Spectral Instabilities in Water Wave Models

by

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Dedicated to my parents

Certificate

This is to certify that the thesis entitled **"Spectral Instabilities in Water Wave Models"** being submitted by **"Ms. Bhavna"** to the **Indraprastha Institute of Information Technology Delhi**, for the award of the Degree of **Doctor of Philosophy** is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

This thesis investigates the spectral instabilities of various nonlinear water wave models through rigorous analytical techniques. Focusing on three fundamental types of instabilities, modulational instability, transverse instability, and high-frequency instability, the work provides a unified spectral framework to study how small perturbations evolve and potentially destabilize wave solutions in dispersive systems.

We begin by analyzing modulational instability, wherein a periodic traveling wave becomes unstable to long-wavelength perturbations. Using perturbation theory and spectral analysis, we characterize conditions under which modulational instability arises in generalized Ostrovsky equations. The effect of dispersion, nonlinearity, and surface tension is examined in detail.

The study then turns to transverse instability, where planar wave trains destabilize due to perturbations in the transverse direction. We consider rotation-modified and surface tension-influenced variants of the Kadomtsev–Petviashvili (KP) equation, the rotation-modified KP equation, and the KD equation, and identify parameter regimes leading to transverse spectral instabilities.

Finally, we explore high-frequency instability, focusing on the behavior of the spectrum. We demonstrate how high-frequency perturbations can induce instabilities in small-amplitude periodic traveling waves.

Altogether, the results contribute to a deeper understanding of how wave coherence is affected by perturbations of various scales and directions. The insights gained have potential implications for the stability of wave patterns in physical settings such as oceanography, fluid mechanics, and nonlinear optics.

Contents

List of Symbols

Symbol	**Meaning**
x	Spatial variable with values in $\mathbb{R}$
t	Time variable with values in $\mathbb{R}^+$
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{R}$	Set of real numbers
$L^2(\mathbb{R})$	Space of square integrable functions on $\mathbb{R}$.
$L^2(\mathbb{T})$	Space of 2π-periodic square integrable functions
$L^2_0(\mathbb{T})$	space of 2π-periodic square-integrable functions with of zero-mean
$\langle \cdot, \cdot \rangle$	$L^2(\mathbb{T})$-inner product
μ	Floquet exponent
a	amplitude parameter
T	Surface Tension
$u(x,t)$	Typically a velocity depending on x and t
$\mathcal{M}$	A multiplier operator
$\mathcal{M}_k$	A multiplier operator restricted to space of periodic functions

Research Publications

Published Papers

I have the following published papers:

1. Bhavna, Mathew A. Johnson, and Ashish Kumar Pandey. *Modulational Instability in the Ostrovsky Equation and Related Models.* **SIAM Journal on Mathematical Analysis**, 2024, https://epubs.siam.org/doi/full/10.1137/23M1608665

2. Bhavna, Ashish Kumar Pandey, and Sudhir Singh. *Transverse spectral instabilities in Konopelchenko–Dubrovsky equation.* **Studies in Applied Mathematics**, 2023. https://doi.org/10.1111/sapm.12617

3. Bhavna, Atul Kumar, and Ashish Kumar Pandey. *Transverse spectral instability in generalized Kadomtsev-Petviashvili equation.* **Proceedings of the Royal Society A**, 2022. https://doi.org/10.1098/rspa.2021.0693

4. Bhavna, Atul Kumar, and Ashish Kumar Pandey. *High-frequency instabilities of the Ostrovsky equation.* **Water Waves**, 2022. https://doi.org/10.1007/s42286-021-00054-0

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Introduction

Water wave models have long served as fundamental tools for understanding a wide variety of nonlinear dispersive phenomena in fluid mechanics, oceanography, and related physical systems. These models describe the dynamics of wave propagation and capture the interplay between nonlinearity and dispersion, essential to wave formation, stability, and evolution. Among such models, particular interest lies in the behavior of periodic and solitary traveling wave solutions and their response to small perturbations. This thesis focuses on the spectral (in)stability of such waveforms in various water wave models through the lens of modulational instability, transverse instability, and high-frequency instability.

Periodic traveling waves are spatially periodic solutions that retain a fixed shape while traveling at a constant speed. Typically written as $u(x,t) = \phi(x-ct)$, where ϕ is a periodic function and c is the wave speed, these solutions represent regular wave trains observed in fluid systems. They are central to the study of nonlinear dispersive equations and provide a natural setting for analyzing wave modulation and stability. Understanding their spectral properties under small perturbations is essential for identifying the mechanisms that lead to either the persistence or breakdown of these coherent structures.

A periodic traveling wave solution is said to be spectrally stable if small perturbations evolve without leading to exponential growth in time. This contrasts with spectral stability, where small disturbances don't lead to rapid growth, and the wave maintains its structure. The spectral instability of periodic traveling waves can be studied through several mathematical tools, such as linearization, Floquet-Bloch theory, Krein signatures, and linear stability analysis. Each method plays a crucial role in understanding how small perturbations evolve over time and whether they lead to instability. Here's a more systematic breakdown:

1. General Form of the Equation

We begin with the general form of a nonlinear wave equation describing the system:

$$\frac{\partial u}{\partial t} + f(u) = 0,$$

where $u(x,t)$ represents the wave profile and $f(u)$ is a nonlinear function. For a periodic traveling wave solution, we look for solutions of the form $u(x,t) = u_0(x-ct)$, where c is the wave speed. Small perturbations $\epsilon(x,t)$ are introduced around this solution:

$$u(x,t) = u_0(x-ct) + \epsilon(x,t).$$

After moving to a co-moving frame $\xi = x-ct$, the linearized equation for the perturbation ϵ is

$$\frac{\partial \epsilon}{\partial t} + \mathcal{L}\epsilon = 0,$$

where $\mathcal{L}$ is a linear operator derived from the nonlinear dynamics. This equation approximately governs the evolution of small perturbations.

2. Need for Floquet-Bloch Theory

Periodic traveling waves give rise to linear operators $\mathcal{L}$ with periodic coefficients, and their stability is determined by analyzing such operators. To handle this, Floquet–Bloch theory is employed. The central idea is to exploit the periodicity of the background wave and reduce the spectral problem to a family of simpler problems on a single period of the wave. Perturbations are assumed to take the form

$$\epsilon(x,t) = e^{\lambda t} e^{i\mu x} \hat{\phi}(x), \quad \text{with} \quad \hat{\phi}(x+L) = \hat{\phi}(x),$$

where $\lambda \in \mathbb{C}$, $\mu \in [-\pi/L, \pi/L)$ is the *Floquet parameter* (or Bloch wave number), and L is the period of the wave. This ansatz captures the quasi-periodic nature of perturbations by allowing them to oscillate at a frequency μ while maintaining periodicity in $\hat{\phi}(x)$. For each fixed μ, this leads to a linear eigenvalue problem on a compact domain (length L) for the modified operator $\mathcal{L}_\mu = e^{-i\mu x} \mathcal{L} e^{i\mu x}$. The total spectrum of the original linearized operator $\mathcal{L}$, acting on $L^2(\mathbb{R})$, is then given by the union of the spectra of $\mathcal{L}_\mu$ as μ ranges over the Brillouin zone $[-\pi/L, \pi/L)$:

$$\text{spec}(\mathcal{L}) = \bigcup_{\mu \in [-\pi/L, \pi/L)} \text{spec}(\mathcal{L}_\mu).$$

This construction converts the spectral problem from a global one on the real line to a parametrized family of spectral problems on a bounded interval, and reveals that the spectrum of $\mathcal{L}$ is in general *continuous.* Floquet–Bloch theory is thus essential in studying the spectral (in)stability of periodic traveling waves, providing a structured and computationally tractable framework.

3. Krein Signatures

Krein signatures are used to study the spectrum of the linearized operator, particularly when analyzing the stability of the wave in regions *away from the origin.* In systems with Hamiltonian or reversible structure, eigenvalues typically appear in symmetric pairs, and Krein signatures help determine which of these eigenvalues are susceptible to instabilities.

The Krein signature assigns a sign to eigenvalues based on the sign of the energy associated with their eigenfunctions. When eigenvalues with opposite Krein signatures collide under parameter variations, they may move off the imaginary axis into the complex plane, leading to spectral instability. Thus, Krein signature theory helps predict such instabilities by identifying which eigenvalues are "dangerous", those that can cause growth when perturbed.

Although the origin is often associated with symmetries or conserved quantities, Krein signatures remain a valuable tool both near and away from the origin, particularly for tracking non-zero eigenvalue collisions that may lead to instability.

4. Linear Stability Analysis

After obtaining the linearized operator $\mathcal{L}$, the evolution of a perturbation ϵ is governed by

$$\frac{\partial \epsilon}{\partial t} + \mathcal{L}\epsilon = 0.$$

Seeking solutions of the form

$$\epsilon(x,t) = e^{\lambda t} e^{i\mu x} \hat{\phi}(x), \quad \hat{\phi}(x+L) = \hat{\phi}(x),$$

where $\lambda \in \mathbb{C}$ and $\mu \in [-\pi/L, \pi/L)$ is the Floquet exponent, reduces the PDE to the spectral problem

$$\mathcal{L}\phi = \lambda\phi,$$

where $\phi(x) = e^{i\mu x}\hat{\phi}(x)$. Thus, the stability analysis of a periodic traveling wave requires determining the spectrum of $\mathcal{L}$. To proceed, one studies how the operator $\mathcal{L}$ acts on a suitable class of functions ϕ, often taken from a Hilbert space like $L^2(\mathbb{R})$, and seeks all $\lambda \in \mathbb{C}$ such that the equation above has nontrivial solutions.

In the periodic setting, the coefficients in $\mathcal{L}$ are periodic, so the classical eigenvalue problem is replaced by a family of Bloch-type problems indexed by the Floquet parameter $\mu \in [-\pi/L, \pi/L)$. For each fixed μ, we solve:

$$\mathcal{L}_\mu \phi = \lambda\phi, \quad \text{with} \quad \phi(x+L) = e^{i\mu L}\phi(x),$$

where $\mathcal{L}_\mu$ is the restriction of $\mathcal{L}$ to quasi-periodic functions. This leads to a discrete spectrum for each μ, and the union of these spectra over all μ gives the full spectrum of $\mathcal{L}$ on $L^2(\mathbb{R})$. This process converts the original spectral problem into a band structure, capturing the continuous spectrum of the linearized operator.

For each μ, one computes the eigenfunctions ϕ_μ and determines how $\mathcal{L}_\mu$ acts on them. The resulting eigenvalues $\lambda(\mu)$ describe how perturbations with quasi-momentum μ grow or decay over time. If for any μ, the spectrum contains an eigenvalue with $\text{Re}(\lambda) > 0$, the wave is spectrally unstable. If all eigenvalues lie on the imaginary axis, the wave is spectrally stable (though not necessarily nonlinearly stable).

In Hamiltonian or symmetric systems, the spectrum exhibits symmetry about both the real and imaginary axes. This symmetry constrains how eigenvalues can move: they cannot drift independently into one half of the complex plane. Instead, eigenvalues appear in symmetric quartets $\{\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}\}$. As a result, the emergence of an unstable eigenvalue necessarily brings its counterparts, providing a clear indication of spectral instability.

After establishing the general framework for spectral stability, it is important to classify the different mechanisms through which instabilities can arise. Depending on the nature of the perturbation and the system's structure, periodic traveling waves may exhibit modulational, high-frequency, or transverse instabilities. Each type corresponds to a distinct spectral behavior and has different physical implications. We describe these instabilities below.

Types of Spectral Instabilities

Once the spectral stability framework is established, one can classify instabilities into several types depending on the nature of perturbations. The most common types include modulational instability, high-frequency instability, and transverse instability.

Modulational Instability. Modulational instability refers to the growth of long-wavelength (low-frequency) perturbations and is typically associated with the interaction between a carrier wave and its slow modulations. It manifests in the spectral analysis as eigenvalues with positive real part emerging near the origin of the spectral plane. More precisely, one tracks the behavior of eigenvalues $\lambda(\mu)$ for small values of the Floquet parameter μ. If $\mathrm{Re}(\lambda(\mu)) > 0$ for some $\mu \approx 0$, the wave is said to be modulationally unstable.

To illustrate, consider the well-studied Korteweg–de Vries (KdV) equation

$$u_t - \beta u_{xxx} + \left(u^2\right)_x = 0,$$

where β is the dispersion coefficient, which determines the type of dispersion. The KdV equation models the unidirectional propagation of weakly nonlinear, long-wavelength water waves; see [55] and references therein. While the KdV equation captures long-wave phenomena such as solitary and periodic traveling waves, it fails to describe important high-frequency behaviors like wave breaking and peaking. This limitation stems from its linear dispersion relation, which poorly approximates the full water wave dispersion outside the long-wavelength regime.

Indeed, the non-dimensional phase speed of unidirectional water waves expands as

$$c_{ww}(k) := \sqrt{\frac{\tanh(k)}{k}} = 1 - \frac{1}{6}k^2 + \mathcal{O}(k^4), \quad |k| \ll 1,$$

so the KdV equation's dispersion relation matches only up to $\mathcal{O}(k^2)$ in this regime. To better capture the full dispersion while retaining the shallow water nonlinearity, Whitham proposed the equation

$$u_t + \beta \mathcal{M}_{ww} u_x + \left(u^2\right)_x = 0, \tag{1.0.1}$$

where $\mathcal{M}_{ww}$ is a Fourier multiplier with symbol

$$\widehat{\mathcal{M}_{ww} f}(k) = c_{ww}(k)\hat{f}(k),$$

and β is the dispersion coefficient, which determines the type of dispersion. This model, now known as the *Whitham equation*, retains the full linear dispersion of water waves and thus was conjectured by Whitham to better predict phenomena such as wave breaking and peaking. Recent work has confirmed these features: equation (2.2.1) has been shown to exhibit wave breaking [37] and peaking [19, 23].

Moreover, the Whitham equation supports modulational instability of small-amplitude periodic traveling waves, as demonstrated in [38]. Related numerical studies on large-amplitude waves further confirm these findings [68]. While the precise derivation of (2.2.1) from the full water wave equations remains a topic of study, recent comparisons show that it outperforms traditional models like KdV and BBM in the intermediate- and short-wave regimes [7, 61]. Taken together, these results indicate that the Whitham equation provides a powerful framework for capturing modulational and high-frequency features of water waves beyond the long-wave approximation.

High-Frequency Instability. High-frequency instability refers to the growth of perturbations with large wavenumbers and is often related to the spectral properties of the linearized operator at large Floquet exponents. Unlike modulational instability, which concerns the low-frequency behavior near the origin of the spectral plane, high-frequency instability typically involves eigenvalues emerging far from the origin, often near the edges of the Brillouin zone. It is especially relevant in dispersive models where the linear phase speed fails to match the full physical dispersion for short waves.

Mathematically, these instabilities arise when the spectral bands of the Bloch operator $\mathcal{L}_\mu$ collide at high frequencies or when they become non-symmetric due to the nonlocal nature of the dispersion operator. The phenomenon is particularly prominent in models where the dispersion relation introduces strong nonlinearity in the frequency domain, such as in equations involving fractional or pseudodifferential operators.

Transverse Instability. Transverse instability refers to the instability of one-dimensional traveling waves with respect to perturbations that vary in a transverse direction, that is, orthogonal to the direction of wave propagation. This phenomenon is especially important in understanding the full two-dimensional behavior of solutions originally derived from one-dimensional models. Consider a wave $u(x,t)$ and a perturbation of the form

$$\epsilon(x,y,t) = e^{ily}\tilde{\epsilon}(x,t),$$

where y is the transverse variable and l is the transverse wavenumber. The linearized stability problem then becomes a family of spectral problems parameterized by l. For each l, one studies the spectrum of the resulting operator to determine whether eigenvalues with positive real part appear. If such eigenvalues exist for some l, the wave is transversely unstable. This type of instability is central in multi-dimensional extensions of integrable systems, such as the Kadomtsev–Petviashvili (KP) equation or the two-dimensional nonlinear Schrödinger equation, and is associated with phenomena such as wave disintegration or transverse pattern formation.

The transverse instability of solitary waves of the Korteweg–de Vries (KdV) equation within the Kadomtsev–Petviashvili (KP) framework was first investigated by Kadomtsev and Petviashvili [48]. They showed that solitary waves are transversely *stable* in the *KP-II* equation (negative dispersion) but become *unstable* in the *KP-I* equation (positive dispersion) to long-wavelength transverse perturbations—even though these solutions are stable in the corresponding one-dimensional problem.

The transverse stability of *periodic cnoidal wave* solutions of the KdV equation within the KP setting was studied in [70], where the authors found instability results for the KP-I case and established transverse stability for KP-II. Subsequent work by Johnson and Zumbrun [47] considered the transverse instability of periodic waves for the KP–generalized KdV (KP–gKdV) equations. Their analysis focused on periodic perturbations in the direction of wave propagation and long-wavelength perturbations in the transverse direction. They developed an *orientation index* based on comparing the low- and high-frequency behavior of the periodic Evans function.

In the case of *small periodic waves*, Haragus [32] studied the transverse stability for more general classes of perturbations within the KP–KdV equation. More recently, in [35], the authors rigorously proved the *transverse spectral stability* of one-dimensional periodic traveling waves of the KP-II equation with respect to two-dimensional perturbations that are bounded in the direction of propagation.

In a broader context, transverse instability of periodic waves in the KP-I and nonlinear Schrödinger equations has been investigated in [30], while the transverse instability of solitary waves in various water-wave models has been examined in [29, 63, 65, 66]. These studies emphasize the importance of dimensionality in understanding stability phenomena and highlight the rich structure of instability mechanisms in nonlinear dispersive wave equations.

In the first part of this thesis, we investigate modulational instability in generalized Ostrovsky-type equations. These equations model unidirectional wave propagation in rotating fluids, incorporating both dispersive and rotational effects. Our analysis derives a modulational instability index for small-amplitude periodic waves, depending on the phase and group velocities of the linearized equation. Notably, we consider general Fourier multipliers, enabling the study of a broad class of dispersion relations beyond classical KdV-type models. We show that small-amplitude periodic waves in these systems become modulationally unstable when the wave number exceeds a critical threshold, which we explicitly compute for various choices of dispersion.

We next turn to transverse spectral instability, where we assess the response of one-dimensional periodic traveling waves to perturbations that vary in a second, transverse spatial direction. This class of instability is especially relevant in two-dimensional generalizations of unidirectional models, such as the Kadomtsev-Petviashvili (KP) and Konopelchenko-Dubrovsky (KD) equations. In these models, we study the transverse instability of small-amplitude wave trains and establish conditions under which instability arises depending on the wavelength and structure of the perturbations. Our analysis covers KP-type generalizations with different dispersive terms, including fractional, intermediate long wave, and Whitham-type dispersion, as well as the RMKP (rotation-modified KP) and KD equations. We demonstrate that these systems exhibit transverse instability in certain parameter regimes even when the base wave is stable to one-dimensional perturbations.

The final focus of the thesis is on high-frequency instabilities, which involve perturbations with large wave numbers. These instabilities are more subtle and arise due to the collision of nonzero eigenvalues in the spectral plane. We study these phenomena in the context of the Ostrovsky equation and show that such collisions lead to spectral instabilities that are distinct from the modulational type. Using Floquet theory and Krein signature analysis, we provide a complete description of these high-frequency instabilities and classify all possible eigenvalue collisions that result in spectral growth.

Collectively, this thesis builds a comprehensive spectral framework for understanding instabilities in nonlinear water wave models. The inclusion of rotation, surface tension, and nonlocal dispersive effects enriches the models and brings them closer to real-world scenarios encountered in geophysical flows. Our results contribute to the growing body of literature that seeks to rigorously classify stability properties of nonlinear wave solutions in multidimensional dispersive systems.

The remainder of this thesis is organized as follows:

Chapter 1 reviews the background on water wave models and sets up the spectral stability problem. Chapter 2 presents our results on modulational instability for generalized Ostrovsky-type equations. Chapter 3 investigates transverse instabilities in KP-type models, including gKP, KD, and RMKP equations. Chapter 4 focuses on high-frequency spectral instabilities in the Ostrovsky equation. Chapter 5 summarizes the key findings and discusses potential directions for future work.

This work brings together tools from spectral theory, perturbation analysis, and nonlinear dispersive PDEs to provide a unified understanding of spectral instability mechanisms in nonlinear wave propagation.

2

Modulational Instability

2.1 Introduction

This chapter focuses on the *modulational instability* of small-amplitude periodic traveling waves in dispersive water wave models with rotational effects. Modulational instability, also known as the *Benjamin–Feir instability*, describes the amplification of long-wavelength perturbations and is a fundamental mechanism that can destabilize coherent wave trains in nonlinear dispersive systems.

In the context of water wave theory, modulational instability is classically observed in models like the Whitham equation, where full dispersion is retained, unlike the Korteweg–de Vries (KdV) equation which suppresses this phenomenon due to its simplified dispersion relation. In the presence of Coriolis effects, captured by the Ostrovsky equation, the modulation of periodic wave trains reveals even more intricate behavior depending on the choice of dispersion.

This chapter is based on my joint work with Mathew A. Johnson and Ashish Kumar Pandey, published in the *SIAM Journal on Mathematical Analysis* (2024), titled *"Modulational Instability in the Ostrovsky Equation and Related Models"* [3].

We investigate a general class of dispersion-modified Ostrovsky equations, incorporating a Fourier multiplier to allow flexible modeling of physical dispersion. The model includes various well-known equations (Whitham, Intermediate Long Wave, Benjamin-Ono, fractional KdV) as special cases, and our analysis provides a unified framework to study modulational instability in these settings.

The main contributions of this chapter are as follows:

- We rigorously construct small-amplitude, periodic traveling wave solutions via Lyapunov–Schmidt reduction.

- We derive an explicit modulational instability index, formulated in terms of phase and group velocities, which determines the stability of periodic wave trains.

- We verify this index analytically and numerically for several specific models, including the classical and Whitham-Ostrovsky equations, and demonstrate how rotation alone can trigger modulational instability even in otherwise stable regimes.

- We also analyze how the inclusion of surface tension influences this instability index, and determine the critical thresholds in wavenumber that mark the onset of instability.

This work provides one of the first general, rigorous spectral stability analyses of small periodic waves in generalized Ostrovsky-type equations, and confirms prior formal and numerical predictions under a firm analytical foundation. It also connects with classical criteria such as the Lighthill condition, which we show is equivalent to our index in the small-amplitude regime.

The analysis here not only contributes to a deeper understanding of wave modulation in dispersive fluids but also sets the stage for later chapters, where we explore other types of instabilities, such as transverse and high-frequency instabilities, in related models.

2.2 The Model

The well-studied Korteweg-de Vries (KdV) equation

$$u_t + \beta u_{xxx} + \left(u^2\right)_x = 0,$$

which is a canonical model for the unidirectional propagation of weakly nonlinear, small amplitude water waves in the long wavelength regime: see [55] and references therein. Here, β is the dispersion coefficient. It is well known-however, that while the KdV equation explains well long-wave phenomena in a channel of water – traveling solitary and periodic waves, for example – it fails to exhibit many high-frequency phenomena such as wave breaking – the evolutionary formation of bounded solutions with infinite gradients – and peaking – the existence of bounded, steady solutions with a singular point, such as a peak or cusp. This of course should not be surprising, as the phase velocity associated with the linear part of the KdV equation poorly approximates that of the water wave equations outside the long wavelength regime. Indeed, the (non-dimensional) phase speed for unidirectional water waves can be shown to expand as

$$c_{ww}(k) := \sqrt{\frac{\tanh(k)}{k}} = 1 - \frac{1}{6}k^2 + \mathcal{O}(k^4), \quad |k| \ll 1,$$

where here k is the wavenumber of the wave. Thus, in the long wavelength regime $|k| \ll 1$ the water wave phase speed $c_{ww}(k)$ agrees up to $\mathcal{O}(k^2)$ with the phase speed associated to the KdV equation.

In an effort to find a simple mathematical equation that could explain water wave phenomena outside of the long-wavelength regime, Whitham proposed the model

$$u_t + \beta \mathcal{M}_{ww} u_x + \left(u^2\right)_x = 0, \tag{2.2.1}$$

where here $\mathcal{M}_{ww}$ is a Fourier multiplier with symbol

$$\widehat{\mathcal{M}_{ww} f}(k) = c_{ww}(k)\hat{f}(k).$$

The model (2.2.1), now referred to as the Whitham equation, balances both the full phase speed for unidirectional water waves with a canonical shallow water nonlinearity and hence, Whitham conjectured, one might expect it to be capable of predicting both breaking and peaking of water waves. This has recently been seen to be the case. Indeed, (2.2.1) has recently been seen to exhibit both wave breaking [37] and peaking [19, 23]. Additionally, (2.2.1) was shown in [38] to bear out the famous Benjamin-Feir, or modulational, instability of small amplitude periodic traveling waves: see also the related numerical work in [68] on the stability of large amplitude periodic waves. Taken together, it seems clear that, regardless of its rigorous connection to the full water wave problem[1], the dispersion generalized model (2.2.1) admits many interesting high-frequency features known to exist in the full water wave problem.

Given the success of (2.2.1) in describing water wave phenomena outside of the long-wavelength regime, a number of recent works have aimed to study such models that incorporate additional physical effects such as surface tension, constant vorticity, as well as models allowing for bidirecitonal wave propagation: see, for example, [12, 13, 20, 21, 39, 45, 46] and references therein. Interestingly, these works that focus on stability may often be carried out by considering a *dispersion generalized Whitham equation*

$$u_t + \beta \mathcal{M} u_x + \left(u^2\right)_x = 0, \tag{2.2.2}$$

where here the Fourier multiplier $\mathcal{M}$

$$\widehat{\mathcal{M} f}(k) = m(k)\hat{f}(k), \tag{2.2.3}$$

need only have a symbol $m(k)$ that satisfies a few simple non-degeneracy, smoothness and growth assumptions. Results for specific models, or incorporating specific physical properties, can then be immediately ascertained by substituting for $\mathcal{M}$ a specific Fourier multiplier whose symbol agrees with the associated phase velocity. For example, with the choice $m(k) = 1 - |k|^{\alpha}(\alpha > 1)$ (2.2.2) is the Fractional KdV (fKdV) equation, $m(k) = 1 - |k|$ is the Benjamin-Ono (BO) equation, $m(k) = k \coth k$ is the Intermediate Long wave (ILW) equation and $m(k) = \sqrt{\tanh k/k}$ is the Whitham equation (2.2.1).

Continuing in this spirit, this current work seeks to study the existence and stability of

[1]The relevance of the Whitham equation as a model for water waves has recently been studied in [7, 61], where it was found to perform better than the KdV and BBM equations in describing surface water waves in the intermediate and short wave regime.

periodic traveling wave solutions in the following dispersion-generalized Ostrovsky (gOst) equation

$$\left(u_t + \beta \mathcal{M} u_x + (u^2)_x\right)_x - \gamma u = 0, \ \ \gamma > 0, \ \ \beta \in \mathbb{R} \setminus \{0\}, \tag{2.2.4}$$

where here the Fourier multiplier $\mathcal{M}$

$$\widehat{\mathcal{M} f}(k) = m(k) \hat{f}(k), \tag{2.2.5}$$

which satisfies the following assumptions:

Hypothesis 2.2.1. *The multiplier $m(k)$ in (2.2.5) is assumed to satisfy the following:*

(H1) m is real valued, even and without loss of generality, $m(0) = 1$;

(H2) there exists constants $C_1, C_2 > 0$ and $\alpha \geq -1$ such that

$$C_1 k^\alpha \leq m(k) \leq C_2 k^\alpha,$$

for $k \gg 1$.

Additionally, throughout we will assume the frequencies $k > 0$ considered satisfy the following:

(H3) for each fixed $n = 2, 3, \ldots$ we have

$$k^2 \left(m(kn) - m(k)\right) \neq \frac{\gamma(n^2 - 1)}{\beta n^2},$$

for all $k > 0$.

Remark 2.2.2. *Hypotheses (H1)-(H3) above will be essential for the proof of the existence of small-amplitude periodic traveling waves of (2.2.4): see Section 2.4 below. In particular, we note that hypothesis (H3) rules out the resonance between the fundamental mode and a higher harmonic. For more details, see the discussion directly preceding the statement of Theorem 2.4.1.*

Also, we note in the case $m(k) = 1 - k^2$, corresponding to the classical Ostrovsky equation, the non-resonance condition (H3) holds for all $k > 0$ if $\beta < 0$ while if $\beta > 0$ it requires that

$$k \notin \left\{ \left(\frac{\gamma}{\beta n^2}\right)^{1/4} : n \in \mathbb{N}, \quad n \geq 2 \right\},$$

thus excluding a countable number of frequencies tending to zero as $n \to \infty$.

The model (2.2.4) can thus be thought of as an extension of (2.2.1) that incorporates rotational effects. Notice (2.2.4) recovers the standard Ostrovsky equation by choosing $\mathcal{M} = \partial_x^2$,

$$(u_t + \beta u_{xxx} + (u^2)_x)_x = \gamma u. \tag{2.2.6}$$

Further, one can take $m(k) = 1 - |k|^\alpha$, $m(k) = 1 - |k|$, $m(k) = k\coth(k)$ and $m(k) = \sqrt{\tanh(k)/k}$ to produce variants of the fKdV, BO, ILW, and Whitham equations that incorporates rotating background fluids. Further, note that since $m(k)$ is even, if $u(x,t)$ satisfies (2.2.4) for a particular choice of β and γ, then $-u(x,-t)$ satisfies (2.2.4) with β and γ replaced by $-\beta$ and $-\gamma$ respectively. Due to this symmetry, we restrict ourselves to the case of $\gamma > 0$.

2.3 Main Results

To state our main result, note that the dispersion relation associated with (2.2.4) is given by

$$\omega(k) = \beta k m(k) + \frac{\gamma}{k}, \tag{2.3.1}$$

while the corresponding phase and group velocities are given by

$$c_p(k) := \frac{\omega(k)}{k} = \beta m(k) + \frac{\gamma}{k^2} \quad \text{and} \quad c_g(k) := \frac{d\omega}{dk} = \beta(m(k) + km'(k)) - \frac{\gamma}{k^2}, \tag{2.3.2}$$

respectively. We now state our main result, providing a criterion governing the modulational instability of periodic traveling waves of the generalized Ostrovsky equation (2.2.4) which is as follows.

Theorem 2.3.1 (Modulational instability index). *A $2\pi/k$-periodic traveling wave of (2.2.4) with sufficiently small amplitude is modulationally unstable if*

$$\Delta(k) := (c_p(k) - c_p(2k))\frac{dc_g(k)}{dk} < 0, \tag{2.3.3}$$

where c_p and c_g are phase and group velocities respectively given in (2.3.2). It is modulationally stable if $\Delta(k) > 0$.

Remark 2.3.2. *Our precise definition of modulational stability and instability is provided in Definition 2.5.4 in Section 2.5 below.*

Remark 2.3.3. *We note that the instability condition (2.3.3) is precisely the same as the well-known Lighthill criteria for modulational instability of small-amplitude periodic traveling wave solutions in the context of the Ostrovsky equation. Indeed, in [28] the authors use formal asymptotic methods (so-called modulation theory) to show that if u is a small amplitude, weakly nonlinear periodic solution of the Ostrovsky equation with an asymptotic expansion of the form*

$$u(x,t) = \left(Ae^{i\theta} + c.c.\right) + \left(A_2 e^{2i\theta} + c.c.\right) + \dots,$$

where c.c. denotes the complex conjugate of the preceding term, $\theta = kx - \widetilde{\omega}(|A|,k)t$, where $\widetilde{\omega}(|A|,k)$ is the nonlinear dispersion relation, and $|A| = \varepsilon \ll 1$ is slowly varying (in space and time) with $A_2 = \mathcal{O}(\varepsilon^2)$, then the leading order term A satisfies the effective nonlinear Schrödinger (NLS) equation

$$iA_t + \frac{1}{2}\omega''(k)A_{XX} - \omega_2(k)|A|^2 A = 0, \tag{2.3.4}$$

where here $X = x - c_g(k)t$, $\omega(k)$ is the dispersion relation (2.3.1) for the Ostrovsky equation and where ω_2 is identified as the $\mathcal{O}(\varepsilon^2)$ correction to the dispersion relation

$$\widetilde{\omega}(|A|,k) = \omega(k) + \varepsilon^2\omega_2 + \mathcal{O}(\varepsilon^4),$$

in the amplitude of the wave. The Lighthill criteria then says that the small weakly nonlinear periodic traveling wave solution is modulationally unstable provided that the NLS equation (2.3.4) is focusing, i.e. provided that

$$\omega''(k)\omega_2(k) < 0. \tag{2.3.5}$$

In Remark 2.4.3 below, we will show that the Lighthill condition (2.3.5) agrees precisely with the rigorous modulational instability index (2.3.3) stated in Theorem 2.3.1. In this way, in the case of the Ostrovsky equation, our work can be seen as a rigorous justification of the (formal) Lighthill condition (2.3.5) at the level of the (rigorous) spectral stability *of the underlying small-amplitude periodic traveling wave. Of course, our analysis also applies to the wider class of dispersion generalized Ostrovsky equations given in* (2.2.4).

From Theorem 2.3.1, $\Delta(k)$ can change sign using two mechanisms - first when $c_p(k) = c_p(2k)$, that is, phase velocities of first and second harmonic coincide, and second when $\frac{dc_g(k)}{dk} = 0$, that is, group velocity has a critical value. As an immediate corollary of Theorem 2.3.1, we obtain modulational instability in the Ostrovsky equation (2.2.6).

Corollary 2.3.4 (Modulational instability)**.** *For a fixed* $\gamma > 0$*, a* $2\pi/k$*-periodic traveling wave of the Ostrovsky equation* (2.2.6) *with sufficiently small amplitude is modulationally unstable if* $k > k_c$ *where*

$$\begin{cases} k_c = \left(\dfrac{\gamma}{3|\beta|}\right)^{1/4} & \text{if } \beta > 0, \\ k_c = \left(\dfrac{\gamma}{4|\beta|}\right)^{1/4} & \text{if } \beta < 0, \end{cases}$$

and it is modulationally stable otherwise.

Figure 2.1 displays the spectrum obtained by numerically computing the eigenvalues of the linearized operator around the small-amplitude periodic wave, using Floquet–Bloch theory. The eigenvalues are plotted in the complex plane, with the real part on the horizontal axis and the imaginary part on the vertical axis [16].

For $\beta = 1$ and $\gamma = 1$, the value of k_c in Theorem 2.3.1 is approximately 0.76 which agrees with critical wavenumber obtained by Whitfield and Johnson in [74]. See also Figure 2.1 where modulational instability and stability is demonstrated numerically for cases $k > k_c$ and $k < k_c$, respectively. In addition, Whitfield and Johnson mention that the instability is caused as group velocity has a critical value at k_c which in accordance with our analysis for $\beta > 0$. In our analysis, we will also see when $\beta < 0$ that modulational instability is caused by coinciding phase velocities of first and second harmonic at k_c.

Remark 2.3.5. *From Corollary 2.3.4 we see that, regardless of the sign of* β*, the critical frequency* k_c *satisfies*

$$k_c \sim \gamma^{1/4}, \quad \text{for} \quad 0 < \gamma \ll 1.$$

Thus, one may think this suggests that all periodic traveling wave solutions of the KdV equation (corresponding essentially to $\gamma = 0$*) exhibit a modulational instability. This*

(a) (b)

Figure 2.1: A numerical determination of the $\lambda \in \mathbb{C}$ with $|\lambda| \ll 1$ obtained from the linearized problem for the classical Ostrovsky equation (corresponding to $m(k) = 1 - k^2$). In both figures, we took $\beta = 1$ and $\gamma = 1$, for which Corollary 2.3.4 gives $k_c = 0.76$. Taking an amplitude $a = 0.05$, in (a) we took $k = 0.8 > k_c$ as predicted by Theorem 2.3.1. Similarly, in (b) we took $k = 0.72 < k_c$, and so the spectral stability in a neighbourhood of the origin is again consistent with Theorem 2.3.1.

is, in fact not the case, as it is actually known that all periodic traveling waves in the KdV equation are modulationally stable (in fact, spectrally stable to general localized and bounded perturbations) [8, 10, 11, 33]. This emphasizes the singular nature of the $\gamma \to 0^+$ *limit, and results about the associated* $\gamma = 0$ *model can not be directly inferred from taking* $\gamma \to 0^+$ *in our analysis.*

2.4 Asymptotically small-amplitude periodic traveling waves

To begin, we seek *periodic traveling wave* solutions of (2.2.4). Here and throughout our work, we will always assume the symbol $m(\cdot)$ associated with the Fourier multiplier $\mathcal{M}$ satisfies (H1)-(H3) in Hypothesis 2.2.1. To this end we make a change of variables $z := k(x - ct)$, where $k > 0$ is the wave number and c is the speed of the wave, and note that if u is a $2\pi/k$-periodic traveling wave solution of (2.2.4) then the function

$\eta(z) := u(k(x-ct),t)$ is a 2π-periodic solution of

$$-ck^2\eta'' + \beta k^2 \mathcal{M}_k \eta'' + k^2(\eta^2)'' - \gamma\eta = 0, \tag{2.4.1}$$

where here $\mathcal{M}_k$ is a Fourier multiplier satisfying

$$\mathcal{M}_k e^{inz} = m(kn)e^{inz} \quad \text{for} \quad n \in \mathbb{Z}.$$

Note that $m(k)$ is assumed to be real-valued and even. Consequently, (2.4.1) is invariant under translation $(z \mapsto z + z_0)$ and $z \mapsto -z$ and therefore, we may assume that η is even. Also, since (2.4.1) does not possess scaling invariance, we may not a priori assume that $k = 1$. In fact, the modulational instability results obtained below depend on the wavenumber k of the background periodic wave.

