the map $(\phi_n, \mathrm{d}\phi_n)$.

Proof. In our explicit construction ϕ_n defined by (6.47), if u_0 is even, ϕ_n acts as $i_n \circ \tilde{\phi}_n$ on $\tilde{I}_{v_0}^{u_0,u_1}$. In Proposition 6.17 and Remark 6.18, the mapping behaviour of $\tilde{\phi}_n$ is discussed explicitly. We also note that $\mathrm{d}_p\tilde{\phi}_n = \mathrm{id}$ for any base point $p \in I_{v_0,v_1,v_2}^{u_0,u_1,u_2,u_3,u_4}$. Therefore we conclude

$$(\tilde{\phi}_n, \mathrm{d}\tilde{\phi}_n)(\tilde{I}_{v_0}^{u_0,u_1} \times T_j) = \bigcup_{u_2=1}^{k_n^5-2} \bigcup_{v_1=2q_n}^{2k_n^6 q_n - 2q_n - 1} \bigcup_{u_3,u_4,v_2=1}^{k_n^5-2} I_{u_2,v_1,v_2}^{u_0,u_1,k_n^5-v_0-1,u_3,u_4} \times T_j.$$

In Proposition 6.19, the third property defines the map i_n as a composition of a translation and a rotation by $\frac{u_1 \cdot \pi}{k_n}$ on $I_{u_2,v_1,v_2}^{u_0,u_1,k_n^5-v_0-1,u_3,u_4}$. Additionally, the second statement in Proposition 6.19 establishes that the image of $I_{u_2,v_1,v_2}^{u_0,u_1,k_n^5-v_0-1,u_3,u_4}$, for all $u_3, v_2, v_1 \in \mathbb{Z}$, under i_n is contained in $\check{I}_{u_2}^{u_0,u_1,k_n^5-v_0-1}$. Consequently, we obtain the following containment:

$$\begin{aligned}&(i_n, \mathrm{d}i_n)\left(\bigcup_{u_2=1}^{k_n^5-2} \bigcup_{v_1=2q_n}^{2k_n^6 q_n - 2q_n - 1} \bigcup_{u_3,u_4,v_2=1}^{k_n^5-2} I_{u_2,v_1,v_2}^{u_0,u_1,k_n^5-v_0-1,u_3,u_4} \times T_j\right)\\ &\subseteq \bigcup_{u_2=1}^{k_n^5-2} \check{I}_{u_2}^{u_0,u_1,k_n^5-v_0-1} \times T_k,\end{aligned}$$

where $k \equiv j + u_1 \mod k_n$. This implies the first statement and the first part of the second statement of the lemma.
Together with (6.40), the first statement of the lemma, Remark 6.15, and the fact

that the map ϕ_n is measure-preserving, we get

$$\mu\left(\left(\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{u_2}^{u_0,u_1,k_n^5-v_0-1}\right)\Big\backslash \phi_n\big(\tilde{I}_{v_0}^{u_0,u_1}\big)\right) \leq \mu\left(\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{u_2}^{u_0,u_1,k_n^5-v_0-1}\right) - \mu\left(\phi_n\big(\tilde{I}_{v_0}^{u_0,u_1}\big)\right)$$
$$\leq \frac{8}{k_n^{11} q_n}.$$

Moreover, using the first part of second statement of the lemma, we have

$$\mu\left(\pi_M\left((\phi_n, \mathrm{d}\phi_n)\big(\tilde{I}_{v_0}^{u_0,u_1} \times T_j\big) \triangle \left(\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{u_2}^{u_0,u_1,k_n^5-v_0-1} \times T_k\right)\right)\right)$$
$$\leq \mu\left(\left(\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{u_2}^{u_0,u_1,k_n^5-v_0-1}\right)\Big\backslash \phi_n\big(\tilde{I}_{v_0}^{u_0,u_1}\big)\right) \leq \frac{8}{k_n^{11} q_n}.$$

Analogously, we examine the effect of ϕ_n^{-1}, which is defined by $\tilde{\phi}_n^{-1} \circ i_n^{-1}$, on $\tilde{I}_{v_0}^{u_0,u_1}$ for an even value of u_0. With Proposition 6.19, we conclude

$$\bigcup_{u_2=1}^{k_n^5-2} \big(i_n, \mathrm{d}i_n\big)^{-1}\left(\bigcup_{v_1=2q_n}^{2k_n^6 q_n-2q_n-1} \bigcup_{u_3,u_4,v_2=1}^{k_n^5-2} I_{v_0,v_1,v_2}^{u_0,u_1,u_2,u_3,u_4} \times T_j\right) \subseteq \bigcup_{u_2=1}^{k_n^5-2} \check{I}_{v_0}^{u_0,u_1,u_2} \times T_{k'},$$

where $k' \equiv -u_1 + j \mod k_n$. Considering Remark 6.18, which discusses the action of $\tilde{\phi}_n$ on the elements $\check{I}_{v_0}^{u_0,u_1,u_2} \in \eta_n$, and in addition to the fact that $\mathrm{d}_p\tilde{\phi}_n^{-1} = \mathrm{id}$ for any base point $p \in \check{I}_{v_0}^{u_0,u_1,u_2}$, we conclude

$$(\tilde{\phi}_n, \mathrm{d}\tilde{\phi}_n)^{-1}\left(\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{v_0}^{u_0,u_1,u_2} \times T_{k'}\right) \subseteq \bigcup_{u_2=1}^{k_n^5-2} \check{I}_{k_n^5-u_2-1}^{u_0,u_1,v_0} \times T_{k'}.$$

Altogether, this yields the third statement as

$$(\tilde{\phi}_n, \mathrm{d}\tilde{\phi}_n)^{-1} \circ \big(i_n, \mathrm{d}i_n\big)^{-1}(\tilde{I}_{v_0}^{u_0,u_1} \times T_j) \subseteq \bigcup_{u_2=1}^{k_n^5-2} \check{I}_{k_n^5-u_2-1}^{u_0,u_1,v_0} \times T_{k'}.$$

Similar to the proof of estimate (6.51), we obtain the estimate (6.52). $\square$

Remark 6.33. Note that the partial partition $\hat{\eta}_n$ defined in section 6.10.1 satisfies the required property **(D4)** for the partial partition $\hat{\eta}_n$ of $\mathbb{P}\mathrm{T}M$ in section 6.6. Specifically, with our explicit partition of $\hat{\eta}_n$, we have $\tilde{I}_{n,l} = \tilde{I}_{v_0}^{u_0,u_1}$ and $\hat{I}_{n,j} = \bigcup_{u_1=0}^{k_n-1} \tilde{I}_{v_0}^{u_0,u_1} \times T_j$, for $u_0 \in \{0,\dots,q_n-1\}, v_0 \in \{1,\dots,k_n^5-2\}$, and $u_1, j \in \{0,\dots,k_n-1\}$. Since the map g_n^{-1}, defined in Proposition 6.1, acts as a translation in the horizontal direction on each $\tilde{I}_{v_0}^{u_0,u_1}$ by $\frac{[nq_n^\sigma]v_0}{k_n^5}$, we have $g_n^{-1}(\tilde{I}_{v_0}^{u_0,u_1}) = \tilde{I}_{v_0}^{u_0',u_1'}$ for some u_0' and u_1'.
Furthermore, applying ϕ_n^{-1} to these elements and using the first statement of Lemma 6.32, we obtain

$$h_n^{-1}(\tilde{I}_{v_0}^{u_0,u_1}) = \phi_n^{-1} \circ g_n^{-1}(\tilde{I}_{v_0}^{u_0,u_1}) \subseteq \cup_{u_2=1}^{k_n^5-2} \check{I}_{k_n^5-u_2-2}^{u_0',u_1',v_0},$$

with the estimate

$$\mu(h_n^{-1}(\tilde{I}_{v_0}^{u_0,u_1}) \triangle [\cup_{u_2=1}^{k_n^5-2} \check{I}_{k_n^5-u_2-2}^{u_0',u_1',v_0}]) \leq \frac{16}{k_n^5} \mu(\tilde{I}_{v_0}^{u_0,u_1}). \tag{6.53}$$

Along with Remark 6.14, which states that each $\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{k_n^5-u_2-2}^{u_0',u_1',v_0}$ is almost covered by the elements $\tilde{I}_{v_0}^{u_0} \in \tilde{\eta}_n$, let

$$\bar{\Lambda}_{\tilde{I}_{v_0}^{u_0',u_1'}} = \left\{ \tilde{I}_{b_0}^{a_0,a_1} \subseteq \bigcup_{u_2=1}^{k_n^5-2} \check{I}_{k_n^5-u_2-2}^{u_0',u_1',v_0} : \tilde{I}_{b_0}^{a_0} \in \tilde{\eta}_n, \tilde{I}_{b_0}^{a_0} = \bigcup_{a_1=0}^{k_n-1} \tilde{I}_{b_0}^{a_0,a_1} \right\}$$

be a collection of such sets, and the following estimate holds:

$$\mu(\cup_{u_2=1}^{k_n^5-2} \check{I}_{k_n^5-u_2-2}^{u_0',u_1',v_0} \triangle [\cup_{\tilde{I}_{b_0}^{a_0,a_1} \in \bar{\Lambda}_{\tilde{I}_{v_0}^{u_0',u_1'}}} \tilde{I}_{b_0}^{a_0,a_1}]) \leq \frac{16}{k_n^5} \mu(\tilde{I}_{v_0}^{u_0',u_1'}).$$

Furthermore, by applying the triangle inequality with equation (6.53), we can derive the following estimation:

$$\mu(h_n^{-1}(\tilde{I}_{v_0}^{u_0,u_1}) \triangle [\cup_{\tilde{I}_{b_0}^{a_0,a_1} \in \bar{\Lambda}_{\tilde{I}_{v_0}^{u_0',u_1'}}} \tilde{I}_{b_0}^{a_0,a_1}]) \leq \frac{32}{k_n^5} \mu(\tilde{I}_{v_0}^{u_0,u_1}),$$

which verifies the property **(D4)** for the elements of $\tilde{\eta}_n$.

Lemma 6.34. *Consider the map $\Phi_n = \phi_n \circ R_{\alpha_{n+1}}^{m_n} \circ \phi_n^{-1}$. The map $(\Phi_n, \mathrm{d}\Phi_n)$ is $(\gamma, \delta, \varepsilon_1, \varepsilon_2)$-distributing the elements $\hat{I}_{u_0,v_0,j} \in \hat{\eta}_n$ in the sense of Definition 6.10 with $\gamma = \frac{1}{2k_n^5 q_n}, \delta = \frac{1}{k_n^4}, \varepsilon_1 = \varepsilon_2 = \frac{1}{n^5}$.*

Proof. For any $\hat{I}_{u_0,v_0,j} = \cup_{u_1=0}^{k_n-1} \tilde{I}_{v_0}^{u_0,u_1} \times T_j \in \hat{\eta}_n$, we consider two cases for the values of u_0 to understand the action of $(\Phi_n, \mathrm{d}\Phi_n)$. If u_0 is even, we apply Lemma 6.32. If u_0 is odd, we use the fact that the maps $(\phi_n, \mathrm{d}\phi_n)$ act as the identity inside the "good domains" and Remark 6.31.