For fixed β and γ, define the operator[2] $F : H^{\alpha+2}_{\text{even}}(\mathbb{T}) \times \mathbb{R} \times \mathbb{R}^+ \to L^2(\mathbb{T})$ as

$$F(\eta, c; k) = -ck^2\eta'' + \beta k^2 \mathcal{M}_k \eta'' + k^2(\eta^2)'' - \gamma\eta.$$

Note that F is well defined by a elementary Sobolev embedding argument. We seek a non-trivial 2π-periodic solution η of (2.4.1) in $H^{\alpha+2}_{\text{even}}(\mathbb{T})$ with $c \in \mathbb{R}$ and $k > 0$ such that

$$F(\eta, c; k) = 0.$$

Note that if $\eta \in H^{\alpha+2}(\mathbb{T})$ solves (2.4.1) then by a Sobolev inequality we have $\mathcal{M}_k\eta'' \in H^{\alpha}(\mathbb{T})$. Therefore, by (H2) in Hypothesis 2.2.1, $\eta \in H^{2(\alpha+1)}(\mathbb{T})$. By a bootstrap argument it follows that solutions $\eta \in H^{\alpha+2}(\mathbb{T})$ of (2.4.1) necessarily satisfy $\eta \in H^{\infty}(\mathbb{T})$.

Now, to study (2.4.1) note that $F(0, c; k) = 0$ for all $c \in \mathbb{R}$ and $k > 0$ and

$$\partial_\eta F(0, c; k) = -ck^2\partial_z^2 + \beta k^2 \mathcal{M}_k \partial_z^2 - \gamma,$$

so that, in particular, for each $c \in \mathbb{R}$, $k > 0$ and $n \in \mathbb{N}$ we have

$$\partial_\eta F(0, c; k)\cos(nz) = \big(c(kn)^2 - \beta(kn)^2 m(kn) - \gamma\big)\cos(nz).$$

[2]Here, α is as in (H2).

It follows that

$$\ker\left(\partial_\eta F(0, c_0; k)\right) = \mathrm{span}\left(\cos(z)\right),$$

provided that

$$c = c_0 = \frac{\gamma}{k^2} + \beta m(k),$$

and that $k > 0$ is chosen so that that[3]

$$m(kn) - m(k) \neq \frac{\gamma(n^2-1)}{\beta n^2} \quad \text{for any} \quad n \in \mathbb{N}, \quad n \geq 2, \tag{2.4.2}$$

i.e. k should be chosen so that (H3) holds. Using a Lyapunov-Schmidt argument, one can thus establish the existence of a one-parameter family of non-trivial, even solutions $(\eta(a;k)(\cdot), c(a;k))$ of $F(\eta, c; k)$ bifurcating from $\eta \equiv 0$ and $c = c_0$ and defined for $|a| \ll 1$ provided that $k > 0$ satisfies the non-resonance condition (2.4.2). This existence argument is elementary and follows the same lines as those in [22, 44], and is hence omitted here. A key feature of the solutions $(\eta(a;k)(\cdot), c(a;k))$, however, is that they depend analytically on the parameter a for $|a| \ll 1$. Exploiting that fact, the next result further establishes analytic expansions for these solutions valid for all $|a| \ll 1$.

Theorem 2.4.1. *Suppose that the symbol $m(\cdot)$ in (2.2.5) satisfies hypotheses (H1)-(H2). For all wavenumbers $k > 0$ satisfying hypothesis (H3) there exists a one parameter family of solutions of (2.4.1) given by $u(x,t) = \eta(a;k)(k(x - c(a;k)t))$ for $a \in \mathbb{R}$ and $|a|$ sufficiently small; $\eta(a;k)(\cdot)$ is 2π-periodic, even and smooth in its argument, and $c(a;k)$ is even in a; $\eta(a;k)(\cdot)$ and $c(a;k)$ depend analytically on a and k. Moreover,*

$$\eta(a;k)(z) = a\cos(z) + a^2 A_2 \cos 2z + a^3 A_3 \cos 3z + O(a^4), \tag{2.4.3}$$

and

$$c(a;k) = c_0 + a^2 c_2 + O(a^4), \tag{2.4.4}$$

as $a \to 0$, where

$$c_0 = \frac{\gamma}{k^2} + \beta m(k),$$

[3] While the choice of c_0 guarantees that $\cos(z)$ is in the kernel of $\partial_\eta F(0, c; k)$, the restriction on k guarantees $\partial_\eta F(0, c_0; k)\cos(nz) \neq 0$ for any $n \in \mathbb{N}$ with $n \geq 2$. That is, the restriction on k guarantees that the kernel of $\partial_\eta F(0, c_0; k)$ is simple.

and

$$c_2 = A_2, \quad A_2 = \frac{2k^2}{3\gamma + 4\beta k^2(m(k) - m(2k))} \quad \text{and} \quad A_3 = \frac{9k^2 A_2}{8\gamma + 9\beta k^2(m(k) - m(3k))}.$$

Proof. As this argument is standard, here we only sketch the details. The existence of the solutions $(\eta(a;k)(\cdot), c(a;k))$ for a fixed $k > 0$ follows by an elementary Lyapunov-Schmidt argument: for similar arguments, see, for example, [22]. As a result, for a fixed $k > 0$ it follows that the solutions $(\eta(a,k)(\cdot), c(a,k))$ are analytic in a for $|a| \ll 1$ and hence may be expanded as[4]

$$\begin{cases} \eta(a;k)(z) = a\cos(z) + a^2 w_2(k)(z) + a^3 w_3(k)(z) + \mathcal{O}(a^4), \\ \quad c(a;k) = c_0 + c_2(k)a^2 + c_4(k)a^4 + \mathcal{O}(a^6). \end{cases} \tag{2.4.5}$$

Substituting these expansions into the profile equation (2.4.1) yields a hierarchy of compatability conditions indexed by the order of the small parameter a. By construction, the first non-trivial equation occurs at $\mathcal{O}(a^2)$, which reads

$$\partial_\eta F(0, c; k)\eta_2 = 2k^2 \cos 2z, \tag{2.4.6}$$

Requiring, via the Fredholm Alternative, that the right-hand side of (2.4.6) be orthogonal to the kernel of the (symmetric) operator $\partial_\eta F(0, c; k)$ yields the stated equation for $c_2 = c_2(k)$, and then subsequently solving (2.4.6) results in the stated formula for the function η_2. Continuing to higher order equations in a, using again the Fredholm alternative and subsequently solving the resulting equation, yields the remainder of the formulae. We omit the remainder of the details, and instead refer the interested reader to [44][Lemma 2.1], for instance. □

Remark 2.4.2. *We note, in particular, that the solutions $\eta(a;k)(z)$ constructed above all satisfy a mean-zero constraint, i.e., they satisfy $\int_0^{2\pi} \eta(z)dz = 0$. This of course is a property of all localized or periodic traveling wave solutions of the generalized Ostrovsky equation considered here, as integrating (2.2.4) over a period (for periodic traveling waves) or on the whole line (for localized traveling waves) yields the mean-zero requirement since $\gamma \neq 0$.*

[4]The relation $c(a;k) = c(-a;k)$ follows from the details of the Lyapunov-Schmidt argument discussed above. See, for example, [22, 44].

Remark 2.4.3. *From Theorem 2.4.1, we see that the nonlinear dispersion relation* $\omega(a;k) = kc(a;k)$ *can be expanded as*

$$\omega(a;k) = \omega(k) + a^2 \frac{2k^3}{3\gamma + 4\beta k^2 (m(k) - m(2k))} + \mathcal{O}(a^4).$$

Noting that

$$c_p(k) - c_p(2k) = \frac{3\gamma}{4k^2} + \beta\left(m(k) - m(2k)\right),$$

and that $\frac{dc_g}{dk}(k) = \omega''(k)$ *by definition, it follows that the Lighthill condition given in* (2.3.5) *can be expressed as*

$$\omega''(k)\omega_2(k) = \frac{k}{2\left(c_p(k) - c_p(2k)\right)} \cdot \frac{dc_g}{dk}(k),$$

and hence the sign of the rigorous modulational instability index from Theorem 2.3.1 agrees exactly with the Lighthill condition (2.3.5).

2.5 Modulational Instability Index

Throughout this section, let $\eta = \eta(x;a,k)$ with $k > 0$ and $|a| \ll 1$ be a small amplitude $2\pi/k$-periodic traveling wave solution of (2.2.4) with wave speed $c = c(a;k)$, whose existence follows from Theorem 2.4.1 above. The goal of this section is to study the modulational stability of the wave η.

Linearizing (2.2.4) about η in the spatial frame of reference $z = k(x - ct)$ we arrive at the linear evolution equation

$$k(v_t - ckv_z + \beta k\mathcal{M}_k v_z + 2k(\eta v)_z)_z = \gamma v,$$

governing the perturbation $v(z,t)$. We seek a solution of the form $v(z,t) = e^{\frac{\lambda}{k}t}\tilde{v}(z)$, $\lambda \in \mathbb{C}$ and $\tilde{v} \in L^2(\mathbb{R})$ to arrive at the equation

$$\mathcal{T}^{\lambda}_{k,a}\tilde{v} := (\lambda\partial_z + k^2\partial_z^2(-c + \beta\mathcal{M}_k + 2\eta) - \gamma)\tilde{v} = 0, \tag{2.5.1}$$

where here $\mathcal{T}^{\lambda}_{k,a} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is considered as a closed, densely defined linear operator. The operator $\mathcal{T}^{\lambda}_{k,a}$ introduced here is precisely the one used in the numerical computations presented in Figure 2.1.

Definition 2.5.1. *The periodic traveling wave solution* η *of* (2.2.4) *is spectrally stable with respect to square integrable perturbations if the operator* $\mathcal{T}^{\lambda}_{k,a}$ *is invertible on* $L^2(\mathbb{R})$ *for every* $\lambda \in \mathbb{C}$ *with* $\Re(\lambda) > 0$. *Otherwise, it is deemed to be spectrally unstable.*

Remark 2.5.2. *Note that since* (2.5.1) *is invariant under the transformation* $(v, \lambda) \mapsto (\bar{v}, \bar{\lambda})$, *as well as the transformation* $(z, \lambda) \mapsto (-z, -\lambda)$, *the set of* $\lambda \in \mathbb{C}$ *where the operator* $\mathcal{T}^{\lambda}_{k,a}$ *fails to be invertible is symmetric with respect to reflections about the real and imaginary axes. Consequently,* η *is spectrally stable if and only if* $\mathcal{T}^{\lambda}_{k,a}$ *is invertible for all* $\lambda \in \mathbb{C}$ *with* $\Re(\lambda) \neq 0$.

Since the coefficients of the operator $\mathcal{T}^{\lambda}_{k,a}$ are 2π-periodic, we can use the Floquet theory such that all solutions of (2.5.1) in $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$ are of the form $\tilde{v}(z) = e^{i\mu z}V(z)$ where $\mu \in (-1/2, 1/2]$ is the Floquet exponent and V is a 2π-periodic function. As a result, we get the following.

Lemma 2.5.3. *The linear operator* $\mathcal{T}^{\lambda}_{k,a}$ *is invertible with bounded inverse on* $L^2(\mathbb{R})$ *if and only if the linear operators*

$$\mathcal{T}^{\lambda}_{k,a,\mu} := \lambda(\partial_z + i\mu) + k^2(\partial_z + i\mu)^2(-c + \beta e^{i\mu z}\mathcal{M}_k e^{-i\mu z} + 2\eta) - \gamma, \tag{2.5.2}$$

acting in $L^2(\mathbb{T})$ *with domain* $H^{\alpha+2}(\mathbb{T})$ *are invertible for all* $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right]$. *Moreover,* $\mathcal{T}^{\lambda}_{k,a,\mu}$ *is invertible in* $L^2(\mathbb{T})$ *if and only if zero is not an* $L^2(\mathbb{T})$*-eigenvalue of* $\mathcal{T}^{\lambda}_{k,a,\mu}$.

Proof. We refer readers to [31, 32, 44] for detailed proofs in similar situations. It is straightforward to adapt those results to our case. □

Lemma 2.5.3 reduces the invertibility problem (2.5.1) in $L^2(\mathbb{R})$ to a one-parameter family of invertibility problems

$$\mathcal{T}^{\lambda}_{k,a,\mu}\phi := (\lambda(\partial_z + i\mu) + k^2(\partial_z + i\mu)^2(-c + \beta e^{i\mu z}\mathcal{M}_k e^{-i\mu z} + 2\eta) - \gamma)\phi = 0, \tag{2.5.3}$$

in $L^2(\mathbb{T})$ indexed by $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right]$. From Definition 2.5.1, it follows that the periodic traveling wave η is spectrally unstable with respect to square integrable perturbations in $L^2(\mathbb{R})$ if and only if for some $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ the operator $\mathcal{T}^{\lambda}_{k,a,\mu}$ acting on $L^2(\mathbb{T})$ has non-trivial kernel for some $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. While determining the full set of λ for which the operator $\mathcal{T}^{\lambda}_{k,a,\mu}$ is invertible for all $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ is already a daunting task, in this work our focus is on a particular subclass of the possible instabilities.

Definition 2.5.4. *A periodic traveling wave solution $\eta(a;k)$ of (2.2.4) is said to be modulationally stable if the associated linear operators $\mathcal{T}^{\lambda}_{a,k,\mu}$ is invertible on $L^2(\mathbb{T})$ for all $|(\lambda,\mu)| \ll 1$ with $\Re(\lambda) \neq 0$. Otherwise, the solution $\eta(a;k)$ is modulationally unstable.*

Remark 2.5.5. *In essence, the above states that the periodic traveling wave $\eta(a;k)$ is modulationally stable if the operator $\mathcal{T}^{\lambda}_{a,k}$ is invertible for all λ off the imaginary axis in a sufficiently small neighborhood of $\lambda = 0$. It follows that if the solution η is modulationally unstable, then it is indeed spectrally unstable in the sense of Definition 2.5.1. However, being modulationally stable does not automatically imply spectral stability since the latter requires that $\mathcal{T}^{\lambda}_{a,k}$ is invertible for* all *$\lambda \in \mathbb{C}$ with $\Re(\lambda) \neq 0$, not just invertible for such λ near the origin.*

Remark 2.5.6. *We note that modulational instabilities are a fundamental feature of many nonlinear systems, including those arising in the modeling of nonlinear optics as well as surface water waves. The connection between such "spectral" modulational instabilities, as described above, and the dynamic instability of periodic traveling wave solutions to slow modulations (via Whitham's theory of modulations) has been studied in many works, including [9, 42, 43]. In addition to the references mentioned above, the reader can consult [75] for a mathematical and physical discussion.*

We now list some properties of the operator $\mathcal{T}^{\lambda}_{k,a,\mu}$, which may be readily verified by direct calculation.

Proposition 2.5.7 (Symmetric Property)**.** *The operator $\mathcal{T}^{\lambda}_{k,a,\mu}$ acting on $L^2(\mathbb{T})$ has following properties:*

1. $\mathcal{T}^{\lambda}_{k,a,\mu}(z) = \overline{\mathcal{T}^{-\bar{\lambda}}_{k,a,\mu}(-z)}$,
2. $\mathcal{T}^{\lambda}_{k,a,\mu} = \overline{\mathcal{T}^{\bar{\lambda}}_{k,a,-\mu}}$.

We now set forth the study of the $L^2(\mathbb{T})$-kernel of the operator $\mathcal{T}^{\lambda}_{k,a,\mu}$ for $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ and $|a|$ sufficiently small. Note that, thanks to Proposition 2.5.7(2), it is sufficient to take $\mu \in [0, 1/2]$. Since k is fixed, in what follows, we denote $\mathcal{T}^{\lambda}_{k,a,\mu}$ by $\mathcal{T}^{\lambda}_{a,\mu}$.

We begin by discussing the case $a = 0$, corresponding to the trivial solution $\eta = 0$. A straightforward Fourier calculation yields that

$$\mathcal{T}_{0,\mu}^{\lambda} e^{inz} = \left(i\lambda(n+\mu) + \gamma((n+\mu)^2 - 1) + \beta k^2 (n+\mu)^2 (m(k) - m(k(n+\mu)))\right) e^{inz} = 0, \tag{2.5.4}$$

for all $n \in \mathbb{Z}$ and $\mu \in [0, 1/2]$. The kernel of $\mathcal{T}_{0,\mu}^{\lambda}$ is thus non-trivial when

$$\lambda = i\left(\gamma\left(n + \mu - \frac{1}{n+\mu}\right) + n\beta k^2 (m(k) - m(k(n+\mu)))\right) =: i\Omega_{n,\mu}, \quad n \in \mathbb{Z}, \tag{2.5.5}$$

and hence the trivial solution $\eta = 0$ of (2.2.4) is spectrally stable to square integrable perturbations as expected. Moreover, for $|a|$ small, because of Proposition 2.5.7, values of λ in (2.5.5) will bifurcate to leave imaginary axis only when two of them collide on imaginary axis. Therefore, we obtain instability for small $|a|$ if for some λ in (2.5.5), the kernel of $\mathcal{T}_{0,\mu}^{\lambda}$ is at least two-dimensional. Since we are only concerned with modulational (in)stability, we will only consider small values of $|\mu|$ and $|\lambda|$.

In particular, notice that the two values $\Omega_{1,\mu}$ and $\Omega_{-1,\mu}$ collide at $\lambda = 0$ when $\mu = 0$: for all other $n \in \mathbb{N}$ and μ we have $\Omega_{n,\mu} \neq 0$. Furthermore, the two-dimensional generalized kernel for $\mathcal{T}_{0,0}^{0}$ can be continued into a two-dimensional critical subspace

$$\Sigma_{0,\mu} = \ker\left(\mathcal{T}_{0,\mu}^{i\Omega_{1,\mu}}\right) \oplus \ker\left(\mathcal{T}_{0,\mu}^{i\Omega_{-1,\mu}}\right),$$

with (μ-independent) orthogonal basis

$$\phi_1(z) = \cos(z), \quad \phi_2(z) = \sin(z). \tag{2.5.6}$$

For all other values of $n \neq -1, 1$, the kernel of $\mathcal{T}_{0,0}^{i\Omega_{n,0}}$ is one-dimensional and therefore, from Proposition 2.5.7(1), can not lead to instability for $|(a,\mu)| \ll 1$. We thus aim to track how the values $\Omega_{\pm 1,0}$ bifurcate from the origin for $|(a,\mu)| \ll 1$. To this end, we note that, for $|a|$ and μ small, the operator $\mathcal{T}_{a,\mu}^{\lambda}$ is a perturbation of $\mathcal{T}_{0,0}^{\lambda}$ with

$$\|\mathcal{T}_{a,\mu}^{\lambda} - \mathcal{T}_{0,0}^{\lambda}\|_{L^2(\mathbb{T}) \to L^2(\mathbb{T})} = O(|a| + |\mu|),$$

uniformly in operator norm as $|a|, \mu \to 0$. Consequently, for $|(a,\mu)| \ll 1$ there can be only two values $\lambda = i\Omega_{\pm 1,a,\mu}$ in a sufficiently small neighborhood of the origin where the operator $\mathcal{T}^{\lambda}_{a,\mu}$ fails to be invertible and, further, the functions

$$(a,\mu) \mapsto \Omega_{\pm 1,a,\mu},$$

are analytic in (a,μ) for $|(a,\mu)| \ll 1$ and limit back to $\Omega_{\pm 1,0} = 0$ as $(a,\mu) \to (0,0)$. Further, one has a two-dimensional critical subspace

$$\Sigma_{a,\mu} = \ker\left(\mathcal{T}^{i\Omega_{1,a,\mu}}_{a,\mu}\right) \oplus \ker\left(\mathcal{T}^{i\Omega_{-1,a,\mu}}_{a,\mu}\right),$$

which is an analytical continuation of that found at $a = 0$ above.

Our goal is now to track the critical values $\lambda(a,\mu) = i\Omega_{\pm 1,a,\mu}$ for $|(a,\mu)| \ll 1$. To this end, our strategy is essentially to project the operator equation $\mathcal{T}^{\lambda}_{a,\mu} v = 0$ onto the two-dimensional critical subspace $\Sigma_{a,\mu}$ above. More precisely, we will compute a suitable basis $\{\phi_j(z;a,\mu)\}_{j=1,2}$ for $\Sigma_{a,\mu}$ and then compute the 2×2 matrix

$$\mathcal{B}^{\lambda}_{a,\mu} = [B_{ij}]^{2}_{i,j=1} \quad \text{with} \quad B_{ij} = \frac{\langle \mathcal{T}^{\lambda}_{a,\mu}\phi_i(z;a,\mu), \phi_j(z;a,\mu)\rangle}{\langle \phi_i(z;a,\mu), \phi_i(z;a,\mu)\rangle}. \tag{2.5.7}$$

The critical values $\lambda(a,\mu)$ are then found by solving the algebraic equation

$$\det(\mathcal{B}^{\lambda}_{a,\mu}) = 0, \tag{2.5.8}$$

for the variable λ. It remains then to find a suitable basis for the critical subspace $\Sigma_{a,\mu}$, and then to compute the appropriate projections above.

To compute a basis for $\Sigma_{a,\mu}$ that is compatible with $\Sigma_{0,\mu}$, note that differentiating the profile equation (2.4.1) with respect to z and a gives

$$\mathcal{T}^{0}_{a,0}(\partial_z \eta) = 0, \quad \mathcal{T}^{0}_{a,0}(\partial_a \eta) = 0.$$

Using the expansions in Theorem 2.4.1, we thus obtain a normalized basis for the critical subspace $\Sigma_{a,0}$, i.e. the generalized kernel of $\mathcal{T}^{0}_{a,0}$, as

$$\phi_1(z;a,0) = -\frac{1}{a}(\partial_z \eta)(z) = \sin z + 2aA_2 \sin 2z + 3a^2 A_3 \sin 3z + O(a^4), \tag{2.5.9}$$

and

$$\phi_2(z;a,0) = (\partial_a\eta)(z) = \cos z + 2aA_2\cos 2z + 3a^2A_3\cos 3z + O(a^4), \qquad (2.5.10)$$

where here the values A_j are as in Theorem 2.4.1. These functions provide asymptotic extensions for the generalized kernel of $\mathcal{T}^0_{0,0}$ and, in fact, they provide an asymptotic extension for the (μ-independent) basis of the critical subspace $\Sigma_{0,\mu}$ provided in (2.5.6). By spectral perturbation theory, it follows that the functions $\phi_1(\cdot;a,0)$ and $\phi_2(\cdot;a,0)$ continue into a μ-dependent basis for the critical subspace $\Sigma_{a,\mu}$ for $|(a,\mu)| \ll 1$. We note, however, that as in [38, 39, 44], the variations in the basis functions $\phi_j(\cdot;a,\mu)$ does not play a role in the asymptotic calculation below as they contribute only to higher-order terms than what are needed here. Thus, below, the calculations are done with the μ-independent basis $\phi_j(\cdot;a,0)$.

Continuing our strategy, we now compute the action of $\mathcal{T}^\lambda_{a,\mu}$ on the critical subspace $\Sigma_{a,\mu}$. Here $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the $L^2(\mathbb{T})$- inner product defined as

$$\langle f,g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(z)\overline{g}(z)\;dz = \sum_{n\in\mathbb{Z}} \widehat{f}_n\overline{\widehat{g}_n}, \qquad (2.5.11)$$

where here, for a given $h \in L^2(\mathbb{T})$,

$$\widehat{h}_n = \frac{1}{2\pi}\int_0^{2\pi} h(z)e^{-inx}dx,$$

denotes the nth Fourier coefficient of h. Now, for μ and $|a|$ sufficiently small, we expand $\mathcal{T}^\lambda_{a,\mu}$ using Baker-Campbell-Hausdorff formula as

$$\mathcal{T}^\lambda_{a,\mu} = T_{0,a} + i\mu T_{1,a} - \frac{\mu^2}{2}T_{2,a} + O(\mu^3), \qquad (2.5.12)$$

as $\mu \to 0$ where

$$\begin{aligned}
T_{0,a} &:= \mathcal{T}^\lambda_{a,0} = T_0 + 2k^2\partial z^2(a\cos z + a^2A_2\cos 2z) + \mathcal{O}(a^3),\\
T_{1,a} &:= [\mathcal{T}^\lambda_{a,0}, z] = T_1 + 4k^2\partial z(a\cos z + a^2A_2\cos 2z) + \mathcal{O}(a^3),\\
T_{2,a} &:= [[\mathcal{T}^\lambda_{a,0}, z], z] = T_2 + 4k^2(a\cos z + a^2A_2\cos 2z) + \mathcal{O}(a^3),
\end{aligned}$$

and

$$T_0 = \mathcal{T}^{\lambda}_{0,0}, \quad T_1 = [T_0, z] \quad \text{and} \quad T_2 = [[T_0, z], z].$$

Note that $T_{1,a}$ and $T_{2,a}$ are well defined in $L^2(\mathbb{T})$. Now, to find the action of T_0, T_1 and T_2 on the generalized kernel, we use the expansion $\mathcal{T}^{\lambda}_{0,\mu} e^{inz}$ rather than computing tedious Fourier series expansions. Moreover,

$$\mathcal{T}^{\lambda}_{0,\mu}(\cos nz) = \mathcal{T}^{\lambda}_{0,\mu}\left(\frac{e^{inz} + e^{-inz}}{2}\right) \qquad \text{and} \qquad \mathcal{T}^{\lambda}_{0,\mu}(\sin nz) = \mathcal{T}^{\lambda}_{0,\mu}\left(\frac{e^{inz} - e^{-inz}}{2i}\right).$$

Consequently,

$$\begin{aligned}
T_0(\cos nz) =& - n\lambda \sin nz + \gamma(n^2 - 1)\cos nz + \beta k^2 n^2 (m(k) - m(kn)) \cos nz,\\
T_1(\cos nz) =& \lambda \cos nz + 2n\gamma \sin nz + \beta k^2(-n^2 k m'(kn) + 2n(m(k)\\
&- m(kn)) \sin nz,\\
T_2(\cos nz) =& - 2\gamma \cos nz - 2\beta k^2(m(k) - m(kn) - 2nkm'(kn)\\
&- \frac{n^2k^2}{2} m''(kn)) \cos nz,
\end{aligned}$$

and

$$\begin{aligned}
T_0(\sin nz) =& n\lambda \cos nz + \gamma(n^2 - 1)\sin nz + \beta k^2 n^2 (m(k) - m(kn)) \sin nz,\\
T_1(\sin nz) =& \lambda \sin nz - 2n\gamma \cos nz - \beta k^2(-n^2 k m'(kn) + 2n(m(k)\\
&- m(kn)) \cos nz,\\
T_2(\sin nz) =& - 2\gamma \sin nz - 2\beta k^2(m(k) - m(kn) - 2nkm'(kn)\\
&- \frac{n^2k^2}{2} m''(kn)) \sin nz.
\end{aligned}$$

Using these, we get

$$\begin{aligned}
\left\langle \mathcal{T}^{\lambda}_{a,\mu}\phi_1, \phi_1 \right\rangle =& \frac{1}{2}(i\mu\lambda(1 + 4a^2A_2^2) - \frac{\mu^2}{2}(-2\gamma(1 + a^2A_2^2) + \beta k^4(m''(k)\\
&+ 16a^2A_2^2 m''(2k)) + 4\beta k^3(m'(k) + 8a^2A_2^2 m'(2k)))) + \mathcal{O}(\mu^3 + a^3),\\
\left\langle \mathcal{T}^{\lambda}_{a,\mu}\phi_1, \phi_2 \right\rangle =& \frac{1}{2}(\lambda(1 + 8a^2A_2^2) + i\mu(-2\gamma(1 + 2a^2A_2^2)\\
&+ \beta k^3(m'(k) + 16a^2A_2^2 m'(2k))) + \mathcal{O}(\mu^3 + a^3),
\end{aligned}$$

$$\begin{aligned}\langle \mathcal{T}^{\lambda}_{a,\mu}\phi_2,\phi_1\rangle =&\frac{1}{2}(-\lambda(1+8a^2A_2^2)+i\mu(2\gamma(1+2a^2A_2^2)-4a^2k^2A_2^2-\beta k^3(m'(k)+16a^2A_2^2m'(2k)))\\ &+\mathcal{O}(\mu^3+a^3),\\ \langle \mathcal{T}^{\lambda}_{a,\mu}\phi_2,\phi_2\rangle =&\frac{1}{2}(-2a^2k^2A_2+i\mu\lambda(1+4a^2A_2^2)-\frac{\mu^2}{2}(-2\gamma(1+a^2A_2^2)+4a^2k^2A_2+\beta k^4(m''(k)\\ &+16a^2A_2^2m''(2k))+4\beta k^3(m'(k)+8a^2A_2^2m'(2k))))+\mathcal{O}(\mu^3+a^3).\end{aligned}$$

Using the above obtained expressions, along with the expansion (2.5.12), it follows that the matrix $\mathcal{B}^{\lambda}_{a,\mu}$ in (2.5.7) can be expanded for sufficiently small μ and $|a|$ as

$$\begin{aligned}\mathcal{B}^{\lambda}_{a,\mu} =&\frac{\lambda}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}+\frac{a^2}{2}\begin{pmatrix}0 & 8\lambda A_2^2\\ -8\lambda A_2^2 & -2k^2A_2\end{pmatrix}+\frac{i\mu}{2}\begin{pmatrix}\lambda & -2\gamma+\beta k^3m'(k)\\ 2\gamma-\beta k^3m'(k) & \lambda\end{pmatrix}\\ &+\frac{i\mu a^2}{2}\begin{pmatrix}4\lambda A_2^2 & -4\gamma A_2^2+16\beta k^3A_2^2m'(2k)\\ 4\gamma A_2^2-4k^2A_2^2-16\beta k^3A_2^2m'(2k) & 4\lambda A_2^2\end{pmatrix}\\ &-\frac{\mu^2}{2}(-2\gamma+\beta k^4m''(k)+4\beta k^3m'(k))\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ &-\frac{\mu^2a^2}{2}\left((-2\gamma A_2^2+16\beta k^4A_2^2m''(2k)+32\beta k^3A_2^2m'(2k))\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & 4k^2A_2\end{pmatrix}\right)\\ &+\mathcal{O}(\mu^3+a^3).\end{aligned}$$

To study the two critical values $\lambda(a,\mu)$ bifurcating from the $(\lambda,a,\mu)=(0,0,0)$ state, we recall from (2.5.8) that we must study the roots of the polynomial

$$\det(\mathcal{B}^{\lambda}_{a,\mu})=b_0(a,\mu)+ib_1(a,\mu)\lambda+b_2(a,\mu)\lambda^2,$$

where the coefficient functions b_j, defined for $|(a,\mu)|\ll 1$, depend smoothly on a and μ. From the symmetry property in Proposition 2.5.7(1), we further see that the functions b_j are real-valued for $j=0,1,2$. Similarly, Proposition 2.5.7(2) implies that b_0 and b_2 are even functions of μ while b_1 is odd in μ. Note also that since the values $\Omega_{1,\mu}$ and $\Omega_{-1,\mu}$ collide at $\lambda=0$ when $\mu=0$, we see that $b_0=\mathcal{O}(\mu^2)$ for $|(a,\mu)|\ll 1$. It follows that

$$b_j(a,\mu)=d_j(a,\mu)\mu^{2-j},\quad j=0,1,2,$$

where the functions d_j are real-valued functions depending smoothly on a and μ for

$|(a, \mu)| \ll 1$. Setting $\lambda = i\mu X$ it follows that

$$\det(\mathcal{B}^{\lambda}_{a,\mu}) = \mu^2 \left(d_0(a,\mu) - d_1(a,\mu)X - d_2(a,\mu)X^2\right) =: \mu^2 Q(a,\mu,X).$$

The underlying wave is thus modulationally unstable if the polynomial Q admits roots with non-zero imaginary parts, while if it modulationally stable if Q admits two distinct real roots.

To determine the reality of the roots of Q, it is sufficient to study its discriminant $\mathcal{D}_{a,\mu}$, which can be directly expanded as

$$\mathcal{D}_{a,\mu} = \mu^2(\beta k^3(km''(k) + 2m'(k)) + 2\gamma)^2 + a^2\frac{k^4}{2}\left(\frac{2\gamma + \beta k^3(km''(k) + 2m'(k))}{3\gamma + 4\beta k^2(m(k) - m(2k))}\right) + \mathcal{O}\left(a^2\left(a^2 + \mu^2\right)\right).$$

It follows that the asymptotically small background periodic traveling waves $\eta(\cdot; a, k)$ are modulationally stable provided that $\mathcal{D}_{a,\mu} > 0$ for $0 < |\mu| \ll 1$ and modulationally unstable if $\mathcal{D}_{a,\mu} < 0$ for $0 < |\mu| \ll 1$. In particular, we note that for a fixed small $|a|$ we can choose $0 < \mu_0 \ll 1$ sufficiently small such that $\mathcal{D}_{a,\mu} < 0$ for all $0 < |\mu| \ll \mu_0$, indicating modulational instability of the background wave, provided that

$$\frac{2\gamma + \beta k^3(km''(k) + 2m'(k))}{3\gamma + 4\beta k^2(m(k) - m(2k))} < 0, \tag{2.5.13}$$

while one can similarly guarantee $\mathcal{D}_{a,\mu} > 0$ for all $0 < |\mu| \ll 1$ provided that the expression in (2.5.13) is strictly positive.

To complete the proof of Theorem 2.3.1, it remains simply to note by the calculations in Remark 2.4.3, along with the observation that

$$\frac{dc_g}{dk}(k) = \frac{2\gamma}{k^3} + \beta\left(2m'(k) + km''(k)\right),$$

that the quantity in (2.5.13) can be rewritten as

$$\frac{2\gamma + \beta k^3(km''(k) + 2m'(k))}{3\gamma + 4\beta k^2(m(k) - m(2k))} = \frac{4}{k^3\left(c_p(k) - c_p(2k)\right)}\frac{dc_g}{dk}(k).$$

It follows that the sign of the expression in (2.5.13) agrees precisely with that of the modulational instability index $\Delta(k)$ in Theorem 2.3.1, completing the proof.

2.6 Application to Specific Models

In this section, we apply the general result from Theorem 2.3.1 to a number of specific models. When possible, we compare our results to previously known results. Specifically, we first apply our results to the classical Ostrovsky equation (2.2.6) as well as the fractional Ostrovsky equation. We then consider the Whitham-Ostrovsky equation, where the associated one-dimensinoal equation encodes the full dispersion relation from the Euler equations for uni-directional surface water waves, as well as an Ostrovsky variant of the well-studied Intermediate Long-Wave equation. Our final examples go further to consider the effects of adding capillary effects into the the classical Ostrovsky and Whitham-Ostrovsky equations.

We note that for each of the examples considered in this section, the corresponding modulational stability of small amplitude periodic traveling waves in the non-rotational version of the equation given by (2.2.2) has been previously studied: see, for example, the works [33, 38, 39, 44]. As such, the results presented below essentially study the effect of rotation (as modeled by the Ostrovsky equation) on these previously studied models.

2.6.1 Classical Ostrovsky Equation

As a first application, we apply Theorem 2.3.1 to the classical Ostrovsky equation (2.2.6). Note that (2.2.6) corresponds to our generalized-dispersion model (2.2.4) with the choice

$$\mathcal{M} = 1 + \partial_x^2, \quad \text{i.e.} \quad m(k) = 1 - k^2. \tag{2.6.1}$$

The symbol $m(k)$ clearly satisfies Hypotheses 2.2.1 (H1), (H2) ($\alpha = 2$, $C_1 = 1$ and $C_2 = 2$), and (H3) (m is strictly decreasing for $k > 0$). Consequently, we obtain asymptotically small periodic traveling wave solutions from Theorem 2.4.1 along with asymptotic expansions provided explicitly by substituting $m(k) = 1 - k^2$ into (2.4.3).

In this case, the corresponding phase and group velocities are given explicitly by

$$c_p(k) = \beta\left(1 - k^2\right) + \frac{\gamma}{k^2} \quad \text{and} \quad c_g(k) = \beta\left(1 - 3k^2\right) - \frac{\gamma}{k^2}.$$

The qualitative properties clearly depend on the sign of β. For $\beta > 0$, the group velocity c_g attains a global maxima (for $k > 0$) at $k = k_c = \left(\frac{\gamma}{3|\beta|}\right)^{1/4}$ and is monotonically increasing for $k \in (0, k_c)$ and monotonically decreasing for $k > k_c$. Further, in this case the phase speed $c_p(k)$ is strictly decreasing for $k > 0$, and hence in this case one has

$$\Delta(k) > 0 \quad \text{for} \quad k \in (0, k_c) \quad \text{and} \quad \Delta(k) < 0 \quad \text{for} \quad k > k_c.$$

This establishes the modulational instability and stability result in Corollary 2.3.4 in the case $\beta > 0$. As noted in the Introduction, in the case $\beta = 1$ and $\gamma = 1$ our result agrees with that derived by Whitfield and Johnson in [74].

Similarly, when $\beta < 0$ the group velocity is strictly increasing for all $k > 0$, while $c_p(k) - c_p(2k)$ changes signs exactly once, from positive to negative, at $k_c = \left(\frac{\gamma}{4|\beta|}\right)^{1/4}$. This establishes Corollary 2.3.4 in the case $\beta < 0$. Notice, in particular, that the mechanisms accounting for the modulational instabilities in the case $\beta > 0$ are different from those in the $\beta < 0$ case.

2.6.2 Fractional Ostrovsky Equation

The Ostrovsky-fractional KdV equation

$$\left(u_t + \beta(1 - |\partial_x|^\delta)u_x + (u^2)_x\right)_x - \gamma u = 0, \;\; \gamma > 0, \;\; \beta \in \mathbb{R} \setminus \{0\}, \tag{2.6.2}$$

can be obtained from (2.2.4) by choosing $m(k) = 1 - |k|^\delta$, $\delta > 1/2$. In this case, the symbol $m(k)$ clearly satisfies Hypotheses 2.2.1 (H1), (H2) ($\delta = \alpha$, $C_1 = 1$, and $C_2 = 2$), and (H3) (m is strictly decreasing for $k > 0$). As above, we can obtain asymptotically small amplitude periodic travleing wave solutions of the Ostrovsky-fKdV equation from Theorem 2.4.1 by substituting $m(k) = 1 - |k|^\delta$. Applying precisely the same reasoning as in the previous section for the classical Ostrovsky equation, we obtain the following result.

Corollary 2.6.1. *For a fixed $\gamma > 0$, a $2\pi/k$-periodic traveling wave of the Ostrovsky-fKdV equation (2.6.2) with sufficiently small amplitude is modulationally unstable if $k > k_c$,* *where*

$$\begin{cases} k_c = \left(\dfrac{2\gamma}{\delta(1+\delta)|\beta|}\right)^{1/(2+\delta)} & \text{if } \beta > 0, \\ k_c = \left(\dfrac{3\gamma}{4(2^\delta - 1)|\beta|}\right)^{1/(2+\delta)} & \text{if } \beta < 0. \end{cases}$$

and it is modulationally stable otherwise.

Proof. The proof is same as the Ostrovsky equation in Section 2.6.1. □

2.6.3 Whitham-Ostrovsky Equation

Continuing as above, we may consider an Ostrovsky variant of the well-studied Whitham equation (2.2.1) by choosing

$$m(k) = \sqrt{\frac{\tanh k}{k}}. \tag{2.6.3}$$

The symbol $m(k)$ clearly satisfies Hypotheses 2.2.1 (H1), (H2) ($\alpha = -1/2$, $C_1 = 1$ and $C_2 = 2$), and (H3) (m is strictly decreasing for $k > 0$). As above, we can obtain asymptotically small amplitude periodic traveling wave solutions of the Ostrovsky-fKdV equation from Theorem 2.4.1. In this case, our Theorem 2.3.1 gives the following result.

Corollary 2.6.2. *For a fixed $\gamma > 0$, a $2\pi/k$-periodic traveling wave of the Whitham-Ostrovsky equation (2.2.6) with sufficiently small amplitude is modulationally unstable if $k > k_c$, where k_c is the unique real solution of following equations*

$$\begin{cases} k^3\left(k\dfrac{d^2}{dk^2}\left(\sqrt{\frac{\tanh k}{k}}\right) + 2\dfrac{d}{dk}\left(\sqrt{\frac{\tanh k}{k}}\right)\right) = -\dfrac{2\gamma}{|\beta|} & \text{if } \beta > 0, \\ \\ k^2\left(\sqrt{\frac{\tanh k}{k}} - \sqrt{\frac{\tanh 2k}{2k}}\right) = \dfrac{3\gamma}{4|\beta|} & \text{if } \beta < 0, \end{cases}$$

and it is modulationally stable otherwise.

Proof. The proof is same as the Ostrovsky equation in Section 2.6.1. □

We note that numerics indicate that the functions

$$k \mapsto -k^3\left(k\frac{d^2}{dk^2}\left(\sqrt{\frac{\tanh k}{k}}\right) + 2\frac{d}{dk}\left(\sqrt{\frac{\tanh k}{k}}\right)\right), \tag{2.6.4}$$

and

$$k \mapsto k^2 \left(\sqrt{\frac{\tanh k}{k}} - \sqrt{\frac{\tanh 2k}{2k}} \right), \tag{2.6.5}$$

are both equal to zero at $k = 0$ and are both monotonically increasing for $k > 0$, tending to infinity as $k \to \infty$: see Figure 2.2. Consequently, it is clear in each case $\beta > 0$ and $\beta < 0$ there is a unique $k = k_c(\beta) > 0$ where the conditions in Corollary 2.6.2 are satisfied. Further, they are both $\mathcal{O}(k^4)$ for $|k| \ll 1$ and hence, for the Whitham-Ostrovsky equation, the critical frequency satisfies $k_c \sim \gamma^{1/4}$ for $0 < \gamma \ll 1$. Note that for the one-dimensional Whitham equation for water waves, corresponding to (2.2.2) with the choice (2.6.3), the corresponding asymptotically small waves exhibit a modulational instability for $k > \widetilde{k}_c \approx 1.146$. Again, this demonstrates a singularity of the Whitham-Ostrovsky equation in the limit $\gamma \to 0^+$: see Remark 2.3.5 in the Introduction.

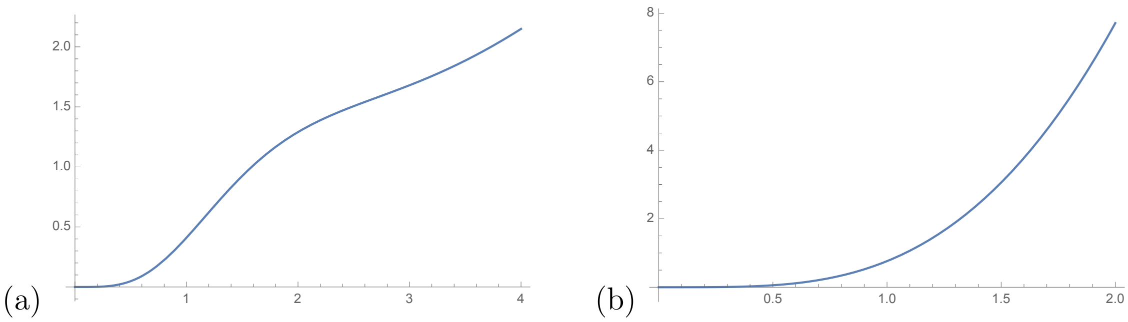


FIGURE 2.2: Plots of the functions (a) (2.6.4) and (b) (2.6.5) associated with the Whitham-Ostrovsky equation.

2.6.4 ILW-Ostrovsky Equation

The Intermediate Long Wave (ILW) equation is given by

$$u_t + \beta \mathcal{M} u_x + \left(u^2\right)_x = 0,$$

where here $\mathcal{M}$ is a Fourier multiplier with symbol $m(k) = k \coth(k)$. The ILW is well-known to describe long internal gravity waves in a two-layer stratified fluid, with the lower layer having a large finite depth. By adding rotational effects, we can obtain an ILW-Ostrovsky equation by making the choice $m(k) = k \coth(k)$ in (2.2.4). The symbol $m(k)$ clearly satisfies Hypotheses 2.2.1 (H1), (H2) ($\alpha = 2$, $C_1 = 1$ and $C_2 = 2$), and (H3) (m is strictly increasing for $k > 0$). As above, we can obtain asymptotically small amplitude periodic travleing wave solutions of the ILW-Ostrovsky equation from Theorem 2.4.1. Again applying the same reasoning as in Section 2.6.1, we obtain the following result.

Corollary 2.6.3. *For a fixed $\gamma > 0$, a $2\pi/k$-periodic traveling wave of the ILW-Ostrovsky equation* (2.2.6) *with sufficiently small amplitude is modulationally unstable if $k > k_c$, where k_c is the unique real solution of following equations*

$$\begin{cases} k^2(2k\coth 2k - k\coth k) - \dfrac{3\gamma}{4|\beta|} = 0 & \text{if } \beta > 0, \\ \\ k^3(-4k\operatorname{csch}^2 k + 2k^2\coth k\operatorname{csch}^2 k + 2\coth k) - \dfrac{2\gamma}{|\beta|} = 0 & \text{if } \beta < 0, \end{cases}$$

and it is modulationally stable otherwise.

Proof. The proof is same as the classical Ostrovsky equation in Section 2.6.1. Note that while we lack an explicit formula for the critical frequency k_c, it can of course be numerically approximated. □

We note that, similar to the above example, numerics indicate that the functions

$$k \mapsto k^2(2k\coth 2k - k\coth k), \tag{2.6.6}$$

and

$$k \mapsto k^3(-4k\operatorname{csch}^2 k + 2k^2\coth k\operatorname{csch}^2 k + 2\coth k), \tag{2.6.7}$$

both vanish at $k = 0$ and are monotonically increasing for $k > 0$, tending to infinity as $k \to \infty$: see Figure 2.3. As above, it follows that for each case $\beta > 0$ and $\beta < 0$ there is a unique $k = k_c(\beta) > 0$ where the conditions in Corollary 2.6.3 are satisfied. Further, both functions are $\mathcal{O}(k^4)$ for $|k| \ll 1$. As such, for the ILW-Ostrovsky equation, we again see the scaling relation $k_c \sim \gamma^{1/4}$ for $0 < \gamma \ll 1$ as observed for both the classical Ostrovsky and Whitham-Ostrovsky equations considered above.

2.6.5 Effects of Surface Tension on Modulational instability

We conclude our study by further adding capillary effects to the classical Ostrovsky and Whitham-Ostrovsky equations considered above. Note that it is known that incorporating such capillary effects into even the non-rotational models drastically affects the

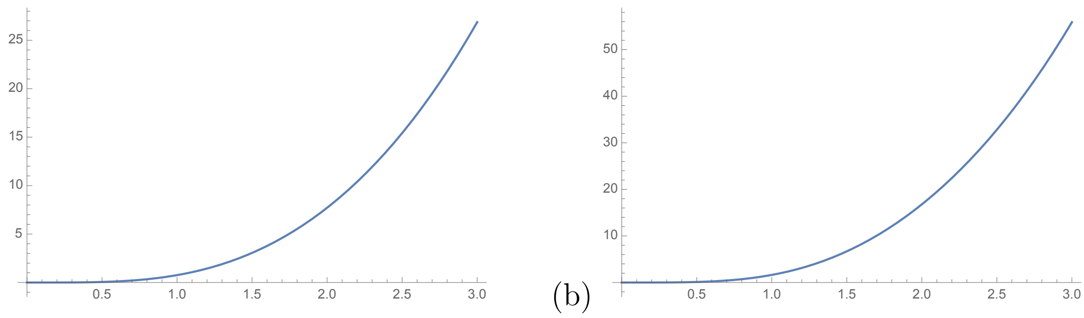
(a) (b)


Figure 2.3: Plots of the functions (a) (2.6.6) and (b) (2.6.7) associated with the ILW-Ostrovsky equation.

modulational instability of asymptotically small waves: see, for example, [39]. As such, we find it worthwhile to report on the effects adding surface tension to the rotational models considered in our work.

2.6.5.1 Classical Ostrovsky Equation with Surface Tension

Upon normalization, in the presence of surface tension in the classical Ostrovsky equation, the dispersive symbol in (2.2.5) will change to

$$m(k,T) = 1 - k^2 + 3Tk^2, \tag{2.6.8}$$

where here $T \geq 0$ is the constant coefficient of surface tension. Note that for $T = 0$, (2.6.8) reduces to (2.6.1), which was studied in Section 2.6.1 above.

Using Theorem 2.3.1, we have following results on the effects of surface tension on modulational stability in the Ostrovsky equation. As one might expect, the results depend on the sign of β as well as whether the parameter T satisfies $0 < T < 1/3$ or $T > 1/3$.

Corollary 2.6.4. *In the case $\beta > 0$, a sufficiently small $2\pi/k$-periodic traveling wave of the full-dispersion Ostrovsky equation* (2.2.2) *with dispersive symbol $m(k) = m(k;T)$ as in* (2.6.8) *is modulationally unstable if $k > k_c(T)$, where*

$$\begin{cases} k_c(T) = \left(\dfrac{\gamma}{3|\beta|(1-3T)}\right)^{1/4} & \text{if } T < 1/3, \\ k_c(T) = \left(\dfrac{\gamma}{4|\beta|(3T-1)}\right)^{1/4} & \text{if } T > 1/3, \end{cases}$$

and it is modulationally stable otherwise.

For the same equation, but in the case $\beta < 0$, a sufficiently small $2\pi/k$-periodic traveling wave is modulationally unstable if $k > k_c(T)$, where

$$\begin{cases} k_c(T) = \left(\dfrac{\gamma}{4|\beta|(1-3T)}\right)^{1/4} & \textit{if } T < 1/3, \\ k_c(T) = \left(\dfrac{\gamma}{3|\beta|(3T-1)}\right)^{1/4} & \textit{if } T > 1/3, \end{cases}$$

and it is modulationally stable otherwise.

For $T = 0$, that is, when capillary effects are absent, value of $k_c(0)$ of course agrees with k_c in Corollary 2.3.4. For $T > 0$, we describe the modulational instability through the Figure 2.4, where we took $\frac{\gamma}{\beta} = 1$ and $\frac{\gamma}{\beta} = -1$ for the computations[5]. In k-$k\sqrt{T}$ plane, two curves are corresponding to each mechanism splitting the plane into two regions of stability and two regions of instability. Any fixed $T > 0$ corresponds to a line passing through the origin of slope $\sqrt{T}$. For $T \neq 1/3$, the line through the origin only crosses one curve at a time producing one interval of stable wave numbers and one interval of unstable wave numbers. The result becomes inconclusive for $T = 1/3$.

2.6.5.2 Whitham-Ostrovsky Equation with Surface Tension

To incorporate the effects of surface tension into the Whitham-Ostrovsky equation, we replace the dispersive symbol $m(k)$ in (2.6.3) by

$$m(k,T) = \sqrt{\frac{\tanh k}{k}(1+Tk^2)}, \tag{2.6.9}$$

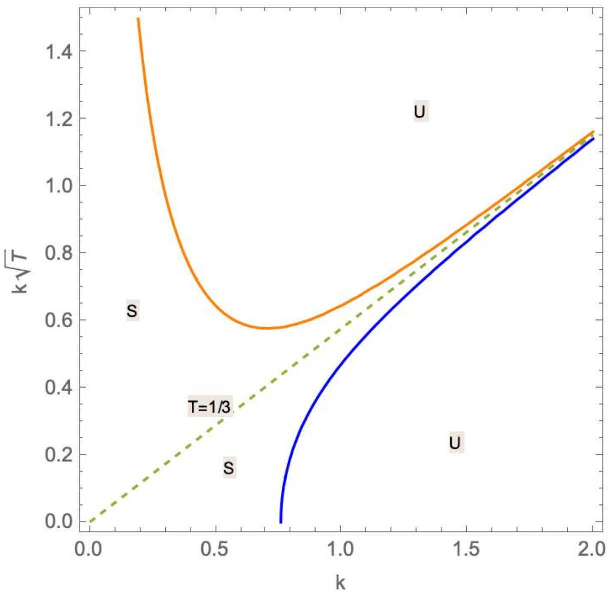
where $T \geq 0$ is the (properly normalized) coefficient of surface tension. Note that when $T = 0$, (2.6.9) of course reduces to (2.6.3), and hence we will only consider the case $T > 0$ here.

First, note that for each fixed $T > 0$ the symbol $m(\cdot, T)$ satisfies

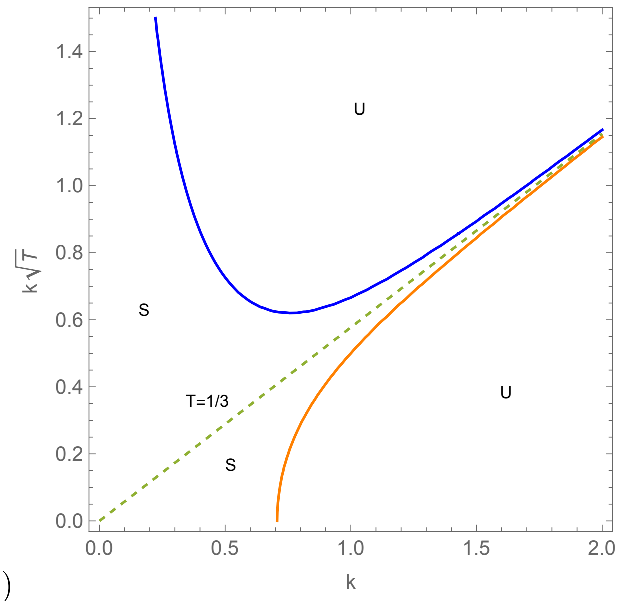
$$m(0,T) = 1 \quad \text{and} \quad \lim_{|k|\to\infty} m(k,T) = \infty.$$

Further, it is readily seen that $m(\cdot, T)$ is strictly monotone for $k > 0$ provided $T > 1/3$, while for $0 < T < 1/3$ there exists a unique $k^*(T)$ such that $m(\cdot, T)$ is monotone

[5]Note that while the results for the cases $\frac{\gamma}{\beta} = \pm 1$ are qualitatively similar, the exact stability boundary curves in each figure are slightly different.



(a)



(b)

FIGURE 2.4: Stability diagrams for sufficiently small, periodic wave trains of Ostrovsky equation with surface tension (see Corollary 2.6.4). Here, for the computations we took (a) $\frac{\gamma}{\beta} = 1$ and (b) $\frac{\gamma}{\beta} = -1$. In both figures, "S" and "U" denote stable and unstable regions. Solid curves represent roots of the stability condition (2.3.3). Specifically, the solid orange curve represents the coots of $c_p(k) - c_p(2k)$ while the solid blue curve represents the roots of $\frac{dc_g(k)}{dk}$. (Color available online.)

decreasing on $(0, k^*(T))$ and monotone increasing for $(k^*(T), \infty)$. In this way, the critical surface tension $T = 1/3$ is often used to differentiate between the "weak" $(0 < T < 1/3)$ and "strong" $(T > 1/3)$ surface tension cases.

According to Theorem 2.3.1, whenever factors $c_p(k) - c_p(2k)$ and $\frac{dc_g(k)}{dk}$ of the modulational instability index $\Delta(k)$ are zero, stability changes, where $c_p(k)$ and $c_g(k)$ are defined in (2.3.2). To show the explicit dependence on surface tension, we replace $c_p(k)$ and $c_g(k)$, by $c_p(k, T)$ and $c_g(k, T)$, respectively. Notice in particular that the roots (in the frequency k) of the factors $c_p(k, T) - c_p(2k, T)$ and $\frac{dc_g(k,T)}{dk}$ are precisely the same for each fixed value of the ratio

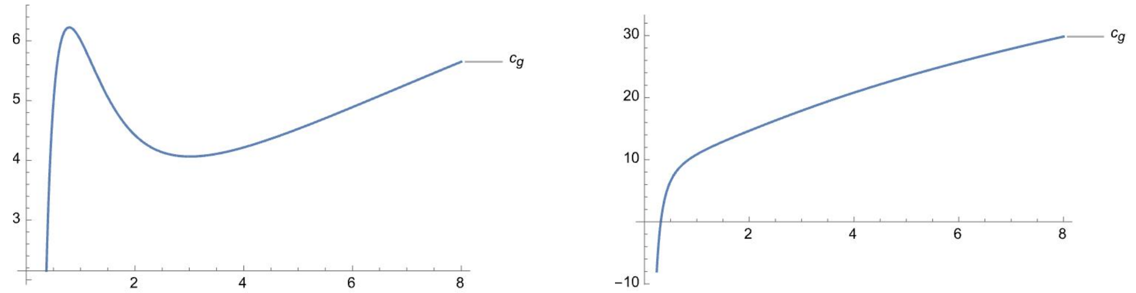
$$\sigma := \frac{\gamma}{\beta}.$$

Of course, since γ is always positive, the sign of σ agrees with that of β. Below, we will consider the cases $\sigma > 0$ and $\sigma < 0$ separately.

Case $\sigma > 0$:

It is straightforward to see that for each $\sigma > 0$, there exists a unique $T = T_c(\sigma)$

such that the group velocity c_g attains a local maxima and a local minima for $0 < T < T_c(\sigma)$ and is monotonic for $T > T_c(\sigma)$. For instance, for $\sigma = 0.1$, $T_c \approx 0.132$ and therefore, plots of $c_g(k, T)$ vs. k for $T = 0.02 < T_c$ and $T = 0.5 > T_c$ (see Figure 2.5) confirm the monotonocity property of c_g. Moreover, for any $T > 0$, $c_p(k, T) - c_p(2k, T)$ is changing its sign only once, see Figure 2.6, for instance. Therefore, for $0 < T <$



((a)) $T = 0.02$ ((b)) $T = 0.5$

FIGURE 2.5: Graph of function c_g vs. k for $\sigma = 0.1$ for which $T_c \approx 0.132$.

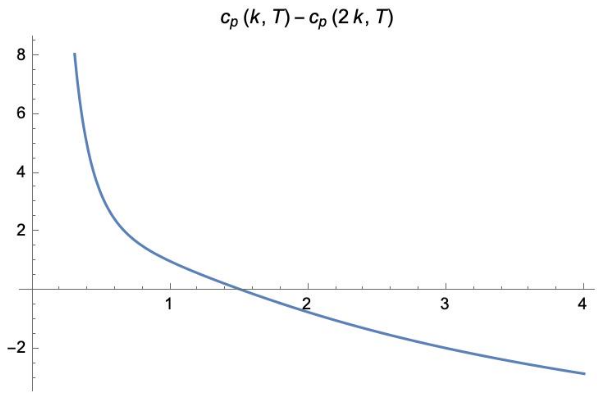


FIGURE 2.6: $c_p(k, T) - c_p(2k, T)$ vs k for $\sigma = 0.1$ and $T = 0.2$.

$T_c(\sigma)$, there exist three critical wavenumbers $0 < k_1 < k_2 < k_3$ such that sufficiently small $2\pi/k$-periodic traveling wave is modulationally stable for $k \in (0, k_1) \cup (k_2, k_3)$ and modulationally unstable for $k \in (k_1, k_2) \cup (k_3, \infty)$. On the other hand, for $T > T_c(\sigma)$ there is only one critical wave number $k_c > 0$ such that sufficiently small $2\pi/k$-periodic traveling wave is modulationally stable for $0 < k < k_c(T)$ and modulationally unstable for $k > k_c(T)$. The graph of T_c vs. σ is shown in Figure 2.10.

In [49] and [18], the effects of surface tension on modulational instability in the full water wave problem has been shown in $k - k\sqrt{T}$ plane. We produce a similar plot for $\sigma = 0.1$ in Figure 2.7. In k-$k\sqrt{T}$ plane, two curves are corresponding to each mechanism splitting the plane into three regions of stability and three regions of instability. For

$\sigma = 0.1$, this can be easily seen that $T_c \approx 0.132$ such that for $0 < T < 0.132$, the line crosses both the curves producing two intervals of stable wave numbers and two intervals of unstable wave numbers. On the other hand, for $T > 0.132$, the line through the origin crosses only one curve producing one interval of stable wave numbers and one interval of unstable wave numbers.

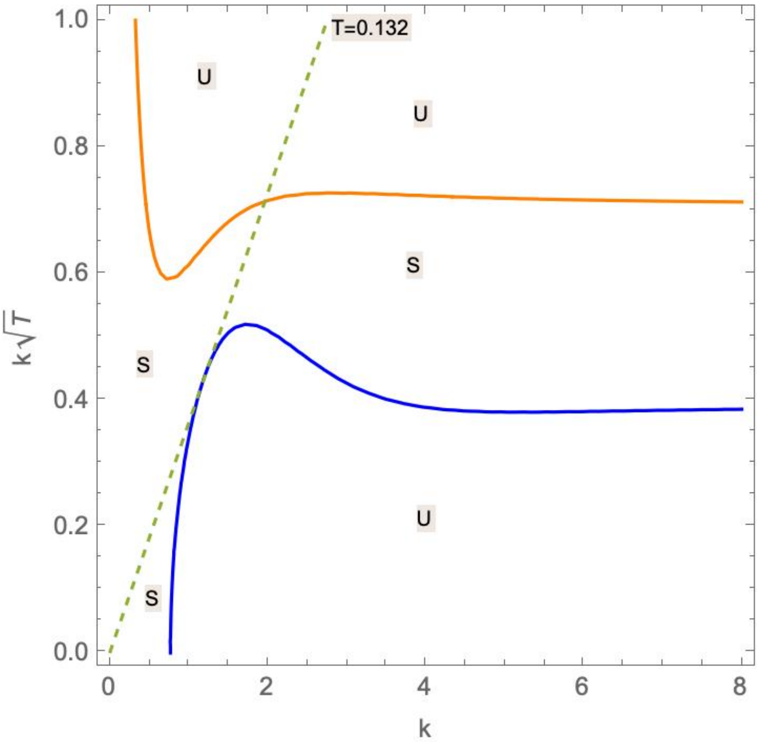
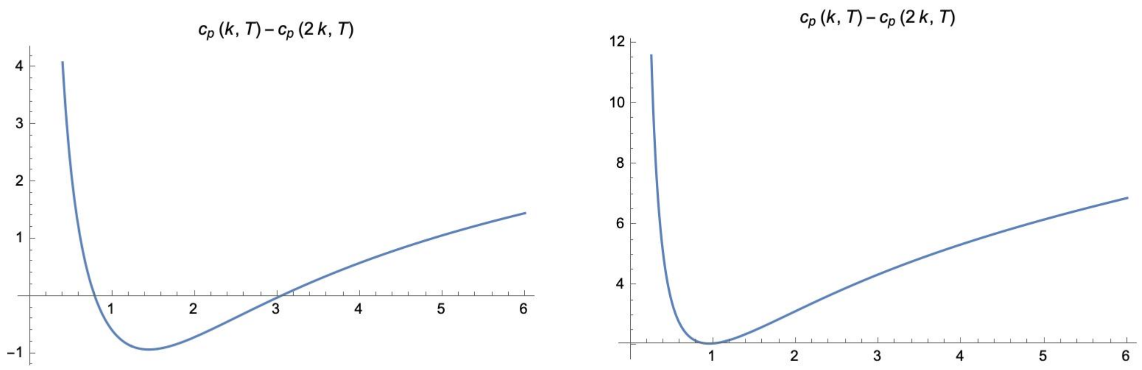


Figure 2.7: Stability diagram for sufficiently small, periodic wave trains Whitham-Ostrovsky equation for $\sigma = 0.1$. "S" and "U" denote stable and unstable regions. As in Figure 2.4, the solid orange curve represent roots of $c_p(k) - c_p(2k)$ and the solid blue curve represent roots of $\frac{dc_g}{dk}$. (Color available online.) Note in this case we have $T_c \approx 0.132$.

Case $\sigma < 0$:

For each $\sigma < 0$, there exists $T = T_c(\sigma)$ such that for $0 < T < T_c(\sigma)$, $c_p(k, T) - c_p(2k, T)$ changes its sign at two wavenumbers resulting in two critical wavenumbers, and for $T > T_c(\sigma)$, $c_p(k, T) - c_p(2k, T)$ does not change its sign (see Figure 2.8). On the other hand, for each $T > 0$, c_g attains extremum only once (see Figure 2.9) producing one critical wavenumber. Therefore, for $0 < T < T_c(\sigma)$, there exist three critical wavenumbers such that sufficiently small $2\pi/k$-periodic traveling wave is modulationally stable for $k \in (0, k_{c_1}) \cup (k_{c_2}, k_{c_3})$ and modulationally unstable for $k \in (k_{c_1}, k_{c_2}) \cup (k_{c_3}, \infty)$. For

$T > T_c(\sigma)$, there is only one critical wave number k_c such that sufficiently small $2\pi/k$-periodic traveling wave is modulationally stable for $0 < k < k_c(T)$ and modulationally unstable for $k > k_c(T)$. The graph of T_c vs. σ is shown in Figure 2.10.



((a)) $T = 0.02$ ((b)) $T = 0.5$

FIGURE 2.8: Graph of function $c_p(k,T) - c_p(2k,T)$ vs. k for $\sigma = -0.1$ for which $T_c = 0.141$.

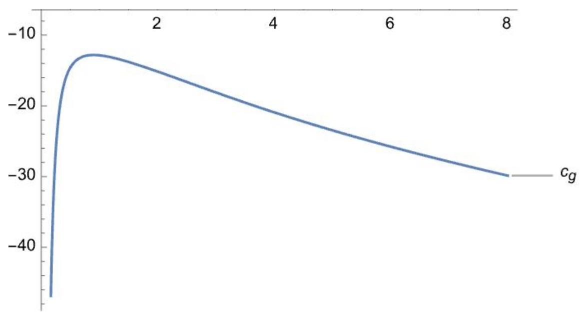


FIGURE 2.9: c_g vs k for $\sigma = -0.1$ and $T = 0.5$.

For $\sigma = -0.1$, we can see this behavior through Figure 2.11. In the k-$k\sqrt{T}$ plane, two curves correspond to each mechanism, splitting the plane into three regions of stability and three regions of instability. For $\sigma = -0.1$, $T_c \approx 0.141$ such that for $0 < T < 0.141$, the line crosses both the curves, producing two intervals of stable wave numbers and two intervals of unstable wave numbers. On the other hand, for $T > 0.141$, the line through the origin crosses only one curve, producing one interval of stable wave numbers and one interval of unstable wave numbers.

Remark 2.6.5. *For every $\sigma < 0$, there is a value of $0 < T = T_s < T_c(\sigma)$ corresponding to the intersection of two curves in the Figure 2.11 for which there is only one interval of stable and unstable wavenumbers contrary to other values of T in the interval $(0, T_c)$.*

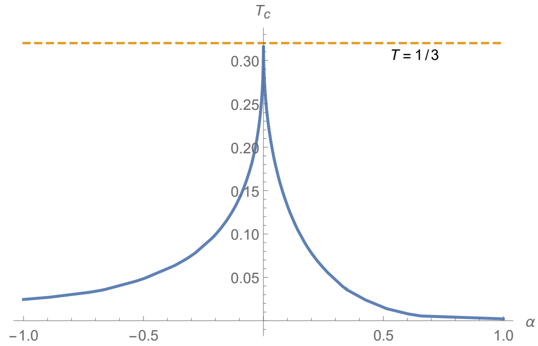


Figure 2.10: T_c vs σ.

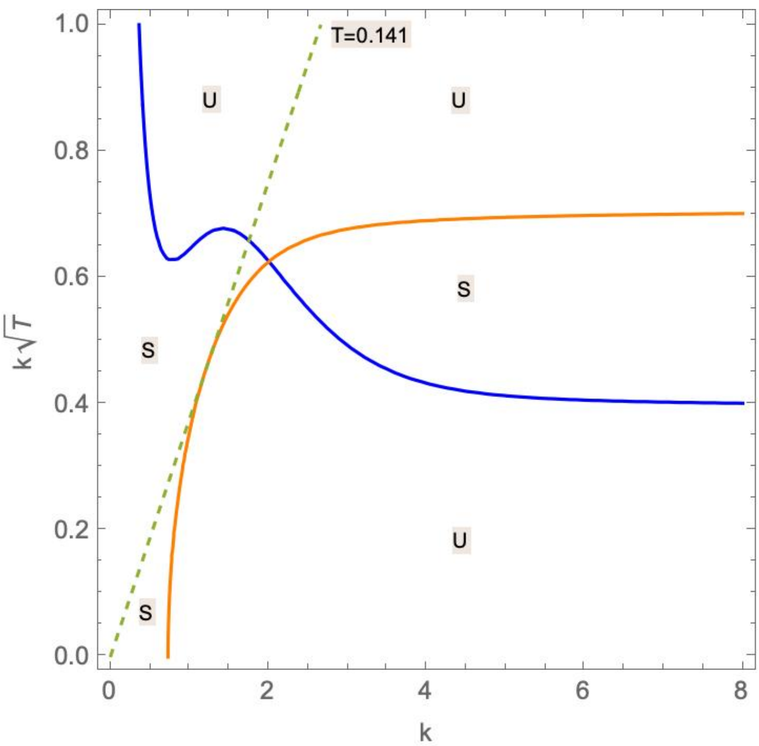


Figure 2.11: Stability diagram for sufficiently small, periodic wave trains of Whitham-Ostrovsky equation for $\sigma = -0.1$. "S" and "U" denote stable and unstable regions. Orange curve represent roots of $c_p(k,T) - c_p(2k,T) = 0$ and blue curve represent roots of $\dfrac{dc_g}{dk} = 0$.

3

Transverse Instability

3.1 Introduction

This chapter investigates the *transverse spectral instability* of periodic traveling waves in various dispersive water wave models. Transverse instability refers to the destabilization of one-dimensional wave trains due to perturbations that act in a direction orthogonal to the primary wave propagation. This mechanism is central in understanding multidimensional phenomena such as wave disintegration and the breakdown of planar structures in nonlinear dispersive systems.

In contrast to modulational instability, which involves longitudinal perturbations, transverse instability captures the essential two-dimensional character of many physical systems, especially in oceanography and plasma physics. Mathematically, the analysis involves linearizing about a one-dimensional periodic wave and studying the resulting spectral problem as a function of the transverse wavenumber.

This chapter brings together results from three works that examine transverse instability in increasingly complex and physically relevant models. In all three parts of this chapter, we examine the transverse spectral stability of one-dimensional periodic traveling wave solutions with respect to two-dimensional perturbations. These perturbations are always assumed to be periodic in the transverse (y) direction, and may be either periodic or localized in the longitudinal (x) direction. When the perturbation is periodic in (x), it is taken to be co-periodic with the underlying wave. In contrast, non-periodic perturbations in (x) are considered to be square-integrable over the real line. The transverse perturbations are further categorized by their wavelength: we distinguish between long-wavelength (modulational) and short- or finite-wavelength (high-frequency) perturbations, and we analyze the spectral behavior under each type.

- The first part focuses on the *generalized Kadomtsev–Petviashvili (gKP) equation*, extending classical results by incorporating general dispersion, including Whitham-, BO-, and ILW-type effects. This work is based on my paper with Atul Kumar and Ashish Kumar Pandey, published in the *Proceedings of the Royal Society A* (2022) [5].

- The second part addresses the *Konopelchenko–Dubrovsky (KD) equation*, a higher-order multidimensional integrable system. Here, we investigate both periodic and localized transverse perturbations, establishing sharp thresholds for instability. This section is based on my joint work with Ashish Kumar Pandey and Sudhir Singh, published in *Studies in Applied Mathematics* (2023) [6].

- The final and most extensive part examines the *rotation-modified Kadomtsev–Petviashvili (RMKP) equation* and several related models. This study introduces rotation into the KP framework, yielding rich instability structures depending on the physical parameters. This material is drawn from my recent work titled *"Transverse Spectral Instabilities in Rotation-Modified Kadomtsev–Petviashvili Equation and Related Models"* (with Ashish Kumar Pandey and Anastassiya Semenova), submitted for publication.

Across these models, we rigorously establish conditions under which transverse instabilities arise for both periodic and localized perturbations. Our methods involve Floquet theory, spectral perturbation analysis tailored to different classes of dispersion.

Collectively, these studies provide a comprehensive spectral analysis of transverse instabilities across a wide class of nonlinear dispersive equations. The results not only generalize earlier formal and numerical findings but also introduce new examples of instability in previously unexplored models, offering deeper insight into the role of dimension and dispersion in nonlinear wave dynamics.

We begin our study of transverse instability with the generalized Kadomtsev–Petviashvili (gKP) equation, which captures a broad class of two-dimensional extensions of one-dimensional dispersive models.

3.2 Transverse spectral instability in generalized Kadomtsev-Petviashvili equation

We propose the generalized Kadomtsev-Petviashvili (gKP) equation

$$(u_t + \mathcal{M}u_x - uu_x)_x + \sigma u_{yy} = 0, \tag{3.2.1}$$

in which $u(x, y, t)$ depends upon the spatial variables $x, y \in \mathbb{R}$, and the temporal variable $t \in \mathbb{R}$, $\mathcal{M}$ is a multiplier operator given by the symbol $m(k)$ as

$$\widehat{\mathcal{M}f}(k) = m(k)\widehat{f}(k), \tag{3.2.2}$$

and σ is equal to either 1 or -1. The multiplier symbol $m(k)$ in (3.2.2) should satisfy the Hypothesis 2.2.1 and m is strictly monotonic for $k > 0$.

The gKP equation (3.2.1) is a generalization to the Kadomtsev-Petviashvili (KP) equation [48]

$$(u_t - u_{xxx} - uu_x)_x + \sigma u_{yy} = 0, \tag{3.2.3}$$

where $\mathcal{M} = -\partial_x^2$. The KP equation is a natural extension to two spatial dimensions of the well-known Korteweg-de Vries (KdV) equation

$$u_t - u_{xxx} - uu_x = 0. \tag{3.2.4}$$

The KP equation (3.2.3) with $\sigma = -1$ (negative dispersion) is called the KP-II equation, whereas the one with $\sigma = 1$ (positive dispersion) is called the KP-I equation. Along the same lines, the gKP equation can be thought of as an extension to two spatial dimensions of the equation

$$u_t + \mathcal{M}u_x - uu_x = 0. \tag{3.2.5}$$

The gKP equation in the form (3.2.1) first appears in [69], where the existence and properties of its localized solitary waves were studied. The function $m(k)$ determines the dispersive properties of the model and has already been discussed in detail in Chapter 2, where various choices corresponding to physically relevant equations were introduced. Here, we adopt the same naming convention: depending on whether $m(k)$ corresponds to the fractional KdV, Benjamin–Ono, ILW, or Whitham-type dispersion, we refer to the resulting two-dimensional equation as KP-fKdV, KP-BO, KP-ILW, or KP-Whitham, respectively. We append '-I' or '-II' to indicate whether the transverse parameter σ is set to -1 (KP-I) or $+1$ (KP-II).

Dispersion relation

Assuming a plane-wave solution of the form

$$u(x,t) = e^{i(kx-\omega t+\ell y)},$$

for the linear part

$$(u_t + \mathcal{M}u_x)_x + \sigma u_{yy} = 0,$$

of the gKP equation (3.2.1), we arrive at the dispersion relation given by the phase velocity

$$v_p(k) = \frac{\omega}{k} = m(k) + \sigma\frac{\ell^2}{k^2}. \tag{3.2.6}$$

From (3.2.6), we observe that phase velocity v_p is monotonic if $\sigma = 1$ and m is decreasing or $\sigma = -1$ and m is increasing, while in other two combinations it changes its behavior along local extremum.