Case 1: If u_0 is even, we have for each fixed $u_1 \in \{0, \dots, k_n - 1\}$,

$$
\begin{aligned}
&(\Phi_n, \mathrm{d}\Phi_n)(\tilde{I}_{v_0}^{u_0,u_1} \times T_j) \\
&= (\phi_n, \mathrm{d}\phi_n) \circ (R_{\alpha_{n+1}}, \mathrm{d}R_{\alpha_{n+1}})^{m_n} \circ (\phi_n, \mathrm{d}\phi_n)^{-1} \left(\tilde{I}_{v_0}^{u_0,u_1} \times T_j \right) \\
&\subseteq (\phi_n, \mathrm{d}\phi_n) \circ (R_{\alpha_{n+1}}, \mathrm{d}R_{\alpha_{n+1}})^{m_n} \left(\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{k_n^5-u_2-1}^{u_0,u_1,v_0} \times T_{k'} \right),
\end{aligned}
$$

where $k' \equiv -u_1 + j \mod k_n$. Using the Remark 6.31 and Lemma 6.32, and the fact that $(\phi_n, \mathrm{d}\phi_n)$ acts as the identity in the domain $(\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{k_n^5-u_2-1}^{u_0+1,u_1,v_0} \times T_{k'})$, we can conclude

$$
(\Phi_n, \mathrm{d}\Phi_n)(\tilde{I}_{v_0}^{u_0,u_1} \times T_j) \cap \bigcup_{u_2=1}^{k_n^5-2} \check{I}_{k_n^5-u_2-1}^{u_0+1,u_1,v_0} \times T_{k'} \neq \emptyset,
$$

and the following estimate

$$
\mu\left(\pi_M \left((\Phi_n, \mathrm{d}\Phi_n)(\tilde{I}_{v_0}^{u_0,u_1} \times T_j) \triangle \bigcup_{u_2=1}^{k_n^5-2} \check{I}_{k_n^5-u_2-1}^{u_0+1,u_1,v_0} \times T_{k'} \right) \right) \leq \frac{9}{k_n^{11} q_n}.
$$

Case 2: If u_0 is odd, we have for each fixed $u_1 \in \{0, \dots, k_n - 1\}$,

$$
\begin{aligned}
&(\Phi_n, \mathrm{d}\Phi_n)(\tilde{I}_{v_0}^{u_0,u_1} \times T_j) \\
&= (\phi_n, \mathrm{d}\phi_n) \circ (R_{\alpha_{n+1}}, \mathrm{d}R_{\alpha_{n+1}})^{m_n} \circ (\phi_n, \mathrm{d}\phi_n)^{-1} \big(\tilde{I}_{v_0}^{u_0,u_1} \times T_j \big) \\
&= (\phi_n, \mathrm{d}\phi_n) \circ (R_{\alpha_{n+1}}, \mathrm{d}R_{\alpha_{n+1}})^{m_n} \big(\tilde{I}_{v_0}^{u_0,u_1} \times T_j \big).
\end{aligned}
$$

Similar to Case 1, using Remark 6.31 and Lemma 6.32, we have

$$(\Phi_n, \mathrm{d}\Phi_n)(\tilde{I}_{v_0}^{u_0,u_1} \times T_j) \bigcap \bigcup_{u_2=1}^{k_n^5-2} \check{I}_{u_2}^{u_0+1,u_1,k_n^5-v_0-1} \times T_k \neq \emptyset,$$

and the following estimate

$$\mu\left(\pi_M\left(((\Phi_n, \mathrm{d}\Phi_n)(\tilde{I}_{v_0}^{u_0,u_1} \times T_j)) \triangle (\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{u_2}^{u_0+1,u_1,k_n^5-v_0-1} \times T_k)\right)\right) \leq \frac{9}{k_n^{11} q_n},$$

where $k \equiv u_1 + j \mod k_n$.

Note that the θ widths of the sets $(\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{u_2}^{u_0+1,u_1,k_n^5-v_0-1})$ and $(\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{k_n^5-u_2-1}^{u_0+1,u_1,v_0})$ are at most $\frac{1}{2k_n^6 q_n}(1 - \frac{1}{k_n^5})$. Consequently, both cases imply that the set $\Phi_n(\tilde{I}_{v_0}^{u_0,u_1})$, for all $u_1 \in \{0, \ldots, k_n - 1\}$, is contained within a set whose θ width is at most $\frac{1}{2k_n^6 q_n} - \frac{1}{2k_n^{11} q_n} + \frac{9}{k_n^{11} q_n}$. Hence, we can choose $\gamma = \frac{1}{2k_n^5 q_n}$.

Furthermore, note that $\mu(\tilde{I}_{v_0}^{u_0,u_1}) \geq \frac{1}{2k_n^6 q_n}\left(1 - \frac{16}{k_n^5}\right)$. The transformation Φ_n is measure-preserving, and $\Phi_n(\tilde{I}_{v_0}^{u_0,u_1})$ has θ width of at most $\frac{1}{2k_n^6 q_n}\left(1 + \frac{4}{k_n^5}\right)$, i.e.

$$\lambda\left(\pi_\theta(\Phi_n(\tilde{I}_{v_0}^{u_0,u_1}))\right) \leq \frac{1}{2k_n^6 q_n}\left(1 + \frac{4}{k_n^5}\right).$$

This implies that $\lambda\left(\pi_r\big(\Phi_n(\tilde{I}_{v_0}^{u_0,u_1})\big)\right) \geq \left(1 - \frac{20}{k_n^5}\right)$, allowing us to choose $\delta = \frac{1}{k_n^4}$.

Denote $J_{u_1} = \pi_r\big(\Phi_n(\tilde{I}_{v_0}^{u_0,u_1})\big)$, $J'_{u_1} = \pi_r(\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{u_2}^{u_0+1,u_1,k_n^5-v_0-1})$, and

$$A_\theta^{u_1} = \pi_\theta\left(\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{u_2}^{u_0+1,u_1,k_n^5-v_0-1}\right) = \pi_\theta(\check{I}_{u_2}^{u_0+1,u_1,k_n^5-v_0-1}).$$

Thus we can express $\bigcup_{u_2=1}^{k_n^5-2} \check{I}_{u_2}^{u_0+1,u_1,k_n^5-v_0-1} = A_\theta^{u_1} \times J'_{u_1}$.

Analogous to Cases 1 and 2, using Lemma 6.32 and Remark 6.33, we obtain the following estimate:

$$\mu\left(\Phi_n(\tilde{I}_{v_0}^{u_0,u_1})\triangle\bigcup_{u_2=1}^{k_n^5-2}\check{I}_{u_2}^{u_0+1,u_1,k_n^5-v_0-1}\right)\leq\frac{9}{k_n^{11}q_n}\leq\frac{36}{k_n^5}\mu\left(\bigcup_{u_2=1}^{k_n^5-2}\check{I}_{u_2}^{u_0+1,u_1,k_n^5-v_0-1}\right),\tag{6.54}$$

for any $\tilde{J}\subseteq J'_{u_1}\cap J_{u_1}$, we can conclude that

$$\left(1-\frac{36}{k_n^5}\right)\lambda(A_\theta^{u_1})\lambda(\tilde{J})\leq\mu(\Phi_n(\tilde{I}_{v_0}^{u_0,u_1})\cap(\mathbb{T}\times\tilde{J}))\leq\left(1+\frac{36}{k_n^5}\right)\lambda(A_\theta^{u_1})\lambda(\tilde{J}).$$

Equation (6.54) also yields

$$\begin{aligned}|\lambda(A_\theta^{u_1})\lambda(J_{u_1})-\mu(\Phi_n(\tilde{I}_{v_0}^{u_0,u_1}))|&=\left|\mu\left(\bigcup_{u_2=1}^{k_n^5-2}\check{I}_{u_2}^{u_0+1,u_1,k_n^5-v_0-1}\right)-\mu(\Phi_n(\tilde{I}_{v_0}^{u_0,u_1}))\right|\\&\leq\mu\left(\Phi_n(\tilde{I}_{v_0}^{u_0,u_1})\triangle\bigcup_{u_2=1}^{k_n^5-2}\check{I}_{u_2}^{u_0+1,u_1,k_n^5-v_0-1}\right)\\&\leq\frac{9}{k_n^{11}q_n}\leq\frac{36}{k_n^5}\mu(\tilde{I}_{v_0}^{u_0,u_1}).\end{aligned}\tag{6.55}$$

For the last inequality, we used $\frac{1}{2k_n^6q_n}(1-\frac{16}{k_n^5})<\mu(\tilde{I}_{v_0}^{u_0,u_1})$ that yields $\frac{1}{\mu(\tilde{I}_{v_0}^{u_0,u_1})}\leq 2k_n^6q_n(1-\frac{16}{k_n^5})^{-1}\leq 4k_n^6q_n$.
Using the triangle inequality with the estimates in (6.54) and (6.55), we obtain

$$\begin{aligned}&|\mu(\Phi_n(\tilde{I}_{v_0}^{u_0,u_1})\cap(\mathbb{T}\times\tilde{J}))\lambda(J_{u_1})-\lambda(\tilde{J})\mu(\tilde{I}_{v_0}^{u_0,u_1})|\\&\leq\lambda(J_{u_1})|\mu(\Phi_n(\tilde{I}_{v_0}^{u_0,u_1})\cap(\mathbb{T}\times\tilde{J}))-\lambda(\tilde{J})\lambda(A_\theta^{u_1})|\\&\quad+\lambda(\tilde{J})|\lambda(A_\theta^{u_1})\lambda(J_{u_1})-\mu(\Phi_n(\tilde{I}_{v_0}^{u_0,u_1}))|\\&\leq\frac{36}{k_n^5}\lambda(\tilde{J})\lambda(A_\theta^{u_1})\lambda(J_{u_1})+\frac{36}{k_n^5}\lambda(\tilde{J})\mu(\tilde{I}_{v_0}^{u_0,u_1})\\&\leq\frac{108}{k_n^5}\lambda(\tilde{J})\mu(\tilde{I}_{v_0}^{u_0,u_1})\\&\leq\frac{1}{n^5}\lambda(\tilde{J})\mu(\tilde{I}_{v_0}^{u_0,u_1}).\end{aligned}$$

Thus, we can choose $\varepsilon_1 = \frac{1}{n^5}$.
Next, we show that $(\Phi_n, \mathrm{d}\Phi_n)$ distributes $\hat{I}_{u_0,v_0,j} = \cup_{u_1=0}^{k_n-1} \tilde{I}_{v_0}^{u_0,u_1} \times T_j \in \hat{\eta}_n$ in the tangent direction in the sense of Definition 6.10: For any $j, k \in \{0, \dots, k_n - 1\}$, there exist a unique $u_1 \in \{0, \dots, k_n - 1\}$ satisfying $k \equiv u_1 + j \mod k_n$, and $\tilde{J} \subseteq J_{u_1} = \pi_r(\Phi_n(\tilde{I}_{v_0}^{u_0,u_1}))$ such that

$$(\Phi_n, \mathrm{d}\Phi_n)(\tilde{I}_{v_0}^{u_0,u_1} \times T_j) \cap (\mathbb{T} \times \tilde{J} \times T_k) \neq \emptyset,$$

and building on (6.55) we get the following estimate

$$\left|\mu\left(\pi_M\left(\tilde{I}_{v_0}^{u_0,u_1} \times T_j \cap (\Phi_n, \mathrm{d}\Phi_n)^{-1}(\mathbb{T} \times \tilde{J} \times T_k)\right)\right)\lambda(J_{u_1}) - \lambda(\tilde{J})\mu(\tilde{I}_{v_0}^{u_0,u_1})\right|$$
$$\leq \frac{1}{n^5}\lambda(\tilde{J})\mu(\tilde{I}_{v_0}^{u_0,u_1}),$$

which allows us to choose $\varepsilon_2 = \frac{1}{n^5}$.
Altogether, this shows that $(\Phi_n, \mathrm{d}\Phi_n)$ is $(\frac{1}{2k_n^5 q_n}, \frac{1}{k_n^4}, \frac{1}{n^5}, \frac{1}{n^5})$-distributing every element $\hat{I}_{u_0,v_0,j} \in \hat{\eta}_n$. □

Lemma 6.35. *Consider a sequence of partial partitions $(\hat{\eta}_n)_{n\in\mathbb{N}}$ and $(\tilde{\eta}_n)_{n\in\mathbb{N}}$ of the form as described in (6.50) and (6.40), respectively, and the diffeomorphism g_n from Proposition 6.1. Furthermore, let $(H_n)_{n\in\mathbb{N}}$ be a sequence of smooth measure-preserving diffeomorphism satisfying the condition* **(P1)**. *Consider the partial partitions*

$$\hat{\nu}_n = \{\Gamma_{u_0,v_0,j} = (H_{n-1}, \mathrm{d}H_{n-1}) \circ (g_n, \mathrm{d}g_n)(\hat{I}_{u_0,v_0,j}) \quad : \quad \hat{I}_{u_0,v_0,j} \in \hat{\eta}_n\}. \tag{6.56}$$

Then $\hat{\nu}_n \to \epsilon$.