Small amplitude periodic traveling waves

Seeking y-independent traveling wave solution of (3.2.1) of the form $u(x, y, t) = U(x - ct)$, where $c \in \mathbb{R}$ is the speed of propagation, then U satisfies the following

$$(-cU' + \mathcal{M}U' - UU')' = 0.$$

Here, the prime denotes differentiation with respect to the traveling-wave variable $\xi = x - ct$, i.e., $U' = \frac{dU}{d\xi}$.

Integrating this, we get

$$\mathcal{M}U = cU + \frac{U^2}{2} + b + dx,$$

in which b and d are arbitrary constants. Since we are interested in periodic solutions, we can set $d = 0$ and the equation becomes

$$\mathcal{M}U = cU + \frac{U^2}{2} + b. \tag{3.2.7}$$

Let U be a $2\pi/k$-periodic function in x. Then, $\eta(z) := U(x)$ with $z := kx$ is a 2π-periodic function in z. For each $k > 0$, a family of small amplitude 2π-periodic and smooth solutions $\eta(k, a, b)(z)$ exists at $c = c(k, a, b)$, see [38, Proposition 2.2] for more details. Moreover,

$$\begin{cases} \eta(k,a,b)(z) =(1-m(k))b + a\cos z + a^2(A_0 + A_2\cos 2z) + a^3 A_3 \cos 3z + O(a^4 + ab + b^2), \\ \quad c(k,a,b) =m(k) - (1-m(k))b + a^2 c_2 + O(a^4 + ab + b^2), \end{cases} \tag{3.2.8}$$

where

$$A_0 = \frac{1}{4(1-m(k))}, \quad A_2 = \frac{1}{4(m(2k)-m(k))}, \quad A_3 = \frac{A_2}{2(m(3k)-m(k))} \text{ and } c_2 = -A_0 - \frac{A_2}{2}. \tag{3.2.9}$$

Hypotheses 2.2.1 H1 and H2 are used to prove the existence of $\eta(k, a, b)(z)$ and $c(k, a, b)$, see [38, Proposition 2.2] for the proof. Starting now, we denote $\eta(k, a, b)$ and $c(k, a, b)$ as η and c respectively.

Background

Clearly, a solution of (3.2.5) is a y-independent solution of the gKP equation (3.2.1). The stability (or instability) of such a one-dimensional solution of (3.2.1) with respect to perturbations which are two-dimensional is generally termed as *transverse stability (or instability)*.[1] The transverse instability of solitary waves of the KdV in the KP equation was first conducted by Kadomtsev, and Petviashvili [48], where it was found that such solutions are stable to transverse perturbations in the case of negative dispersion ($\sigma = -1$), while they are unstable to long-wavelength transverse perturbations in the case of positive dispersion ($\sigma = 1$) even though they are stable in the corresponding one-dimensional problem. The transverse stability of cnoidal wave solutions of KdV in the KP equation has been studied in [70] where authors obtain some instability results for KP-I equation and prove transverse stability for KP-II equation. Johnson and Zumbrun [47] have studied transverse instability of periodic waves for the KP-gKdV equation, with respect to periodic perturbations in the direction of propagation and of long wavelength in the transverse direction. They have constructed an orientation index by comparing the low and high-frequency behavior of the periodic Evans functions. Mariana Haragus [32] has also studied the transverse stability of KP-KdV equations, but the author restricted to the case of small periodic waves and considered transverse stability for more general perturbations for the KP-KdV equation. Recently in [35], authors have proved transverse spectral stability of one-dimensional periodic traveling waves of KP-II equation with respect to two-dimensional perturbations which are bounded in the direction of propagation of wave. Transverse instability of periodic waves of KP-I and Schrödinger equations have been studied in [30]. Transverse instability of solitary wave solutions of various water-wave models have also been explored by several authors, see [29, 63, 65, 66].

Notations

In this section, the value of σ and monotonic nature of the symbol m in (3.2.2) appear numerous times, and we use the shorthand notation in Table 3.1 for them.

[1]The definition of transverse stability can be different in different articles depending on what is the nature of the underlying stability analysis, for example, orbital or spectral stability.

		m Increasing	m Decreasing
σ	1	$(1, \uparrow)$	$(1, \downarrow)$
	-1	$(-1, \uparrow)$	$(-1, \downarrow)$

Table 3.1: Notations for different values of σ and monotonic nature of m.

3.2.1 Main Results

Our main results are following theorems depicting the transverse stability and instability of small amplitude periodic traveling waves (3.2.8) of (3.2.1) depending upon the nature of the two-dimensional perturbation in x- and y-directions.

Theorem 3.2.1 (Transverse stability). *Assume that small amplitude periodic traveling waves (3.2.8) of (3.2.1) are spectrally stable in $L^2(\mathbb{T})$ as a solution of the corresponding y-independent one-dimensional equation. Then, for any a sufficiently small, $k > 0$, and m satisfying Hypotheses 2.2.1, periodic traveling waves (3.2.8) of (3.2.1) are transversely stable with respect to two-dimensional perturbations which are periodic in the direction of propagation of the wave and of*

1. *finite and short wavelength in the transverse direction if $\sigma = 1$ with monotonically increasing $m(k)$ and $\sigma = -1$ with monotonically decreasing $m(k)$.*

2. *long wavelength in the transverse direction if $\sigma = 1$ with monotonically decreasing $m(k)$ and $\sigma = -1$ with monotonically increasing $m(k)$.*

Theorem 3.2.2 (Transverse instability). *For any a sufficiently small, $k > 0$, and m satisfying Hypotheses 2.2.1, periodic traveling waves (3.2.8) of (3.2.1) are transversely unstable with respect to two-dimensional perturbations which are*

1. *periodic in the direction of propagation of the wave and of long wavelength in the transverse direction if $\sigma = 1$ with monotonically increasing $m(k)$ and $\sigma = -1$ with monotonically decreasing $m(k)$.*

2. *non-periodic (localized or bounded) in the direction of propagation of the wave and of finite wavelength in the transverse direction if $\sigma = 1$ with monotonically increasing $m(k)$ and $\sigma = -1$ with monotonically decreasing $m(k)$.*

Consequently, by applying these theorems, we obtain transverse stability and instability results for KP-fKdV-I, KP-fKdV-II, KP-ILW-I, KP-ILW-II, KP-Whitham-I, and

KP-Whitham-II equations conditioned on the spectral stability of periodic traveling waves with respect to one-dimensional perturbations.

Corollary 3.2.3 (Transverse stability vs. instability of KP-fKdV). *For any a sufficiently small and $k > 0$,*

1. (a) *periodic traveling waves (3.2.8) of the KP-fKdV-I equation are transversely stable with respect to two-dimensional perturbations, which are periodic in the direction of propagation of the wave and of finite and short wavelength in the transverse direction.*

   (b) *periodic traveling waves (3.2.8) of the KP-fKdV-II equation are transversely stable with respect to two-dimensional perturbations, which are periodic in the direction of propagation of the wave and of long wavelength in the transverse direction.*

2. *periodic traveling waves (3.2.8) of KP-fKdV-I equation are transversely unstable with respect to two-dimensional perturbations, which are*

   (a) *periodic in the direction of propagation of the wave and of long-wavelength in the transverse direction, and*

   (b) *non-periodic in the direction of propagation of the wave and of finite wavelength in the transverse direction.*

Corollary 3.2.4 (Transverse stability vs. instability of KP-ILW). *For any a sufficiently small and $k > 0$,*

1. (a) *periodic traveling waves (3.2.8) of the KP-ILW-I equation are transversely stable with respect to two-dimensional perturbations, which are periodic in the direction of propagation of the wave and of finite and short wavelength in the transverse direction.*

   (b) *periodic traveling waves (3.2.8) of the KP-ILW-II equation are transversely stable with respect to two-dimensional perturbations, which are periodic in the direction of propagation of the wave and of long wavelength in the transverse direction.*

2. *periodic traveling waves (3.2.8) of KP-ILW-I equation are transversely unstable with respect to two-dimensional perturbations, which are*

   (a) *periodic in the direction of propagation of the wave and of long-wavelength in the transverse direction, and*

   (b) *non-periodic in the direction of propagation of the wave and of finite wavelength in the transverse direction.*

Corollary 3.2.5 (Transverse stability vs. instability of KP-Whitham)**.** *For any a sufficiently small and $k > 0$,*

1. (a) *periodic traveling waves (3.2.8) of the KP-Whitham-II equation are transversely stable with respect to two-dimensional perturbations, which are periodic in the direction of propagation of the wave and of finite and short wavelength in the transverse direction.*

   (b) *periodic traveling waves (3.2.8) of the KP-Whitham-I equation are transversely stable with respect to two-dimensional perturbations, which are periodic in the direction of propagation of the wave and of long wavelength in the transverse direction.*

2. *periodic traveling waves (3.2.8) of KP-Whitham-II equation are transversely unstable with respect to two-dimensional perturbations, which are*

   (a) *periodic in the direction of propagation of the wave and of long-wavelength in the transverse direction, and*

   (b) *non-periodic in the direction of propagation of the wave and of finite wavelength in the transverse direction.*

The remainder of this section is devoted to the detailed proofs of the above theorems.

3.2.2 Linearization

Linearizing the gKP equation (3.2.1) about its one-dimensional periodic traveling wave η in (3.2.8) and using change of variables, abusing notation, $x \to kz$, $t \to kt$, and $y \to ky$, we arrive at

$$v_{tz} - cv_{zz} + \mathcal{M}_k v_{zz} - (\eta v)_{zz} + \sigma v_{yy} = 0. \tag{3.2.10}$$

For $v(z,y,t) = e^{\lambda t + i\ell y} V(z)$, we obtain

$$\lambda V_z - cV_{zz} + \mathcal{M}_k V_{zz} - (\eta V)_{zz} - \sigma \ell^2 V = 0,$$

which can be rewritten as

$$\mathcal{T}_a(\lambda,\ell)V := (\lambda\partial_z - \partial_z^2(c - \mathcal{M}_k + \eta) - \sigma\ell^2)V = 0. \tag{3.2.11}$$

We assume that $2\pi/k$-periodic traveling wave solution $u(x,y,t) = \eta(k(x-ct))$ of (3.2.1) is a stable solution of the one-dimensional equation (3.2.5) where η and c are as in (3.2.8). We then say that the periodic wave η in (3.2.8) is transversely spectrally stable with respect to two-dimensional periodic perturbations (resp. non-periodic (localized or bounded perturbations)) if the gKP operator $\mathcal{T}_a(\lambda,\ell)$ acting in $L^2(\mathbb{T})$ (resp. $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$) with domain $H^{\alpha+2}(\mathbb{T})$ (resp. $H^{\alpha+2}(\mathbb{R})$ or $C_b^{\alpha+2}(\mathbb{R})$), where α is in Hypothesis 2.2.1 H2, is invertible, for any $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$ and any $\ell \neq 0$.

Depending on the space in which we are studying the invertibility of $\mathcal{T}_a(\lambda,\ell)$, we split our study into periodic ($L^2(\mathbb{T})$) and non-periodic perturbations ($L^2(\mathbb{R})$ or $C_b(\mathbb{R})$). Also, depending upon the values of ℓ we distinguish two different regimes: long-wavelength transverse perturbations, when $|\ell| \ll 1$ and short or finite wavelength transverse perturbations, otherwise.

3.2.3 Periodic Perturbations

In this section, we study transverse stability with respect to two-dimensional perturbations, which are co-periodic in the direction of the propagation of the wave. Therefore, we check if the operator $\mathcal{T}_a(\lambda,\ell)$ acting in $L^2(\mathbb{T})$ is invertible, for any $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$ and any $\ell \neq 0$. We reformulate the invertibility problem for this particular case.

Proposition 3.2.6. *The following statements are equivalent:*

1. *$\mathcal{T}_a(\lambda,\ell)$ acting in $L^2(\mathbb{T})$ with domain $H^{\alpha+2}(\mathbb{T})$ is not invertible.*

2. *The restriction of $\mathcal{T}_a(\lambda,\ell)$ to the subspace $L_0^2(\mathbb{T})$ of $L^2(\mathbb{T})$ is not invertible, where*

$$L_0^2(\mathbb{T}) = \left\{ f \in L^2(\mathbb{T}) \ : \ \int_0^{2\pi} f(z)\, dz = 0 \right\}.$$

3. *λ belongs to the spectrum of the operator $\mathcal{A}_a(\ell)$ acting in $L_0^2(\mathbb{T})$ with domain $H^{\alpha+1}(\mathbb{T})\cap L_0^2(\mathbb{T})$, where*

$$\mathcal{A}_a(\ell) = \partial_z(c - \mathcal{M}_k + \eta) + \sigma\ell^2\partial_z^{-1}.$$

We refer to [32, Lemma 4.1, Corollary 4.2] for a detailed proof in a similar situation. Proposition 3.2.6 reduces the invertibility problem of $\mathcal{T}_a(\lambda, \ell)$ to the study of the spectrum of $\mathcal{A}_a(\ell)$ acting on $L_0^2(\mathbb{T})$ with domain $H^{\alpha+1}(\mathbb{T}) \cap L_0^2(\mathbb{T})$. The operator $\mathcal{A}_a(\ell)$ acting on $L_0^2(\mathbb{T})$ has a compact resolvent so that its spectrum consists of isolated eigenvalues with finite multiplicity. In addition, the spectrum of $\mathcal{A}_a(\ell)$ is symmetric with respect to both the real and imaginary axes.

A straightforward calculation reveals that

$$\mathcal{A}_0(\ell)e^{inz} = i\omega_{n,\ell}e^{inz} \quad \text{for all} \quad n \in \mathbb{Z} \setminus \{0\}, \tag{3.2.12}$$

where

$$\omega_{n,\ell} = n(m(k) - m(kn)) - \frac{\sigma\ell^2}{n}. \tag{3.2.13}$$

Consequently, $L_0^2(\mathbb{T})$-spectrum of $\mathcal{A}_0(\ell)$ consists of purely imaginary eigenvalues of finite multiplicity. Since

$$\|\mathcal{A}_a(\ell) - \mathcal{A}_0(\ell)\| = O(|a|),$$

as $a \to 0$ uniformly in the operator norm. A standard perturbation argument then guarantees the spectrum of $\mathcal{A}_a(\ell)$ and $\mathcal{A}_0(\ell)$ will stay close for $|a|$ small. Recalling that the spectrum of $\mathcal{A}_a(\ell)$ is symmetric with respect to the imaginary axis, it follows then that for $|a|$ small when eigenvalues of $\mathcal{A}_a(\ell)$ bifurcate from the imaginary axis they must bifurcate in pairs resulting from collisions of eigenvalues of $\mathcal{A}_0(\ell)$ on the imaginary axis. For $p \neq q \in \mathbb{Z} \setminus \{0\}$, the two eigenvalues $i\omega_{p,\ell}$ and $i\omega_{q,\ell}$ collide for some $\ell = \ell_c$ when

$$\omega_{p,\ell_c} = \omega_{q,\ell_c}. \tag{3.2.14}$$

The linear operator $\mathcal{A}_a(\ell)$ can be decomposed as

$$\mathcal{A}_a(\ell) = J_a(\ell),$$

where

$$J = \partial_z \quad \text{and} \quad \mathcal{L}_a(\ell) = c - \mathcal{M}_k + \eta + \sigma\ell^2\partial_z^{-2}.$$

The operator J is skew-adjoint whereas the operator $\mathcal{L}_a(\ell)$ is self-adjoint. The *Krein signature* κ_n of an eigenvalue $i\omega_{n,\ell}$ of $\mathcal{A}_0(\ell)$ is defined as

$$\kappa_n = \text{sgn}(\langle_0(\ell)e^{inz}, e^{inz}\rangle) = \text{sgn}\left(m(k) - m(kn) - \frac{\sigma\ell^2}{n^2}\right), \quad n \in \mathbb{Z}\setminus\{0\}, \qquad (3.2.15)$$

where sgn is the signum function which determines the sign of a real number. A pair of eigenvalues leave imaginary axis after collision only if their Krein signatures κ_n are opposite. We have the following lemma.

Lemma 3.2.7. *For any $|a|$ sufficiently small, there exists a $\ell_a > 0$ such that for all $|\ell| > \ell_a$, the spectrum of $\mathcal{A}_a(\ell)$ is purely imaginary if (σ, m) is $(1, \uparrow)$ or $(-1, \downarrow)$, where these notations are explained in Table 3.1.*

Proof. For $(\sigma, m) = (1, \uparrow)$, κ_n is negative for all $n \in \mathbb{Z}\setminus\{0\}$ and for $(\sigma, m) = (-1, \downarrow)$, κ_n is positive for all $n \in \mathbb{Z}\setminus\{0\}$ for all $k > 0$. Therefore, Krein signatures of all eigenvalues remain same in both cases implying that eigenvalues will not bifurcate from the imaginary axis even if there is a collision away from the origin for $|a|$ sufficiently small. The collision at the origin may possibly lead to bifurcation away from the imaginary axis for sufficiently small ℓ (in fact, this is actually the case, see Lemma 3.2.10). Therefore, there exists an ℓ_a depending on a such that for all $|\ell| > \ell_a$, the spectrum of $\mathcal{A}_a(\ell)$ is purely imaginary. □

It follows from Lemma 3.2.7 that the only collision that may lead to instability for $(\sigma, m) = (1, \uparrow)$ or $(-1, \downarrow)$ is the collision at the origin between $\omega_{1,0}$ and $\omega_{-1,0}$. Since this collision takes place at $\ell = 0$, the perturbation analysis will take place in the regime $|\ell| \ll 1$. In other words, the underlying transverse perturbations are of long wavelength. The other regime is of finite and short-wavelength perturbations. We split our further analysis into these two regimes.

3.2.3.1 Finite and short-wavelength transverse perturbations

We start the analysis of the spectrum of $\mathcal{A}_a(\ell)$ with the values of ℓ away from the origin, $|\ell| \geq \ell_0$, for some $\ell_0 > 0$, i.e., finite and short wavelength transverse perturbations. Using Lemma 3.2.7, there are no collisions of eigenvalues that may lead to instability for

$|\ell| \geq \ell_0 > 0$ if $(\sigma, m) = (1, \uparrow)$ or $(-1, \downarrow)$. Therefore, we restrict our attention to other two cases, $(\sigma, m) = (1, \downarrow)$ or $(-1, \uparrow)$.

Let eigenvalues $i\omega_{p,\ell}$ and $i\omega_{q,\ell}$, $p \neq q$, collide at $\ell = \ell_c > 0$. From (3.2.15), Krein signatures κ_p and κ_q are opposite at $\ell = \ell_c$ when $pq < 0$, i.e. p and q should be of opposite parity. A direct calculation shows that if $(\sigma, m) = (1, \downarrow)$ or $(-1, \uparrow)$ then $i\omega_{p,\ell}$ and $i\omega_{-q,\ell}$ collide when

$$\ell^2 = \ell_{p,q}^2 = \frac{\sigma pq}{p+q}(p(m(k) - m(kp)) + q(m(k) - m(kq))) > 0.$$

for all $p, q \in \mathbb{N}$ except $(p, -q) = (1, -1)$.

Let Δ denotes the distance between indices p and $-q$ of colliding eigenvalues. For $\Delta = 1$ and 2 there are no pairs of eigenvalues which can lead to instability. For $\Delta = 3$, there are two such pairs of colliding eigenvalues, $\{\omega_{-1,\ell}, \omega_{2,\ell}\}$ and $\{\omega_{1,\ell}, \omega_{-2,\ell}\}$ which can lead to instability. In what follows, we shall do instability analysis for $\Delta n = 3$ and check whether the pair of potentially unstable eigenvalues indeed lead to instability or not. Let for some $n \in \mathbb{Z}$, we have

$$0 \neq \omega_{n,\ell_c} = \omega_{n+3,\ell_c} = \omega \ (say), \tag{3.2.16}$$

for some $\ell^2 = \ell_c^2 > 0$. Therefore, $i\omega$ is an eigenvalue of $\mathcal{A}_0(\ell_c)$ of multiplicity two with an orthonormal basis of eigenfunctions $\{e^{inz}, e^{i(n+3)z}\}$. For $|\ell - \ell_c|$ and $|a|$ sufficiently small, let $\lambda_{n,a,\ell}$ and $\lambda_{n+3,a,\ell}$ be eigenvalues of $\mathcal{A}_a(\ell)$ bifurcating from $i\omega$ with an orthonormal basis of eigenfunctions $\{\phi_{n,a,\ell}(z), \phi_{n+3,a,\ell}(z)\}$. Note that $\lambda_{n,0,\ell_c} = \lambda_{n+3,0,\ell_c} = i\omega$ with $\phi_{n,0,\ell_c}(z) = e^{inz}$ and $\phi_{n+3,0,\ell_c}(z) = e^{i(n+3)z}$. Let

$$\lambda_{n,a,\ell} = i\omega + i\delta_{n,a,\ell} \qquad \textit{and} \qquad \lambda_{n+3,a,\ell} = i\omega + i\delta_{n+3,a,\ell}. \tag{3.2.17}$$

We are interested in the location of $\delta_{n,a,\ell}$ and $\delta_{n+3,a,\ell}$ for $|\ell - \ell_c|$ and $|a|$ sufficiently small.

We start with the following expansions of eigenfunctions[14]

$$\phi_{a,n,\ell}(z) = e^{inz} + a\phi_{n,1} + a^2\phi_{n,2} + a^3\phi_{n,3} + O(a^4), \tag{3.2.18}$$

$$\phi_{a,n+3,\ell}(z) = e^{i(n+3)z} + a\phi_{n+3,1} + a^2\phi_{n+3,2} + a^3\phi_{n+3,3} + O(a^4). \tag{3.2.19}$$

We use orthonormality of $\phi_{a,n,\ell}(z)$ and $\phi_{a,n+3,\ell}(z)$ to find that

$$\phi_{n,1} = \phi_{n,2} = \phi_{n,3} = \phi_{n+3,1} = \phi_{n+3,2} = \phi_{n+3,3} = 0.$$

Using expansions of η and c in (3.2.8), we expand $\mathcal{A}_a(\ell)$ in a as

$$\mathcal{A}_a(\ell) = \mathcal{A}_0(\ell) + a\partial_z(\cos z) + a^2\partial_z\left(-\frac{A_2}{2} + A_2\cos 2z\right) + a^3\partial_z(A_3\cos 3z) + O(a^4). \tag{3.2.20}$$

Now, to trace the bifurcation of the eigenvalues from the point of the collision on the imaginary axis for $|\ell - \ell_c|$ and $|a|$ sufficiently small, we compute the action of $\mathcal{A}_a(\ell)$ and identity operators on the extended eigenspace $\{\phi_{a,n,\ell}(z), \phi_{a,n+3,\ell}(z)\}$ and arrive at

$$\mathcal{B}_a(\ell) = \begin{pmatrix} i\omega - ia^2\dfrac{A_2}{2}n - \dfrac{i\sigma a}{n} & ia^3\dfrac{A_3}{2}(n+3) \\ ia^3\dfrac{A_3}{2}n & i\omega - ia^2\dfrac{A_2}{2}(n+3) - \dfrac{i\sigma a}{n+3} \end{pmatrix} + O(a^4),$$

where $a = \ell^2 - \ell_c^2$ and $\mathcal{I}_a$, 2×2 identity matrix, respectively. To locate δ, we compute

$$\det(\mathcal{B}_a(\ell) - (i\omega + i\delta)\mathcal{I}_a) = 0, \tag{3.2.21}$$

and arrive at a quadratic in δ

$$\delta^2 + \delta\left(\sigma a\left(\frac{1}{n} + \frac{1}{n+3}\right) + \frac{a^2A_2}{2}((n+3)+n)\right) \\ + \frac{a^4A_2^2 n(n+3)}{4} + \frac{\sigma^2a^2}{n(n+3)} + O(a^2|a| + |a|^5) = 0.$$

A direct computation shows that the discriminant of the above quadratic is

$$\text{disc}_a(a) = \frac{9\sigma^2a^2}{n^2(n+3)^2} + \frac{9a^4A_2^2}{4} + O(a^2|a| + |a|^5).$$

Note that, for $|a|$ and $|a|$ sufficiently small, the leading term in the discriminant is always positive irrespective of the values of n, σ and m. Therefore, we do not observe any instability for the $\Delta = 3$ case by performing the perturbation calculation up to the fourth power of the amplitude parameter a.

Remark 3.2.8. *A similar instability analysis can be carried out for any $\Delta \geq 4$. But to* *explicitly obtain all coefficients, we will need higher powers of a in the expansion of the operator $\mathcal{A}_{a,\mu}$ and we will need to calculate more terms in the expansion of solution w which we do not pursue here. But for a fixed $\Delta \geq 4$, the matrix $\mathcal{B}_a(\ell)$ would take the form*

$$\mathcal{B}_a(\ell) = \begin{pmatrix} i\omega - \dfrac{i\sigma a}{n} + in(\alpha_2 a^2 + \alpha_4 a^4 + \dots) & i(n+\Delta)\beta a^{\Delta} \\ in\beta a^{\Delta} & i\omega - \dfrac{i\sigma a}{n+\Delta} + i(n+\Delta)(\alpha_2 a^2 + \alpha_4 a^4 + \dots) \end{pmatrix} + O(a^{\Delta+1}),$$

and $\mathcal{I}_a$ would be 2×2 identity matrix. Then, the resulting discriminant would look like

$$\mathrm{disc}_a(a) = \frac{\sigma^2 \Delta^2 a^2}{n^2(n+\Delta)^2} + \Delta^2 \alpha_2^2 a^4 + O(a^2|a| + |a|^5),$$

which is positive for sufficiently small $|a|$ and $|a|$ leading to stability in a sufficiently small neighbourhood of $\ell = \ell_c$ and $a = 0$.

3.2.3.2 Long wavelength transverse perturbations

In all four cases $(\sigma, m) = (1, \uparrow)$, $(1, \downarrow)$, $(-1, \uparrow)$, and $(-1, \downarrow)$, there is a collision at the origin of eigenvalues $i\omega_{-1,\ell}$ and $i\omega_{1,\ell}$ at $\ell = 0$. Since m is monotonic for $k > 0$, the remaining eigenvalues at $\ell = 0$ are all simple, purely imaginary, and located outside the open ball $B(0; |m(k) - m(2k)|)$. The perturbation analysis to locate the bifurcation of these eigenvalues for small ℓ and a will correspond to long wavelength transverse perturbations. The following lemma ensures that for sufficiently small ℓ and a, bifurcating eigenvalues from the origin are separated from the rest of the spectrum by a non-zero distance.

Lemma 3.2.9. *The following properties hold, for any ℓ and a sufficiently small.*

1. *The spectrum of $\mathcal{A}_a(\ell)$ decomposes as*

$$\mathrm{spec}_0(\mathcal{A}_a(\ell)) \cup \mathrm{spec}_1(\mathcal{A}_a(\ell)),$$

with

$$\mathrm{spec}_0(\mathcal{A}_a(\ell)) \subset B(0; R/3), \quad \mathrm{spec}_1(\mathcal{A}_a(\ell)) \subset \mathbb{C} \setminus \overline{B(0; R/2)},$$

where $R = |m(k) - m(2k)|$.

2. *The spectral projection* $\Pi_a(\ell)$ *associated with* $\mathrm{spec}_0(\mathcal{A}_a(\ell))$ *satisfies* $\|\Pi_a(\ell) - \Pi_0(0)\| = O(\ell^2 + |a|)$.

3. *The spectral subspace* $\mathcal{X}_a(\ell) = \Pi_a(\ell)(L^2_0(0, 2\pi))$ *is two dimensional.*

The proof of these properties is similar to [32, Lemma 4.7]. In the following lemma, we show that for sufficiently small ℓ and a, two eigenvalues in $\mathrm{spec}_0(\mathcal{A}_a(\ell))$ leave imaginary axis if $(\sigma, m) = (1, \uparrow)$ or $(-1, \downarrow)$ but remain on imaginary axis if $(\sigma, m) = (1, \downarrow)$ or $(-1, \uparrow)$.

Lemma 3.2.10. *Assume ℓ and a are sufficiently small. For $(\sigma, m) = (1, \uparrow)$ or $(-1, \downarrow)$, there exists $\ell_a^2 = \sigma A_2 a^2 + O(a^4) > 0$, such that*

1. *for any $\ell^2 \geq \ell_a^2$, the spectrum of $\mathcal{A}_a(\ell)$ is purely imaginary,*

2. *for any $\ell^2 < \ell_a^2$, the spectrum of $\mathcal{A}_a(\ell)$ is purely imaginary, except for a pair of simple real eigenvalues with opposite signs.*

For $(\sigma, m) = (1, \downarrow)$ or $(-1, \uparrow)$, the spectrum of $\mathcal{A}_a(\ell)$ is purely imaginary.

Proof. Consider the decomposition of the spectrum of $\mathcal{A}_a(\ell)$ in Lemma 3.2.9. The eigenvalues in $\mathrm{spec}_0((\mathcal{A}_a(\ell))$ are the eigenvalues of the restriction of $\mathcal{A}_a(\ell)$ to the two-dimensional spectral subspace $\mathcal{X}_a(\ell)$. We determine the location of these eigenvalues by computing successively a basis of $\mathcal{X}_a(\ell)$, the 2×2 matrix representing the action of $\mathcal{A}_a(\ell)$ on this basis, and the eigenvalues of this matrix. Note that for $a = 0$, $\mathcal{X}_0(\ell)$ is spanned by $\{\cos z, \sin z\}$. Moreover, a direct calculation shows that zero is an $L^2_0(\mathbb{T})$-eigenvalue of $\mathcal{A}_a(0)$ of multiplicity two with eigenfunctions $(\partial_b c)(\partial_a \eta) - (\partial_a c)(\partial_b \eta))(z; k, a, 0)$ and $\partial_z \eta(z; k, a, 0)$. We use expansions of η and c in (3.2.8) to calculate expansion of a basis $\{\phi_1, \phi_2\}$ for $\mathcal{X}_a(\ell)$ for small a and ℓ as

$$\begin{aligned}\phi_1(z) &:= \frac{1}{(m(k) - 1)}((\partial_b c)(\partial_a \eta) - (\partial_a c)(\partial_b \eta))(z; k, a, 0) \\ &:= \cos z + 2aA_2 \cos 2z + 3a^2 A_3 \cos 2z + O(a^3), \end{aligned} \tag{3.2.22}$$

$$\phi_2(z) := -\frac{1}{a}\partial_z \eta(z; k, a, 0) = \sin z + 2aA_2 \sin 2z + 3a^2 A_3 \sin 3z + O(a^3). \tag{3.2.23}$$

We use expansion of $\mathcal{A}_a(\ell)$ in (3.2.20) to find actions of $\mathcal{A}_a(\ell)$ and identity operator on $\{\phi_1, \phi_2\}$ as

$$\mathbf{B}_a(\ell) = \begin{pmatrix} 0 & \sigma\ell^2 - A_2 a^2 \\ -\sigma\ell^2 & 0 \end{pmatrix} + O(|a|(\ell^2 + a^2)),$$

and

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O(a^4), \tag{3.2.24}$$

for $|a|$ and $|\ell|$ sufficiently small. To locate the two eigenvalues bifurcating from the origin, we examine the characteristic equation $\det(\mathbf{B}_a(\ell) - \lambda\mathbf{I}) = 0$, which leads to

$$\lambda^2 + \sigma\ell^2(\sigma\ell^2 - A_2 a^2) + O(|a|\ell^2(\ell^2 + a^2)) = 0.$$

From which we conclude that

$$\lambda^2 = -\ell^2(\ell^2 - \sigma a^2 A_2) + O(|a|\ell^2(\ell^2 + a^2)).$$

For $\ell = a = 0$, we get zero as a double eigenvalue, which agrees with our calculation. For $(\sigma, m) = (1, \downarrow)$ and $(-1, \uparrow)$, we obtain two purely imaginary eigenvalues for all ℓ and a sufficiently small. For $(\sigma, m) = (1, \uparrow)$ and $(-1, \downarrow)$, we obtain two purely imaginary eigenvalues when $\ell^2 \geq \ell_a^2$, and real eigenvalues, with opposite signs when $\ell^2 < \ell_a^2$ where

$$\ell_a^2 = \sigma A_2 a^2 + O(a^4).$$

Now, we study $\text{spec}_1(\mathcal{A}_a(\ell))$ for ℓ and a sufficiently small. For $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$, as ℓ is sufficiently small, κ_n in (3.2.15) depends upon behavior of $m(k)$, which is monotonic for all $k > 0$. Hence, k_n is negative for all n and σ when m is monotonically increasing, and k_n is positive for all n and σ when m is monotonically decreasing. This implies that even if eigenvalues in $\text{spec}_1(\mathcal{A}_0(\ell))$ collide, they remain on the imaginary axis and for all ℓ and a sufficiently small $\text{spec}_1(\mathcal{A}_a(\ell))$ is a subset of the imaginary axis. This proves the lemma. □

3.2.4 Non-Periodic Perturbations

In this section, we will study two-dimensional perturbations which are non-periodic (localized or bounded) in the direction of the propagation of the wave. For non-periodic perturbations, we study the invertibility of $\mathcal{T}_a(\lambda, \ell)$ in (3.2.11) acting in $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$ (with domain $H^{\alpha+2}(\mathbb{R})$ or $C_b^{\alpha+2}(\mathbb{R})$), for $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, and $\ell \in \mathbb{R}$, $\ell \neq 0$. Since coefficients of $\mathcal{T}_a(\lambda, \ell)$ are periodic functions, using Floquet Theory, all solutions of (3.2.11) in $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$ are of the form $V(z) = e^{i\mu z}\tilde{V}(z)$ where $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ is the Floquet exponent and $\tilde{V}$ is a 2π-periodic function, see [31] for a similar situation. This replaces the study of invertibility of the operator $\mathcal{T}_a(\lambda, \ell)$ in $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$ by the study of invertibility of a family of Bloch operators in $L^2(\mathbb{T})$ parameterized by the Floquet exponent μ. We present the precise reformulation in the following lemma.

Lemma 3.2.11. *The linear operator $\mathcal{T}_a(\lambda, \ell)$ is invertible in $L^2(\mathbb{R})$ if and only if the linear operators*

$$\mathcal{T}_{a,\mu}(\lambda, \ell) = \lambda(\partial_z + i\mu) - (\partial_z + i\mu)^2(c - \mathcal{M}_k + \eta) - \sigma\ell^2,$$

acting in $L^2(\mathbb{T})$ with domain $H^{\alpha+2}(\mathbb{T})$ are invertible for all $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right]$.

We refer to [32, Lemma 5.1] for a detailed proof in the similar situation. The fact that the operators $\mathcal{T}_{a,\mu}(\lambda, \ell)$ act in $L^2(\mathbb{T})$ implies that these operators have only point spectrum. Note that $\mu = 0$ corresponds to the periodic perturbations which we have already investigated, so now we would restrict ourselves to the case of $\mu \neq 0$. The operator $\partial_z + i\mu$ is invertible in $L^2(\mathbb{T})$. Using this, we have the following result.

Lemma 3.2.12. *The operator $\mathcal{T}_{a,\mu}(\lambda, \ell)$ is not invertible in $L^2(\mathbb{T})$ for some $\lambda \in \mathbb{C}$ and $\mu \neq 0$ if and only if $\lambda \in \operatorname{spec}(\mathcal{A}_a(\ell, \mu))$, $L^2(\mathbb{T})$-spectrum of the operator,*

$$\mathcal{A}_a(\ell, \mu) := (\partial_z + i\mu)(c - \mathcal{M}_k + \eta) + \sigma\ell^2(\partial_z + i\mu)^{-1}.$$

Note that the operator $(\partial_z + i\mu)^{-1}$ becomes singular, as $\mu \to 0$, so replacing the study of the invertibility of $\mathcal{T}_{a,\mu}(\lambda, \ell)$ by the study of the spectrum of $\mathcal{A}_a(\ell, \mu)$ is not suitable for small μ. To avoid this, we only study the spectrum of $\mathcal{A}_a(\ell, \mu)$ for $|\mu| > \epsilon > 0$. Also, for $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ and $\mu \neq 0$, since the spectrum $\operatorname{spec}(\mathcal{A}_a(\ell, \mu))$ is symmetric with respect to the imaginary axis, and $\mathrm{spec}(\mathcal{A}_a(\ell,\mu)) = \mathrm{spec}(-\mathcal{A}_a(\ell,-\mu))$, we can restrict our study to $\mu \in \left(0, \frac{1}{2}\right]$.

We will study the $L^2(\mathbb{T})$-spectra of linear operators $\mathcal{A}_a(\ell,\mu)$ for $|a|$ sufficiently small. It is straightforward to establish the estimate

$$\|\mathcal{A}_a(\ell,\mu) - \mathcal{A}_0(\ell,\mu)\| = O(|a|),$$

as $a \to 0$ uniformly for $\mu \in \left(0, \frac{1}{2}\right]$ in the operator norm. Therefore, In order to locate the spectrum of $\mathcal{A}_a(\ell,\mu)$, we need to determine the spectrum of $\mathcal{A}_0(\ell,\mu)$. A simple calculation yields that

$$\mathcal{A}_0(\ell,\mu))e^{inz} = i\omega_{n,\ell,\mu}e^{inz}, \quad n \in \mathbb{Z},$$

where

$$\omega_{n,\ell,\mu} = (n+\mu)(m(k) - m(k(n+\mu))) - \frac{\sigma\ell^2}{n+\mu}.$$

As in the previous section, the linear operator $\mathcal{A}_a(\ell,\mu)$ can be decomposed as

$$\mathcal{A}_a(\ell) = J_\mu \mathcal{L}_a(\ell,\mu),$$

where

$$J_\mu = \partial_z + i\mu \quad \text{and} \quad \mathcal{L}_a(\ell,\mu) = c - \mathcal{M}_k + \eta + \sigma\ell^2(\partial_z + i\mu)^{-2}.$$

The operator J_μ is skew-adjoint, whereas the operator $\mathcal{L}_a(\ell)$ is self-adjoint. As defined in (3.2.15), the Krein signature, $\kappa_{n,\mu}$ of an eigenvalue $i\omega_{n,\ell,\mu}$ in $\mathrm{spec}(\mathcal{A}_0(\ell,\mu)))$ is

$$\kappa_{n,\mu} = \mathrm{sgn}\left(m(k) - m(k(n+\mu)) - \frac{\sigma\ell^2}{(n+\mu)^2}\right), \quad n \in \mathbb{Z}. \tag{3.2.25}$$

Therefore, a necessary condition for bifurcation of colliding eigenvalues, $i\omega_{p,\ell,\mu}$ and $i\omega_{q,\ell,\mu}$, from the imaginary axis is $(p+\mu_0)(q+\mu_0) < 0$, where μ_0 is the value of the floquet exponent where eigenvalues $i\omega_{p,\ell,\mu}$ and $i\omega_{q,\ell,\mu}$ collide. As in the previous subsection, we split the analysis in this subsection into finite and short wavelength, and long wavelength transverse perturbations.