Proof. By construction $\hat{\eta}_n$ is a partial partition of $\mathbb{P}\mathrm{T}M$ and $\hat{\eta}_n \to \epsilon$, i.e. for every $n \in \mathbb{N}$, $\hat{\eta}_n$ consists of countable disjoint sets and covers a set of almost full measure,

$$\lim_{n\to\infty} \bar{\mu}\left(\bigcup_{(u_0,v_0,j)\in\Lambda_n} \hat{I}_{u_0,v_0,j}\right) = 1,$$

where Λ_n denotes the set of permissible indices in (6.50). By definition, an element

$\hat{I}_{u_0,v_0,j} \in \hat{\eta}_n$ is of the form $\hat{I}_{u_0,v_0,j} = \cup_{u_1=0}^{k_n-1} \tilde{I}_{v_0}^{u_0,u_1} \times T_j$.
Denote $\Gamma_{u_0,v_0,j} = \cup_{u_1=0}^{k_n-1} \Gamma_{v_0,j}^{u_0,u_1}$ and $\Gamma_{v_0,j}^{u_0,u_1} = (H_{n-1}, \mathrm{d}H_{n-1}) \circ (g_n, \mathrm{d}g_n)(\tilde{I}_{v_0}^{u_0,u_1} \times T_j)$. Using the estimates in Lemma 6.6, we can build on Remark 6.30 to conclude that the collection of elements $\Gamma_{u_0,v_0,j} \in \hat{\nu}_n$ covers a set of almost full measure:

$$\begin{aligned}
&\lim_{n\to\infty} \bar{\mu}\left(\bigcup_{u_0,v_0,j\in\Lambda_n} \Gamma_{u_0,v_0,j} \right) \\
&= \lim_{n\to\infty} \sum_{u_0,v_0,j\in\Lambda_n} \sum_{u_1=0}^{k_n-1} \bar{\mu}\left((H_{n-1}, \mathrm{d}H_{n-1}) \circ (g_n, \mathrm{d}g_n)(\tilde{I}_{v_0}^{u_0,u_1} \times T_j) \right) \\
&\geq \lim_{n\to\infty} \sum_{u_0,v_0,j\in\Lambda_n} \sum_{u_1=0}^{k_n-1} \left(1 - \frac{35}{k_n^4}\right) \cdot \frac{1}{k_n} \mu(\tilde{I}_{v_0}^{u_0,u_1}) \\
&\geq \lim_{n\to\infty} \left(1 - \frac{35}{k_n^4}\right) \mu\left(\bigcup_{u_0,v_0,j\in\Lambda_n} \bigcup_{u_1=0}^{k_n-1} \tilde{I}_{v_0}^{u_0,u_1} \right) \\
&\geq \lim_{n\to\infty} \left(1 - \frac{35}{k_n^4}\right)\left(1 - \frac{25}{k_n^3}\right) \geq \lim_{n\to\infty} \left(1 - \frac{80}{k_n^3}\right).
\end{aligned}$$

Further note that, in Proposition 6.1, the map g_n acts as the translation in the θ-direction on $\mathbb{T}^1 \times [\frac{j+2\varepsilon_n}{a_n}, \frac{j+1-2\varepsilon_n}{a_n}]$ by a factor $[nq_n^\sigma] \cdot \frac{j}{k_n^5}$ and its action is trivial in the tangent direction. Since $\tilde{I}_{v_0}^{u_0,u_1} \subset \mathcal{G}_n$, it follows that each $g_n(\tilde{I}_{v_0}^{u_0,u_1})$ is contained in a rectangle of θ width $\frac{1}{2k_n^6 q_n} + \frac{[nq_n^\sigma]}{k_n^5}$ and height $\frac{1}{k_n^5}$. Hence, for $\sigma < 0.5$, and using the fact that $\mathrm{d}g_n$ acts as the identity function on the spherical coordinates with base points $x \in \mathcal{G}_n$ (see (6.7)), we have $\mathrm{diam}((g_n, \mathrm{d}g_n)(\tilde{I}_{v_0}^{u_0,u_1} \times T_j)) \leq \frac{3}{k_n}$. Combining this with condition **(P1)**, we can conclude that

$$\mathrm{diam}\left((H_{n-1}, \mathrm{d}H_{n-1}) \circ (g_n, \mathrm{d}g_n)(\tilde{I}_{v_0}^{u_0,u_1} \times T_j) \right) < \frac{1}{n^2}.$$

Therefore, $\mathrm{diam}(\Gamma_{v_0,j}^{u_0,u_1}) \to 0$ for all $\Gamma_{v_0,j}^{u_0,u_1} \in \hat{\nu}_n$ as $n \to \infty$ and consequently $\hat{\nu}_n \to \epsilon$. □

6.11 Application of the criterion for weak mixing

Proof of Theorem 1.6. For any Liouville number $\alpha \in \mathbb{R}$ we can consider a sequence of diffeomorphisms f_n constructed by equations (6.1) and (6.2) with explicit conjugation maps as in Section 6.4.1 and 6.7.2. Lemma 6.24 and 6.25 guarantee the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ to a limit diffeomorphism $f \in \mathcal{A}_\alpha(\mathbb{T}^2)$ and the proximity condition $d_1(f_n, f) < \frac{1}{2^n}$ for all $n \in \mathbb{N}$.

To establish the weak mixing property of the limiting map $(f, \mathrm{d}f)$ with respect to the invariant measure $\bar{\mu}$ we will apply Proposition 6.11. We consider the sequence of the decomposition $\hat{\eta}_n$ and $\hat{\nu}_n$, and the associated natural numbers (m_n) as described in equations (6.50), (6.56), and Section (6.10.2), respectively. Note that $\hat{\nu}_n \to \epsilon$ by Remark (6.30), $\hat{\eta}_n \to \epsilon$ by Lemma 6.35, and the diffeomorphism $(\Phi_n, \mathrm{d}\Phi_n)$ is $(\frac{1}{2k_n^5 q_n}, \frac{1}{k_n^4}, \frac{1}{n^5}, \frac{1}{n^5})$-distributing every $\hat{I}_n \in \hat{\eta}_n$, as verified Lemma 6.34.

Hence, all the conditions of Proposition 6.11 are satisfied and, consequentially, the map $(f, \mathrm{d}f)$ is weakly mixing with respect to the measures $\bar{\mu}$. Similarly, employing the same combinatorics and the construction of conjugation maps with minor modifications, we can replicate the same construction for any other compact two-dimensional connected manifold M admitting a nontrivial circle action. For more details, we refer to [19, Section 2.4]. $\square$

7

Analytic Weak Mixing Projectivized Derivative Extension

7.1 Introduction

In this chapter, we delve into the extreme ergodic properties of the projectivized derivative extension, with a focus on weak mixing. We explore this aspect in the analytic category using the Approximation by Conjugation method. We present an example of an analytic diffeomorphism whose projectivized derivative extension exhibits weak mixing with respect to a invariant smooth measure. For the torus case, we explore an analytic approximation technique that allows us to closely approximate an analytic diffeomorphism to a smooth diffeomorphism. Additionally, we extend the smooth weak mixing criterion for the projectivized derivative extension to the analytic setting. To demonstrate the weak mixing property of the projectivized derivative extension, we provide a suitable distribution of elements under the explicit choice of conjugation maps and the approximation techniques that meet the necessary requirements of the criterion. In particular, we prove Theorem 1.7.

7.2 Outline of the proof

We employ a similar approach to the smooth construction outlined in Chapter 6 for weak mixing in the projectivized derivative extension, along with an analytic approximation result. We consider a sequence of partial partitions ζ_n of 2-torus converging to a point decomposition. For the analytic constructions, we utilize a close approximation of an analytic diffeomorphism to a smooth diffeomorphism by employing the approximation technique outlined in Sections 7.3 and 7.8. Compared to [33], this approximation method is tailored for Hamiltonian diffeomorphisms and enables the breakdown of a smooth Hamiltonian diffeomorphism into a finite series of one-dimensional linear flows, each closely approximated by analytic flows. These approach ensure a precise level of approximation.

Additionally, we introduce the conjugation map $\hat{h}_n = \hat{g}_n \circ \hat{\phi}_n$, where $\hat{g}_n$ and $\hat{\phi}_n$ represent the analytic diffeomorphisms obtained by applying analytic approximations to the smooth maps g_n and ϕ_n defined in chapter 6, respectively. This map acts as an "almost isometry" on the elements within $I \in \zeta_n$, thereby establishing the existence of an invariant measurable Riemannian metric on $\mathbb{P}\mathrm{T}M$, as discussed in Section 7.6, following a similar concept as described in [33].

Subsequently, we demonstrate the convergence of the scheme $\hat{f}_n$ in the analytic topology, and modifications are made to adapt the weak mixing criterion for the analytic projectivized derivative extension, as detailed in Sections 7.5 and 7.4, respectively. Finally, we apply the weak mixing criterion in Section 7.7 for analytic construction.

7.3 Analytic AbC scheme

On the torus $\mathbb{T}^2$ we execute an approximation by conjugation method in the analytic category, where at step n in the inductive construction process we consider

$$\hat{f}_n = \hat{H}_n \circ R_{\alpha_{n+1}} \circ \hat{H}_n^{-1}, \text{ where } \hat{H}_n = \hat{h}_1 \circ \ldots \circ \hat{h}_n, \tag{7.1}$$

with conjugation maps

$$\hat{h}_n = \hat{g}_n \circ \hat{\phi}_n. \tag{7.2}$$

This time, we choose the sequence $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \alpha_n + \frac{1}{k_n \cdot l_n \cdot q_n^2}$ with parameters k_n, l_n satisfying the conditions **(P1)** – **(P4)** as defined in Section 6.4.

Our conjugation maps $\hat{\phi}_n$ and $\hat{g}_n$ are real analytic diffeomorphisms that commute with R_{α_n} and admit entire complexifications of their lifts of the torus. These maps closely resemble the combinatorial behavior of the conjugation maps ϕ_n and g_n in the smooth case. To establish the existence of such analytic conjugation maps $\hat{\phi}_n$ and $\hat{g}_n$, we can use the following approximation result in Theorem 7.1. It builds on the analytic approximation theorem presented in [7, Theorem 1.8] for the annulus. Here, we derive analogous analytic approximation results for the 2-torus. Its proof is presented in the Section 7.8.

Theorem 7.1 (Analytic Approximation Result)**.** *For any $\epsilon > 0$ and any smooth hamiltonian diffeomorphism $h \in Ham^\infty(\mathbb{T}^2, \mu)$, there exists a real analytic map $\tilde{h} \in Ham^\omega_\infty(\mathbb{T}^2, \mu)$ such that $d_r(h, \tilde{h} \restriction_{\mathbb{T}^2}) < \epsilon$ for all $r \in \mathbb{N}$, and the map $\tilde{h}^{-1} \in Ham^\omega_\infty(\mathbb{T}^2, \mu)$.*

Corollary 7.2. *Let $\epsilon > 0$ and $h \in Ham^\infty(\mathbb{T}^2, \mu)$ be $(\frac{1}{q}, 0)$-periodic. Then there exists $(\frac{1}{q}, 0)$-periodic $\hat{h} \in Ham^\omega_\infty(\mathbb{T}^2, \mu)$ such that $\hat{h}^{-1} \in Ham^\omega_\infty(\mathbb{T}^2, \mu)$ and the following properties hold:*

1. *$d_r(h, \hat{h}|_{\mathbb{T}^2}) < \epsilon$ for $r = 1, 2$;*

2. *$\sup_{x \in \mathbb{T}^2} d((h, \mathrm{d}h)(x), (\hat{h} \restriction_{\mathbb{T}^2}, \mathrm{d}\hat{h})(x)) < \epsilon$.*

Proof of Corollary 7.2. Since $h \in \mathrm{Ham}^\infty(\mathbb{T}^2, \mu)$ is $(\frac{1}{q}, 0)$-periodic, there exists a

$\mathbb{Z}^2$-periodic map $\bar{h} = (\bar{h}_1, \bar{h}_2) \in \mathrm{Ham}^\infty(\mathbb{T}^2, \mu)$ such that

$$h\left(x + \frac{a}{q}, y\right) = \left(\frac{1}{q}\bar{h}_1(qx, y) + \frac{a}{q}, \bar{h}_2(qx, y)\right), \quad \text{where } x \in \left[0, \frac{1}{q}\right).$$

We apply Theorem 7.1 to get $\tilde{h} = (\tilde{h}_1, \tilde{h}_2) \in \mathrm{Ham}^\omega_\infty(\mathbb{T}^2, \mu)$ approximating $\bar{h}$ with precision ϵ/q^2. Then we introduce the map $\hat{h} \in \mathrm{Ham}^\omega_\infty(\mathbb{T}^2, \mu)$ as

$$\hat{h}\left(x + \frac{a}{q}, y\right) = \left(\frac{1}{q}\tilde{h}_1(qx, y) + \frac{a}{q}, \tilde{h}_2(qx, y)\right), \quad \text{where } x \in \left[0, \frac{1}{q}\right).$$

Clearly, $\hat{h}$ is $(\frac{1}{q}, 0)$-periodic and satisfies the required properties. □

For our explicit construction, we pick the proximity parameters as follows:

$$\epsilon_n = \frac{\mathfrak{d}_n}{2k_n^8 q_n^2 \cdot \|DH_{n-1}\|_0^2 \cdot (2|||\phi_n|||_2 + 1)} \quad \text{with} \quad \mathfrak{d}_n = \frac{1}{2^{n^2+1} \cdot n^2 \cdot l_n}. \tag{7.3}$$

Here, we will have to choose $\mathfrak{d}_n$ sufficiently small to meet specific conditions, depending upon H_{n-1}, stated in Sections 6.8 and 6.9. This can be achieved by choosing a sufficiently large l_n. By applying Corollary 7.2 we obtain the maps $\hat{\phi}_n$ and $\hat{g}_n$ in $\mathrm{Diff}^\omega_\infty(\mathbb{T}^2, \mu)$ approximating ϕ_n and g_n, respectively, and satisfying the following conditions:

- $d_r(\hat{\phi}_n \restriction_{\mathbb{T}^2}, \phi_n) < \epsilon_n$ and $d_r(\hat{g}_n \restriction_{\mathbb{T}^2}, g_n) < \epsilon_n$ for all $r \in \mathbb{N}$,
- $\sup_{x \in \mathbb{T}^2} d((\phi_n, \mathrm{d}\phi_n)(x), (\hat{\phi}_n \restriction_{\mathbb{T}^2}, \mathrm{d}\hat{\phi}_n)(x)) < \epsilon_n,$ and
- $\sup_{x \in \mathbb{T}^2} d((g_n, \mathrm{d}g_n)(x), (\hat{g}_n \restriction_{\mathbb{T}^2}, \mathrm{d}\hat{g}_n)(x)) < \epsilon_n.$

7.4 Criterion for analytic weak mixing

We build on our criterion for weak mixing deduced in the previous chapter and show that the weak mixing property still holds for real-analytic AbC diffeomorphisms obtained by utilizing sufficiently close analytic approximations of the conjugation maps from the smooth constructions. We start by showing that such approximations still give good distribution properties in the sense of Definition 6.10.