3.2.4.1 Finite and short-wavelength transverse perturbations

We start the analysis of the spectrum of $\mathcal{A}_a(\ell,\mu)$ with the values of ℓ away from the origin, $|\ell| \geq \ell_0$, for some $\ell_0 > 0$. We further split the analysis into two cases depending on the value of σ and nature of m.

1. $\boxed{(\sigma, m) = (1, \uparrow) \text{ or } (-1, \downarrow)}$

   It is straightforward to verify that for $(\sigma, m) = (1, \uparrow)$ and $(-1, \downarrow)$, $\kappa_{n,\mu}$ is negative for all $n \in \mathbb{Z} \setminus \{-1, 0\}$ and positive for all $n \in \mathbb{Z} \setminus \{-1, 0\}$, respectively, for all $k > 0$ and $\mu \in \left(0, \frac{1}{2}\right]$. However, Krein signatures of $i\omega_{-1,\ell,\mu}$ and $i\omega_{0,\ell,\mu}$ can be positive as well as negative depending upon the values of k, ℓ and μ. Therefore, collision of $i\omega_{-1,\ell,\mu}$ and $i\omega_{0,\ell,\mu}$ with each other or with any other eigenvalue may lead to instability. Another straightforward calculation reveals that eigenvalues $i\omega_{-1,\ell,\mu}$ and $i\omega_{0,\ell,\mu}$ collide when

$$\ell^2 = \ell_c^2 = \sigma\mu(1-\mu)[(1-\mu)(m(k)-m(k(1-\mu)))+\mu(m(k)-m(k\mu))] > 0, \quad (3.2.26)$$

   for all $\mu \in \left(0, \frac{1}{2}\right]$ while there are no collisions between pairs $\{i\omega_{0,\ell,\mu}, i\omega_{-n,\ell,\mu}\}$, $n \in \mathbb{N}\setminus\{1\}$ and $\{i\omega_{-1,\ell,\mu}, i\omega_{m,\ell,\mu}\}$, $m \in \mathbb{N}$. Therefore, we are left with only one pair which may bifurcate into potentially unstable eigenvalues which is $\{\omega_{-1,\ell,\mu}, \omega_{0,\ell,\mu}\}$. We further perform perturbation calculations to obtain the following result.

Lemma 3.2.13. *Assume that $\mu \in \left(0, \frac{1}{2}\right]$ and consider ℓ_c^2 given in (3.2.26). For $(\sigma, m) = (1, \uparrow)$ or $(-1, \downarrow)$ and $|a|$ sufficiently small, there exists $a_a(\mu) = \mu^{3/2}(1-\mu)^{3/2}|a| + O(a^2) > 0$ such that*

(a) for $|\ell^2 - \ell_c^2| \geq a_a(\mu)$, the spectrum of $\mathcal{A}_a(\ell,\mu)$ is purely imaginary,

(b) for $|\ell^2 - \ell_c^2| < a_a(\mu)$, the spectrum of $\mathcal{A}_a(\ell,\mu)$ is purely imaginary, except for a pair of complex eigenvalues with opposite nonzero real parts.

Proof. There exists a curve $\ell = \ell_c$ given in (3.2.26) along which

$$\omega(\ell_c, \mu) := \omega_{-1,\ell_c,\mu} = \omega_{0,\ell_c,\mu}.$$

Furthermore,

$$\phi_{0,-1}(z) = e^{-iz} \qquad \textit{and} \qquad \phi_{0,0}(z) = 1, \tag{3.2.27}$$

forms the corresponding eigenspace for $\mathcal{A}_0(\ell_c, \mu)$ associated with the two eigenvalues. Let

$$i\omega(\ell_c, \mu) + i\delta_{a,\ell,-1} \qquad \textit{and} \qquad i\omega(\ell_c, \mu) + i\delta_{a,\ell,0}, \tag{3.2.28}$$

be the eigenvalues of $\mathcal{A}_a(\ell_c, \mu)$ bifurcating from $i\omega_{-1,\ell_c,\mu}$ and $i\omega_{0,\ell_c,\mu}$ respectively for $|a|$ and $|\ell - \ell_c|$ small. Let $\{\phi_{a,\ell,-1}(z), \phi_{a,\ell,0}(z)\}$ be the extended eigenspace associated with two bifurcating eigenvalues. Following [14], we can take,

$$\phi_{a,-1,\ell}(z) = e^{-iz} + a\phi_{-1} + O(a^2), \tag{3.2.29}$$

$$\phi_{a,0,\ell}(z) = 1 + a\phi_0 + O(a^2). \tag{3.2.30}$$

Using constraint of orthonormality on the eigenfunctions $\phi_{a,\ell,-1}$ and $\phi_{a,\ell,0}$, we obtain

$$\phi_{-1} = \phi_0 = 0.$$

To locate eigenvalues, we calculate matrix representations of $A_a(\ell_c, \mu)$ and identity operators on $\{\phi_{a,-1,\ell}(z), \phi_{a,0,\ell}(z)\}$ for $|a|$ and $|\ell - \ell_c|$ small and find that

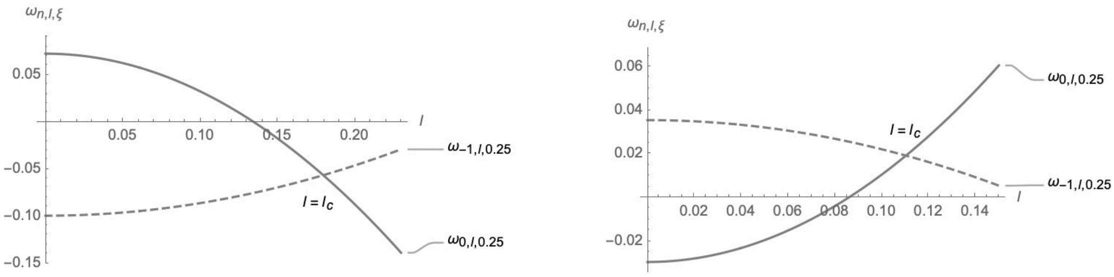
$$\mathbf{B}_a(\ell, \mu) = \begin{pmatrix} i\omega(\ell_c, \mu) - i\frac{\sigma a}{\mu - 1} & \frac{i}{2}a\mu \\ \frac{i}{2}(\mu - 1)a & i\omega(\ell_c, \mu) - i\frac{\sigma\varepsilon}{\mu} \end{pmatrix} + O(|a|(|\varepsilon| + |a|)),$$

where $\varepsilon = \ell^2 - \ell_c^2$ and $\mathbf{I}_a$ is the 2×2 identity matrix. Solving the characteristic equation $\det(\mathbf{B}_a(\ell, \mu) - \lambda \mathbf{I}_a) = 0$ for λ of the form

$$\lambda = i\omega(\ell_c, \mu) + i\delta,$$

leads to the polynomial equation

$$\delta^2 + \delta\sigma\varepsilon\left(\frac{1}{\mu - 1} + \frac{1}{\mu} + O(a^2)\right) - \frac{1}{4}\mu(\mu - 1)a^2 + \frac{\varepsilon^2}{\mu(\mu - 1)} + O(a^2(|\varepsilon| + a^2)) = 0.$$



((a)) KP-ILW-I ((b)) KP-Whitham-II

FIGURE 3.1: Collision of eigenvalues at $\ell = \ell_c$ for $k = 1$ and $\mu = 0.25$.

A direct computation shows that the discriminant of this polynomial is

$$\mathrm{disc}_a(\varepsilon, \mu) = \frac{\varepsilon^2}{\mu^2(\mu-1)^2} + \mu(\mu-1)a^2 + O(a^2(|\varepsilon| + a^2)).$$

For any a sufficiently small, there exists

$$\varepsilon_a(\mu) = \mu^{3/2}(1-\mu)^{3/2}|a| + O(a^2) > 0,$$

such that the two eigenvalues of $\mathcal{A}_a(\ell, \mu)$ are purely imaginary when $|\ell^2 - \ell_c^2| \geq \varepsilon_a(\mu)$ and complex with opposite nonzero real parts when $|\ell^2 - \ell_c^2| < \varepsilon_a(\mu)$, which proves the lemma. □

Figure 3.1 depicts a collision of pair of eigenvalues in KP-ILW-I and KP-Whitham-II equations which lead to instability according to Lemma 3.2.13.

2. $(\sigma, m) = (1, \downarrow)$ or $(-1, \uparrow)$

In contrast to the previous case, for $(\sigma, m) = (1, \downarrow)$ or $(-1, \uparrow)$, there are infinitely many pairs of eigenvalues which collide with each other. A pair $\{\omega_{p,\ell,\mu}, \omega_{-q,\ell,\mu}\}$ collide with each other at

$$\ell^2 = \ell_{p,q}^2 = \frac{\sigma(p+\mu)(q-\mu)}{p+q}((p+\mu)(m(k)-m(k(p+\mu)))+(q-\mu)(m(k)-m(k(q-\mu)))),$$

for all $\mu \in (0, 1/2]$, and for all $p \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{N}$ except $(p, q) = (0, 1)$. A Krein signature analysis tells that for each pair of colliding eigenvalues; there are intervals of ℓ where they have opposite Krein signatures. Therefore, all such collisions may lead to instability. An additional analysis is required in order to detect the bifurcating eigenvalues which indeed leave the imaginary axis and become unstable. In what follows, we examine two pairs $\{\omega_{-1,\ell,\mu}, \omega_{1,\ell,\mu}\}$ and $\{\omega_{0,\ell,\mu}, \omega_{-2,\ell,\mu}\}$ whose indices differ by two. Let $i\omega_{n,\ell,\mu}$ and $i\omega_{n+2,\ell,\mu}$ be such two eigenvalues for some $n \in \mathbb{Z}$. Assume that these eigenvalues collide at $\ell = \ell_c$, that is

$$0 \neq \omega_{n,\ell_c} = \omega_{n+2,\ell_c} = \omega \ (say). \tag{3.2.31}$$

Furthermore,

$$\phi_{0,n}(z) = e^{inz} \qquad and \qquad \phi_{0,n+2}(z) = e^{i(n+2)z}, \tag{3.2.32}$$

form a basis for the corresponding eigenspace for $\mathcal{A}_0(\ell_c, \mu)$ generated by the two eigenvalues. Let

$$i\omega + i\delta_{a,n} \qquad \text{and} \qquad i\omega + i\delta_{a,n+2}, \tag{3.2.33}$$

be the eigenvalues of $\mathcal{A}_a(\ell_c, \mu)$ bifurcating from $i\omega_{n,\ell_c,\mu}$ and $i\omega_{n+2,\ell_c,\mu}$ respectively, for $|a|$ small. Let $\{\phi_{a,n}(z), \phi_{a,n+2}(z)\}$ be a orthonormal basis for the corresponding eigenspace. We assume the following expansions

$$\phi_{a,n,\ell}(z) = e^{inz} + a\phi_{n,1} + a^2\phi_{n,2} + O(a^3), \tag{3.2.34}$$

$$\phi_{a,n+2,\ell}(z) = e^{i(n+2)z} + a\phi_{n+2,1} + a^2\phi_{n+2,2} + O(a^3). \tag{3.2.35}$$

We use orthonormality of $\phi_{a,n,\ell}$ and $\phi_{a,n+2,\ell}$ to find that

$$\phi_{n,1} = \phi_{n,2} = \phi_{n+2,1} = \phi_{n+2,2} = 0.$$

Next, we compute the action of $\mathcal{A}_a(\ell, \mu)$ and identity operators on the extended eigenspace $\{\phi_{a,n,\ell}(z), \phi_{a,n+2,\ell}(z)\}$ for $|\ell - \ell_c|$ and $|a|$ small. We arrive at

$$\mathcal{B}_a(\ell) = \begin{pmatrix} i\omega - ia^2\dfrac{A_2}{2}(n+\mu) - \dfrac{i\sigma\varepsilon}{n+\mu} & ia^2\dfrac{A_2}{2}(n+2+\mu) \\ ia^2\dfrac{A_2}{2}(n+\mu) & i\omega - ia^2\dfrac{A_2}{2}(n+2+\mu) - \dfrac{i\sigma\varepsilon}{n+2+\mu} \end{pmatrix} + O(a^2(|\varepsilon|+|a|)),$$

where $|\varepsilon| = |\ell^2 - \ell_c^2|$ and $\mathcal{I}_a$, the 2×2 identity matrix, respectively. Solving the characteristic equation $|\mathcal{B}_a(\ell) - (i\omega + i\delta)\mathcal{I}_a| = 0$ leads to

$$|\mathcal{B}_a(\ell) - (i\omega + i\delta)\mathcal{I}_a| = \delta^2 + \delta\left(\sigma\varepsilon\left(\frac{1}{n+\mu} + \frac{1}{n+2+\mu}\right) + \frac{a^2A_2}{2}((n+2+\mu)+(n+\mu))\right) + \frac{\sigma^2\varepsilon^2}{(n+\mu)(n+2+\mu)} + O(a^2(|\varepsilon| + |a^3|)) = 0.$$

A direct computation shows that the discriminant of this quadratic in μ is

$$\text{disc}_a(\varepsilon) = \frac{4\sigma^2\varepsilon^2}{(n+\mu)^2(n+2+\mu)^2} + a^4A_2^2(n+1+\mu)^2 + O(a^2(|\varepsilon| + |a^3|)),$$

which implies that for $|\varepsilon|$ and $|a|$ sufficiently small, the leading term in the discriminant is always positive irrespective of the values of n, μ, σ and m. Therefore, we do not observe any instability for the $\Delta = 2$ case by performing the perturbation calculation up to the fourth power of the amplitude parameter a.

Remark 3.2.14. *A remark along the lines of Remark 3.2.8 should hold true here for $\Delta \geq 3$. We do not present explicit calculations as it require higher power of a in solution η.*

3.2.4.2 Long wavelength transverse Perturbations

We now consider the spectrum of $\mathcal{A}_a(\ell,\mu)$ for $|\ell|$ small. Recall that μ is away from the origin, and we have taken $|\mu| > \epsilon$ for some small but fixed $\epsilon > 0$. Because of this, the collision at the origin and collisions away from the origin are well separated for $\mathcal{A}_0(\ell,\mu)$. This separation persists for small $|a|$ using perturbation arguments. More precisely, we have the following lemma.

Lemma 3.2.15. *For any ℓ and a sufficiently small, the spectrum of $\mathcal{A}_a(\ell,\mu)$ decomposes as*

$$\text{spec}(\mathcal{A}_a(\ell,\mu)) = \text{spec}_0(\mathcal{A}_a(\ell,\mu)) \cup \text{spec}_1(\mathcal{A}_a(\ell,\mu)),$$

with

$$dist(\text{spec}_0(\mathcal{A}_a(\ell,\mu)), \text{spec}_1(\mathcal{A}_a(\ell,\mu))) \geq |(m(k) - m(k(1+\epsilon))| > 0.$$

Let $\ell^* > 0$ be the smallest positive value of ℓ for which a collision of eigenvalues of $\mathcal{A}_0(\ell,\mu)$ takes place. Note that such an ℓ^* exists because the collision at the origin takes place only for $\ell = 0$, and other collisions are well separated from this. Now, for any ℓ with

$|\ell| < \ell^*$, there are no collisions between the eigenvalues of $\mathcal{A}_0(\ell, \mu)$ since $|\mu| > \epsilon > 0$. This persists for small values of a using perturbation arguments, and we have the following lemma.

Lemma 3.2.16. *Assume that $\mu \in \left(0, \frac{1}{2}\right]$. There exists $\ell^* > 0$ and $a^* > 0$ such that the spectrum of $\mathcal{A}_a(\ell, \mu)$ is purely imaginary, for any ℓ and a satisfying $|\ell| < \ell^*$ and $|a| < a^*$.*

3.2.5 Proof of main results and applications

We shall prove theorem 3.2.1 and 3.2.2 by using all the results obtained in Sections 3.2.3 and 3.2.4.

Proof of Theorem 3.2.1. We have assumed that $2\pi/k$-periodic traveling wave solution $u(x, y, t) = \eta(k(x - ct))$ of (3.2.1) is a stable solution of the one-dimensional equation (3.2.5) where η and c are as in (3.2.8). Lemma 3.2.7 says that the spectrum of $\mathcal{A}_a(\ell)$ is purely imaginary if (σ, m) is $(1, \uparrow)$ or $(-1, \downarrow)$ for all $|\ell| > \ell_a > 0$, which implies that the small amplitude periodic traveling waves (3.2.8) of (3.2.1) are transversely stable with respect to two-dimensional perturbations which are periodic in the direction of propagation of the wave and of finite or short wavelength in the transverse direction if σ and m in (3.2.1) satisfy $(\sigma, m) = (1, \uparrow)$ or $(-1, \downarrow)$. Lemma 3.2.10 says that the spectrum of $\mathcal{A}_a(\ell)$ is purely imaginary if (σ, m) is $(1, \downarrow)$ or $(-1, \uparrow)$ for all $|\ell| < |\ell_a^*|$, which implies that the small amplitude periodic traveling waves (3.2.8) of (3.2.1) are transversely stable with respect to two-dimensional perturbations which are periodic in the direction of propagation of the wave and of long wavelength in the transverse direction if σ and m in (3.2.1) satisfy $(\sigma, m) = (1, \downarrow)$ or $(-1, \uparrow)$. □

Proof of Theorem 3.2.2. Lemma 3.2.10 says that there exist ℓ_a such that the spectrum of $\mathcal{A}_a(\ell)$ is purely imaginary, except for a pair of simple real eigenvalues with opposite signs if (σ, m) is $(1, \uparrow)$ or $(-1, \downarrow)$ for all $|\ell^2| < |\ell_a^2|$, which implies that the small amplitude periodic traveling waves (3.2.8) of (3.2.1) are transversely unstable with respect to two-dimensional perturbations which are periodic in the direction of propagation and of long wavelength in the transverse direction if $(\sigma, m) = (1, \uparrow)$ or $(-1, \downarrow)$. Moreover, Lemma 3.2.13 provide an interval of finite wavenumbers in the transverse direction for which the spectrum of $\mathcal{A}_a(\ell, \mu)$ have a pair of complex eigenvalues with opposite nonzero real parts when $(\sigma, m) = (1, \uparrow)$ or $(-1, \downarrow)$. These findings imply that the small amplitude periodic traveling waves (3.2.8) of (3.2.1) are transversely unstable with respect to two-dimensional perturbations which are non-periodic in the direction of propagation and of finite wavelength in the transverse direction if $(\sigma, m) = (1, \uparrow)$ or $(-1, \downarrow)$. □

We discuss implications of 3.2.1 and 3.2.2 on KP-fKdV, KP-BO, KP-ILW, and KP-Whitham equations.

3.2.5.1 KP-fKdV and KP-BO Equations

The KP-fKdV equation is obtained from (3.2.1) by taking

$$m(k) = 1 + |k|^{\beta}, \qquad \beta > 1/2.$$

The symbol $m(k)$ clearly satisfies Hypotheses 2.2.1 $H1$, $H2$ ($\alpha = \beta$, $C_1 = 1$, and $C_2 = 2$), and $H3$ (m is strictly increasing for $k > 0$). The two-parameter family of periodic solutions can be obtained from (3.2.8) and (3.2.9) by replacing $m(k)$ with $1 + |k|^{\beta}$. We have obtained transverse stability and instability of these solutions in Corollary 3.2.3 using Theorems 3.2.1 and 3.2.2. Note that KP-BO equation corresponds to KP-fKdV equation with $\beta = 1$. Therefore, Corollary 3.2.3 hold true for the KP-BO equation as well.

For $\beta = 2$, KP-fKdV equation reduces to the KP equation (3.2.3). As results in [8, 34] show that all periodic traveling waves of the KdV equation are spectrally stable in $L^2(\mathbb{T})$, from Corollary 3.2.3, small-amplitude periodic traveling waves (3.2.8) of KP-I (and KP-II resp.) are transversely stable with respect to two-dimensional perturbations which are periodic in the direction of propagation of the wave and of finite or short-wavelength (and long-wavelength resp.) in the transverse direction. These stability results agree with results in [32, 35, 47, 70]. The transverse instability results obtained for KP-I in Corollary 3.2.3 agrees with [32].

3.2.5.2 KP-ILW Equation

The KP-ILW equation is obtained from (3.2.1) by taking,

$$m(k) = k \coth k.$$

The symbol $m(k)$ satisfies Hypotheses 2.2.1 $H1$, $H2$ ($\alpha = 2$, $C_1 = 1$, and $C_2 = 2$), and $H3$ (m is strictly increasing for $k > 0$). The two-parameter family of periodic solutions can be obtained from (3.2.8) and (3.2.9) by replacing $m(k)$ with $k \coth k$. We have discussed the transverse stability and instability of these solutions in Corollary 3.2.4 by using Theorem 3.2.1 and 3.2.2.

3.2.5.3 KP-Whitham Equation

The KP-Whitham equation is obtained from (3.2.1) by taking,

$$m(k) = \sqrt{\frac{\tanh k}{k}}.$$

The symbol $m(k)$ satisfies Hypotheses 2.2.1 $H1$, $H2$ with $\alpha = -\frac{1}{2}$, $C_1 = 1$, and $C_2 = 2$, and $H3$ as m is strictly decreasing for $k > 0$. The two-parameter family of periodic solutions can be obtained from (3.2.8) and (3.2.9) by replacing $m(k)$ with $\sqrt{\frac{\tanh k}{k}}$. We have discussed the transverse stability and instability of these solutions in Corollary 3.2.5 by using Theorem 3.2.1 and 3.2.2.

3.3 Transverse Spectral Instabilities in Konopelchenko-Dubrovsky Equation

We consider the (2+1)-dimensional Konopelchenko-Dubrovsky (KD) equation [50, 52]

$$\begin{cases} u_t - u_{xxx} - 6\ell u u_x + \dfrac{3}{2}\phi^2 u^2 u_x - 3v_y + 3\phi u_x v = 0, \\ u_y = v_x, \end{cases} \tag{3.3.1}$$

where, $u = u(x, y, t)$, $v = v(x, y, t)$, the subscripts denote partial differentiation, ℓ and ϕ are real parameters, defining the magnitude of nonlinearity in wave propagation, modelled for stratified shear flow, the internal and shallow-water waves and the plasmas [77], can also be regarded as combined KP and modified KP equation [80], or generalized (2+1)D Gardner equation [51].

Models

The (1+1)-dimensional reduction of the KD equation (3.3.1) is the Gardner equation[54, 78]

$$u_t - u_{xxx} - 6\ell u u_x + \frac{3}{2}\phi^2 u^2 u_x = 0, \tag{3.3.2}$$

which is example of the generalized Korteweg-de-Vries equation(gKdV)[53], that is

$$u_t + (g(u) - u_{xx})_x = 0,$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth real function. Gardner equation (3.3.2) can be reduced to KdV and modified KdV equations for $\phi = 0$ and $\ell = 0$, respectively.

For $\phi = 0$, (3.3.1) is Kadomtsev-Petviashvili (KP) equation with negative dispersion [48]

$$(u_t - u_{xxx} - 6\ell u u_x)_x - 3u_{yy} = 0, \tag{3.3.3}$$

which is also known as KP-II equation. Modified KP-II [71] (say, mKP-II) reads from (3.3.1) for $\ell = 0$,

$$\left(u_t - u_{xxx} + \frac{3}{2}\phi^2 u^2 u_x\right)_x - 3u_{yy} = 0. \tag{3.3.4}$$

Integrability

The KD equation (3.3.1) is integrable [50, 59, 80]. Integrability is an useful property to have for evolution equations, especially in higher dimensions. It gives sufficient freedom to explore the equation through different aspects. It also helps significantly to observe nonlinear coherent structures like rogue waves, breathers, solitons and elliptic waves in the systems [57, 58, 64, 76, 79]. Some well-known integrable water wave models are classical KdV, KP, and Schrödinger equations. The KD equation (3.3.1), like similar evolution equations in (2+1) dimension, for example, Kadomtsev-Petviashvili equation, Davey-Stewarson equation, and the three-wave equations, is solvable through Inverse Scattering Transform (IST) [50]. It is among the few nonlinear evolution equations which are completely integrable in different settings. Notably, the considered KD equation is also integrable in the Painlevé sense and solvable through IST [50, 51, 77].

Dispersion Relation

Assuming a plane-wave solution of the form

$$u(x,y,t) = e^{i(kx-\Omega t+\ell y)},$$

for the linear part

$$(u_t - u_{xxx})_x - 3u_{yy} = 0,$$

of the KD equation (3.3.1), we arrive at the dispersion relation

$$\Omega(k) = k^3 - \frac{3\ell^2}{k}.$$

Small amplitude periodic traveling waves

The y-independent periodic traveling wave solution of the KD equation (3.3.1) that are also solutions of the Gardner equation (3.3.2), is of the form

$$\begin{pmatrix} u(x,y,t) \\ v(x,y,t) \end{pmatrix} = \begin{pmatrix} u(x-ct) \\ v(x-ct) \end{pmatrix},$$

for some $c \in \mathbb{R}$. Under this assumption, we arrive at

$$\begin{cases} -cu_x - u_{xxx} + \dfrac{\phi^2}{2}(u^3)_x - 3\ell(u^2)_x + 3\phi u_x v = 0, \\ v_x = 0, \end{cases} \tag{3.3.5}$$

which implies $v = b_1$, where b_1 is an arbitrary constant. Substituting $v = b_1$ and integrating, (3.3.5) is reduced to

$$-cu - u_{xx} + \frac{\phi^2}{2}u^3 - 3\ell u^2 + 3\phi u b_1 = b_2, \tag{3.3.6}$$

where $b_1, b_2 \in \mathbb{R}$. Let u be a $2\pi/k$-periodic function of its argument, for some $k > 0$. Then, $\eta(z) := u(x)$ with $z = kx$, is a 2π-periodic function in z, satisfying

$$-c\eta - k^2\eta_{zz} + \frac{\phi^2}{2}\eta^3 - 3\ell\eta^2 + 3\phi\eta b_1 = b_2. \tag{3.3.7}$$

For a fixed ϕ and ℓ, let $F : H^2(\mathbb{T}) \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to L^2(\mathbb{T})$ be defined as

$$F(\eta, c; k, b_1, b_2) = -c\eta - k^2\eta_{zz} + \frac{\phi^2}{2}\eta^3 - 3\ell\eta^2 + 3\phi\eta b_1 - b_2.$$

We try to find a solution $\eta \in H^2(\mathbb{T})$ of

$$F(\eta, c; k, b_1, b_2) = 0. \tag{3.3.8}$$

For any $c \in \mathbb{R}$, $k > 0$, $b_1, b_2 \in \mathbb{R}$ and $|b_1|, |b_2|$ sufficiently small, note that

$$\eta_0(c, k, b_1, b_2) = -\frac{1}{c}b_2 + O((b_1 + b_2)^2), \tag{3.3.9}$$

make a constant solution of (3.3.8). Note that $\eta_0 \equiv 0$ if $b_1 = b_2 = 0$. If non-constant solutions of (3.3.8) bifurcate from $\eta_0 \equiv 0$ for some $c = c_0$ then $\ker(\partial_\eta F(0, c_0; k, 0, 0))$ is non-trivial. Note that

$$\ker(\partial_\eta F(0, c_0; k, 0, 0)) = \ker(-c_0 - k^2\partial_z^2) = \text{span}\{e^{\pm iz}\},$$

provided that $c_0 = k^2$.

The periodic traveling waves of (3.3.2) exists (see, [9]), and by following Lyapunov schmidt procedure, their small amplitude expansion is as follows.

Theorem 3.3.1. *For any $k > 0$, $b_1, b_2 \in \mathbb{R}$ and $|b_1|, |b_2|$ sufficiently small, a one parameter family of solutions of (3.3.1), denoted by*

$$\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} \eta(a, b_1, b_2)(z) \\ v(z) \end{pmatrix},$$

where $z = k(x - c(a, b_1, b_2)t)$, $|a|$ sufficiently small, $\eta(a, b_1, b_2)(z)$ is smooth, even and 2π-periodic in z and c is even in a, is given by

$$\begin{cases} \eta(a, b_1, b_2)(z) = -\dfrac{1}{k^2}b_2 + a\cos z + a^2(A_0 + A_2\cos 2z) + a^3 A_3\cos 3z + O(a^4 + a^2(b_1 + b_2)^2), \\ v(z) = b_1, \\ c(a, b_1, b_2) = k^2 + 3\phi b_1 + \dfrac{3\ell}{k^2}b_2 + a^2 c_2 + O(a^4 + a^2(b_1 + b_2)^2), \end{cases} \tag{3.3.10}$$

where

$$A_0 = -\frac{3\ell}{2k^2}, \quad A_2 = \frac{\ell}{2k^2}, \quad A_3 = -\frac{\phi^2}{64k^2} + \frac{3\ell^2}{16k^4}, \quad \text{and} \quad c_2 = \frac{3\phi^2}{8} + \frac{15\ell^2}{2k^2}. \tag{3.3.11}$$

3.3.1 Main Results

Our main result is the following theorem showing that the small periodic traveling waves (3.3.10) of (3.3.1) possess modulational transverse instabilities with respect to periodic perturbations.

Theorem 3.3.2. *For a fixed $\ell \in \mathbb{R}$ and $\phi \neq 0$, sufficiently small amplitude periodic traveling waves (3.3.10) of KD equation suffers modulational transverse instabilities with respect to periodic perturbations if*

$$k > 2\left|\frac{\ell}{\phi}\right| \quad \text{and} \quad |\gamma| < k|a|\sqrt{\left|\frac{\phi^2}{4} - \frac{\ell^2}{k^2}\right|} + O(a(\gamma + a)).$$

Theorem 3.3.3. *Assume that small amplitude periodic traveling waves (3.3.10) of (3.3.1) are spectrally stable in $L^2(\mathbb{T})$ as a solution of the corresponding y-independent one-dimensional equation. Then, for any a sufficiently small, $\ell \in \mathbb{R}$, $\phi \in \mathbb{R}$, and $k > 0$, periodic traveling waves (3.3.10) of (3.3.1) are transversely stable with respect to two-dimensional perturbations which are either mean-zero periodic or non-periodic (localized or bounded) in the direction of propagation and of finite or short wavelength in the transverse direction.*

As a consequence of this theorem, for all $k > 0$, periodic traveling waves (3.3.10) of the mKP-II equation suffer modulational transverse instability with respect to periodic perturbations which is in accordance with results in [47]. Also, in the limit $\phi \to 0$, there is no modulational transverse instability for KP-II equation by Theorem 3.3.2 which again agrees with results in [5, 32, 36, 47, 70]. We also investigate all potentially unstable nodes which can lead to high-frequency transverse instabilities with respect to both, periodic as well as non-periodic perturbations.

We now turn to the detailed analysis required to establish the results stated above.

3.3.2 Linearization

Linearizing (3.3.1) about its one-dimensional periodic traveling wave solution $\begin{pmatrix} \eta \\ v \end{pmatrix}$ given in (3.3.10), and considering the perturbations to $\begin{pmatrix} \eta \\ v \end{pmatrix}$ of the form

$$\begin{pmatrix} \eta \\ v \end{pmatrix} + \epsilon \begin{pmatrix} \zeta \\ \psi \end{pmatrix} + O(\epsilon^2) \quad \text{for} \quad 0 < |\epsilon| \ll 1, \tag{3.3.12}$$

we arrive at,

$$\begin{cases} \zeta_t - kc\zeta_z - k^3\zeta_{zzz} - 6k\ell(\eta\zeta)_z + \dfrac{3}{2}\phi^2 k(\eta^2\zeta)_z - 3\psi_y + 3\phi k\eta_z\psi + 3\phi k b_1 \zeta_z = 0, \\ \zeta_y - k\psi_z = 0. \end{cases} \tag{3.3.13}$$

We seek a solution of the form $\begin{pmatrix} \zeta(z,t,y) \\ \psi(z,t,y) \end{pmatrix} = e^{\lambda t + i\ell y} \begin{pmatrix} \zeta(z) \\ \psi(z) \end{pmatrix}$, $\lambda \in \mathbb{C}$, $\ell \in \mathbb{R}$, of (3.3.13), which leads to

$$\begin{cases} \lambda\zeta - kc\zeta_z - k^3\zeta_{zzz} - 6k\ell(\eta\zeta)_z + \dfrac{3}{2}\phi^2 k(\eta^2\zeta)_z - 3i\ell\psi + 3\phi k\eta_z\psi + 3\phi k b_1\zeta_z = 0, \\ i\ell\zeta - k\psi_z = 0. \end{cases} \tag{3.3.14}$$

We can reduce this system of equations into

$$\mathcal{Q}_{a,b_1,b_2}(\lambda,\ell)\psi := \left(k \left(\lambda - kc\partial_z - k^3\partial_z^3 - 6k\ell\partial_z(\eta\cdot) + \frac{3}{2}\phi^2 k\partial_z(\eta^2\cdot) \right) \partial_z + 3\ell^2 + 3\phi k(i\ell\eta_z + kb_1\partial_z^2) \right) \psi = 0. \tag{3.3.15}$$

Definition 3.3.4. *(Transverse (in)stability) Assuming that $2\pi/k$-periodic traveling wave solution $u(x,y,t) = \eta(k(x-ct))$ of (3.3.1) is a stable solution of the one-dimensional equation (3.3.2) where η and c are as in (3.3.10), we say that the periodic wave η in (3.3.10) is transversely spectrally stable with respect to two-dimensional periodic perturbations (resp. non-periodic (localized or bounded perturbations)) if the KD operator $\mathcal{Q}_{a,b_1,b_2}(\lambda,\ell)$ acting in $L^2(\mathbb{T})$ (resp. $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$) is invertible, for any $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$ and any $\ell \neq 0$ otherwise it is deemed transversely spectrally unstable.*

We split the study of invertibility of $\mathcal{Q}_{a,b_1,b_2}(\lambda,\ell)$ into periodic ($L^2(\mathbb{T})$) and non-periodic perturbations ($L^2(\mathbb{R})$ or $C_b(\mathbb{R})$). In further study, we assume $b_1 = b_2 = 0$. For nonzero b_1 and b_2, one may explore in like manner. However, the calculation becomes lengthy and tedious.

3.3.3 Periodic Perturbations

In this section, we are considering perturbations which are periodic in z, that is, in the direction of the propagation of wave. We check the invertibility of the operator $\mathcal{Q}_{a,b_1,b_2}(\lambda,\ell)$ acting in $L^2(\mathbb{T})$ for any $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$ and any $\ell \neq 0$. We use the notation $\mathcal{Q}_a(\lambda,\ell)$ for $\mathcal{Q}_{a,b_1,b_2}(\lambda,\ell)$ for simplicity. We convert the invertibility problem

$$\mathcal{Q}_a(\lambda,\ell)\psi = 0; \quad \psi \in L^2(\mathbb{T}),$$

into a spectral problem which requires invertibility of ∂_z. Since ∂_z is not invertible in $L^2(\mathbb{T})$, we restrict the problem to mean-zero subspace $L^2_0(\mathbb{T})$, defined in (2.5.11), of $L^2(\mathbb{T})$. Since $L^2_0(\mathbb{T}) \subset L^2(\mathbb{T})$, if the operator $\mathcal{Q}_a(\lambda,\ell)$ is not invertible for some $\lambda \in \mathbb{C}$ in $L^2_0(\mathbb{T})$ implies that the operator $\mathcal{Q}_a(\lambda,\ell)$ is not invertible in $L^2(\mathbb{T})$ as well for the same $\lambda \in \mathbb{C}$. The operator $\mathcal{Q}_a(\lambda,\ell)$ acting on $L^2_0(\mathbb{T})$ has a compact resolvent so that the spectrum consists of isolated eigenvalues with finite multiplicity. Therefore, $\mathcal{Q}_a(\lambda,\ell)$ is invertible in $L^2_0(\mathbb{T})$ if and only if zero is not an eigenvalue of $\mathcal{Q}_a(\lambda,\ell)$. Using this and invertibility of ∂_z in $L^2_0(\mathbb{T})$, we have the following result.

Lemma 3.3.5. *The operator $\mathcal{Q}_a(\lambda,\ell)$ is not invertible in $L^2_0(\mathbb{T})$ for some $\lambda \in \mathbb{C}$ if and only if $\lambda \in \operatorname{spec}_{L^2_0(\mathbb{T})}(\mathcal{H}_a(\ell))$, that is, $L^2_0(\mathbb{T})$-spectrum of the operator, where*

$$\mathcal{H}_a(\ell) := ck\partial_z + k^3\partial_z^3 + 6k\ell\partial_z(\eta) - \frac{3}{2}\phi^2 k\partial_z\left(\eta^2\right) - \frac{3\ell^2}{k}\partial_z^{-1} - i3\phi\ell\eta_z\partial_z^{-1}.$$

Proof. The operator $\mathcal{Q}_a(\lambda,\ell)$ is not invertible in $L^2_0(\mathbb{T})$ for some $\lambda \in \mathbb{C}$, if and only if zero is an eigenvalue of $\mathcal{Q}_a(\lambda,\ell)$. Moreover, for a $\varphi \in L^2_0(\mathbb{T})$, $\mathcal{Q}_a(\lambda,\ell)\varphi = 0$ if and only if $\mathcal{H}_a(\ell)\varphi = \lambda\varphi$. The proof follows trivially. □

Next, we analyze the spectrum of the operator $\mathcal{H}_a(\ell)$ acting in $L^2_0(\mathbb{T})$ with domain $H^3(\mathbb{T}) \cap L^2_0(\mathbb{T})$. Since η is an even function, η_z is an odd function and therefore, the spectrum of $\mathcal{H}_a(\ell)$ is not symmetric with respect to the reflection through the origin.