Lemma 7.3. *Let $\varepsilon > 0$ and $f, g \in \mathrm{Diff}^\infty(\mathbb{T}^2, \mu)$ with $d_0(f, g) < \varepsilon$. Moreover, let η be a partial partition of $\mathbb{T}^2$. Consider two subsets $A, B \subset \mathbb{T}^2$ such that $A \subset B$, and dist$(\partial A, \partial B) > \varepsilon$, where dist$(A, B) = \inf_{x\in A, y\in B} \inf_{k\in\mathbb{Z}^2} \|x - y + k\|$, where $\|\cdot\|$ denotes the Euclidean norm, and ∂A denotes the boundary of A. Then the following relations hold:*

1. *If $f(x) \in A$, then $g(x) \in B$, and $\mu(f(I) \cap A) \le \mu(g(I) \cap B)$ for any $I \in \eta$.*
2. *If $g(x) \in A$, then $f(x) \in B$, and $\mu(g(I) \cap A) \le \mu(f(I) \cap B)$ for any $I \in \eta$.*

Proof. The proof can be derived directly from the definitions. $\square$

Lemma 7.4. *Consider $\phi_n \in \mathrm{Diff}^\infty(\mathbb{T}^2, \mu)$ and $\hat{\phi}_n \in \mathrm{Diff}^\omega_\infty(\mathbb{T}^2, \mu)$ satisfying $d_2(\phi_n, \hat{\phi}_n \restriction_{\mathbb{T}^2}) < \epsilon_n$, where $\epsilon_n > 0$ and $\mathfrak{d}_n > 0$ are given by (7.3). Denote $\Phi_n = \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$ and $\hat{\Phi}_n = \hat{\phi}_n \circ R^{m_n}_{\alpha_{n+1}} \circ \hat{\phi}_n^{-1}$. If the map $(\Phi_n, \mathrm{d}\Phi_n)$ is $(\gamma, \delta, \varepsilon_1, \varepsilon_2)$-distributing the elements $I \in \hat{\eta}_n$ in the sense of Definition 6.10, then $(\hat{\Phi}_n, \mathrm{d}\hat{\Phi}_n)$ is $(\gamma', \delta', \varepsilon_1', \varepsilon_2')$-distributing the elements $I \in \hat{\eta}_n$ with $\gamma' = \gamma + \frac{1}{2^n}$, $\delta' = \delta + \frac{1}{2^n}$, $\varepsilon_1' = 2 \cdot \varepsilon_1 + \frac{3}{2^n}$, and $\varepsilon_2' = 2 \cdot \varepsilon_2 + \frac{3}{2^n}$.*

Proof. By using the triangle inequality and applying the mean value theorem, we can deduce the following proximity statement:

$$
\begin{aligned}
d_1(\Phi_n, \hat{\Phi}_n \restriction_{\mathbb{T}^2}) &= d_1(\phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}, (\hat{\phi}_n \circ R^{m_n}_{\alpha_{n+1}} \circ \hat{\phi}_n^{-1}) \restriction_{\mathbb{T}^2}) \\
&\le d_1(\phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ (\hat{\phi}_n^{-1}) \restriction_{\mathbb{T}^2}, (\hat{\phi}_n \circ R^{m_n}_{\alpha_{n+1}} \circ \hat{\phi}_n^{-1}) \restriction_{\mathbb{T}^2}) \\
&\quad + d_1(\phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}, \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ (\hat{\phi}_n^{-1}) \restriction_{\mathbb{T}^2}) \\
&\le \max(|||\phi_n|||_1, \left|\left|\left|\hat{\phi}_n\right|\right|\right|_1) \cdot d_1(\phi_n, \hat{\phi}_n \restriction_{\mathbb{T}^2}) + |||\phi_n|||_2 \cdot d_1(\phi_n, \hat{\phi}_n \restriction_{\mathbb{T}^2}) \\
&\le (|||\phi_n|||_2 + \epsilon_n) \cdot d_1(\phi_n, \hat{\phi}_n \restriction_{\mathbb{T}^2}) + |||\phi_n|||_2 \cdot d_1(\phi_n, \hat{\phi}_n \restriction_{\mathbb{T}^2}) \\
&\le (2|||\phi_n|||_2 + \epsilon_n) \cdot d_1(\phi_n, \hat{\phi}_n \restriction_{\mathbb{T}^2}) \\
&\le (2|||\phi_n|||_2 + 1) \cdot \epsilon_n \le \mathfrak{d}_n.
\end{aligned} \tag{7.4}
$$

With the definition of $(\Phi_n, \mathrm{d}\Phi_n)$ being $(\gamma, \delta, \varepsilon_1, \varepsilon_2)$-distributing, it satisfies four conditions: The first two conditions show that $\Phi_n(\tilde{I}_{n,l})$ is contained within $[c_l, c_l + \gamma] \times \mathbb{T}^1$ for some constant $c_l \in \mathbb{T}^1$ and that there exists a set J_l such that

$J_l \subseteq \pi_r(\Phi_n(\tilde{I}_{n,l}))$ with $1-\delta \leq \lambda(J_l) \leq 1$ for all $l \in \{0,\dots,k_n-1\}$. We denote $J_l' = J_l \cap \pi_r(\hat{\Phi}_n(\tilde{I}_{n,l}))$, which satisfies $\lambda(J_l') > 1-(\delta+2\cdot\mathfrak{d}_n)$. With $\mathfrak{d}_n < \frac{1}{2^{n+1}}$, this allows us to choose $\gamma' = \gamma + \frac{1}{2^n}$ and $\delta' = \delta + \frac{1}{2^n}$.
For any subset $\tilde{J} \subset J_l'$, we consider two subsets, $\tilde{J}_1$ and $\tilde{J}_2$, inside J_l such that $\tilde{J}_1 \subset \tilde{J} \subset \tilde{J}_2$. Additionally, we require $\lambda(\tilde{J}_i \triangle \tilde{J}) \leq \frac{1}{2^n}\cdot\lambda(\tilde{J})$ for $i=1,2$, and for sufficiently large n that $\mathrm{dist}(\partial\tilde{J}_i, \partial\tilde{J}) > \mathfrak{d}_n$ for $i=1,2$.
Given the other conditions in the definition of $(\Phi_n, \mathrm{d}\Phi_n)$ as $(\gamma,\delta,\varepsilon_1,\varepsilon_2)$ distributing, we can conclude that for any $\tilde{J}_i \subset J_l$ for $i=1,2$, and for all $l \in \{0,\dots,k_n-1\}$

$$|\mu(\tilde{I}_{n,l} \cap \Phi_n^{-1}(\mathbb{T}^1 \times \tilde{J}_i)\lambda(J_l)) - \lambda(\tilde{J}_i)\mu(\tilde{I}_{n,l})| \leq \varepsilon_1\lambda(\tilde{J}_i)\mu(\tilde{I}_{n,l}), \tag{7.5}$$

and for any $j,k \in \{0,1,\dots,k_n-1\}$, there exists a $l \in \{0,1,\dots,k_n-1\}$ such that $k \equiv l+j \mod k_n$, for $\tilde{J}_i \subset J_l' \subseteq J_l$, and $T_k = \left[\frac{k}{k_n}, \frac{k+1}{k_n}\right]$ we have

$$\begin{aligned}&\left|\mu\left(\pi_M\left(\tilde{I}_{n,l,j} \cap (\Phi_n, \mathrm{d}\Phi_n)^{-1}(\mathbb{T}^1 \times \tilde{J}_i \times T_k)\right)\right)\lambda(J_l) - \lambda(\tilde{J}_i)\mu(\tilde{I}_{n,l})\right| \\ &\leq \varepsilon_2\lambda(\tilde{J}_i)\mu(\tilde{I}_{n,l}).\end{aligned}$$

Let us denote the following subsets of $\mathbb{T}^2$ as $A_1 = \mathbb{T}^1 \times \tilde{J}_1$, $A = \mathbb{T}^1 \times \tilde{J}$, and $A_2 = \mathbb{T}^1 \times \tilde{J}_2$, which satisfy the following conditions: $A_1 \subset A \subset A_2$, $\mu(A_i \triangle A) \leq \mu(\tilde{J}_i \triangle \tilde{J}) \leq \frac{1}{2^n}\cdot\mu(A)$, and $\mathrm{dist}(\partial A_i, \partial A) > \mathrm{dist}(\partial\tilde{J}_i, \partial\tilde{J}) > \mathfrak{d}_n$ for $i=1,2$. With Lemma 7.3, we can establish the following relations: if $I \in \Phi_n^{-1}(A_1)$, then $I \in \hat{\Phi}_n^{-1}(A)$, and if $I \in \hat{\Phi}_n^{-1}(A)$, then $I \in \Phi_n^{-1}(A_2)$. This implies

$$\mu(\tilde{I}_{n,l} \cap \Phi_n^{-1}(A_1))\lambda(J_l') \leq \mu(\tilde{I}_{n,l} \cap \hat{\Phi}_n^{-1}(A))\lambda(J_l') \leq \mu(\tilde{I}_{n,l} \cap \Phi_n^{-1}(A_2))\lambda(J_l'). \tag{7.6}$$

Altogether with equation (7.5) and relation $|\lambda(J_l') - \lambda(J_l)| \leq 2\mathfrak{d}_n\lambda(J_l)$ it holds:

$$(1-2\mathfrak{d}_n)(1-\varepsilon_1)\mu(\tilde{I}_{n,l})\lambda(\tilde{J}_1) \leq \mu(\tilde{I}_{n,l} \cap \hat{\Phi}_n^{-1}(A))\lambda(J_l'); \tag{7.7}$$

$$\mu(\tilde{I}_{n,l} \cap \hat{\Phi}_n^{-1}(A))\lambda(J_l') \leq (1+2\mathfrak{d}_n)(1+\varepsilon_1)\mu(\tilde{I}_{n,l})\lambda(\tilde{J}_2). \tag{7.8}$$

Furthermore, we can obtain the following estimates with $\mathfrak{d}_n \leq \frac{1}{2^{n+1}}$:

$$\begin{aligned}
&\mu(\tilde{I}_{n,l} \cap \hat{\Phi}_n^{-1}(A))\lambda(J_l') - \mu(\tilde{I}_{n,l})\lambda(\tilde{J}) \\
&\leq (\varepsilon_1 + 2\mathfrak{d}_n(1+\varepsilon_1))\mu(\tilde{I}_{n,l})\lambda(\tilde{J}) + (1+2\mathfrak{d}_n)(1+\varepsilon_1)\mu(\tilde{I}_{n,l})(\lambda(\tilde{J}_2) - \lambda(\tilde{J})) \\
&\leq (\varepsilon_1 + 2\mathfrak{d}_n(1+\varepsilon_1))\mu(\tilde{I}_{n,l})\lambda(\tilde{J}) + (1+2\mathfrak{d}_n)(1+\varepsilon_1)\mu(\tilde{I}_{n,l})\lambda(\tilde{J}_2 \triangle \tilde{J}) \\
&\leq \left(\varepsilon_1 + 2\mathfrak{d}_n(1+\varepsilon_1) + (1+2\mathfrak{d}_\mathfrak{n})(1+\varepsilon_1)\frac{1}{2^n}\right)\mu(\tilde{I}_{n,l})\lambda(\tilde{J}) \\
&\leq \left(2 \cdot \varepsilon_1 + \frac{3}{2^n}\right)\mu(\tilde{I}_{n,l})\lambda(\tilde{J}).
\end{aligned}$$

Similarly, we deduce the following estimate using (7.8):

$$\mu(\tilde{I}_{n,l} \cap \hat{\Phi}_n^{-1}(A))\lambda(J_l') - \mu(\tilde{I}_{n,l})\lambda(\tilde{J}) \geq -\left(2 \cdot \varepsilon_1 + \frac{3}{2^n}\right)\mu(\tilde{I}_{n,l})\lambda(\tilde{J}).$$

Altogether, we obtain that

$$|\mu(\tilde{I}_{n,l} \cap \hat{\Phi}_n^{-1}(\mathbb{T}^1 \times \tilde{J}))\lambda(J_l') - \mu(\tilde{I}_{n,l})\lambda(\tilde{J})| \leq \left(2 \cdot \varepsilon_1 + \frac{3}{2^n}\right)\mu(\tilde{I}_{n,l})\lambda(\tilde{J}).$$