Moreover, the operator $\mathcal{H}_a(\ell)$ is not real, therefore the spectrum of $\mathcal{H}_a(\ell)$ is not symmetric with respect to the real axis as well. The spectrum of $\mathcal{H}_a(\ell)$ inherits following symmetry property.

Lemma 3.3.6. *The spectrum of $\mathcal{H}_a(\ell)$ is symmetric with respect to the reflection through the imaginary axis.*

Proof. We consider $\mathcal{R}$ to be the reflection through the imaginary axis defined as follows

$$\mathcal{R}\psi(z) = \overline{\psi(-z)}.$$

Assume λ is the eigenvalue of $\mathcal{H}_a(\ell)$ with an associated eigenvector φ, then we have

$$\mathcal{H}_a(\ell)\varphi = \lambda\varphi. \tag{3.3.16}$$

Since

$$(\mathcal{H}_a(\ell)\mathcal{R}\psi)(z) = \mathcal{H}_a(\ell)(\mathcal{R}\psi(z)) = \mathcal{H}_a(\ell)\overline{\psi(-z)} = -(\overline{\mathcal{H}_a(\ell)\psi})(-z) = -(\mathcal{R}\mathcal{H}_a(\ell)\psi)(z),$$

therefore, $\mathcal{H}_a(\ell)$ anti-commutes with $\mathcal{R}$. Using (3.3.16), we arrive at

$$\mathcal{H}_a(\ell)\mathcal{R}\varphi = -\mathcal{R}\mathcal{H}_a(\ell)\varphi = -\overline{\lambda}\mathcal{R}\varphi.$$

We conclude from here that if λ is an eigenvalue of $\mathcal{H}_a(\ell)$ with associated eigenvector φ, then $-\overline{\lambda}$ is also an eigenvalue of $\mathcal{H}_a(\ell)$ with associated eigenvector $\mathcal{R}\varphi$. $\square$

As a consequence of the symmetry of the spectrum obtained in Lemma 3.3.6, we obtain instability if there is an eigenvalue of $\mathcal{H}_a(\ell)$ off the imaginary axis. A straightforward calculation reveals that

$$\mathcal{H}_0(\ell)e^{inz} = i\Omega_{n,\ell}e^{inz} \quad \text{for all} \quad n \in \mathbb{Z}^* := \mathbb{Z} \setminus \{0\}, \tag{3.3.17}$$

where

$$\Omega_{n,\ell} = k^3 n\left(1 - n^2\right) + \frac{3\ell^2}{kn}. \tag{3.3.18}$$

Therefore, the $L_0^2(\mathbb{T})$-spectrum of $\mathcal{H}_0(\ell)$ is given by

$$\text{spec}_{L_0^2(\mathbb{T})}(\mathcal{H}_0(\ell)) = \{i\Omega_{n,\ell}; n \in \mathbb{Z}^*\},$$

which implies $\text{spec}_{L_0^2(\mathbb{T})}(\mathcal{H}_0(\ell))$ consists of purely imaginary eigenvalues of finite multiplicity. This is because the coefficients of the operator $\mathcal{H}_0(\ell)$ are real, which should be the case, since zero-amplitude solutions are spectrally stable.

Spectra of $\mathcal{H}_a(\ell)$ and $\mathcal{H}_0(\ell)$ remain close for $|a|$ small as

$$||\mathcal{H}_a(\ell) - \mathcal{H}_0(\ell)|| \to 0 \text{ as } a \to 0,$$

in the operator norm. Due to the symmetry in Lemma 3.3.6, for $|a|$ sufficiently small, bifurcation of eigenvalues of $\mathcal{H}_a(\ell)$ from imaginary axis can happen only when a pair of eigenvalues of $\mathcal{H}_0(\ell)$ collide on the imaginary axis. Let $n \neq m \in \mathbb{Z}^*$, a pair of eigenvalues $i\Omega_{n,\ell}$ and $i\Omega_{m,\ell}$ of $\mathcal{H}_0(\ell)$ collide for some $\ell = \ell_c$ when

$$\Omega_{n,\ell_c} = \Omega_{m,\ell_c}. \tag{3.3.19}$$

We list all the collisions in the following lemma.

Lemma 3.3.7. *For a fix $\Delta \in \mathbb{N}$, eigenvalues $\Omega_{n,\ell}$ and $\Omega_{n+\Delta,\ell}$ of the operator $\mathcal{H}_0(\ell)$ collide for all $n \in (-\Delta, 0) \cap \mathbb{Z}$ at some $\ell = \ell_c(k)$. All such collisions take place away from the origin in the complex plane except when Δ is even and $n = -\Delta/2$ in which case eigenvalues $\Omega_{n,\ell}$ and $\Omega_{-n,\ell}$ collide at the origin.*

Proof. Without any loss of generality, consider $m > n$ and $m = n + \Delta$ with $\Delta \in \mathbb{N}$ in the collision condition (3.3.19) then we obtain

$$3\ell_c^2 = k^4 n(n+\Delta)(-3n^2 - 3n\Delta - \Delta^2 + 1), \tag{3.3.20}$$

which can be rewritten as

$$3\ell_c^2 = -k^4[3n^2(n+\Delta)^2 + n(n+\Delta)(\Delta^2 - 1)]. \tag{3.3.21}$$

The above equation implies that collision between n and $n + \Delta$ takes place if and only if $n(n + \Delta) < 0$, that is, $-\Delta < n < 0$. Observe that $\Omega_{n,\ell_c} = \Omega_{-n,\ell_c} = 0$ for $\ell_c^2 =$

$\dfrac{k^4n^2(n^2-1)}{3}$. Therefore, Ω_{n,ℓ_c} and $\Omega_{n+\Delta,\ell_c}$ collide at the origin when Δ is even and $n=-\Delta/2$. All other collisions are away from the origin. Hence, the lemma. $\square$

From (3.3.21), assume $\ell^2=-\dfrac{k^4}{3}f(n)g(n)$, where $f(n)=n(n+\Delta)$ and $g(n)=3n^2+3n\Delta+\Delta^2-1$. For a fixed $\Delta\in\mathbb{N}$, $f(n)\geq f(-\Delta/2)$ and $g(n)\geq g(-\Delta/2)$ for all $n\in(-\Delta,0)\cap\mathbb{Z}$. And $f(n)g(n)\leq f(-\Delta/2)g(-\Delta/2)$ for all $n\in(-\Delta,0)\cap\mathbb{Z}$. Also, $f(n)g(n)\geq-\dfrac{(\Delta^2-1)^2}{12}$. Therefore $\dfrac{k^4}{48}\Delta^2(\Delta^2-4)\leq\ell^2\leq\dfrac{k^4}{36}(\Delta^2-1)^2$. Collision for $\{n,n+\Delta\}=\{-1,1\}$ occur at $\ell=0$ and all other collision mentioned in Lemma 3.3.7 occur for $\ell^2\in\left[\dfrac{k^4}{48}\Delta^2(\Delta^2-4),\dfrac{k^4}{36}(\Delta^2-1)^2\right]$ with $\dfrac{k^4}{48}\Delta^2(\Delta^2-4)>0$.

Depending upon the investigation of different type of instabilities, we split our further analysis into following subsections.

3.3.3.1 Modulational transverse instabilities

Throughout this subsection, we work in the regime $|\ell|\ll 1$, that is, with respect to long wavelength perturbations. From Lemma 3.3.7, when $\ell=0$, there is a collision among the eigenvalues $i\Omega_{1,0}$ and $i\Omega_{-1,0}$ at the origin while all other eigenvalues, on the other hand, remain simple and purely imaginary. Since

$$\|\mathcal{H}_a(\ell)-\mathcal{H}_0(\ell)\|=O(|a|),$$

as $a\to 0$ uniformly in the operator norm. A standard perturbation argument assures that the spectrum of $\mathcal{H}_a(\ell)$ and $\mathcal{H}_0(\ell)$ will stay close for $|a|$ and $|\ell|$ small. Therefore, we may write that

$$\operatorname{spec}(\mathcal{H}_a(\ell))=\operatorname{spec}_0(\mathcal{H}_a(\ell))\cup\operatorname{spec}_1(\mathcal{H}_a(\ell)),$$

for a and ℓ sufficiently small where $\operatorname{spec}_0(\mathcal{H}_a(\ell))$ contains two eigenvalues bifurcating continuously in a from $i\Omega_{1,0}$ and $i\Omega_{-1,0}$ while $\operatorname{spec}_1(\mathcal{H}_a(\ell))$ consists of infinitely many simple eigenvalues. Further, we investigate if the pair of eigenvalues in $\operatorname{spec}_0(\mathcal{H}_a(\ell))$ bifurcate away from the imaginary axis and contribute to modulational transverse instabilities.

For $a=0$, $\operatorname{spec}_0(\mathcal{H}_0(\ell))=\{i\Omega_{-1,0},i\Omega_{1,0}\}$ with eigenfunctions $\{e^{-iz},e^{iz}\}$. We choose the real basis $\{\cos z,\sin z\}$. We calculate expansion of a basis $\{\psi_1,\psi_2\}$ for the eigenspace corresponding to the eigenvalues of $\operatorname{spec}_0(\mathcal{H}_a(\ell))$ in $L^2_0(\mathbb{T})$ by using expansions of η and

c in (3.3.10), for small a and ℓ as

$$\psi_1(z) = \cos z + 2aA_2 \cos 2z + 3a^2 A_3 \cos 3z + O(a^4),$$
$$\psi_2(z) = \sin z + 2aA_2 \sin 2z + 3a^2 A_3 \sin 3z + O(a^4).$$

We have the following expression for $\mathcal{H}_a(\ell)$ after expanding and using η and c

$$\begin{aligned}\mathcal{H}_a(\ell) = \mathcal{H}_0(\ell) + ka^2\left(c_2 + 6\ell A_0 - \frac{3}{4}\phi^2\right)\partial z + \left(6k\ell a - 3\phi^2 kA_0a^3 - \frac{3}{2}\phi^2 kA_2a^3\right)\partial_z(\cos z) + \\ ka^2\left(6\ell A_2 - \frac{3}{4}\phi^2\right)\partial_z(\cos 2z) + \left(6k\ell a^3 A_3 - \frac{3}{2}\phi^2 ka^3 A_2\right)\partial_z(\cos 3z) + i3\ell\phi(a\sin z + \\ 2a^2A_2 \sin 2z + 3a^3A_3 \sin 3z)\partial_z^{-1} + O(a^4).\end{aligned} \tag{3.3.22}$$

In order to locate the bifurcating eigenvalues for $|a|$ sufficiently small, we calculate the action of $\mathcal{H}_a(\ell)$ on the extended eigenspace $\{\psi_1(z), \psi_2(z)\}$ viz.

$$\mathcal{T}_a(\ell) = \left[\frac{\langle \mathcal{H}_a(\ell)\psi_i(z), \psi_j(z)\rangle}{\langle \psi_i(z), \psi_i(z)\rangle}\right]_{i,j=1,2} \text{ and } \mathcal{I}_a = \left[\frac{\langle \psi_i(z), \psi_j(z)\rangle}{\langle \psi_i(z), \psi_i(z)\rangle}\right]_{i,j=1,2}. \tag{3.3.23}$$

We use expansion of $\mathcal{H}_a(\ell)$ in (3.3.22) to find actions of $\mathcal{H}_a(\ell)$ and identity operator on $\{\psi_1, \psi_2\}$, and arrive at

$$\mathcal{T}_a(\ell) = \begin{pmatrix} 0 & -\dfrac{3\ell^2}{k} + 3a^2k\left(\dfrac{\phi^2}{4} - \dfrac{\ell^2}{k^2}\right) \\ \dfrac{3\ell^2}{k} & 0 \end{pmatrix} + O(a^2(\ell + a)).$$

To locate where these two eigenvalues are bifurcating from the origin, we analyze the characteristic equation $|\mathcal{T}_a(\ell) - \lambda\mathcal{I}| = 0$, where $\mathcal{I}_a$ is 2×2 identity matrix. From which we conclude that

$$\lambda = \pm\frac{3|\ell|}{k}\sqrt{\Lambda} + O(a(\ell + a)), \tag{3.3.24}$$

where

$$\Lambda = -\ell^2 + a^2k^2\left(\frac{\phi^2}{4} - \frac{\ell^2}{k^2}\right) + O(a^2(\ell + a)). \tag{3.3.25}$$

For $\ell = a = 0$, we get zero as a double eigenvalue, which agrees with our calculation. For

ℓ and a sufficiently small, we obtain two eigenvalues which have non-zero real part with opposite sign when

$$\ell^2 < a^2k^2\left(\frac{\phi^2}{4} - \frac{\ell^2}{k^2}\right) + O(a^2(\ell + a)), \tag{3.3.26}$$

which is possible only for

$$k > 2\left|\frac{\ell}{\phi}\right|.$$

Hence the theorem 3.3.2.

3.3.3.2 High-frequency transverse instabilities

Here, we work in the regime $|\ell| > |\ell_0|$, that is, with respect to finite or short wavelength perturbations. We need to examine whether eigenvalues which collide away from origin, indeed bifurcate from the imaginary axis or not. Note that, there is no collision for $\Delta = 1$ and 2.

3.3.3.3 (In)stability analysis for $\Delta = 3$

For $\Delta = 3$, we have two pairs of colliding eigenvalues $(\Omega_{-2,\ell_1}, \Omega_{1,\ell_1})$ and $(\Omega_{-1,\ell_2}, \Omega_{2,\ell_2})$. Moreover, $\ell_1 = \ell_2 = \ell_c$(say) and $\ell_c^2 = \frac{4}{3}k^4$. We further check if both of these pairs lead to instability. For some $n \in \mathbb{Z}^*$, assume that the eigenvalues $i\Omega_{n,\ell}$ and $i\Omega_{n+3,\ell}$ collide at $\ell = \ell_c$, that is

$$i\Omega_{n,\ell_c} = i\Omega_{n+3,\ell_c} = i\Omega(say). \tag{3.3.27}$$

The associated eigenvectors are $\{e^{inz}, e^{i(n+3)z}\}$ and we choose $\psi_{0,n}(z) = e^{inz}$ and $\psi_{0,n+3}(z) = e^{i(n+3)z}$ as basis for the corresponding eigenspace of $\mathcal{H}_0(\ell_c)$ generated by the two eigenvalues. Let $i\Omega + i\nu_{a,n}$ and $i\Omega + i\nu_{a,n+3}$ be the eigenvalues of $\mathcal{H}_a(\ell)$ bifurcating from $i\Omega_{n,\ell_c}$ and $i\Omega_{n+3,\ell_c}$ respectively, for $|a|$ and $|\ell - \ell_c|$ sufficiently small. We need to locate $\nu_{a,n}$ and $\nu_{a,n+3}$ for $|a|$ small as if any of them is located on right half complex plane, then we obtain high-frequency transverse instabilities. Let $\{\psi_{a,n}(z), \psi_{a,n+3}(z)\}$ be orthonormal basis for the corresponding eigenspace. Using the following representation for the expansion of eigenfunctions[14]

$$\psi_{a,r}(z) = e^{irz} + a\psi_{r,1} + a^2\psi_{r,2} + a^3\psi_{r,3} + O(a^4), \tag{3.3.28}$$

for $r = n, n+3$. Through the condition of orthogonality of eigenfunctions, we obtain,

$$\psi_{r,1} = \psi_{r,2} = \psi_{r,3} = 0,$$

for $r = n, n+3$. Assume $\varepsilon = \ell^2 - \ell_c^2$, sufficiently small. We use the expansion of $\mathcal{H}_a(\ell)$ in (3.3.22) to find actions of $\mathcal{H}_a(\ell)$ and identity operator on $\{\psi_{a,n,\ell}(z), \psi_{a,n+3,\ell}(z)\}$ as

$$\begin{aligned}\mathcal{T}_a(\ell) &= \begin{pmatrix} \dfrac{\langle \mathcal{H}_a(\ell)\psi_{a,n}, \psi_{a,n}\rangle}{\langle \psi_{a,n}, \psi_{a,n}\rangle} & \dfrac{\langle \mathcal{H}_a(\ell)\psi_{a,n}, \psi_{a,n+3}\rangle}{\langle \psi_{a,n}, \psi_{a,n}\rangle} \\ \dfrac{\langle \mathcal{H}_a(\ell)\psi_{a,n+3}, \psi_{a,n}\rangle}{\langle \psi_{a,n+3}, \psi_{a,n+3}\rangle} & \dfrac{\langle \mathcal{H}_a(\ell)\psi_{a,n+3}, \psi_{a,n+3}\rangle}{\langle \psi_{a,n+3}, \psi_{a,n+3}\rangle} \end{pmatrix} \\ &= \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} + O(a^4),\end{aligned}$$

where

$$\begin{aligned}
T_{11} &= i\Omega + \frac{i3\varepsilon}{kn} - inka^2\left[3\ell A_2 + \frac{3}{8}\phi^2\right], \\
T_{12} &= i3(n+3)k\ell a^3 A_3 - \frac{i3(n+3)}{4}\phi^2 k a^3 A_2 - \frac{9i}{2n}\ell\phi a^3 A_3, \\
T_{21} &= i3nk\ell a^3 A_3 - \frac{i3n}{4}\phi^2 k a^3 A_2 + \frac{9i}{2(n+3)}\ell\phi a^3 A_3, \\
T_{22} &= i\Omega + \frac{i3\varepsilon}{k(n+3)} - i(n+3)ka^2\left[3\ell A_2 + \frac{3}{8}\phi^2\right].
\end{aligned}$$

We compute

$$\det(\mathcal{T}_a(\ell) - (i\Omega + i\nu)\mathcal{I}_a) = 0, \tag{3.3.29}$$

where $\mathcal{I}_a$ is 2×2 identity matrix, and arrive at

$$\begin{aligned}\nu^2 - \nu\left[\frac{3\varepsilon}{k}\left(\frac{1}{n} + \frac{1}{n+3}\right) - \left(3\ell A_2 + \frac{3}{8}\phi^2\right)ka^2[n + (n+3)]\right] + \frac{9\varepsilon^2}{n(n+3)k^2} \\ + n(n+3)a^4k^2\left(3\ell A_2 + \frac{3}{8}\phi^2\right)^2 + O(a^2(|\varepsilon| + |a^3|)) = 0.\end{aligned}$$

A direct computation shows that the discriminant of the above quadratic is

$$\mathrm{disc}_a(\varepsilon) = \frac{81\varepsilon^2}{k^2n^2(n+3)^2} + 9k^2a^4\left(3\ell A_2 + \frac{3}{8}\phi^2\right)^2 + O(a^2(|\varepsilon| + |a^3|)).$$

For sufficiently small $|\varepsilon|$ and $|a|$, the leading term in the discriminant remains always positive, regardless of the values of n, ℓ and ϕ. As a result, there is no instability for $\Delta = 3$ situation for sufficiently small a.

3.3.3.4 (In)stability analysis for $\Delta \geq 4$

For some $n \in \mathbb{Z}^*$ and a fixed $\Delta \geq 4$, we have

$$i\Omega_{n,\ell_c} = i\Omega_{n+\Delta,\ell_c} = i\Omega. \tag{3.3.30}$$

We expand $\mathcal{H}_a(\ell)$ in a as

$$\begin{aligned}\mathcal{H}_a(\ell) =& \mathcal{H}_0(\ell) + (\beta_2 a^2 + \beta_4 a^4 + \dots)\partial_z + \alpha_1 a\partial_z(\cos z) + \cdots + \alpha_\Delta a^\Delta \partial_z(\cos(\Delta z)) \\ &+ (i\delta_1 a \sin z + \cdots + i\delta_\Delta a^\Delta \sin(\Delta z))\partial_z^{-1}.\end{aligned}$$

To explicitly obtain the values of all unknown coefficients in the expansion of $\mathcal{H}_a(\ell)$, we require coefficients of higher powers of a in the expansion of solution η. Calculating higher coefficients is difficult as the coefficients of the solution do not seem to have any obvious symmetry. Therefore, we pursue the instability analysis without calculating the unknown coefficients explicitly.

Following the same steps as in the previous subsection, we arrive at

$$\mathcal{T}_a(\ell) = \begin{bmatrix} i\Omega + \dfrac{i3\varepsilon}{kn} + in(\beta_2 a^2 + \beta_4 a^4 + \dots) & \dfrac{ia^\Delta}{2}\left(n\alpha_\Delta - \dfrac{\delta_\Delta}{(n+\Delta)}\right) \\ \dfrac{ia^\Delta}{2}\left((n+\Delta)\alpha_\Delta + \dfrac{\delta_\Delta}{n}\right) & i\Omega + \dfrac{i3\varepsilon}{k(n+\Delta)} + i(n+\Delta)(\beta_2 a^2 + \beta_4 a^4 + \dots) \end{bmatrix} + O(a^{\Delta+1}).$$

The resulting discriminant of the characteristic equation $\det(\mathcal{T}_a(\ell) - (i\Omega + i\nu)\mathcal{I}_a) = 0$ is

$$\mathrm{disc}_a(\varepsilon) = \frac{9\Delta^2\varepsilon^2}{k^2n^2(n+\Delta)^2} + \Delta^2\beta_2^2 a^4 + O(a^2(|\varepsilon| + |a^3|)).$$

which is positive for sufficiently small $|\varepsilon|$ and $|a|$ which implies that no eigenvalue of $\mathcal{H}_a(\ell)$ is bifurcating from the imaginary axis due to collision. As a consequence of this, the following result follows.

Lemma 3.3.8. *For all $k > 0$, periodic traveling waves (3.3.10) of KD equation does not possess any high-frequency transverse instability with respect to mean-zero periodic perturbations in a sufficiently small neighbourhood of $\ell = \ell_c$ and $a = 0$, irrespective of the values of ℓ and ϕ.*

3.3.4 Non-Periodic Perturbations

In this section, we aim to study the invertibility of $\mathcal{Q}_a(\lambda, \ell)$ acting in $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$ (with domain $H^4(\mathbb{R})$ or $C_b^4(\mathbb{R})$), for $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, and $\ell \in \mathbb{R}$, $\ell \neq 0$. In $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$, the operator $\mathcal{Q}_a(\lambda, \ell)$ no longer has point isolated spectrum; rather, it has continuous spectrum. Thus, we rely upon the Floquet Theory such that all solutions of (3.3.15) in $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$ are of the form $\psi(z) = e^{i\mu z}\Psi(z)$ where $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ is the Floquet exponent and $\Psi(z)$ is a 2π-periodic function, see [31] for a similar situation. By following same arguments as in the proof of [31, Proposition A.1], we can infer that the study of the invertibility of $\mathcal{Q}_a(\lambda, \ell)$ in $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$ is equivalent to the invertibility of the linear operators $\mathcal{Q}_{a,\mu}(\lambda, \ell)$ in $L^2(\mathbb{T})$ with domain $H^4(\mathbb{T})$, for all $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, where

$$\begin{aligned}\mathcal{Q}_{a,\mu}(\lambda, \ell) = &\left(\lambda - kc(\partial_z + i\mu) - k^3(\partial_z + i\mu)^3 - 6k\ell(\partial_z + i\mu)(\eta)\right)\left(k(\partial_z + i\mu)\right) \\ &+ \left(\frac{3}{2}\phi^2 k(\partial_z + i\mu)(\eta^2)\right)\left(k(\partial_z + i\mu)\right) + 3\ell^2 + i3k\ell\phi\eta_z.\end{aligned}$$

Since $\mu = 0$ corresponds to the periodic perturbations which we have already investigated, so now we would restrict ourselves to the case of $\mu \neq 0$. The $L^2(\mathbb{T})$-spectra of operator $\mathcal{Q}_{a,\mu}(\lambda, \ell)$ consist of eigenvalues of finite multiplicity. Therefore, $\mathcal{Q}_{a,\mu}(\lambda, \ell)$ is invertible in $L^2(\mathbb{T})$ if and only if zero is not an eigenvalue of $\mathcal{Q}_{a,\mu}(\lambda, \ell)$. Using this and invertibility of $\partial_z + i\mu$, we have the following result.

Lemma 3.3.9. *The operator $\mathcal{Q}_{a,\mu}(\lambda, \ell)$ is not invertible in $L^2(\mathbb{T})$ for some $\lambda \in \mathbb{C}$ and $\mu \neq 0$ if and only if $\lambda \in \mathrm{spec}_{L^2(\mathbb{T})}(\mathcal{H}_a(\ell, \mu))$, $L^2(\mathbb{T})$-spectrum of the operator,*

$$\mathcal{H}_a(\ell, \mu) := kc(\partial_z + i\mu) + k^3(\partial_z + i\mu)^3 + 6k\ell(\partial_z + i\mu)(\eta)$$

$$-\frac{3}{2}\phi^2 k(\partial_z + i\mu)\left(\eta^2\right) - \left(\frac{3\ell^2}{k} + i3\phi\ell\eta_z\right)(\partial_z + i\mu)^{-1}.$$

Proof. The proof is similar to Lemma 3.3.5. □

We will study the $L^2(\mathbb{T})$-spectra of linear operators $\mathcal{H}_a(\ell,\mu)$ for $|a|$ sufficiently small, and for $|\mu| > \delta > 0$ since the operator $(\partial_z + i\mu)^{-1}$ becomes singular, as $\mu \to 0$. Note that the spectrum of $\mathcal{H}_a(\ell,\mu)$ is not symmetric with respect to the reflection through the real axis or origin. Instead, we have the following symmetry.

Lemma 3.3.10. *The spectrum of $\mathcal{H}_a(\ell,\mu)$ is symmetric with respect to the reflection through the imaginary axis for all $\mu \in \left(-\frac{1}{2},\frac{1}{2}\right] \setminus \{0\}$.*

Proof. The proof is similar to Lemma 3.3.6. □

A standard perturbation argument assures that $\text{spec}_{L^2(\mathbb{T})}(\mathcal{H}_a(\ell,\mu))$ and $\text{spec}_{L^2(\mathbb{T})}(\mathcal{H}_0(\ell,\mu))$ will stay close for $|a|$ sufficiently small. Therefore, to locate the spectrum of $\mathcal{H}_a(\ell,\mu)$, we need to determine the spectrum of $\mathcal{H}_0(\ell,\mu)$. A simple calculation yields that

$$\mathcal{H}_0(\ell,\mu))e^{inz} = i\Omega_{n,\ell,\mu}e^{inz}, \quad n \in \mathbb{Z},$$

where

$$\Omega_{n,\ell,\mu} = k^3(n+\mu)\left(1-(n+\mu)^2\right) + \frac{3\ell^2}{k(n+\mu)}.$$

Therefore, the $L^2(\mathbb{T})$-spectrum of $\mathcal{H}_0(\ell,\mu)$ is given by

$$\text{spec}_{L^2(\mathbb{T})}(\mathcal{H}_0(\ell,\mu)) = \{i\Omega_{n,\ell,\mu}; n \in \mathbb{Z}, \mu \in (-1/2, 1/2] \setminus \{0\}\}. \tag{3.3.31}$$

Since if $\lambda \in \text{spec}_{L^2(\mathbb{T})}(\mathcal{H}_0(\ell,\mu))$ then $\bar{\lambda} \in \text{spec}_{L^2(\mathbb{T})}(\mathcal{H}_0(\ell,-\mu))$, therefore, it is enough to consider $\mu \in (0,1/2]$. Let $n \neq m \in \mathbb{Z}$, a pair of eigenvalues $i\Omega_{n,\ell,\mu}$ and $i\Omega_{m,\ell,\mu}$ of $\mathcal{H}_0(\ell,\mu)$ collide for some $\ell = \ell_c$ and $\mu \in (0,1/2]$ when

$$\Omega_{n,\ell_c,\mu} = \Omega_{m,\ell_c,\mu}. \tag{3.3.32}$$

We list all the collisions in the following lemma.

Lemma 3.3.11. *For a fix $\Delta \in \mathbb{N}$, eigenvalues $\Omega_{n,\ell,\mu}$ and $\Omega_{n+\Delta,\ell,\mu}$ of the operator $\mathcal{H}_0(\ell,\mu)$ collide for all $n \in [-\Delta,-1] \cap \mathbb{Z}$ along a curve $\ell = \ell_c(\mu)$, $\mu \in (0,1/2]$; except $\{n, n+\Delta\} = \{-1, 0\}$. All such collisions take place away from the origin in the complex plane except when Δ is odd and $n = -(\Delta+1)/2$ in which case eigenvalues $\Omega_{n,\ell,\mu}$ and $\Omega_{-n-1,\ell,\mu}$ collide at the origin for $\ell = \ell_c(1/2)$.*

Proof. Without loss of generality, assume that $m > n$ and $m = n + \Delta$ with $\Delta \in \mathbb{N}$. Then from collision condition (3.3.32), we obtain

$$3\ell^2 = -k^4[3(n+\mu)^2(n+\mu+\Delta)^2 + (n+\mu)(n+\mu+\Delta)(\Delta^2-1)]. \tag{3.3.33}$$

This implies that collision between n and $n+\Delta$ takes place only if $(n+\mu)(n+\mu+\Delta) < 0$, that is, $-\Delta \le n < 0$. In order to check for which $n \in [-\Delta, 0)$, there is indeed a collision, assume $n = -s$, $s \in \mathbb{N}$ such that $-s+\mu+\Delta > 0$. From collision condition (3.3.32), we get

$$3\ell^2\left(\frac{1}{s-\mu} + \frac{1}{-s+\mu+\Delta}\right) = k^4[(s-\mu)((s-\mu)^2-1) + (-s+\mu+\Delta)((-s+\mu+\Delta)^2-1)]. \tag{3.3.34}$$

There exist such ℓ satisfying (3.3.32) for all s and $-s+\Delta$, except $\{-s, -s+\Delta\} = \{-1, 0\}$. Hence the lemma. □

Note that $\Omega_{n,\ell,\mu} = 0$ at $\ell^2 = -\dfrac{k^4}{3}(n+\mu)^2(1-(n+\mu)^2)$. $\Omega_{n,\ell_c,\mu} = \Omega_{n+\Delta,\ell_c,\mu}$=0 for a fixed ℓ_c is possible only for $\Delta = -2n-1$, $\mu = 1/2$. Therefore, $\Omega_{n,\ell_c,\mu}$ and $\Omega_{n+\Delta,\ell_c,\mu}$ collide at the origin for $n = -(\Delta+1)/2$, for all $n \in [-\Delta,-1] \cap \mathbb{Z}$, $\mu = 1/2$ and $\ell_c^2 = \dfrac{k^4(2n+1)^2(4n^2+4n-3)}{48}$; except the pair $\{n, n+\Delta\} = \{-1, 0\}$. All other collisions are away from the origin.

From (3.3.33), assume $\ell^2 = -\dfrac{k^4}{3}d(n)h(n)$, where $d(n) = (n+\mu)(n+\mu+\Delta)$ and $h(n) = 3(n+\mu)^2 + 3(n+\mu)\Delta + \Delta^2 - 1$.

$$d(n) = (n+\mu)(n+\mu+\Delta) = (n+\mu)^2 + \Delta(n+\mu) = \left(n+\mu+\frac{\Delta}{2}\right)^2 - \frac{\Delta^2}{4}, \tag{3.3.35}$$

$$h(n) = 3(n+\mu)^2 + 3(n+\mu)\Delta + \Delta^2 - 1 = 3\left(n+\mu+\frac{\Delta}{2}\right)^2 + \frac{\Delta^2-4}{4}. \tag{3.3.36}$$

From (3.3.35) and (3.3.36), for a fixed $\Delta \in \mathbb{N}$, $f(n) \ge -\dfrac{\Delta^2}{4}$ and $g(n) \ge \dfrac{\Delta^2-4}{4}$ for all $n \in \mathbb{Z}$. Collision for $\Delta = 2$ occur for $\ell^2 \geq \frac{k^4}{4}\mu^3(2-\mu) > 0$ and all other collision mentioned in Lemma 3.3.11 occur for $\ell^2 \geq \frac{k^4}{48}\Delta^2(\Delta^2-4) > 0$. Also $\ell^2 \leq \frac{k^4}{36}(\Delta^2-1)^2$ for all $\Delta \in \mathbb{N}$. Therefore, Collision for $\Delta = 2$ occur for $\frac{k^4}{4}\mu^3(2-\mu) \leq \ell^2 \leq \frac{k^4}{4}$ and all other collisions occur for $\ell^2 \in \left[\frac{k^4}{48}\Delta^2(\Delta^2-4), \frac{k^4}{36}(\Delta^2-1)^2\right]$ with $\frac{k^4}{48}\Delta^2(\Delta^2-4) > 0$.

Since if $\lambda \in \operatorname{spec}_{L^2(\mathbb{T})}(\mathcal{H}_0(\ell,\mu))$ then $\bar{\lambda} \in \operatorname{spec}_{L^2(\mathbb{T})}(\mathcal{H}_0(\ell,-\mu))$, there will be collision between conjugate of eigenvalues mentioned in Lemma 3.3.11, for all $\mu \in (-1/2, 0)$. Table 3.2 summarizes all the collisions between $\Omega_{n,\ell_c,\mu}$ and $\Omega_{n+\Delta,\ell_c,\mu}$ for a given $\Delta \in \mathbb{N}$

$\boldsymbol{\Delta}$	$\boldsymbol{\{n, n+\Delta\}}$
1	none
2	$\{-2,0\},\{-1,1\},\{0,2\}$
≥ 3	$\{-1,\Delta-1\},\{-2,\Delta-2\},\ldots,\{-\Delta+1,1\},\{-\Delta,0\},\{0,\Delta\}$

Table 3.2: All Collisions for non-periodic perturbations.

and for some $\mu \in (-1/2, 1/2]$. More specifically, collisions for $\{-\Delta, 0\}$ occur for all $\mu \in (0, 1/2]$, for $\{0, \Delta\}$ occur for all $\mu \in (-1/2, 0)$, and the remaining collisions occur for all $\mu \in (-1/2, 1/2]$.

The perturbation analysis for the collisions mentioned in Lemma 3.3.11 will be performed with respect to finite or short wavelength perturbations.

3.3.4.1 High-frequency transverse instabilities

Here, we work in the regime $|\ell| > |\ell_0| > 0$. Using Table 3.2, note that, there is no collision for $\Delta = 1$.

3.3.4.2 (In)stability analysis for $\Delta = 2$

For $\Delta = 2$, we have three pairs of colliding eigenvalues $\{\Omega_{-1,\ell,\mu}, \Omega_{1,\ell,\mu}\}$, $\{\Omega_{0,\ell,\mu}, \Omega_{-2,\ell,\mu}\}$ and $\{\Omega_{0,\ell,\mu}, \Omega_{2,\ell,\mu}\}$. We further check if these pairs lead to instability. Let $i\Omega_{n,\ell,\mu}$ and $i\Omega_{n+2,\ell,\mu}$ be such two eigenvalues for some $n \in \mathbb{Z}$. Assume that these eigenvalues collide at $\ell = \ell_c$, that is

$$0 \neq \Omega_{n,\ell_c,\mu} = \Omega_{n+2,\ell_c,\mu} = \Omega \ (say). \tag{3.3.37}$$

That is, $i\Omega$ is an eigenvalue of $\mathcal{H}_0(\ell_c,\mu)$ of multiplicity two with an orthonormal basis of eigenfunctions $\{e^{inz}, e^{i(n+2)z}\}$. Let $i\Omega + i\nu_{a,n}$ and $i\Omega + i\nu_{a,n+2}$ be the eigenvalues of

$\mathcal{H}_a(\ell,\mu)$ bifurcating from $i\Omega_{n,\ell_c,\mu}$ and $i\Omega_{n+2,\ell_c,\mu}$ respectively, for $|a|$ and $|\ell-\ell_c|$ small. Let $\{\varphi_{a,n}(z),\varphi_{a,n+2}(z)\}$ be a orthonormal basis for the corresponding eigenspace. We assume the following expansions

$$\varphi_{a,n}(z) = e^{inz} + a\varphi_{n,1} + a^2\varphi_{n,2} + O(a^3), \tag{3.3.38}$$

$$\varphi_{a,n+2}(z) = e^{i(n+2)z} + a\varphi_{n+2,1} + a^2\varphi_{n+2,2} + O(a^3). \tag{3.3.39}$$

We use orthonormality of $\varphi_{a,n,\ell}$ and $\varphi_{a,n+2,\ell}$ to find that

$$\varphi_{n,1} = \varphi_{n,2} = \varphi_{n+2,1} = \varphi_{n+2,2} = 0.$$

Next, we calculate the action of $\mathcal{H}_a(\ell,\mu)$ on the eigenspace $\{\varphi_{a,n}(z),\varphi_{a,n+2}(z)\}$ for $|\ell-\ell_c|$ and $|a|$ small. We arrive at

$$\mathcal{T}_a(\ell,\mu) = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} + O(a^3(\ell^2+a^2)),$$

where

$$\begin{aligned}
T_{11} &= i\Omega + \frac{i3\varepsilon}{k(n+\mu)} - i(n+\mu)a^2k\left(\frac{3}{8}\phi^2 + \frac{3}{2}\frac{\ell^2}{k^2}\right),\\
T_{12} &= i(n+2+\mu)a^2k\left(\frac{3}{2}\frac{\ell^2}{k^2} - \frac{3}{8}\phi^2\right) - \frac{ia^2 3\ell\phi A_2}{(n+\mu)},\\
T_{21} &= i(n+\mu)a^2k\left(\frac{3}{2}\frac{\ell^2}{k^2} - \frac{3}{8}\phi^2\right) + \frac{ia^2 3\ell\phi A_2}{(n+2+\mu)},\\
T_{22} &= i\Omega + \frac{i3\varepsilon}{k(n+2+\mu)} - i(n+2+\mu)a^2k\left(\frac{3}{8}\phi^2 + \frac{3}{2}\frac{\ell^2}{k^2}\right),
\end{aligned}$$

and $\varepsilon = \ell^2-\ell_c^2$, sufficiently small. Further, we obtained the equation $\det(\mathcal{T}_a(\ell,\mu) - (i\Omega + i\nu)\mathcal{I}_a) = 0$, where $\mathcal{I}_a$ is 2×2 identity matrix, and concluded the discriminant Δ as

$$\begin{aligned}
\Delta = \frac{36\varepsilon^2}{k^2(n+\mu)^2(n+2+\mu)^2} - \frac{36\ell_c^2\phi^2a^4A_2^2}{(n+\mu)(n+2+\mu)} + 9a^4k^2(n+\mu+1)^2\left(\frac{\ell^4}{k^4} + \frac{\phi^4}{16}\right)\\
+ \frac{9a^4\ell^2\phi^2}{2k^2}(1-(n+\mu)(n+2+\mu)) + O(a^2|\varepsilon| + |a|^5).
\end{aligned}$$

Note that all the collisions listed in Table 3.2 for $\Delta = 2$ have $(n+\mu)(n+2+\mu) < 0$ which implies that for $|\varepsilon|$ and $|a|$ sufficiently small, the leading term in the discriminant is always positive for all ℓ and ϕ. Therefore, we do not get any instability for $\Delta = 2$ case for sufficiently small amplitude parameter a.

3.3.4.3 (In)stability analysis for $\Delta \geq 3$

We expand $\mathcal{H}_a(\ell,\mu)$ in a as

$$\begin{aligned}\mathcal{H}_a(\ell,\mu) =& \mathcal{H}_0(\ell,\mu) + (\beta_2 a^2 + \beta_4 a^4 + \dots)(\partial_z + i\mu) + \alpha_1 a(\partial_z + i\mu)(\cos z) + \dots \\ &+ \alpha_\Delta a^\Delta(\partial_z + i\mu)(\cos(\Delta z)) + (i\delta_1 a \sin z + \cdots + i\delta_\Delta a^\Delta \sin(\Delta z))(\partial_z + i\mu)^{-1}.\end{aligned}$$

We use the expansion of $\mathcal{H}_a(\ell,\mu)$ to find actions of $\mathcal{H}_a(\ell,\mu)$ and identity operator on $\{\psi_{a,n}(z), \psi_{a,n+\Delta}(z)\}$ as

$$\mathcal{T}_a(\ell,\mu) = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} + O(a^{\Delta+1}),$$

where

$$\begin{aligned}
T_{11} &= i\Omega + \frac{i3\varepsilon}{k(n+\mu)} + i(n+\mu)(\beta_2 a^2 + \beta_4 a^4 + \dots), \\
T_{12} &= \frac{ia^\Delta}{2}\left((n+\mu)\alpha_\Delta - \frac{\delta_\Delta}{(n+\Delta+\mu)}\right), \\
T_{21} &= \frac{ia^\Delta}{2}\left((n+\Delta+\mu)\alpha_\Delta + \frac{\delta_\Delta}{n+\mu}\right), \\
T_{22} &= i\Omega + \frac{i3\varepsilon}{k(n+\Delta+\mu)} + i(n+\Delta+\mu)(\beta_2 a^2 + \beta_4 a^4 + \dots).
\end{aligned}$$

The resulting discriminant of the characteristic equation $\det(\mathcal{T}_a(\ell,\mu) - (i\Omega + i\nu)\mathcal{I}_a) = 0$ is

$$\mathrm{disc}_a(\varepsilon) = \frac{9\Delta^2\varepsilon^2}{k^2(n+\mu)^2(n+\Delta+\mu)^2} + \Delta^2\beta_2^2 a^4 + O(a^2(|\varepsilon| + |a^3|)),$$

which is positive for sufficiently small $|\varepsilon|$ and $|a|$ which implies that no eigenvalue of $\mathcal{H}_a(\ell)$ is bifurcating from the imaginary axis due to collision. We conclude the following result.

Lemma 3.3.12. *For all $k > 0$, periodic traveling waves (3.3.10) of KD equation do not possess any high-frequency transverse instability with respect to non-periodic perturbations in a sufficiently small neighbourhood of $\ell = \ell_c$ and $a = 0$, irrespective of the values of ℓ and ϕ.*

We now turn to the final section of this chapter, where we examine transverse spectral instabilities in the rotation-modified Kadomtsev–Petviashvili (RMKP) equation and related models.

3.4 Transverse spectral Instabilities in rotation-modified Kadomtsev-Petviashvili equation and related models

The (2+1)-dimensional rotation-modified Kadomtsev-Petviashvili (RMKP) equation [26, 27] is

$$\left(u_t - \beta u_{xxx} + (u^2)_x\right)_x - \gamma u + u_{yy} = 0, \;\; \gamma > 0, \tag{3.4.1}$$

where, $u = u(x, y, t)$, $t \in \mathbb{R}^+$ is a temporal variable, $x, y \in \mathbb{R}$ are spatial variables. Here, x represents the direction of wave propagation and y represents the transverse direction to the motion of the wave. The type of dispersion is determined by the coefficient β. If the coefficient is $\beta < 0$ (negative dispersion), the equation simulates gravity surface waves in a shallow water channel and internal waves in the ocean, whereas if the coefficient is $\beta > 0$ (positive dispersion), the equation represents oblique magneto-acoustic waves or capillary surface waves in plasma. The parameter $\gamma > 0$, which is proportional to the Coriolis force, assesses the effects of rotation. Equation (3.4.1) simplifies to the Kadomtsev-Petviashvili (KP) equation [48] in the situation $\gamma = 0$, or without the effects of rotation.

$$\left(u_t + \beta u_{xxx} + (u^2)_x\right)_x + u_{yy} = 0, \tag{3.4.2}$$

and after removing y-term, it gets converted to the Ostrovsky equation[62]

$$\left(u_t + \beta u_{xxx} + (u^2)_x\right)_x - \gamma u = 0. \tag{3.4.3}$$

On one hand, the RMKP equation (3.4.1) can be viewed as a rotation-modified extension of the classical KP equation (3.4.2), incorporating Coriolis effects into the two-dimensional dispersive framework. On the other hand, it can also be regarded as a two-dimensional generalization of the Ostrovsky equation (3.4.3), which models rotational effects in unidirectional wave propagation. More broadly, the RMKP equation extends the well-known Korteweg–de Vries (KdV) equation by incorporating both transverse spatial dimensions and rotational influences. The KdV equation itself plays a fundamental role in describing nonlinear wave phenomena in weakly dispersive media, and its generalizations such as RMKP capture additional physical features essential for modeling geophysical flows.

One-dimensional periodic traveling waves

Since the y-independent form of (3.4.1) is the Ostrovsky equation (3.4.3), we are interested in small-amplitude periodic traveling waves of (3.4.3). The existence of a one-parameter family of such solutions has been established using the Lyapunov–Schmidt procedure, and their small-amplitude expansions have been derived in our work presented in Chapter 4 (see also [4, Appendix A and Theorem 2.1]). which we briefly describe here. Depending on the sign of β, take into account the following wavenumbers,

1. for $\beta < 0$, all $k > 0$, and
2. for $\beta > 0$, all $k > 0$ but $\left(\frac{\gamma}{\beta n^2}\right)^{1/4}$, $2 \leq n \in \mathbb{N}$.

Consequently, a one parameter family of solutions of (3.4.3) exists for all such wavenumbers k, given by $u(x,t) = \eta(a;k)(k(x - c(a;k)t))$ for $a \in \mathbb{R}$ and $|a|$ sufficiently small; $\eta(a;k)(\cdot)$ is 2π-periodic, even and smooth in its argument, and $c(an;k)$ is even in a; $\eta(a;k)$ and $c(a;k)$ depend analytically on a and k. Moreover,

$$\begin{cases} \eta(a;k)(z) = a\cos(z) + a^2\eta_2\cos 2z + a^3\eta_3\cos 3z + O(a^4), \\ c(a;k) = c_0 + a^2 c_2 + O(a^4), \end{cases} \tag{3.4.4}$$

as $a \to 0$,

$$\eta_2 = \frac{2k^2}{3\gamma - 12\beta k^4}, \quad \eta_3 = \frac{9k^2\eta_2}{8\gamma - 72\beta k^4}, \quad c_0 = \frac{\gamma}{k^2} + \beta k^2, \quad c_2 = \eta_2.$$

Main results

Our main findings are the following theorems, which, depending on the kind of the two-dimensional perturbations in the x- and y-directions, show the transverse stability and instability of small amplitude periodic traveling waves (3.4.4) of (3.4.1).

Theorem 3.4.1 (Transverse instability). *For a fixed $\gamma > 0$ and $\beta > 0$, sufficiently small amplitude periodic traveling waves (3.4.4) of the RMKP equation (3.4.1) are transversely unstable with respect to co-periodic perturbations in the x-direction and, periodic with long wavelength perturbations in the y- direction if*

$$k > \left|\frac{\gamma}{4\beta}\right|^{1/4}.$$

Theorem 3.4.2 (Transverse instability). *For a fixed $\gamma > 0$ and $\beta > 0$, sufficiently small amplitude periodic traveling waves (3.4.4) of the RMKP equation (3.4.1) are transversely unstable with respect to non-periodic (localized or bounded) perturbations in the direction of propagation of the wave and, periodic with finite wavelength perturbations in the transverse direction if*

$$k > \left|\frac{4\gamma}{\beta}\right|^{1/4}.$$

Theorem 3.4.3 (Transverse stability). *Assume that small-amplitude periodic traveling waves(3.4.4) of the Ostrovsky equation (3.4.3) are spectrally stable in $L^2(\mathbb{T})$ with respect to one-dimensional perturbations. Then, for $\beta \leq 0$, $\gamma > 0$, and $k > 0$, periodic traveling waves (3.4.4) of the RMKP equation(3.4.1) are transversely stable with respect to either periodic or non-periodic (localized or bounded) perturbations in the direction of wave propagation and, periodic in the transverse direction.*

Theorem 3.4.4 (Transverse stability). *Assume that small-amplitude periodic traveling waves(3.4.4) of the Ostrovsky equation (3.4.3) are spectrally stable in $L^2(\mathbb{T})$ with respect to one-dimensional perturbations. Then, for $\beta > 0$, $\gamma > 0$, and $k > 0$, periodic traveling waves (3.4.4) of the RMKP equation(3.4.1) are transversely stable with respect to periodic perturbations in the direction of wave propagation and, periodic with finite wavelength in the transverse direction.*

Generalized RMKP Model

By generalizing the dispersion and nonlinearity in the RMKP equation (3.4.1), we put forward the generalized Rotation-Modified Kadomtsev-Petviashvili (gRMKP) equation

$$\left(u_t + \mathcal{M}u_x + \alpha_1(u^2)_x + \alpha_2(u^3)_x\right)_x - \gamma u + u_{yy} = 0, \;\; \gamma > 0. \tag{3.4.5}$$

Here, the multiplier operator $\mathcal{M}$ is represented by the symbol $m(k)$ as

$$\widehat{\mathcal{M}f}(k) = m(k)\widehat{f}(k), \tag{3.4.6}$$

$\alpha_1 \in \{0, 1\}$, $\alpha_2 \in \{-1, 0\}$ and m satisfies Hypotheses 2.2.1 and m is strictly monotonic for $k > 0$. This equation can also be viewed as the extension of the generalized Ostrovsky equation

$$\left(u_t + \mathcal{M}u_x + \alpha_1(u^2)_x + \alpha_2(u^3)_x\right)_x - \gamma u = 0. \tag{3.4.7}$$

For different choices of $m(k)$, α_1, and α_2, we obtain several models of interest. In a similar fashion, one can define the Rotation-Modified Benjamin-Ono (RMBO) equation

$$(u_t + \beta\mathcal{H}u_{xx} + (u^2)_x)_x - \gamma u = 0, \tag{3.4.8}$$

by incorporating rotation effects in the Benjamin-Ono (BO) equation

$$u_t + \beta\mathcal{H}u_{xx} + (u^2)_x = 0. \tag{3.4.9}$$

Here, $\mathcal{H}$ is the Hilbert transform. The RMBO equation represents the long internal waves in a deep rotating fluid [26]. The equation (3.4.8) can be extended to two dimensions as

$$(u_t + \beta\mathcal{H}u_{xx} + (u^2)_x)_x - \gamma u + u_{yy} = 0, \tag{3.4.10}$$

which we call Rotation-Modified Benjamin-Ono Kadomtsev-Petviashvili (RMBO-KP) equation. We consider the following generalization of (3.4.1) and (3.4.10)

$$(u_t + \beta\Lambda^\alpha u_x + (u^2)_x)_x + u_{yy} - \gamma u = 0, \tag{3.4.11}$$

where the psuedo-differential operator $\Lambda^\alpha = (\sqrt{-\partial_x^2})^\alpha$ is defined by its Fourier symbol as $\widehat{\Lambda^\alpha f}(\kappa) = |\kappa|^\alpha \widehat{f}(\kappa)$. Here, $1 \leq \alpha \in \mathbb{R}$. Notice that, $\alpha = 2$ and $\alpha = 1$ in (3.4.11) gives (3.4.1) and (3.4.10) respectively. We say (3.4.11) as RM-fKdV-KP. Similarly, one can also define the RM-Whitham-KP equation as

$$(u_t + \beta \mathcal{M} u_x + (u^2)_x)_x + u_{yy} - \gamma u = 0, \tag{3.4.12}$$

where

$$\widehat{\mathcal{M}f}(k) = \sqrt{\frac{\tanh k}{k}} =: m(k)\widehat{f}(k), \tag{3.4.13}$$

and RMILW-KP generalizing the Intermediate Long wave (ILW) equation as

$$(u_t + \beta \mathcal{M} u_x + (u^2)_x)_x + u_{yy} - \gamma u = 0, \tag{3.4.14}$$

where

$$\widehat{\mathcal{M}f}(k) = k \coth k =: m(k)\widehat{f}(k). \tag{3.4.15}$$

Also, RMG-KP can be defined by generalizing the Gardner equation as follows

$$(u_t - \beta u_{xxx} + (u^2)_x - (u^3)_x)_x + u_{yy} - \gamma u = 0. \tag{3.4.16}$$

RM-mKdV-KP equation can be defined by generalizing the mKdV equation

$$(u_t - \beta u_{xxx} - (u^3)_x)_x + u_{yy} - \gamma u = 0. \tag{3.4.17}$$

For $\beta = 0$, We will call (3.4.1) as Reduced-RMKP equation. All these models discussed above fits into the model (3.4.5).

Seeking for one-dimensional traveling wave solution of (3.4.5) of the form $u(x, y, t) = U(x - c_\mathfrak{g} t)$, where $c \in \mathbb{R}$ is the speed of wave propagation, U satisfies the following

$$-c_\mathfrak{g} U'' + \mathcal{M} U'' + \alpha_1 (U^2)'' + \alpha_2 (U^3)'' - \gamma U = 0. \tag{3.4.18}$$

We look for a periodic solution of (3.4.18), that is, U is a $2\pi/k$-periodic function of its argument where $k > 0$ is the wave number. Taking $z := kx$, the function $\eta_\mathfrak{g}(z) := U(x)$

is periodic with period 2π in z and satisfies

$$k^2(-c_{\mathfrak{g}}\eta_{\mathfrak{g}}'' + \mathcal{M}_k\eta_{\mathfrak{g}}'' + \alpha_1(\eta_{\mathfrak{g}}^2)'' + \alpha_2(\eta_{\mathfrak{g}}^3)'') - \gamma\eta_{\mathfrak{g}} = 0. \tag{3.4.19}$$

Note that (3.4.19) is invariant under $z \mapsto z + z_0$ and $z \mapsto -z$ and hence, we may suppose that $\eta_{\mathfrak{g}}$ is even. Also, observe that (3.4.19) does not possess scaling invariance. We seek a non-trivial 2π-periodic solution $\eta_{\mathfrak{g}}$ of (3.4.19). For fixed $\gamma > 0$, let $F : H^4(\mathbb{T}) \times \mathbb{R} \times \mathbb{R}^+ \to L^2(\mathbb{T})$ be defined as

$$F(\eta_{\mathfrak{g}}, c_{\mathfrak{g}}; k) = k^2(-c_{\mathfrak{g}}\eta_{\mathfrak{g}}'' + \mathcal{M}_k\eta_{\mathfrak{g}}'' + \alpha_1(\eta_{\mathfrak{g}}^2)'' + \alpha_2(\eta_{\mathfrak{g}}^3)'') - \gamma\eta_{\mathfrak{g}}. \tag{3.4.20}$$

It is well defined by a Sobolev inequality. We seek a solution $\eta_{\mathfrak{g}} \in H^4(\mathbb{T})$, $c \in \mathbb{R}$ and $k > 0$ of

$$F(\eta_{\mathfrak{g}}, c_{\mathfrak{g}}; k) = 0.$$

Clearly, $F(0, c; k) = 0$ for all $c_{\mathfrak{g}} \in \mathbb{R}$ and $k > 0$. If non-trivial solutions of $F(\eta_{\mathfrak{g}}, c_{\mathfrak{g}}; k) = 0$ bifurcate from $\eta_{\mathfrak{g}} \equiv 0$ for some $c_{\mathfrak{g}}$ then

$$I_0 := \partial_{\eta_{\mathfrak{g}}} F(0, c_{\mathfrak{g}}; k) = -c_{\mathfrak{g}}k^2\partial_z^2 + k^2\mathcal{M}_k\partial_z^2 - \gamma,$$

from $H^4(\mathbb{T})$ to $L^2(\mathbb{T})$, is not an isomorphism. From a straightforward calculation,

$$I_0 e^{inz} = (c_{\mathfrak{g}}k^2n^2 - k^2n^2m(k) - \gamma)e^{inz} = 0, \quad n \in \mathbb{Z},$$

if and only if

$$c_{\mathfrak{g}} = m(k)n^2 + \frac{\gamma}{k^2n^2}, \quad n \in \mathbb{Z}.$$

Without loss of generality, we take $n = 1$ viz.

$$c_{\mathfrak{g}} = m(k) + \frac{\gamma}{k^2}. \tag{3.4.21}$$

Note that wavenumbers satisfying $k^2\left(m(kn) - m(k)\right) = \dfrac{\gamma(n^2-1)}{n^2}$, $2 \leq n \in \mathbb{N}$, satisfy resonance condition

$$\frac{\gamma}{k^2} + m(k) = \frac{\gamma}{k^2n^2} + m(kn), \tag{3.4.22}$$

of the fundamental mode and nth harmonic, making the kernel of I_0 four-dimensional. With both the kernel and the co-kernel being spanned by $e^{\pm iz}$, I_0 is a Fredholm operator of index zero for all other values of k.

The proof of existence of a one-parameter family of non-trivial solutions of $F(\eta_{\mathfrak{g}}, c_{\mathfrak{g}}; k) = 0$ bifurcating from $\eta_{\mathfrak{g}} \equiv 0$ and $c_{\mathfrak{g}} = c_{\mathfrak{g}_0}$ adheres to the same reasoning as the arguments in Chapter 4. We summarize the existence result for periodic traveling waves of (3.4.5) and their small-amplitude expansion below.

Theorem 3.4.5. *Depending on the nature of m, take into account the following wavenumbers,*

1. *for m decreasing, all $k > 0$, and*
2. *for m increasing, all $k > 0$ but those satisfying $k^2(m(kn) - m(k)) = \dfrac{\gamma(n^2-1)}{n^2}$; $2 \le n \in \mathbb{N}$.*

Consequently, a one parameter family of solutions of (3.4.19) exists for all such wavenumbers k, given by $u(x,t) = \eta_{\mathfrak{g}}(a;k)(k(x - c_{\mathfrak{g}}(a;k)t))$ for $a \in \mathbb{R}$ and $|a|$ sufficiently small; $\eta_{\mathfrak{g}}(a;k)(\cdot)$ is 2π-periodic, even and smooth in its argument, and $c_{\mathfrak{g}}(a;k)$ is even in a; $\eta_{\mathfrak{g}}(a;k)$ and $c_{\mathfrak{g}}(a;k)$ depend analytically on a and k. Moreover,

$$\eta_{\mathfrak{g}}(a;k)(z) = a\cos(z) + a^2\eta_{\mathfrak{g}_2}\cos 2z + a^3\eta_{\mathfrak{g}_3}\cos 3z + O(a^4), \tag{3.4.23}$$

and

$$c_{\mathfrak{g}}(a;k) = c_{\mathfrak{g}_0} + a^2 c_{\mathfrak{g}_2} + O(a^4),$$

as $a \to 0$, where $c_{\mathfrak{g}0}$ is in (4.3.3),

$$\eta_{\mathfrak{g}_2} = \frac{2\alpha_1 k^2}{3\gamma + 4k^2(m(k) - m(2k))}, \quad \eta_{\mathfrak{g}_3} = \frac{9\alpha_1 k^2\eta_{\mathfrak{g}_2} + 9/4\alpha_2 k^2}{8\gamma + 9k^2(m(k) - m(3k))},$$

$$c_{\mathfrak{g}_2} = \alpha_1\eta_{\mathfrak{g}_2} + (3/4)\alpha_2.$$

Starting now, we denote $\eta_{\mathfrak{g}}(k,a)(z)$ and $c_{\mathfrak{g}}(k,a)$ as $\eta_{\mathfrak{g}}$ and $c_{\mathfrak{g}}$ respectively. We study the transverse stability and instability of these waves with respect to two-dimensional perturbations of different nature. We obtain results generalizing Theorems 3.4.1 -3.4.4, which we list down below.

Theorem 3.4.6 (Transverse instability). *For a fixed $\gamma > 0$, sufficiently small amplitude periodic traveling waves (3.4.23) of the gRMKP equation (3.4.5) are transversely unstable with respect to periodic perturbations in both directions and long wavelength perturbations in the transverse direction of propagation when*

1. $\alpha_1 = 1$, $\alpha_2 = 0$

   (a) *m is increasing, and*

   $$|k^2(m(2k) - m(k))| > \frac{3\gamma}{4},$$

   (b) *m is decreasing, No instability.*

2. $\alpha_1 = 0$, $\alpha_2 = -1$; *all m, $\forall k > 0$.*

3. $\alpha_1 = 1$, $\alpha_2 = -1$, *all m, and*

$$\frac{2k^2}{3\gamma + 4k^2(m(k) - m(2k))} < \frac{3}{4}.$$

Theorem 3.4.7 (Transverse instability). *For fixed $\gamma > 0$ and m increasing, sufficiently small amplitude periodic traveling waves (3.4.23) of the gRMKP equation (3.4.5) are transversely unstable with respect to non-periodic (localized or bounded) perturbations in the direction of propagation of the wave and, periodic with finite wavelength perturbations in the transverse direction if*

$$k^2(m(k) - m(k/2)) > 3\gamma.$$

Theorem 3.4.8 (Transverse stability). *Assume that small-amplitude periodic traveling waves of the generalized Ostrovsky equation(3.4.7) are spectrally stable in $L^2(\mathbb{T})$ with respect to one-dimensional perturbations. Then, for m decreasing, $\gamma > 0$, and $k > 0$, periodic traveling waves (3.4.23) of the gRMKP equation(3.4.5) are transversely stable with respect to*

1. *either periodic or non-periodic perturbations in the direction of propagation and periodic perturbations in the transverse direction when $\alpha_2 = 0$,*

2. *non-periodic (localized or bounded) perturbations in the direction of propagation and periodic perturbations in the transverse direction when $\alpha_2 \neq 0$.*

Theorem 3.4.9 (Transverse stability). *Assume that small-amplitude periodic traveling waves of the generalized Ostrovsky equation*(3.4.7) *are spectrally stable in* $L^2(\mathbb{T})$ *with respect to one-dimensional perturbations. Then, for all choices of* m, $\gamma > 0$, *and* $k > 0$, *periodic traveling waves* (3.4.23) *of the gRMKP equation*(3.4.5) *are transversely stable with respect to periodic perturbations in the direction of propagation and, periodic with finite wavelength perturbations in the transverse direction.*

Theorems 3.4.6-3.4.9 have interesting consequences on example equations like RMBO-KP, RM-fKdV-KP, RMG-KP, RM-mKdV-KP, RM-Whitham-KP, and RMILW-KP. We list all these results in Table 3.3.

Equations	**LWTP**			**F/SWTP**		
	Periodic		**Non-periodic**	**Periodic**	**Non-periodic**	
	$\boldsymbol{\beta > 0}$	$\boldsymbol{\beta \leq 0}$	$\boldsymbol{\beta \in \mathbb{R}}$	$\boldsymbol{\beta \in \mathbb{R}}$	$\boldsymbol{\beta > 0}$	$\boldsymbol{\beta \leq 0}$
RMBO-KP	Unstable	Stable	Stable	Stable	Unstable	Stable
RM-fKdV-KP	Unstable	Stable	Stable	Stable	Unstable	Stable
RMG-KP	Unstable	Unstable	Stable	Stable	Unstable	Stable
RM-mKdV-KP	Unstable	Unstable	Stable	Stable	Unstable	Stable
RM-Whitham-KP	Stable	Unstable	stable	stable	Stable	Unstable
RMILW-KP	Unstable	Stable	Stable	Stable	Unstable	Stable

Table 3.3: Transverse stability or instability for different type of equations. Here 'LWTP' stands for *Long wavelength transverse perturbations* and 'F/SWTP' stands for *Finite or short wavelength transverse perturbations.*

In subection 3.4.1, we discuss RMKP equation (3.4.1) and prove Theorems 3.4.1-3.4.4. All results for RMKP equation generalizes for gRMKP equation (3.4.5) and analysis can be lifted from Section 3.4.1 as it is. We discuss this briefly in subsection 3.4.2 and prove Theorems 3.4.6-3.4.9. We discuss applications of Theorems 3.4.6-3.4.9 in Section 3.4.3 and provide a proof for results in Table 3.3 for example equations.

3.4.1 RMKP equation

3.4.1.1 Construction of the spectral problem

In a traveling frame of reference moving with velocity $c \in \mathbb{R}$, with coordinates $(x,t) \to (k(x-ct),t)$, (3.4.1) becomes

$$k(\eta_t - ck\eta_z - \beta k^3\eta_{zzz} + k(\eta^2)_z)_z + \eta_{yy} - \gamma\eta = 0, \tag{3.4.24}$$

where η is the one-dimensional periodic traveling wave solution of (3.4.1), given in (3.4.4). We consider perturbations to η of the form $\eta + \epsilon\eta^* + O(\epsilon^2)$ with $0 < |\epsilon| \ll 1$, we arrive at

$$k(\eta_t^* - ck\eta_z^* - \beta k^3 \eta_{zzz}^* + 2k(\eta\eta^*)_z)_z + \eta_{yy}^* - \gamma\eta^* = 0. \tag{3.4.25}$$

We are looking for a solution of the form

$$\eta^*(z, y, t) = e^{\frac{\lambda}{k}t + i\ell y}\varphi(z), \quad \lambda \in \mathbb{C}, \ \ell \in \mathbb{R}, \tag{3.4.26}$$

to arrive at

$$\mathcal{G}_a(\lambda, \ell)\varphi := (\lambda\partial_z - k^2\partial_z^2(c + \beta k^2\partial_z^2 - 2\eta) - \ell^2 - \gamma)\varphi = 0. \tag{3.4.27}$$

3.4.1.2 Periodic perturbations

In case of periodic perturbations, we investigate the invertibility of the operator $\mathcal{G}_a(\lambda, \ell)$ acting in $L^2(\mathbb{T})$ with domain $H^4(\mathbb{T})$, for $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$, $0 \neq \ell \in \mathbb{R}$. The invertibility problem

$$\mathcal{G}_a(\lambda, \ell)\varphi = 0, \quad \varphi \in L^2(\mathbb{T}),$$

is transformed into a spectral problem which requires invertibility of ∂_z. Since ∂_z is not invertible in $L^2(\mathbb{T})$, we restrict the problem to mean-zero subspace $L_0^2(\mathbb{T})$, defined in (2.5.11), of $L^2(\mathbb{T})$. The study of invertibility of the operator $\mathcal{G}_a(\lambda, \ell)$ in $L^2(\mathbb{T})$ is equivalent to the study of invertibility of the operator $\mathcal{G}_a(\lambda, \ell)$ in $L_0^2(\mathbb{T})$, as shown in the following lemma.

Lemma 3.4.10. *$\mathcal{G}_a(\lambda, \ell)$ acting in $L^2(\mathbb{T})$ with domain $H^4(\mathbb{T})$ is not invertible if and only if λ belongs to $L_0^2(\mathbb{T})$-spectrum of the operator $\mathcal{Q}_a(\ell)$, where*

$$\mathcal{Q}_a(\ell) := k^2\partial_z(c + \beta k^2\partial_z^2 - 2\eta) + (\gamma + \ell^2)\partial_z^{-1}. \tag{3.4.28}$$

Proof. Observe that if $\varphi \in L_0^2(\mathbb{T})$, then $\mathcal{G}_a(\lambda, \ell)\varphi \in L_0^2(\mathbb{T})$ implies that the subspace $L_0^2(\mathbb{T}) \subset L^2(\mathbb{T})$ is $\mathcal{G}_a(\lambda, \ell)$- invariant. Additionally, the spectrum is composed of discrete eigenvalues with finite multiplicity because the operator $\mathcal{G}_a(\lambda, \ell)$ acting on $L^2(\mathbb{T})$ has a compact resolvent. Therefore, $\mathcal{G}_a(\lambda, \ell)$ is not invertible if and only if zero is an eigenvalue of $\mathcal{G}_a(\lambda, \ell)$, that is, if and only if there exists a non-zero $\psi \in H^4(\mathbb{T})$ such that $\mathcal{G}_a(\lambda, \ell)\psi = 0$.

Since $\gamma \neq 0$, we conclude that $\psi \in L^2_0(\mathbb{T})$. Therefore, zero is an eigenvalue of $\mathcal{G}_a(\lambda, \ell)$ acting in $L^2(\mathbb{T})$ if and only if zero is an eigenvalue of the restriction of $\mathcal{G}_a(\lambda, \ell)$ to $L^2_0(\mathbb{T})$. This implies that $\mathcal{G}_a(\lambda, \ell)$ acting in $L^2(\mathbb{T})$ with domain $H^4(\mathbb{T})$ is not invertible if and only if its restriction to the subspace $L^2_0(\mathbb{T})$ is not invertible. Moreover, for a $\varphi \in L^2_0(\mathbb{T})$, $\mathcal{G}_a(\lambda, \ell)\varphi = 0$ if and only if $\mathcal{Q}_a(\ell)\varphi = \lambda\varphi$, where

$$\mathcal{Q}_a(\ell) := k^2\partial_z(c + \beta k^2\partial_z^2 - 2\eta) + (\gamma + \ell^2)\partial_z^{-1}. \tag{3.4.29}$$

Therefore, the operator $\mathcal{G}_a(\lambda, \ell)$ is not invertible in $L^2_0(\mathbb{T})$ for some $\lambda \in \mathbb{C}$ if and only if $\lambda \in \operatorname{spec}_{L^2_0(\mathbb{T})}(\mathcal{Q}_a(\ell))$, that is, $L^2_0(\mathbb{T})$-spectrum of the operator $\mathcal{Q}_a(\ell)$. Hence the lemma. □

We arrive at pseudo-differential spectral problem

$$\mathcal{Q}_a(\ell)\varphi = \lambda\varphi, \tag{3.4.30}$$

where $\varphi \in L^2_0(\mathbb{T})$. With respect to periodic perturbations, we will study the spectrum of the operator $\mathcal{Q}_a(\ell)$ acting in $L^2_0(\mathbb{T})$ with domain $H^3(\mathbb{T}) \cap L^2_0(\mathbb{T})$.

Proposition 3.4.11. *The operator $\mathcal{Q}_a(\ell)$ possess following properties.*

1. *The operator $\mathcal{Q}_a(\ell)$ commutes with the reflection through the real axis.*
2. *The operator $\mathcal{Q}_a(\ell)$ anti-commutes with the reflection through the origin and the imaginary axis.*
3. *The spectrum of $\mathcal{Q}_a(\ell)$ is symmetric with respect to the reflections through origin, real axis, and imaginary axis.*

Proof. We consider $\mathcal{R}_r$, $\mathcal{R}_i$ and $\mathcal{R}_o$ to be the reflections through the real axis, imaginary axis, and the origin, respectively, defined as follows

$$\mathcal{R}_r\psi(z) = \overline{\psi(z)}, \quad \mathcal{R}_i\psi(z) = \overline{\psi(-z)} \quad \text{and} \quad \mathcal{R}_o\psi(z) = \psi(-z). \tag{3.4.31}$$

Assume λ is an eigenvalue of $\mathcal{Q}_a(\ell)$ with an associated eigenvector φ, then we have

$$\mathcal{Q}_a(\ell)\varphi = \lambda\varphi. \tag{3.4.32}$$

Observe that

$$(\mathcal{Q}_a(\ell)\mathcal{R}_r\psi)(z) = \mathcal{Q}_a(\ell)(\mathcal{R}_r\psi(z)) = \mathcal{Q}_a(\ell)\overline{\psi(z)} = (\overline{\mathcal{Q}_a(\ell)\psi})(z) = (\mathcal{R}_r\mathcal{Q}_a(\ell)\psi)(z), \tag{3.4.33}$$

$$(\mathcal{Q}_a(\ell)\mathcal{R}_i\psi)(z) = \mathcal{Q}_a(\ell)(\mathcal{R}_i\psi(z)) = \mathcal{Q}_a(\ell)\overline{\psi(-z)} = -(\overline{\mathcal{Q}_a(\ell)\psi})(-z) = -(\mathcal{R}_i\mathcal{Q}_a(\ell)\psi)(z), \tag{3.4.34}$$

$$(\mathcal{Q}_a(\ell)\mathcal{R}_o\psi)(z) = \mathcal{Q}_a(\ell)(\mathcal{R}_o\psi(z)) = \mathcal{Q}_a(\ell)\psi(-z) = -(\mathcal{Q}_a(\ell)\psi)(-z) = -(\mathcal{R}_o\mathcal{Q}_a(\ell)\psi)(z). \tag{3.4.35}$$

From (3.4.33), $\mathcal{Q}_a(\ell)$ commutes with $\mathcal{R}_r$. From (3.4.34) and (3.4.35), we conclude that $\mathcal{Q}_a(\ell)$ anti-commutes with $\mathcal{R}_i$ and $\mathcal{R}_o$. Using (3.4.32), we arrive at

$$\mathcal{Q}_a(\ell)\mathcal{R}_r\varphi = \mathcal{R}_r\mathcal{Q}_a(\ell)\varphi = \overline{\lambda}\mathcal{R}_r\varphi,$$

$$\mathcal{Q}_a(\ell)\mathcal{R}_i\varphi = -\mathcal{R}_i\mathcal{Q}_a(\ell)\varphi = -\overline{\lambda}\mathcal{R}_i\varphi,$$

$$\mathcal{Q}_a(\ell)\mathcal{R}_o\varphi = -\mathcal{R}_o\mathcal{Q}_a(\ell)\varphi = -\lambda\mathcal{R}_o\varphi.$$

We conclude from here that if λ is an eigenvalue of $\mathcal{Q}_a(\ell)$ with associated eigenvector φ, then $\overline{\lambda}$, $-\overline{\lambda}$ and $-\lambda$ are also eigenvalues of $\mathcal{Q}_a(\ell)$ with associated eigenvectors $\mathcal{R}_r\varphi$, $\mathcal{R}_i\varphi$ and $\mathcal{R}_o\varphi$, respectively. Therefore, the spectrum of $\mathcal{Q}_a(\ell)$ is symmetric with respect to the reflections through the origin, real axis and imaginary axis. □

Here, we check if there exist any $\lambda \in \mathrm{spec}_{L^2_0(\mathbb{T})}(\mathcal{Q}_a(\ell))$ with $\Re(\lambda) > 0$ for some $\ell \neq 0$. As discussed above, $\mathrm{spec}_{L^2_0(\mathbb{T})}(\mathcal{Q}_a(\ell))$ inherits quadrafold symmetry that means an eigenvalue of $\mathcal{Q}_a(\ell)$ away from the imaginary axis guarantees an eigenvalue with positive real part. The spectral analysis for the operator $\mathcal{Q}_a(\ell)$ is based on perturbation arguments in which we consider $\mathcal{Q}_a(\ell)$ as a perturbation of the operator $\mathcal{Q}_0(\ell)$ for $|a|$ sufficiently small. More precisely, for any $d > 0$, there exist $\delta > 0$ such that for any a with $\|a\| \leq \delta$, the spectrum of $\mathrm{spec}_{L^2_0(\mathbb{T})}(\mathcal{Q}_a(\ell))$ satisfies

$$\mathrm{spec}_{L^2_0(\mathbb{T})}(\mathcal{Q}_a(\ell)) \subset \{\lambda \in \mathbb{C}; \mathrm{dist}(\lambda, \mathrm{spec}_{L^2_0(\mathbb{T})}(\mathcal{Q}_0(\ell))) < d\}.$$

To find the spectrum of $\mathcal{Q}_a(\ell)$, we require to find the spectrum of $\mathcal{Q}_0(\ell)$. Since $\mathrm{spec}_{L^2_0(\mathbb{T})}(\mathcal{Q}_0(\ell))$

is a differential operator with periodic coefficients, a straightforward Fourier analysis allows to compute its spectrum explicitly as

$$\mathcal{Q}_0(\ell)e^{inz} = i\Omega_{n,\ell}e^{inz} \quad \text{for all} \quad n \in \mathbb{Z} \setminus \{0\}, \tag{3.4.36}$$

where

$$\Omega_{n,\ell} = \gamma\left(n - \frac{1}{n}\right) + \beta k^4(n - n^3) - \frac{\ell^2}{n}. \tag{3.4.37}$$

Therefore, the $L^2_0(\mathbb{T})$-spectrum of $\mathcal{Q}_0(\ell)$ is given by

$$\text{spec}_{L^2_0(\mathbb{T})}(\mathcal{Q}_0(\ell)) = \{i\Omega_{n,\ell}; n \in \mathbb{Z}^*\}, \tag{3.4.38}$$

which implies $\text{spec}_{L^2_0(\mathbb{T})}(\mathcal{Q}_0(\ell))$ contains purely imaginary eigenvalues with finite algebraic multiplicity, which has to be like this since $a = 0$ represents the zero solution, which is trivially stable. The eigenvalues in (3.4.38) shift around and may depart from the imaginary axis when $|a|$ rises, resulting in spectral instability. Since the spectrum around the real and imaginary axes is symmetric, it follows that for sufficiently small values of $|a|$, eigenvalues of $\mathcal{Q}_a(\ell)$ must bifurcate in pairs as a result of collisions of eigenvalues of $\mathcal{Q}_0(\ell)$ on the imaginary axis.