Analogously, we deduce using the estimate (7.6) that for any pair $j, k \in \{0, \dots, k_n - 1\}$, there exists $l \in \{0, \dots, k_n - 1\}$ such that $k \equiv l + j \mod k_n$, and for $\tilde{J} \subset J_l'$, $T_k = \left[\frac{k}{k_n}, \frac{k+1}{k_n}\right]$ we have

$$\begin{aligned}
&\left|\mu\left(\pi_M\left(\tilde{I}_{n,l,j} \cap (\hat{\Phi}_n, \mathrm{d}\hat{\Phi}_n)^{-1}(\mathbb{T}^1 \times \tilde{J} \times T_k)\right)\right)\lambda(J_l') - \mu(\tilde{I}_{n,l})\lambda(\tilde{J})\right| \\
&\leq \left(2 \cdot \varepsilon_2 + \frac{3}{2^n}\right)\mu(\tilde{I}_{n,l})\lambda(\tilde{J}).
\end{aligned}$$

In summary, this yields that $(\hat{\Phi}_n, \mathrm{d}\hat{\Phi}_n)$ is $(\gamma', \delta', \varepsilon_1', \varepsilon_2')$-distributing $\hat{I}_{n,j} \in \hat{\eta}_n$. □

Lemma 7.5. *Let g_n be as in Proposition 6.1 and satisfy the property of Lemma 6.8 with the map $(\hat{\Phi}_n, \mathrm{d}\hat{\Phi}_n)$ which is supposed to $(\gamma', \delta', \varepsilon_1', \varepsilon_2')$ distribute $\hat{I}_{n,j} \in \hat{\eta}_n$, where $\hat{\eta}_n$ is the partial partition of $\mathbb{P}\mathrm{T}M$ constructed in (6.9). Furthermore, let $\hat{g}_n \in \mathrm{Diff}^\omega_\infty(\mathbb{T}^2, \mu)$ such that $d_k(g_n, \hat{g}_n \restriction_{\mathbb{T}^2}) < \epsilon_n$, where ϵ_n is given by (7.3). Let S_n be a square of side length equal to $\frac{1}{k_n}$ lying in $\mathbb{T}^1 \times [\delta, 1-\delta]$. Denote $\tilde{S}_n = S_n \cap \widetilde{G}_n$*

with $\widetilde{G}_n$ defined in (6.14). For any $k \in \{0, \dots, k_n - 1\}$, denote $\tilde{S}_{n,k} = \tilde{S}_n \times T_k$ with $T_k = \left[\frac{k}{k_n}, \frac{k+1}{k_n}\right]$.

Then for every pair $j, k \in \{0, 1, \dots, k_n - 1\}$ there exist $l \in \{0, 1, \dots, k_n - 1\}$ such that $k \equiv (l + j) \mod k_n$ and it holds:

$$\begin{aligned}
&\left|\mu\left(\pi_M\left(\tilde{I}_{n,l,j} \cap (\hat{\Phi}_n, \mathrm{d}\hat{\Phi}_n)^{-1} \circ (\hat{g}_n, \mathrm{d}\hat{g}_n)^{-1}(\tilde{S}_{n,k})\right)\right) - \mu(\tilde{I}_{n,l})\mu(\tilde{S}_n)\right| \\
\leq &2\left(\varepsilon_2' + \frac{69}{n}\right)\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n).
\end{aligned}$$

Proof. The proof follows a similar line of reasoning as the previous lemma. For notation, let $\tilde{S}_n = S_n \cap \widetilde{G}_n$ and S_n be a square of side length $\frac{1}{k_n}$ lying in $\mathbb{T}^1 \times [\delta', 1 - \delta']$. Now, consider two squares, S_n^1 and S_n^2, such that $S_n^1 \subset S_n \subset S_n^2$, $\mu(S_n^i \triangle S_n) \leq \frac{1}{2^n} \cdot \mu(\tilde{S}_n)$, and $\mathrm{dist}(\partial S_n^i, \partial \tilde{S}_n) > \mathfrak{d}_n$ for $i = 1, 2$.

Denote the cuboid $\tilde{S}_{n,k} = \tilde{S}_n \times T_k$ and $\tilde{S}_{n,k}^i = \tilde{S}_n^i \times T_k$ for $i = 1, 2$ where $\tilde{S}_n^i = S_n^i \cap \widetilde{G}_n$. These cuboids satisfy the following conditions:

$$\tilde{S}_{n,k}^1 \subset \tilde{S}_{n,k} \subset \tilde{S}_{n,k}^2.$$

Using Lemma 6.8 along with Remark 6.9 for the maps $(g_n, \mathrm{d}g_n)$ and $(\hat{\Phi}_n, \mathrm{d}\hat{\Phi}_n)$, characterized by $(\gamma', \delta', \varepsilon_1', \varepsilon_2')$ distribute $\hat{I}_{n,j} = \cup_{l=0}^{k_n-1} \tilde{I}_{n,l,j} \in \hat{\eta}_n$ for the cuboid $\tilde{S}_{n,k}^i$, $i = 1, 2$: for any $j, k \in \{0, 1, \dots, k_n - 1\}$ there exist $l \in \{0, 1, \dots, k_n - 1\}$ such that $k \equiv (l + j) \mod k_n$, and the following estimates holds:

$$\begin{aligned}
&\left|\mu\left(\pi_M\left(\tilde{I}_{n,l,j} \cap (\hat{\Phi}_n, \mathrm{d}\hat{\Phi}_n)^{-1} \circ (g_n, \mathrm{d}g_n)^{-1}(\tilde{S}_{n,k}^i)\right)\right) - \mu(\tilde{I}_{n,l})\mu(\tilde{S}_n^i)\right| \\
\leq &\left(\varepsilon_2' + \frac{68}{n}\right)\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n^i). \qquad (7.9)
\end{aligned}$$

Using the mean value theorem along with $d_1(g_n, \hat{g}_n \restriction_{\mathbb{T}^2}) < \epsilon_n$, and equation (7.3), we have $d_1(\hat{\Phi}_n^{-1} \circ g_n^{-1}, \hat{\Phi}_n^{-1} \circ \hat{g}_n^{-1} \restriction_{\mathbb{T}^2}) < \mathfrak{d}_n$. Moreover, by analyzing the property of Lemma 7.3, we can derive the following relations. First, we have:

$$\begin{aligned}
&\mu\left(\pi_M\left(\tilde{I}_{n,l,j} \cap (\hat{\Phi}_n, \mathrm{d}\hat{\Phi}_n)^{-1} \circ (g_n, \mathrm{d}g_n)^{-1}(\tilde{S}_{n,k}^1)\right)\right) \\
\leq &\mu\left(\pi_M\left(\tilde{I}_{n,l,j} \cap (\hat{\Phi}_n, \mathrm{d}\hat{\Phi}_n)^{-1} \circ (\hat{g}_n, \mathrm{d}\hat{g}_n)^{-1}(\tilde{S}_{n,k})\right)\right), \qquad (7.10)
\end{aligned}$$

and establish the second relation as:

$$\begin{aligned}&\mu\big(\pi_M\big(\tilde{I}_{n,l,j}\cap(\hat{\Phi}_n,\mathrm{d}\hat{\Phi}_n)^{-1}\circ(\hat{g}_n,\mathrm{d}\hat{g}_n)^{-1}(\tilde{S}_{n,k})\big)\big)\\ \leq&\mu\big(\pi_M\big(\tilde{I}_{n,l,j}\cap(\hat{\Phi}_n,\mathrm{d}\hat{\Phi}_n)^{-1}\circ(g_n,\mathrm{d}g_n)^{-1}(\tilde{S}^2_{n,k})\big)\big). \end{aligned} \tag{7.11}$$

Altogether with equation (7.9), it holds:

$$\begin{aligned}&\mu\left(\pi_M\left(\tilde{I}_{n,l,j}\cap(\hat{\Phi}_n,\mathrm{d}\hat{\Phi}_n)^{-1}\circ(\hat{g}_n,\mathrm{d}\hat{g}_n)^{-1}(\tilde{S}_{n,k})\right)\right)-\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n)\\ \leq&\left(\left(\varepsilon_2'+\frac{68}{n}\right)\left(1+\frac{1}{2^n}\right)+\frac{1}{2^n}\right)\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n). \end{aligned} \tag{7.12}$$

Analogously, we obtain

$$\begin{aligned}&\mu\left(\pi_M\left(\tilde{I}_{n,l,j}\cap(\hat{\Phi}_n,\mathrm{d}\hat{\Phi}_n)^{-1}\circ(\hat{g}_n,\mathrm{d}\hat{g}_n)^{-1}(\tilde{S}_{n,k})\right)\right)-\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n)\\ \geq&-\left(\left(\varepsilon_2'+\frac{68}{n}\right)\left(1+\frac{1}{2^n}\right)+\frac{1}{2^n}\right)\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n). \end{aligned} \tag{7.13}$$

With $(1+\frac{1}{2^n})<2$ and $\frac{1}{2^n}<\frac{2}{n}$ in equation (7.12) and (7.13), we have

$$\begin{aligned}&\left|\mu\left(\pi_M\left(\tilde{I}_{n,l,j}\cap(\hat{\Phi}_n,\mathrm{d}\hat{\Phi}_n)^{-1}\circ(\hat{g}_n,\mathrm{d}\hat{g}_n)^{-1}(\tilde{S}_{n,k})\right)\right)-\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n)\right|\\ \leq&2\left(\varepsilon_2'+\frac{69}{n}\right)\mu(\tilde{I}_{n,l})\mu(\tilde{S}_n),\end{aligned}$$

which concludes the claim. □

Proposition 7.6. *(Criterion for analytic weak mixing) Let* $\hat{f}_n=\hat{H}_n\circ R_{\alpha_{n+1}}\circ\hat{H}_n^{-1}$ *be defined by (7.1), such that* $(\hat{f}_n,\mathrm{d}\hat{f}_n)$ *converges to* $(\hat{f},\mathrm{d}\hat{f})$ *and satisfies* $d_1(\hat{f}_n,\hat{f})<\frac{1}{2^n}$ *for all* $n\in\mathbb{N}$*. Additionally we assume that the sequence* $(H_n)_{n\in\mathbb{N}}$ *satisfies condition* **(P1)***, and the map* g_n *is as in Lemma 7.5. Consider a sequence of partial partitions* $(\hat{\eta}_n)_{n\in\mathbb{N}}$*, defined as in (6.9) such that* $(\hat{\eta}_n)\to\epsilon$*. Additionally, we assume that the partial partition* $\hat{\hat{\nu}}_n=\{\Gamma_{n,j}=(\hat{H}_{n-1},\mathrm{d}\hat{H}_{n-1})\circ(\hat{g}_n,\mathrm{d}\hat{g}_n)(\hat{I}_{n,j})\mid\hat{I}_{n,j}\in\hat{\eta}_n\}$*, satisfies* $\hat{\hat{\nu}}_n\to\epsilon$*. Suppose for a sequence* $(m_n)_{n\in\mathbb{N}}$ *and the diffeomorphism* $(\hat{\Phi}_n,\mathrm{d}\hat{\Phi}_n):\mathbb{P}\mathrm{T}M\to\mathbb{P}\mathrm{T}M$ *with* $\hat{\Phi}_n=\hat{\phi}_n\circ R^{m_n}_{\alpha_{n+1}}\circ\hat{\phi}_n^{-1}$ *is* $(\gamma,\delta,\varepsilon_1,\varepsilon_2)$*-distributing* $\hat{I}_{n,j}\in\hat{\eta}_n$ *with* $\gamma<\frac{1}{k_n^3q_n},\delta<\frac{50}{k_n^2},\varepsilon_1,\varepsilon_2<\frac{150}{n}$*. Then*

$(\hat{f}, \mathrm{d}\hat{f}) = \lim_{n\to\infty}(\hat{f}_n, \mathrm{d}\hat{f}_n)$ *is weakly mixing on* $\mathbb{P}\mathrm{T}M$.

Proof. The proof follows along the lines of Proposition 6.11, utilizing Lemma 7.4 and Lemma 7.5. □

7.5 Convergence of sequence $< \hat{f}_n >_{n\in\mathbb{N}}$ in the analytic topology

Lemma 7.7. *There exists a sequence of rational numbers* α_{n+1} *such that the sequence of diffeomorphisms* $\hat{f}_n \in \mathrm{Diff}^\omega_\infty(\mathbb{T}^2, \mu)$*, defined by (7.1), converges to a diffeomorphism* $\hat{f} \in \mathrm{Diff}^\omega_\infty(\mathbb{T}^2, \mu)$*. Moreover, it satisfies the following condition for any natural number* $m \leq q_{n+1}$*,*

1. $d_0((\hat{f}_n)^m, (\hat{f})^m) < \frac{1}{2^n}$,
2. $\sup_{x\in\mathbb{P}\mathrm{T}M} d((\hat{f}_n, \mathrm{d}\hat{f}_n)^m(x), (\hat{f}, \mathrm{d}\hat{f})^m(x)) < \frac{1}{2^n} < \frac{1}{n^2}$.

Proof. Let us fix the sequence $\rho_n = n \in \mathbb{N}$ used for the complex domain Ω_{ρ_n} as defined by (2.10). Note that the sequence of diffeomorphisms $\hat{f}_n$, defined by (7.1), admits an entire complexification. Thus, each nth term of the sequence, $\hat{f}_n \in \mathrm{Diff}^\omega_{\rho_n}(\mathbb{T}^2, \mu)$, where $\rho_n = n$. With the relations $\alpha_{n+1} = \alpha_n + \frac{1}{k_n l_n q_n^2}$ and $\hat{h}_n \circ R_{\alpha_n} = R_{\alpha_n} \circ \hat{h}_n$, we can obtain for $\rho_n = n$:

$$\begin{aligned} d^\omega_n(\hat{f}_n, \hat{f}_{n-1}) &= d^\omega_n(\hat{H}_n \circ R_{\alpha_{n+1}} \circ \hat{H}_n^{-1}, \hat{H}_{n-1}\hat{h}_n \circ \hat{h}_n^{-1} \circ R_{\alpha_n} \circ \hat{H}_{n-1}^{-1}) \\ &= d^\omega_n(\hat{H}_n \circ R_{\alpha_{n+1}} \circ \hat{H}_n^{-1}, \hat{H}_{n-1}\hat{h}_n \circ R_{\alpha_n} \circ \hat{h}_n^{-1} \circ \hat{H}_{n-1}^{-1}) \\ &\leq \|\mathrm{d}\hat{H}_n\| \cdot |\alpha_{n+1} - \alpha_n| \\ &\leq \|\mathrm{d}\hat{H}_n\| \cdot \frac{1}{k_n l_n q_n^2} \leq \frac{1}{2^n}. \end{aligned} \tag{7.14}$$

Since the construction of $\hat{h}_n = \hat{g}_n \circ \hat{\phi}_n$ does not depend on the parameter l_n, the last step can be achieved by choosing sufficiently large $l_n \in \mathbb{N}$ and $k_n \in \mathbb{N}$, while satisfying the additional conditions **(P1)** – **(P4)** stated in Subsection 6.4. These choices ensure that the following estimates hold:

$$l_n > n^2 \cdot k_n^{10} \cdot q_n^2 \cdot \|\mathrm{d}H_{n-1}\|_0, \quad \text{and} \quad \log(q_{n+1}) = \log(k_n \cdot l_n \cdot q_n^2) > \|\mathrm{d}\hat{H}_n\|,$$

and for any natural number $m \leq q_{n+1}$, the following inequality holds:

$$\sup_{x \in \mathbb{P}\mathrm{T}M} d((\hat{f}_n, \mathrm{d}\hat{f}_n)^m(x), (\hat{f}_{n-1}, \mathrm{d}\hat{f}_{n-1})^m(x)) < \frac{1}{2 \cdot k_n} < \frac{1}{2^n}. \tag{7.15}$$

Based on the estimates (7.14), we can conclude that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\sum_{i=N}^{\infty} \frac{1}{2^i} < \varepsilon$, and for any natural numbers $m, n > N$, and $m \leq n$:

$$\begin{aligned} d_m^{\omega}(\hat{f}_n, \hat{f}_m) &\leq \sum_{i=m+1}^{n} d_m^{\omega}(\hat{f}_i, \hat{f}_{i-1}) \\ &\leq \sum_{i=m+1}^{n} d_i^{\omega}(\hat{f}_i, \hat{f}_{i-1}) \\ &\leq \sum_{i=m+1}^{n} \frac{1}{2^i} < \varepsilon. \end{aligned} \tag{7.16}$$

Analogously, we have $d_k^{\omega}(\hat{f}_n, \hat{f}_m) < d_m^{\omega}(\hat{f}_n, \hat{f}_m) < \varepsilon$, for any $k < m \leq n$. This shows that the sequence $(\hat{f}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{Diff}_k^{\omega}(\mathbb{T}^2, \mu)$ for all $k \leq n$. As $\mathrm{Diff}_k^{\omega}(\mathbb{T}^2, \mu)$ is a complete space, there exists a limit diffeomorphism $\hat{f} \in \mathrm{Diff}_k^{\omega}(\mathbb{T}^2, \mu)$ for all $k \leq n$. Furthermore, the estimate (7.16) guarantees $\hat{f} \in \cap_{i=1}^{n} \mathrm{Diff}_i^{\omega}(\mathbb{T}^2, \mu)$.

Additionally, the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ expands the complex domain of analyticity, $\{\Omega_{\rho_n}\}_{n\in\mathbb{N}}$, and exhausts all $\mathbb{C}^2$. Consequently, we can conclude that the limit diffeomorphism, $\lim_{n\to\infty} \hat{f}_n = \hat{f}$, admits an entire complexification, i.e. $\hat{f} \in \mathrm{Diff}_{\infty}^{\omega}(\mathbb{T}^2, \mu)$.

Moreover, we can conclude that for any natural number $m \leq q_{n+1}$:

$$d_0((\hat{f}_n)^m, (\hat{f})^m) \leq \sum_{i=n+1}^{\infty} d_0((\hat{f}_i)^m, (\hat{f}_{i-1})^m) \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} < \frac{1}{2^n}, \tag{7.17}$$

and the sequence $(\hat{f}_n, \mathrm{d}\hat{f}_n)$ defined on $\mathbb{P}\mathrm{T}M$, using equation (7.15), also satisfies :

$$\begin{aligned}
&\sup_{x\in\mathbb{P}\mathrm{T}M} d((\hat{f}_n, \mathrm{d}\hat{f}_n)^m(x), (\hat{f}, \mathrm{d}\hat{f})^m(x)) \\
&\leq \sum_{i=n+1}^{\infty} \sup_{x\in\mathbb{P}\mathrm{T}M} d((\hat{f}_i, \mathrm{d}\hat{f}_i)^m(x), (\hat{f}_{i-1}, \mathrm{d}\hat{f}_{i-1})^m(x)) \\
&\leq \sum_{i=n+1}^{\infty} \frac{1}{2\cdot k_i} < \sum_{i=n+1}^{\infty} \frac{1}{2^i} < \frac{1}{2^n}.
\end{aligned} \tag{7.18}$$

Altogether, this shows that there exists $\hat{f} \in \mathrm{Diff}^{\omega}_{\infty}(\mathbb{T}^2, \mu)$ such that $\lim_{n\to\infty} \hat{f}_n = \hat{f}$, and it satisfies both the estimates of the lemma given by (7.17) and (7.18). $\square$

7.6 Existence of an invariant Riemannian metric

Next, we show that our conjugation maps h_n and $\hat{h}_n$ fulfill the requirements of Proposition 6.3.

Lemma 7.8. *Let the partial partition ζ_n be defined as in subsection 6.7.1.2. The map $\hat{h}_n$ defined by (7.2) satisfy the following for all $I_n \in \zeta_n$:*

$$dev_{I_n}(\hat{h}_n) \leq \frac{2\mathfrak{d}_n}{\|DH_{n-1}\|_0^2} \text{ with } \mathfrak{d}_n \text{ as in equation (7.3).} \tag{7.19}$$

Proof. Note that with our choice of parameters and the fact that $d_1(\phi_n, \hat{\phi}_n) < \epsilon_n$ and $d_1(g_n, \hat{g}_n) < \epsilon_n$ in section 7.3, we obtain

$$\begin{aligned}
d_1(h_n, \hat{h}_n) &\leq d_1(g_n \circ \phi_n, \hat{g}_n \circ \hat{\phi}_n) \\
&\leq (1 + |||g_n|||_2) \cdot \epsilon_n \leq \frac{\mathfrak{d}_n}{\|DH_{n-1}\|_0^2}.
\end{aligned}$$

Additionally, for any unit vector $v = (v_1, v_2)$, we obtain :

$$\begin{aligned}
\Big| \|\mathrm{d}\hat{h}_n(v)\| - \|\mathrm{d}h_n(v)\| \Big| &\leq \|\mathrm{d}\hat{h}_n(v) - \mathrm{d}h_n(v)\| \\
&\leq d_1(h_n, \hat{h}_n) \leq \frac{\mathfrak{d}_n}{\|DH_{n-1}\|_0^2}.
\end{aligned}$$

Referring to Definition 6.8, we can further calculate:

$$\left|\log\|\mathrm{d}\hat{h}_n(v)\| - \log\|\mathrm{d}h_n(v)\|\right| = \left|\log\frac{\|\mathrm{d}\hat{h}_n(v)\|}{\|\mathrm{d}h_n(v)\|}\right| \leq \frac{\frac{\mathfrak{d}_n}{\|DH_{n-1}\|_0^2}}{\min(\|\mathrm{d}\hat{h}_n(v)\|, \|\mathrm{d}h_n(v)\|)}.$$

Here, we used the following relation: if $|a-b| \leq \epsilon$ then it implies $|\log(a/b)| \leq \frac{\epsilon}{\min(a,b)}$ in the subsequent equation. Now, we maximize this over all unit vectors v:

$$\max_{v\in \mathrm{T}I_n, \|v\|=1}\left|\log\frac{\|\mathrm{d}\hat{h}_n(v)\|}{\|\mathrm{d}h_n(v)\|}\right| \leq \frac{\mathfrak{d}_n}{\|DH_{n-1}\|_0^2}\max_{v\in \mathrm{T}I_n, \|v\|=1}\frac{1}{\min(\|\mathrm{d}\hat{h}_n(v)\|, \|\mathrm{d}h_n(v)\|)}.$$

Since $\hat{h}_n$ and h_n are sufficiently close to being isometries, we can assume that $\min(\|\mathrm{d}\hat{h}_n(v)\|, \|\mathrm{d}h_n(v)\|) \geq \frac{1}{2}$.
Consequently, the deviation for $\hat{h}_n$ can be estimated by:

$$\begin{aligned}
\mathrm{dev}_{I_n}(\hat{h}_n) &= \max_{v\in \mathrm{T}I_n, \|v\|=1}\left|\log\|\mathrm{d}\hat{h}_n(v)\|\right| \\
&\leq \max_{v\in \mathrm{T}I_n, \|v\|=1}\left(|\log\|\mathrm{d}h_n(v)\|| + \left|\log\frac{\|\mathrm{d}\hat{h}_n(v)\|}{\|\mathrm{d}h_n(v)\|}\right|\right) \\
&\leq \mathrm{dev}_{I_n}(h_n) + \frac{2\mathfrak{d}_n}{\|DH_{n-1}\|_0^2} \\
&\leq \frac{2\mathfrak{d}_n}{\|DH_{n-1}\|_0^2}.
\end{aligned}$$

This completes the proof. □

Remark 7.9. For any function $\kappa : \mathrm{Diff}^\infty(M,\mu) \to (0,1)$ we can choose the sequence $\mathfrak{d}_n$ in equation (7.3) to decay sufficiently fast so that the condition $\sum_{k\geq n}\delta_k < \kappa(H_{n-1})$ holds for every $n \in \mathbb{N}$. This is ensured by choosing a sufficiently large l_n dependent on the function $\kappa(H_{n-1})$. Along with Lemma 7.8 and Proposition 6.3, we can then conclude that the limit diffeomorphism $\hat{f}$ obtained from the sequence $\hat{f}_n$ constructed as in (7.1) admits an invariant measurable Riemannian metric.

7.7 Application of criterion for weak mixing

Lemma 7.10. *Consider the sequence of maps $H_n, g_n, \hat{H}_n$, and $\hat{g}_n$ as defined in Section 6.4 and 7.3 respectively. Consider the partial partitions $\acute{\nu}_n$ and $\hat{\acute{\nu}}_n$, defined by equation (6.56), and*

$$\hat{\acute{\nu}}_n = \left\{ (\hat{H}_{n-1}, \mathrm{d}\hat{H}_{n-1}) \circ (\hat{g}_n, d\hat{g}_n)(\hat{I}_{n,j}) \quad : \quad \hat{I}_{n,j} \in \acute{\eta}_n \right\}. \tag{7.20}$$

If $\acute{\nu}_n \to \epsilon$, then $\hat{\acute{\nu}}_n \to \epsilon$.

Proof. The proof follows a similar line to the Lemma 6.35, utilizing the following estimate $d_k(g_n, \hat{g}_n) \le \epsilon_n$, and for any $\tilde{I}^{u_0,u_1}_{v_0} \in \tilde{\eta}_n$:

$$\begin{aligned}|\mathrm{diam}(\hat{H}_{n-1} \circ \hat{g}_n(\tilde{I}^{u_0,u_1}_{v_0})) - \mathrm{diam}(H_{n-1} \circ g_n(\tilde{I}^{u_0,u_1}_{v_0}))| &\le d_1(\hat{H}_{n-1} \circ \hat{g}_n \restriction_{\mathbb{T}^2}, H_{n-1} \circ g_n) \\ &\le (1 + |||H_{n-1}|||_0) \cdot \epsilon_n \le \frac{1}{2^n}.\end{aligned}$$

Therefore, for every $\tilde{I}^{u_0,u_1}_{v_0} \in \acute{\eta}_n$, we have $\mathrm{diam}(H_{n-1} \circ g_n(\tilde{I}^{u_0,u_1}_{v_0})) \to 0$, which in turn implies that $\mathrm{diam}(\hat{H}_{n-1} \circ \hat{g}_n(\tilde{I}^{u_0,u_1}_{v_0})) \to 0$ as $n \to \infty$, leading to $\hat{\acute{\nu}}_n \to \epsilon$. □

Proof of Theorem 1.7. The proof follows along the same lines as in the smooth case. Lemma 7.7 ensures the convergence of the sequence of analytic measure-preserving diffeomorphisms $(\hat{f}_n)_{n\in\mathbb{N}}$ constructed via equations (7.1) and (7.2). It also provides the proximity condition $d_1(\hat{f}_n, \hat{f}) < \frac{1}{2^n}$.

The existence of an invariant measure $\bar{\mu}$ with respect to the limiting map $(\hat{f}, \mathrm{d}\hat{f})$ is guaranteed by subsection 6.2.1.1 and Remark 7.9, similar to the smooth case. We then consider the sequence of decompositions $\acute{\eta}_n, \acute{\nu}_n$, and $\hat{\acute{\nu}}_n$, along with the associated natural numbers (m_n), as described in equations (6.50), (6.56), (7.20), and Section 6.10.2, respectively. It is noted that $\acute{\eta}_n \to \epsilon$, $\acute{\nu}_n \to \epsilon$, and $\hat{\acute{\nu}}_n \to \epsilon$, as outlined in Remark 6.30 and Lemmas 6.35 and 7.10. Referring to Lemma 7.4 and Lemma 6.34, we deduce that for any $\hat{I}_n \in \acute{\eta}_n$, the diffeomorphism $(\hat{\Phi}_n, \mathrm{d}\hat{\Phi}_n)$ is $(\gamma', \delta', \varepsilon_1', \varepsilon_2')$-distributing $\hat{I}_n$, where $\gamma' = \gamma + \frac{1}{2^n} \le \frac{1}{k_n^5 q_n}$, $\delta' = \delta + \frac{1}{2^n} \le \frac{2}{k_n^4}$, and $\varepsilon_1' = \varepsilon_2' = \frac{2}{n^5} + \frac{3}{2^n} \le \frac{1}{n}$.

Thus, all the requirements of Proposition 7.6 are fulfilled. Its application yields that the map $(\hat{f}, \mathrm{d}\hat{f})$ is weakly mixing with respect to the measure $\bar{\mu}$. $\square$

7.8 Analytic approximation result

7.8.1 Proof of Theorem 7.1

In this section we establish the required analytic approximation results. Our arguments follows along the lines of [7, Lemma 3.8, 3.9, 3.10] where the analogous results for $\mathbb{A} = \mathbb{T} \times \mathbb{R}$ were obtained. There are not many changes but we still provide the proofs for completeness.

To start we recall that the *Poisson bracket* for two function $f, g \in C^\infty(\mathbb{T}^2, \mathbb{R})$ is defined to be $\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}$, and the symplectic gradient of $f \in C^\infty(\mathbb{T}^2, \mathbb{R})$ is defined as $X_f = \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right)$. If X_f, X_g and $X_{\{f,g\}}$ are the symplectic gradient of f, g and $\{f, g\}$, then we have the following well known fact from symplectic geometry,

$$[X_f, X_g] = X_{\{f,g\}}, \tag{7.21}$$

where the left hand side represents the Lie bracket of vector fields. Additionally we also recall that the commutator of two flows F^t and G^t is defined by

$$[F^t, G^t] = F^{-t} \circ G^{-t} \circ F^t \circ G^t.$$

Lemma 7.11. *For any $\epsilon > 0$ and $H \in C^\infty(\mathbb{T}^2, \mathbb{R})$, there exists a map $\tilde{H} \in C^\infty(\mathbb{T}^2, \mathbb{R})$ such that $d_r(H, \tilde{H}) < \epsilon$ for all $r \in \mathbb{N}$, where $\tilde{H}$ is of the following form:*

$$\tilde{H}(x, y) = H_0(x, y) + \sum_{m=1}^{N} \left(H_{2m-1}(x, y) + H_{2m}(x, y)\right), \tag{7.22}$$

where
$$\begin{aligned}
H_0(x, y) &= C(y), \\
H_{2m}(x, y) &= \left\{\frac{\cos(2\pi m x)}{M_m}, A_m(y)\right\}, \\
H_{2m-1}(x, y) &= \left\{\frac{\sin(2\pi m x)}{M_m}, B_m(y)\right\},
\end{aligned}$$

for $C, A_1, \ldots, A_N, B_1, \ldots, B_N \in C^\omega(\mathbb{T}^1, \mathbb{R})$, *and* $M_m = \sup\{|2\pi m \cdot \exp(2\pi i \cdot m \cdot z)| : z \in [0,1) + i[-\rho_0, \rho_0]\}$ *where* $\rho_0 \in \mathbb{N}$.

Proof. Recall that the first $4N^2$ terms of the two variable Fourier series expansion of $H(x,y)$ can be written as,

$$\sum_{n=0}^{N}\sum_{m=0}^{N} \alpha_{1,n,m}\cos(2\pi n y)\cos(2\pi m x) + \sum_{n=0}^{N}\sum_{m=0}^{N} \alpha_{2,n,m}\sin(2\pi n y)\cos(2\pi m x)$$
$$+ \sum_{n=0}^{N}\sum_{m=0}^{N} \alpha_{3,n,m}\cos(2\pi n y)\sin(2\pi m x) + \sum_{n=0}^{N}\sum_{m=0}^{N} \alpha_{4,n,m}\sin(2\pi n y)\sin(2\pi m x),$$

where $\alpha_{i,n,m} \in \mathbb{R}$ for $1 \le i \le 4, 0 \le n \le N, 0 \le m \le N$. Using the theory of Fourier series (see [27]), there exists an N large enough such that $d_r(H, \tilde{H}) < \epsilon$ for any $r \in \mathbb{N}$. For $1 \le m \le N$, put

$$b_m(y) = \sum_{n=0}^{N}\left(\alpha_{3,n,m}\cos(2\pi n y) + \alpha_{4,n,m}\sin(2\pi n y)\right), \ A_m(y) = -\frac{M_m}{2\pi m}\int_0^y b_m(t)dt,$$
$$a_m(y) = \sum_{n=0}^{N}\left(\alpha_{1,n,m}\cos(2\pi n y) + \alpha_{2,n,m}\sin(2\pi n y)\right), \ B_m(y) = \frac{M_m}{2\pi m}\int_0^y a_m(t)dt,$$
$$H_0(x,y) = C(y) = \sum_{n=0}^{N}\left((\alpha_{1,n,0} + \alpha_{2,n,0})\cos(2\pi n y) + (\alpha_{3,n,0} + \alpha_{4,n,0})\sin(2\pi n y)\right).$$

Then we put for $1 \le m \le N$,

$$H_{2m}(x,y) = \left\{\frac{\cos(2\pi m x)}{M_m}, A_m(y)\right\} = b_m(y)\sin(2\pi m x),$$
$$H_{2m-1}(x,y) = \left\{\frac{\sin(2\pi m x)}{M_m}, B_m(y)\right\} = a_m(y)\cos(2\pi m x).$$

This concludes the claim. $\square$

Lemma 7.12. *With* $\tilde{H}$ *and* N *as in Lemma 7.11, there exists* $4N+1$ *functions* $\{\tau_j\}_{0\le j\le 2N} \in C^\omega(\mathbb{T}^1, \mathbb{R})$ *and* $\{v_j\}_{1\le j\le 2N} \in C^\omega(\mathbb{T}^1, \mathbb{R})$, *such that the time* t *map of*

$\tilde{H}_N$, $\psi^t_{\tilde{H}_N}$, satisfies

$$\psi^{t^2}_{\tilde{H}} = \left[T^t_{2N}, V^t_{2N}\right] \circ \ldots \circ \left[T^t_j, V^t_j\right] \circ \ldots \circ \left[T^t_1, V^t_1\right] \circ T^{t^2}_0 + \mathcal{O}(t^3),$$

as $t \to 0$ in any C^r-topology. Here

$$\begin{aligned} T^t_j(x,y) &= (x + t\tau_j(y), y) \qquad 0 \leq j \leq 2N, \\ V^t_j(x,y) &= (x, y + tv_j(x)) \qquad 1 \leq j \leq 2N. \end{aligned}$$

Proof. For any $f \in C^\infty(\mathbb{T}^2, \mathbb{R})$, let X_f denote the symplectic gradient of f, and ψ^t_f be the flow of X_f. Then with H_j and $\tilde{H}$ as in equation 7.22, we see that $\tilde{H}(x,y) = \sum_{j=0}^{2N} H_j(x,y)$ and hence at the level of symplectic gradients $X_{\tilde{H}} = \sum_{j=0}^{2N} X_{H_j}$. Let $\psi^t_{\tilde{H}}$ be the flow associated with the vector field $X_{\tilde{H}}$. Then for any $(x,y) \in \mathbb{T}^2$ and a small enough t, we can write using Taylor's theorem for $t \mapsto \psi^t_{\tilde{H}}$ around $t = 0$,

$$\begin{aligned} \psi^t_{\tilde{H}}(x,y) &= (x,y) + \frac{d\psi^t_{\tilde{H}}(x,y)}{dt}(t-0) + \mathcal{O}((t-0)^2) \\ \implies \psi^t_{\tilde{H}} &= id + tX_{\tilde{H}} + \mathcal{O}(t^2) \\ \implies \psi^t_{\tilde{H}} &= id + t\sum_{j=0}^{2N} X_{H_j} + \mathcal{O}(t^2). \end{aligned} \tag{7.23}$$

Next we put,

$$\begin{aligned} f_{2m}(x) &= \frac{\cos 2\pi m x}{M_m}, \quad g_{2m}(y) = A_m(y), \qquad 1 \leq m \leq N \\ f_{2m-1}(x) &= \frac{\sin 2\pi m x}{M_m}, \quad g_{2m-1}(y) = B_m(y), \qquad 1 \leq m \leq N \end{aligned}$$

observing that in equation 7.22, we have $H_j = \{f_j, g_j\}$ for $1 \leq j \leq 2N$. Put

$$\begin{aligned} \tau_0(y) &= C'(y) \\ \tau_j(y) &= g'_j(y), \quad v_j(x) = -f'_j(x), \qquad 1 \leq j \leq 2N. \end{aligned}$$

The respective symplectic gradients for f_j and g_j are

$$X_{f_j}(x,y) = (0, v_j(x)) = \frac{\partial V_j^t(x,y)}{\partial t}, \qquad 1 \le j \le 2N$$
$$X_{g_j}(x,y) = (\tau_j(y), 0) = \frac{\partial T_j^t(x,y)}{\partial t}, \qquad 1 \le j \le 2N.$$

This together with equation 7.21 gives us

$$\begin{aligned} X_{H_j} &= X_{\{f_j,g_j\}} = [X_{f_j}, X_{g_j}] \\ &= \left[\frac{\partial V_j^t(x,y)}{\partial t}, \frac{\partial T_j^t(x,y)}{\partial t}\right]. \end{aligned}$$

We use the above together with the definition of the Lie bracket,

$$\begin{aligned} X_{H_j} &= \frac{d}{dt}[T_j^t, V_j^t] \\ \implies id + t^2 X_{H_j} &= [T_j^t, V_j^t] + \mathcal{O}(t^3), \qquad 1 \le j \le N \end{aligned} \tag{7.24}$$

where the last statement holds in any C^r topology for small t.
Going back to equation 7.23 we can in light of equation 7.24 and definition of T_0^t, write it in the following required form,

$$\psi_{\tilde{H}}^{t^2} = \left[T_{2N}^t, V_{2N}^t\right] \circ \ldots \circ \left[T_j^t, V_j^t\right] \circ \ldots \circ \left[T_1^t, V_1^t\right] \circ T_0^{t^2} + \mathcal{O}(t^3).$$

This completes the proof. $\square$

Proof of Theorem 7.1. For any $h \in \mathrm{Ham}^\infty(\mathbb{T}^2, \mu)$, there exists a smooth family of Hamiltonians $(H_t)_{t\in[0,1]}$ and $H_t \in C^\infty(\mathbb{T}^2, \mathbb{R})$ which defines a family $\{f_t\}_{t\in[0,1]}$ such that $f_0 = id$ to $f_1 = h$ and $X_{f_t} = \partial_t f_t$ is the symplectic gradient of H_t. For large M, we can express

$$h = (f_1 \circ f_{(M-1)/M}^{-1}) \circ \ldots \circ (f_{i+1/M} \circ f_{i/M}^{-1}) \circ \ldots \circ (f_{2/M} \circ f_{1/M}^{-1}) \circ (f_{1/M} \circ f_0^{-1}).$$

Note that each $(f_{i+1/M} \circ f_{i/M}^{-1})$, it is a shift by the flow of X_{f_t} from time $t = \frac{i}{M}$ to $t = \frac{i+1}{M}$ and hence, it is $\mathcal{O}\left(\frac{1}{M^2}\right)$- C^r close to the time $t = 1/M$ map $\psi_{H_i}^{1/M}$ of the vector field $X_{f_{i/M}}$, i.e.

$$(f_{i+1/M} \circ f_{i/M}^{-1}) = \psi_{H_i}^{1/M} + \mathcal{O}\left(\frac{1}{M^2}\right).$$

Thus in the C^r topology:

$$h = \psi_{H_{M-1}}^{1/M} \circ \ldots \circ \psi_{H_i}^{1/M} \circ \ldots \psi_{H_1}^{1/M} \circ \psi_{H_0}^{1/M} + \mathcal{O}\left(\frac{1}{M}\right).$$

By applying Lemma 7.11 and 7.12 with $t^2 = \frac{1}{M}$, we can establish the existence of entire time-t maps $\psi_{\tilde{H}_i}^t$ for each $H_i \in C^\infty(\mathbb{T}^2, \mathbb{R})$. These maps are constructed as finite compositions of entire maps in the form of V_j^t or T_j^t, along with the inverse maps of V_j^t and T_j^t. Also, note that the inverse maps of V_j^t and T_j^t are entire functions, defined as translations by shear functions, specifically as

$$(V_j^t)^{-1}(x, y) = (x, y - tv_j(x)), \quad \text{and} \quad (T_j^t)^{-1}(x, y) = (x - t\tau_j(y), y).$$

Furthermore, the maps $\psi_{\tilde{H}i}^{1/M}$ are C^r-close to $\psi_{H_i}^{1/M}$ with a precision of $\mathcal{O}(t^2) = \mathcal{O}\left(\frac{1}{M}\right)$. Consequently, we can define an entire map $\tilde{h} \in \mathrm{Ham}_\infty^\omega(\mathbb{T}^2, \mu)$ as follows:

$$\tilde{h} = \psi_{\tilde{H}_{M-1}}^{1/M} \circ \ldots \circ \psi_{\tilde{H}_0}^{1/M} + \mathcal{O}\left(\frac{1}{M}\right), \tag{7.25}$$

which is C^r close to h up to a precision of $\mathcal{O}(\frac{1}{M})$, i.e. for $t^2 < \epsilon$, we have $d_r(h, \tilde{h} \restriction_{\mathbb{T}^2}) < \epsilon$. Similarily, $\tilde{h}^{-1}$ is also defined as a composition of entire functions, thus $\tilde{h}^{-1} \in \mathrm{Ham}_\infty^\omega(\mathbb{T}^2, \mu)$. $\square$

8

Conclusion and Future direction

8.1 Introduction

In this chapter, we provide a brief conclusion and discuss some interesting open research questions in this direction.

8.2 The Generic and Non-Generic points

In Chapters 3 and 4, we explored the Anosov-Katok method to construct examples of smooth dynamical systems with intricate dynamics. Our study included investigating the presence of a non-ergodic generic measure in these systems and examining various other topological and ergodic characteristics. We also thoroughly analyzed generic and non-generic points within the system, establishing bounds on the sizes of the sets they form. Specifically, we focused on determining bounds on their Hausdorff dimension, which yielded particularly insightful and interesting results. Additionally, we discussed the rigorous selection of geometric and combinatorial criteria essential for achieving our goals.

8.2.1 Future Directions

Problem 1. *Can we choose a set* B *containing all the generic points such that* $dim_H(\mathrm{B}) = \alpha$ *for all* $0 < \alpha < 2$*?*

Problem 2. *Can we choose a generic set* B *of type* $C \times C$*, where* C *is Cantor set on the unit interval, in the above setup of Theorem 1.3?*

Problem 3. *Can we generalize the Theorem 1.3 for a 3-dimensional torus with a choice of a generic set of type*

1. $\mathrm{B} = \mathbb{T}^1 \times C \times C$. *If this is true, the result can be generalized to* $\mathbb{T}^n$.

2. *In fact, can we choose the set* B *as form* $\mathbb{T}^1\times$*"2D-fractal", where 2D fractal is not necessarily the product of two sets like* $C \times C$ *type?*

In this thesis, we discovered the existence of generic points and generic measures in the smooth category by using the Approximation by Conjugation method. Such generic points, by definition, are Birkhoff generic points. This method has the potential to extend the study to include weighted generic points or logarithmic generic points, defined in [5], in both the smooth and analytic categories. It is important to note that Birkhoff generic points imply weighted generic points, but the reverse may not always be true. This study could also lead to many open questions, such as whether there are systems that have weighted generic points but no generic points for the smooth category.

Problem 4. *Does there exists a smooth measure-preserving diffeomorphism such that there are weighted generic points but no birkhoff generic point?*

Problem 5. *Can we choose a set* B *containing all the weighted generic points, in Theorem 1.4, such that* $0 < dim_H(\mathrm{B}) < \alpha$ *for all* $0 < \alpha < 2$*?*

Addressing these problems requires delving into extreme explicit combinatorics, along with exotic setups, and necessitates the development of new tools and techniques to gain control over all the orbits of the space.

8.3 Projectivized Derivative Extension

In Chapter 6 and Chapter 7, we explore ergodic properties of the diffeomorphism along with its differential map with respect to a smooth measure in the projectivization of the tangent bundle. We explore this aspect using the Approximation by Conjugation method for the smooth and analytic category. We present an example of a derivative extension map that exhibits weak mixing with respect to a smooth measure for both smooth and analytic category. We introduce a weak mixing criterion specifically tailored for the projectivized derivative extension for both categories. To demonstrate the weak mixing property of the projectivized derivative extension, we provide a suitable distribution of elements under the explicit choice of conjugation maps and the approximation techniques that meet the necessary requirements of the criterion.

8.3.1 Future Directions

Problem 1. *Does a minimal projectivized derivative extension exist on* $\mathbb{P}\mathrm{T}M$*?*

Problem 2. *Is there a projectivized derivative extension on* $\mathbb{P}TM$ *with precisely two ergodic invariant measures that are absolutely continuous with respect to the Lebesgue measure?*

Addressing this problem would be more challenging, as existing tools only provide control over all orbits of the base elements of the manifold, making it difficult to attain control over the tangent direction. One would require new tools and techniques to gain control over all the orbits within the space $\mathbb{P}\mathrm{T}M$. Developing such techniques could potentially lead to further exciting questions, such as controlling the existence of the number of ergodic invariant measures on the space $\mathbb{P}\mathrm{T}M$, or determining the existence of a uniquely ergodic projectivized derivative extension.

Problem 3. *Can similar results be extended to Theorems 1.6 and 1.7 for higher-dimensional torus or other manifolds ?*

To investigate this, it is necessary to delve into the combinatorics of distributing elements to achieve weak mixing in the base manifold $\mathbb{T}^n$, and to adapt an appropriate analytic approximation technique.

9
Appendix

This chapter serves as a supplementary section and include detailed derivations, proofs and technical details that support the understanding of the main text.

9.1 Some standard results

We present some standard result on the closeness between the maps constructed as conjugation of translations on the torus.

Lemma 9.1. *Let $k \in \mathbb{N}$. For all $\alpha, \beta \in \mathbb{R}$ and all $h \in \mathrm{Diff}^{k+1}(\mathbb{T}^2)$, we have the estimate*

$$d_k(hS_\alpha h^{-1}, hS_\beta h^{-1}) \leq C_k \max\{|||h|||_{k+1}, \left|\left|\left|h^{-1}\right|\right|\right|_{k+1}\}|\alpha - \beta|,$$

where C_k is a constant that depends only on k.

Proof. For $k = 0$ we clearly have the estimates by the mean value theorem. We claim that for $j \in \mathbb{N}^d$ with $|j| = k$ the partial derivative

$$\mathrm{D}_j[h_i S_\alpha h^{-1} - h_i S_\beta h^{-1}]$$

will consist of a sum of terms with each term being the product of a single derivative

$$(\mathrm{D}_a h_i)(S_\alpha h^{-1}) - (\mathrm{D}_a h_i)(S_\beta h^{-1}) \tag{9.1}$$

with $|a| \leq k$, and atmost k partial derivative of the form

$$\mathrm{D}_b h_j^{-1} \tag{9.2}$$

with $|b| \leq k$. This holds for $k = 1$

$$\frac{\partial}{\partial y_j}[h_i S_\alpha h^{-1} - h_i S_\beta h^{-1}] = \sum_{l=1}^{d} \left(\frac{\partial h_i}{\partial y_l} S_\alpha h^{-1} - \frac{\partial h_i}{\partial y_l} S_\beta h^{-1} \right) \frac{\partial h_l^{-1}}{\partial y_j}.$$

Now we proceed by induction. With the product rule we only need to consider the effect of differentiating equations (9.1) and (9.2). Therefore differentiate (9.1) with respect to y_j we obtain

$$\sum_{l=1}^{d} \left(\frac{\partial \mathrm{D}_b h_i}{\partial y_l} S_\alpha h^{-1} - \frac{\partial \mathrm{D}_b h_i}{\partial y_l} S_\beta h^{-1} \right) \frac{\partial h_l^{-1}}{\partial y_j}.$$

which increases the number of terms of the form (9.2) by 1. Differentiating (9.2) we get another term of the form (9.2) but with $|b| \leq k+1$. we estimate

$$\|(\mathrm{D}_a h_i)(S_\alpha h^{-1}) - (\mathrm{D}_a h_i)(S_\beta h^{-1})\|_0 \leq \|\!|h|\!\|_{|a|+1} |\alpha - \beta|$$

$$\|\mathrm{D}_a h_l^{-1}\|_0 \leq \|\!|h|\!\|_{|a|}$$

Use the fact that the inverse maps have the same structure of the partial derivative as above completes the proof. The constant C_k is the number of terms in the sum which only depends on k and not on the map h. □

Proof of Lemma 3.10 Here we show that T_{n+1} is a cauchy sequence in $\text{Diff}^\infty(\mathbb{T}^2,\mu)$ by choosing l_n enough large and hence implies that T_{n+1} converges to some $T \in \text{Diff}^\infty(\mathbb{T}^2,\mu)$. For any $k>0$ we have

$$\begin{aligned}
&d_k(T_{n+1},T_n) = d_k(H_{n+1}S_{\alpha_{n+1}}H_{n+1}^{-1}, H_nS_{\alpha_n}H_n^{-1})\\
&\le d_k(H_{n+1}S_{\alpha_{n+1}}H_{n+1}^{-1}, H_{n+1}S_{\alpha_n}H_{n+1}^{-1}) + d_k(H_{n+1}S_{\alpha_n}H_{n+1}^{-1}, H_nS_{\alpha_n}H_n^{-1})\\
&\le \|DH_n\|_{k+1}d_k(h_{n+1}S_{\alpha_{n+1}}h_{n+1}^{-1}, h_{n+1}S_{\alpha_n}h_{n+1}^{-1}) + \|DH_n\|_{k+1}d_k(h_{n+1}S_{\alpha_n}h_{n+1}^{-1}, S_{\alpha_n}).
\end{aligned}$$

With the fact that $h_{n+1}\circ S_{\alpha_n} = S_{\alpha_n}\circ h_{n+1}$, second term of the inequality becomes zero. Using the Lemma 9.1, for $\alpha=\alpha_{n+1}, \beta=\alpha_n$ and $h=h_{n+1}$, we have

$$d_k(h_{n+1}S_{\alpha_{n+1}}h_{n+1}^{-1}, h_{n+1}S_{\alpha_n}h_{n+1}^{-1}) \le C_k|||h_{n+1}|||_{k+1}|\alpha_{n+1}-\alpha_n|.$$

where $|\alpha_{n+1}-\alpha_n| < \frac{1}{k_nl_nq_n^2}$. As l_n is choosen to be enough large such that it satisfy

$$|\alpha_{n+1}-\alpha_n| < \frac{1}{2^n\|DH_n\|_k},$$

and estimate the closeness between T_{n+1} and T_n as

$$d_k(T_{n+1},T_n) < \frac{1}{2^n}.$$

For fixed k, T_{n+1} is cauchy sequence in C^k topology, hence converges to $T \in \text{Diff}^k(\mathbb{T}^2,\mu)$. This is true for any arbitrary $k>0$ implies the convergence of T_{n+1} in C^∞ topology. Also note that for any $m \le q_{n+1}$,

$$\begin{aligned}
d_0(T_{n+1}^m,T_n^m) &= d_0(H_{n+1}S_{\alpha_{n+1}}^mH_{n+1}^{-1}, H_nS_{\alpha_n}^mH_n^{-1})\\
&\le 2m\cdot||DH_n||_1\cdot|\alpha_{n+1}-\alpha_n| \le \frac{1}{2^n}.
\end{aligned}$$

Then $d_0(T_{n+1}^m,T_n^m) \le \sum_{i=n}^\infty d_0(T_{i+1}^m,T_i^m) = \frac{1}{2^{n-1}}$.

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