Let $n \neq m \in \mathbb{Z}^*$, a pair of eigenvalues $i\Omega_{n,\ell}$ and $i\Omega_{m,\ell}$ of $\mathcal{Q}_0(\ell)$ collide for some $\ell = \ell_c$ when

$$\Omega_{n,\ell_c} = \Omega_{m,\ell_c}. \tag{3.4.39}$$

Without any loss of generality, consider $n < m$ and $m = n + \Theta$ with $\Theta \in \mathbb{N}$ in collision condition (3.4.39), we arrive at

$$\ell^2 = -\gamma J(n, \Theta) + \beta k^4 Y(n, \Theta), \tag{3.4.40}$$

where,

$$J(n, \Theta) = n(n + \Theta) + 1 \quad \text{and} \quad Y(n, \Theta) = 3n^2(n + \Theta)^2 + n(n + \Theta)(\Theta^2 - 1).$$

Analyzing signs of $J(n,\Theta)$ and $Y(n,\Theta)$ leads us to the following result.

Lemma 3.4.12. *For a fixed $\gamma > 0$, $\beta \in \mathbb{R}$ and each $\Theta \in \mathbb{N}$, eigenvalues $\Omega_{n,\ell}$ and $\Omega_{n+\Theta,\ell}$ of the operator $\mathcal{Q}_0(\ell)$ collide for all $n \in \mathbb{Z}^*$ when $\beta > 0$, and all $n \in (-\Theta, 0)$ when $\beta \leq 0$. For $\beta > 0$, all collisions occur in an interval (k^*, ∞) if $n \in (-\infty, -\Theta) \cup (0, \infty)$, in $(0, k^*)$ if $n \in (-\Theta, 0)$, and for all $k > 0$ if $n \in (-2, 0)$ where $k^* = (\gamma J(n,\Theta)/\beta Y(n,\Theta))^{1/4}$. For $\beta \leq 0$, all collisions occur for all $k > 0$. All collisions take place away from the origin in the complex plane except for the pair $\Omega_{n,\ell}$ and $\Omega_{-n,\ell}$.*

Proof. Note that

$$J(n,\Theta) = \left(n + \frac{\Theta}{2}\right)^2 - \frac{\Theta^2 - 4}{4}. \tag{3.4.41}$$

For a fixed $\Theta \in \mathbb{N}$, $J(n_1,\Theta) = J(n_2,\Theta) = 0$, where

$$n_1 = -\frac{\Theta + \sqrt{\Theta^2 - 4}}{2} \quad \text{and} \quad n_2 = -\frac{\Theta - \sqrt{\Theta^2 - 4}}{2}.$$

Note that n_1, n_2 are purely complex for $\Theta = 1$, real and equal for $\Theta = 2$, and real and distinct for $\Theta \geq 3$. Moreover, for $\Theta \geq 3$, $-\Theta < n_1 < -\Theta + 1/2$, and $-1/2 < n_2 < 0$. Combining these, we have, $J(n,\Theta) > 0$ when $n \in (-\infty, -\Theta) \cup (0, \infty)$ for all $\Theta \in \mathbb{N}$, $J(-1, 2) = 0$, and $J(n,\Theta) < 0$ when $n \in (-\Theta, 0)$ for all $\Theta \geq 3$.

Now, rewriting $Y(n,\Theta)$, we obtain

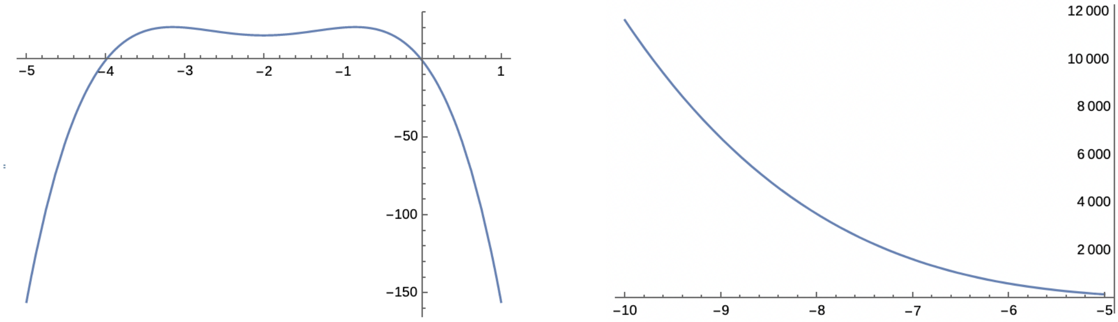
$$\begin{aligned} Y(n,\Theta) =& 3\left(n(n+\Theta) + \frac{\Theta^2 - 1}{6}\right)^2 - \frac{(\Theta^2-1)^2}{12} \\ =& \left(\left(n + \frac{\Theta}{2}\right)^2 - \frac{\Theta^2}{4}\right)\left(3\left(n + \frac{\Theta}{2}\right)^2 + \frac{\Theta^2}{4} - 1\right) \\ =&: Y_1(n,\Theta) Y_2(n,\Theta). \end{aligned}$$

A root analysis similar to $J(n,\Theta)$ on $Y_1(n,\Theta)$ and $Y_2(n,\Theta)$ provides that $Y(n,\Theta) > 0$ when $n \in (-\infty, -\Theta) \cup (0, \infty)$ for all $\Theta \in \mathbb{N}$, $Y(-1, 2) = 0$, and $Y(n,\Theta) < 0$ when $n \in (-\Theta, 0)$ for all $\Theta \geq 3$.

Note that, there exists a $\ell \in \mathbb{R}$ satisfying the collision condition (3.4.40) if

$$X(n,\Theta) := -\gamma J(n,\Theta) + \beta k^4 Y(n,\Theta) > 0. \tag{3.4.42}$$

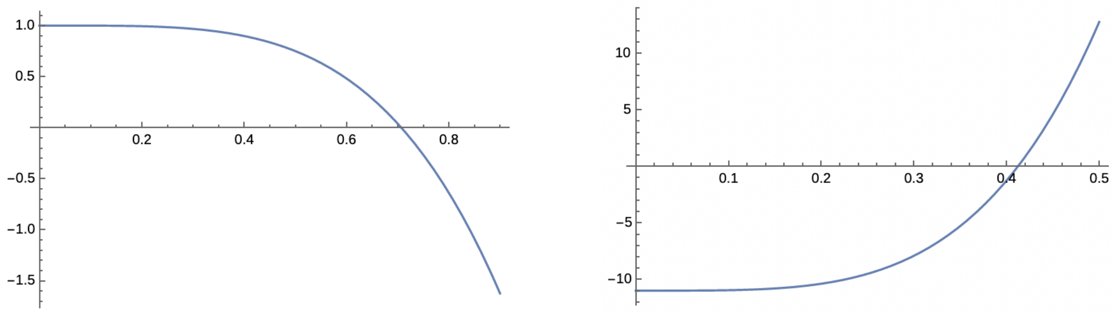
We know that $J(n,\Theta), Y(n,\Theta) > 0$ when $n \in (-\infty, -\Theta) \cup (0, \infty)$ for all $\Theta \in \mathbb{N}$. The



((a)) $X(n,4)$ with $n: -5$ to 1, $\beta=-1$, $\gamma=k=1$

((b)) $X(n,4)$ with $n: -10$ to -5, $\beta=\gamma=k=1$

Figure 3.2: Graph of function $X(n,\Theta)$ vs. n for $\Theta=4$



((a)) $X(-2,3)$ with $k\in(0,0.9)$, $\gamma=\beta=1$

((b)) $X(-5,3)$ with $k\in(0,0.5)$, $\gamma=\beta=1$

Figure 3.3: Graph of function $X(n,\Theta)$ vs. k for $n=-2,-5$ and $\Theta=3$

condition in (3.4.42) does not hold for any $n\in(-\infty,-\Theta)\cup(0,\infty)$ when $\beta<0$. For $\beta>0$, (3.4.42) holds for $n\in(-\infty,-\Theta)\cup(0,\infty)$ only when $k>k^*$. Since $J(n,\Theta),Y(n,\Theta)\le 0$ when $n\in(-\Theta,0)$, there exist $\ell\in\mathbb{R}$ satisfying the collision condition (3.4.40) for all $k>0$ when $\beta\le 0$; and for all $k<k^*$ when $\beta>0$. For instance, see Figures 3.2 and 3.3. In Figure 3.2 (A), we fix $\beta=-1$, $\gamma=k=1$, $\Theta=4$, then $X(n,\Theta)>0$ for $n\in(-4,0)$ and $X(n,\Theta)<0$ otherwise. In Figure 3.2 (B) for $\beta=\gamma=k=1$, $\Theta=4$, $X(n,\Theta)>0$ for all $n\in(-10,-5)$. In Figure 3.3 (A), for $\beta=\gamma=1$, $\Theta=3$, $n=-2$; (3.4.42) holds for $k\in(0,0.707)$ and does not hold otherwise. Also, in Figure 3.3 (B), for $\beta=\gamma=1$, $\Theta=3$, $n=-5$; (3.4.42) holds for $k\in(0.412,\infty)$ and does not hold otherwise. Observe that, $\Omega_{n,\ell_c}=\Omega_{-n,\ell_c}=0$ for $\ell_c^2=(n^2-1)|\gamma-\beta k^4n^2|$. Therefore, Ω_{n,ℓ_c} and $\Omega_{n+\Theta,\ell_c}$ collide at the origin when Θ is even and $n=-\Theta/2$. All other collisions are away from origin. □

For $|a|$ sufficiently small, $\text{spec}_{L^2_0(\mathbb{T})}(\mathcal{Q}_a(\ell))$ contain eigenvalues near to the eigenvalues of $\text{spec}_{L^2_0(\mathbb{T})}(\mathcal{Q}_0(\ell))$ which depend continuously upon a. The opposite sign of the *Krein signatures* of the colliding non-zero eigenvalues is a prerequisite for instability [60]. The linear operator $\mathcal{Q}_a(\ell)$ can be decomposed into the product of a skew-adjoint and a self-adjoint operator as follows

$$\mathcal{Q}_a(\ell) = \mathcal{A}\mathcal{B}_{a,\ell},$$

where $\mathcal{A} = \partial_z$ is skew-adjoint and

$$\mathcal{B}_{a,\ell} = k^2(c + \beta k^2 \partial_z^2 - 2\eta) + (\gamma + \ell^2)\partial_z^{-2},$$

is self-adjoint.

An eigenvalue $\lambda \in i\mathbb{R} \setminus \{0\}$ has negative Krein signature if $\langle \phi, \mathcal{Q}_a(\ell)\phi \rangle < 0$, where ϕ is the corresponding eigenfunction, and it has positive Krein signature if $\langle \phi, \mathcal{Q}_a(\ell)\phi \rangle > 0$. In particular, for sufficiently small a, the Krein signature $\chi_{n,\ell}$ of eigenvalues $i\Omega_{n,\ell}$ in (3.4.37) of $\mathcal{Q}_0(\ell)$ is given by

$$\chi_{n,\ell} = \operatorname{sgn}(\langle \mathcal{B}_{0,\ell} e^{inz}, e^{inz} \rangle) = \operatorname{sgn}\left(\frac{1}{n}\Omega_{n,\ell}\right). \tag{3.4.43}$$

Here, sgn is the signum function which establishes the sign of a real number. If the collision condition (3.4.39) is satisfied for some $n, m \in \mathbb{Z}^*$, then (3.4.43) states that at the collision, the non-zero eigenvalues $i\Omega_{n,\ell}$ and $i\Omega_{m,\ell}$ have opposite Krein signatures if

$$nm < 0, \tag{3.4.44}$$

otherwise they have same Krein signatures at the collision. We rule out several collisions indicated in Lemma 3.4.12 that won't cause transverse instability, using (3.4.44).

Lemma 3.4.13 (Potentially unstable nodes)**.** *Out of all the collisions mentioned in Lemma 3.4.12,*

1. *For $\ell = 0$, $\beta \in \mathbb{R}$, $\{n, n+\Theta\} = \{-1, 1\}$, are potentially unstable with respect to long wavelength transverse perturbations.*

2. *For all $\beta \in \mathbb{R}$; $\{n, n+\Theta\}$, $\ell \neq 0$ with $n \in (-\Theta, 0)$, $\Theta \geq 3$ are potentially unstable with respect to finite or short wavelength transverse perturbations.*

3.4.1.3 Localized or bounded perturbations

We check the invertibility of the operator $\mathcal{G}_a(\lambda,\ell)$ acting in $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$ with domain $H^4(\mathbb{R})$ or $C_b^4(\mathbb{R})$, for $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$, $0 \neq \ell \in \mathbb{R}$. In $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$, the operator $\mathcal{G}_a(\lambda,\ell)$ has continuous spectrum. Since the coefficients of the operator $\mathcal{G}_a(\lambda,\ell)$ are 2π-periodic, we can use the Floquet theory such that all solutions of (3.4.27) in $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$ are of the form $\varphi(z) = e^{i\mu z}\tilde{\varphi}(z)$ where $\mu \in (-1/2, 1/2]$ is the Floquet exponent and $\tilde{\varphi}$ is a 2π-periodic function, see [31] for a comparable circumstance. By following same arguments as in the proof of [31, Proposition A.1], we can deduce that the study of the invertibility of $\mathcal{G}_a(\lambda,\ell)$ in $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$ is equivalent to the invertibility of the linear operators $\mathcal{G}_{a,\mu}(\lambda,\ell)$ in $L^2(\mathbb{T})$ with domain $H^4(\mathbb{T})$, for all $\mu \in (-1/2, 1/2]$, where

$$\mathcal{G}_{a,\mu}(\lambda,\ell) := \lambda(\partial_z + i\mu) - k^2(\partial_z + i\mu)^2(c + \beta k^2(\partial_z + i\mu)^2 - 2\eta) - \ell^2 - \gamma.$$

We will confine ourselves to the case $\mu \neq 0$ because $\mu = 0$ refers to periodic perturbations. The $L^2(\mathbb{T})$-spectra of operator $\mathcal{G}_{a,\mu}(\lambda,\ell)$ consist of eigenvalues of finite multiplicity. Therefore, $\mathcal{G}_{a,\mu}(\lambda,\ell)$ is not invertible in $L^2(\mathbb{T})$ if and only if zero is an eigenvalue of $\mathcal{G}_{a,\mu}(\lambda,\ell)$. For a $\Psi \in L^2(\mathbb{T})$, $\mathcal{G}_{a,\mu}(\lambda,\ell)\Psi = 0$ if and only if $\mathcal{Q}_a(\ell,\mu)\Psi = \lambda\Psi$, where

$$\mathcal{Q}_a(\ell,\mu) := k^2(\partial_z + i\mu)(c + \beta k^2(\partial_z + i\mu)^2 - 2\eta) + (\gamma + \ell^2)(\partial_z + i\mu)^{-1}. \tag{3.4.45}$$

Therefore, the operator $\mathcal{G}_{a,\mu}(\lambda,\ell)$ is not invertible in $L^2(\mathbb{T})$ for some $\lambda \in \mathbb{C}$ and $\mu \neq 0$ if and only if $\lambda \in \operatorname{spec}_{L^2(\mathbb{T})}(\mathcal{Q}_a(\ell,\mu))$, $L^2(\mathbb{T})$-spectrum of the operator. This is straight forward to observe that $\operatorname{spec}_{L^2(\mathbb{T})}(\mathcal{Q}_a(\ell,\mu))$ is not symmetric with respect to the reflections through the real axis and the origin, rather it exhibit following properties.

Proposition 3.4.14. *The operator $\mathcal{Q}_a(\ell,\mu)$ possesses following properties.*

1. *The operator $\mathcal{Q}_a(\ell,-\mu)$ commutes with the reflection through the real axis.*
2. *The operator $\mathcal{Q}_a(\ell,\mu)$ anti-commutes with the reflection through the imaginary axis.*
3. *The operator $\mathcal{Q}_a(\ell,-\mu)$ anti-commutes with the reflection through the origin.*
4. *The spectrum of $\mathcal{Q}_a(\ell,\mu)$ is symmetric with respect to the reflections through the imaginary axis.*

5. *The spectrum of $\mathcal{Q}_a(\ell, -\mu)$ is symmetric with respect to the reflections through the real axis and the origin.*

Proof. Proof is similar to the Proposition 3.4.11. □

As a consequence of the properties mentioned in Proposition 3.4.14, it is sufficient to take $\mu \in (0, 1/2]$. Therefore, we arrive at pseudo-differential spectral problem

$$\mathcal{Q}_a(\ell, \mu)\varphi = \lambda\varphi; \quad \varphi \in L^2(\mathbb{T}) \quad \text{and} \quad \mu \in (0, 1/2]. \tag{3.4.46}$$

In case of non-periodic perturbations, we need to investigate if there exist any $\lambda \in \text{spec}_{L^2(\mathbb{T})}(\mathcal{Q}_a(\ell, \mu))$ with $\Re(\lambda) > 0$ for some $\ell \neq 0$ and $\mu \in (0, 1/2]$. As a consequence of the symmetry of the spectrum derived in Proposition 3.4.14, we obtain instability if there is an eigenvalue of $\mathcal{Q}_a(\ell, \mu)$ off the imaginary axis. We regard $\mathcal{Q}_a(\ell, \mu)$ as a perturbation of the operator $\mathcal{Q}_0(\ell, \mu)$. Consider

$$\mathcal{Q}_a^*(\ell, \mu) = \mathcal{Q}_a(\ell, \mu) - \mathcal{Q}_0(\ell, \mu).$$

This operator is compact in $L^2(\mathbb{T})$, with $\|\mathcal{Q}_a^*(\ell, \mu)\| \to 0$ as $a \to 0$ in the operator norm. Note that this estimate is uniform for $\mu \in (0, 1/2]$. Therefore, spectra of $\mathcal{Q}_a(\ell, \mu)$ and $\mathcal{Q}_0(\ell, \mu)$ remain close for sufficiently small a. To find the spectrum of $\mathcal{Q}_a(\ell, \mu)$, we require to find the spectrum of $\mathcal{Q}_0(\ell, \mu)$. A straightforward calculation reveals that

$$\mathcal{Q}_0(\ell, \mu)e^{inz} = i\Omega_{n,\ell,\mu}e^{inz} \quad \text{for all} \quad n \in \mathbb{Z}, \tag{3.4.47}$$

where

$$\Omega_{n,\ell,\mu} = \gamma\left((n+\mu) - \frac{1}{(n+\mu)}\right) + \beta k^4((n+\mu) - (n+\mu)^3) - \frac{\ell^2}{(n+\mu)}. \tag{3.4.48}$$

Therefore, the $L^2(\mathbb{T})$-spectrum of $\mathcal{Q}_0(\ell, \mu), \mu \in (0, 1/2]$ is given by

$$\text{spec}_{L^2(\mathbb{T})}(\mathcal{Q}_0(\ell, \mu)) = \{i\Omega_{n,\ell,\mu}; n \in \mathbb{Z}\}. \tag{3.4.49}$$

When eigenvalues of $\mathcal{Q}_a(\ell, \mu)$ bifurcate from the imaginary axis for $|a|$ sufficiently small, according to the symmetry of the spectrum around the real and imaginary axes, they must do so in pairs as a result of collisions of eigenvalues of $\mathcal{Q}_0(\ell,\mu)$ on the imaginary axis. Let $n \neq m \in \mathbb{Z}$, a pair of eigenvalues $i\Omega_{n,\ell,\mu}$ and $i\Omega_{m,\ell,\mu}$ of $\mathcal{Q}_0(\ell,\mu)$ collide for some $\ell = \ell_c$ and $\mu \in (0,1/2]$ when

$$\Omega_{n,\ell_c,\mu} = \Omega_{m,\ell_c,\mu}. \tag{3.4.50}$$

Without any loss of generality, consider $n < m$ and $m = n + \Theta$ with $\Theta \in \mathbb{N}$ in collision condition (3.4.50), we arrive at

$$\ell^2 = -\gamma R(n,\mu,\Theta) + \beta k^4 S(n,\mu,\Theta), \tag{3.4.51}$$

where,

$$R(n,\mu,\Theta) = (n+\mu)(n+\mu+\Theta) + 1 \quad \text{and}$$

$$S(n,\mu,\Theta) = 3(n+\mu)^2(n+\mu+\Theta)^2 + (n+\mu)(n+\mu+\Theta)(\Theta^2-1).$$

Lemma 3.4.15. *For a fixed $\gamma > 0$, $\beta \in \mathbb{R}$ and each $\Theta \in \mathbb{N}$, eigenvalues $\Omega_{n,\ell,\mu}$ and $\Omega_{n+\Theta,\ell,\mu}$ of the operator $\mathcal{Q}_0(\ell,\mu)$ collide for all $n \in \mathbb{Z}$ except $n = \{-2,-1\}$ for $\Theta = 2$ when $\beta > 0$, and all $n \in [-\Theta,-1]$ except $n = -1$ for $\Theta = 1$ when $\beta \leq 0$. All collisions take place away from the origin in the complex plane except when Θ is odd and $n = -(\Theta+1)/2$ in which case eigenvalues $\Omega_{n,\ell,\Theta}$ and $\Omega_{-n-1,\ell,\Theta}$ collide at the origin for $\ell = \ell_c(1/2)$. Table 3.4 provides a range of values of k and μ for all such collisions.*

β	Θ	n	k	μ
$\beta > 0$	$\mathbb{N}$	$(-\infty,-\Theta) \cup (0,\infty)$	$(\min(k_\mu),\infty)$	$(0,1/2]$
$\beta > 0$	$\mathbb{N} \setminus \{2\}$	$(-\Theta,-1]$	$(0,\max(k_\mu)]$	$(0,1/2]$
$\beta > 0$	1	$-\Theta$	$(\min(k_\mu),\infty)$	$(0,1/2]$
$\beta > 0$	≥ 3	$-\Theta$	$(0,\max(k_\mu)]$	$\left((\Theta-\sqrt{\Theta^2-4})/2, 1/2\right]$
$\beta \leq 0$	2	$[-\Theta,-1]$	$(\min(k_\mu),\infty)$	$(0,1/2]$
$\beta \leq 0$	≥ 3	$(-\Theta,-1]$	$(0,\infty)$	$(0,1/2]$
$\beta \leq 0$	≥ 3	$-\Theta$	$(\min(k_\mu),\infty)$	$\left(0,(\Theta-\sqrt{\Theta^2-4})/2\right)$
$\beta \leq 0$	≥ 3	$-\Theta$	$(0,\infty)$	$\left[(\Theta-\sqrt{\Theta^2-4})/2, 1/2\right]$

Table 3.4: For a given sign of β and value(s) of Θ, each row lists value(s) of n for which collisions takes place along with the value(s) of k and μ. Here $k_\mu = |\gamma R(n,\mu,\Theta)/\beta S(n,\mu,\Theta)|^{1/4}$.

Proof. The function $R(n,\mu,1), R(n,\mu,2) > 0$ for all $n \in \mathbb{Z}$, $\mu \in (0,1/2]$; while $R(n,\mu,\Theta)$, $\Theta \geq 3$ is zero for $n = -\Theta$, $\mu = \dfrac{\Theta - \sqrt{\Theta^2-4}}{2}$; positive for all $n \in (-\infty,-\Theta-1] \cup [0,\infty)$

for all $\mu \in (0, 1/2]$ and $n = -\Theta$ for all $\mu \in \left(0, \dfrac{\Theta - \sqrt{\Theta^2 - 4}}{2}\right)$, while negative for all $n \in [-\Theta + 1, -1]$ for all $\mu \in (0, 1/2]$ and $n = -\Theta$ for all $\mu \in \left(\dfrac{\Theta - \sqrt{\Theta^2 - 4}}{2}, \dfrac{1}{2}\right]$. The function $S(n, \mu, 1) > 0$ for all $n \in \mathbb{Z}$; and $S(n, \mu, \Theta)$, $\Theta \geq 2$ is positive for all $n \in (-\infty, -\Theta - 1] \cup [0, \infty)$ and negative for all $n \in [-\Theta, -1]$ for all $\mu \in (0, 1/2]$. There exist $\ell \in \mathbb{R}$ satisfying the collision condition (3.4.50) if

$$Z(n, \mu, \Theta) = -\gamma R(n, \mu, \Theta) + \beta k^4 S(n, \mu, \Theta) > 0, \tag{3.4.52}$$

which holds for all $n \in \mathbb{Z}$ except $n = \{-2, -1\}$ for $\Theta = 2$ when $\beta > 0$; and for all $n \in [-\Theta, -1]$ except $n = -1$ for $\Theta = 1$ when $\beta \leq 0$. For instance, see Figure 3.4. Fixing $\beta = -1$, $\gamma = k = 1$; $Z(n, 0.4, 5) > 0$ for $n \in [-5, -1]$ and $Z(n, 0.4, 5) < 0$ otherwise. Also, for $\beta = \gamma = k = 1$; $Z(n, 0.3, 2) > 0$ for all n except $n = -2, -1$. Now see Figure 3.5. For $\beta = \gamma = 1$, $\Theta = 3$, $n = -2$; (3.4.52) holds for $k \in (0, 0.811)$ and does not hold otherwise. Also for $\beta = \gamma = 1$, $k = 0.2$, $n = -\Theta = -4$; (3.4.52) holds for $\mu \in (0.268, 0.5)$ and does not hold otherwise, which agrees with the calculation. Note that $\Omega_{n,\ell,\mu} = 0$ at $\ell^2 = |((n+\mu)^2 - 1)(\gamma - \beta k^4 (n+\mu)^2)|$. $\Omega_{n,\ell_c,\mu} = \Omega_{n+\Theta,\ell_c,\mu}$=0 for a fixed ℓ_c is possible only for $\Theta = -2n - 1$, $\mu = 1/2$. Therefore, $\Omega_{n,\ell_c,\mu}$ and $\Omega_{n+\Theta,\ell_c,\mu}$ collide at the origin for $n = -(\Theta + 1)/2$, for all $n \in [-\Theta, -1] \cap \mathbb{Z}$, $\mu = 1/2$ and $\ell_c^2 = \left|\left(n^2 + n - \dfrac{3}{4}\right)\left(\gamma - \beta k^4 \left(n + \dfrac{1}{2}\right)^2\right)\right|$; except the pair $\{n, n + \Theta\} = \{-1, 0\}$. All other collisions are away from the origin.

□

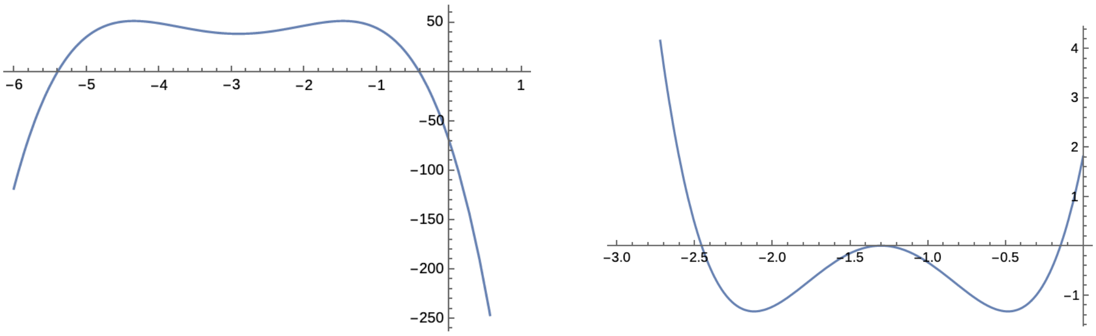
The linear operator $\mathcal{Q}_a(\ell, \mu)$ can be decomposed similarly to the preceding section

$$\mathcal{Q}_a(\ell, \mu) = \mathcal{A}_\mu \mathcal{B}_a(\ell, \mu),$$

where

$$\mathcal{A}_\mu = \partial_z + i\mu \quad \text{and} \quad \mathcal{B}_a(\ell, \mu) = k^2(c + \beta k^2(\partial_z + i\mu)^2 - 2\eta) + (\gamma + \ell^2)(\partial z + i\mu)^{-2}.$$

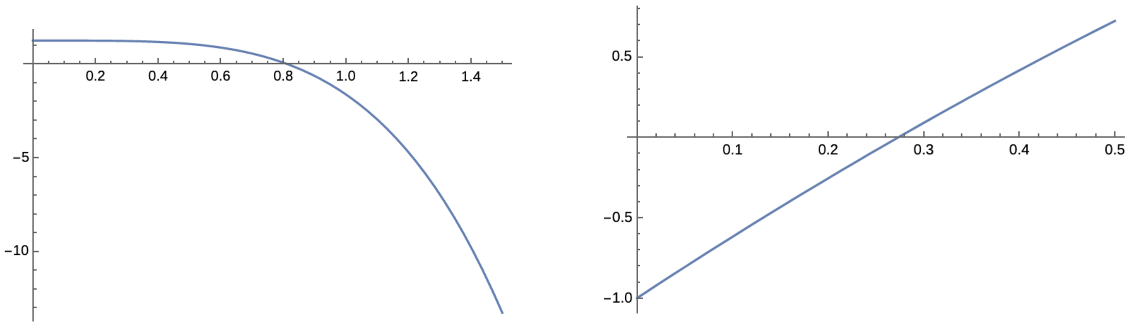
The operator $\mathcal{A}_\mu$ is skew-adjoint, whereas the operator $\mathcal{B}_a(\ell, \mu)$ is self-adjoint. Using the definition in (3.4.43), the Krein signature, $\chi_{n,\mu}$ of an eigenvalue $i\Omega_{n,\ell,\mu}$ in spec($\mathcal{Q}_0(\ell, \mu)$))



((a)) $Z(n, 0.4, 5)$ with $n : -6$ to 1, $\beta = -1$, $\gamma = k = 1$

((b)) $Z(n, 0.3, 2)$ with $n : -3$ to 0, $\beta = \gamma = k = 1$

FIGURE 3.4: Graph of function $Z(n, \mu, \Theta)$ vs. n



((a)) $Z(-2, 0.4, 3)$ with $k \in (0, 1.5)$, $\gamma = \beta = 1$

((b)) $Z(-4, \mu, 4)$ with $\mu \in (0, 0.5)$, $k = 0.2$, $\gamma = \beta = 1$

FIGURE 3.5: Left plot: Graph of function $Z(n, \mu, \Theta)$ vs. k,
Right plot: Graph of function $Z(n, \mu, \Theta)$ vs. μ.

is

$$\chi_{n,\mu} = \operatorname{sgn}\left(\frac{1}{n}\Omega_{n,\ell,\mu}\right), \quad n \in \mathbb{Z}. \tag{3.4.53}$$

Consequently, $(n + \mu_0)(m + \mu_0) < 0$, where μ_0 is the value of the floquet exponent where eigenvalues $i\Omega_{n,\ell,\mu}$ and $i\Omega_{m,\ell,\mu}$ collide, is a necessary condition for the bifurcation of colliding eigenvalues from the imaginary axis.

Lemma 3.4.16 (Potentially unstable nodes)**.** *Out of all the collisions mentioned in Lemma 3.4.15, all $n \in [-\Theta, -1]$ for each $\Theta \in \mathbb{N} \setminus \{2\}$ when $\beta > 0$, and all $n \in [-\Theta, -1]$ for each $\Theta \in \mathbb{N} \setminus \{1\}$ when $\beta \leq 0$ are potentially unstable with respect to finite or short wavelength transverse perturbations.*

3.4.1.4 Long wavelength transverse perturbations:

We start the further analysis with the values of ℓ sufficiently close to origin, that is, $|\ell| \leq \ell_0$ for some $\ell_0 > 0$. Table 3.5 summarizes all the potentially unstable nodes

$\boldsymbol{\Theta}$	**LWTP**				**F/SWTP**		
	Periodic		**Non-periodic**		**Periodic**	**Non-periodic**	
	$\boldsymbol{\beta > 0}$	$\boldsymbol{\beta \leq 0}$	$\boldsymbol{\beta > 0}$	$\boldsymbol{\beta \leq 0}$	$\boldsymbol{\beta \in \mathbb{R}}$	$\boldsymbol{\beta > 0}$	$\boldsymbol{\beta \leq 0}$
1	None	None	None	None	None	$\{-1, 0\}$	None
2	$\{-1, 1\}$	$\{-1, 1\}$	None	None	None	None	$\{-1, 1\}$ $, \{-2, 0\}$
≥ 3	None	None	None	None	$\{-1, \Theta - 1\}$ $\{-2, \Theta - 2\},$ $, \dots$ $\{-\Theta + 1, 1\}$	$\{-1, \Theta - 1\}$ $\{-2, \Theta - 2\},$ $, \dots$ $\{-\Theta + 1, 1\}$ $\{-\Theta, 0\}$	$\{-1, \Theta - 1\}$ $\{-2, \Theta - 2\},$ $, \dots$ $\{-\Theta + 1, 1\}$ $\{-\Theta, 0\}$

Table 3.5: Potentially unstable nodes for a given $\Theta \in \mathbb{N}$. Here 'LWTP' stands for *Long wavelength transverse perturbations* and 'F/SWTP' stands for *Finite or short wavelength transverse perturbations*.

with respect to long wavelength transverse perturbations. For a and ℓ sufficiently small, $\{\Omega_{-1,\ell}, \Omega_{1,\ell}\}$ are pair of eigenvalues bifurcating continuously from $\{\Omega_{-1,0}, \Omega_{1,0}\}$. For $a = 0$, $\{\Omega_{-1,0}, \Omega_{1,0}\}$ are equipped with eigenfunctions $\{e^{-iz}, e^{iz}\}$. We choose the real basis $\{\cos z, \sin z\}$. We calculate expansion of a basis $\{\psi_1, \psi_2\}$ for the eigenspace corresponding to the eigenvalues of $\{\Omega_{-1,\ell}, \Omega_{1,\ell}\}$ in $L^2_0(\mathbb{T})$ by using expansions of η and c in (3.4.4), for small a and ℓ as

$$\psi_1(z) = \cos z + 2aA_2 \cos 2z + O(a^2),$$
$$\psi_2(z) = \sin z + 2aA_2 \sin 2z + O(a^2).$$

We have the following expression for $\mathcal{Q}_a(\ell)$ after expanding and using η and c

$$\mathcal{Q}_a(\ell) = \gamma(\partial_z + \partial_z^{-1}) + \ell^2 \partial_z^{-1} + \beta k^4(\partial_z + \partial_z^3) - 2ak^2\partial_z(\cos z) + O(a^2). \tag{3.4.54}$$

In order to locate the bifurcating eigenvalues for $|a|$ sufficiently small, we calculate the action of $\mathcal{Q}_a(\ell)$ on the extended eigenspace $\{\psi_1(z), \psi_2(z)\}$ viz.

$$\mathcal{T}_a(\ell) = \left[\frac{\langle \mathcal{Q}_a(\ell)\psi_i(z), \psi_j(z)\rangle}{\langle \psi_i(z), \psi_i(z)\rangle}\right]_{i,j=1,2} \text{ and } \mathcal{I}_a = \left[\frac{\langle \psi_i(z), \psi_j(z)\rangle}{\langle \psi_i(z), \psi_i(z)\rangle}\right]_{i,j=1,2}. \tag{3.4.55}$$

We use expansion of $\mathcal{Q}_a(\ell)$ in (3.4.54) to find actions of $\mathcal{Q}_a(\ell)$ and identity operator on $\{\psi_1, \psi_2\}$, and arrive at

$$\mathcal{T}_a(\ell) = \begin{pmatrix} 0 & \ell^2 + 2a^2k^2\eta_2 \\ -\ell^2 & 0 \end{pmatrix} + O(a^2(\ell + a)).$$

To locate where these two eigenvalues are bifurcating from the origin, we analyze the characteristic equation $|\mathcal{T}_a(\ell) - \lambda\mathcal{I}| = 0$, where $\mathcal{I}_a$ is 2×2 identity matrix. From which we conclude that

$$\lambda^2 + \ell^2(\ell^2 + 2a^2k^2\eta_2) = O(|a|\ell^2(\ell^2 + a^2)), \tag{3.4.56}$$

and

$$\lambda^2 = -\ell^2(\ell^2 + 2a^2k^2\eta_2) + O(|a|\ell^2(\ell^2 + a^2)). \tag{3.4.57}$$

For $\ell = a = 0$, we obtain zero as a double eigenvalue, which is consistent with our calculation. For $\beta < 0$, we obtain two purely imaginary eigenvalues for all ℓ and a sufficiently small. For $\beta > 0$, we obtain two real eigenvalues with opposite signs when

$$k > \sqrt[4]{\frac{\gamma}{4\beta}}. \tag{3.4.58}$$

Hence the Theorem 3.4.1.

3.4.1.5 Finite or short wavelength transverse perturbations

Here we work with the values of ℓ away from the origin, $|\ell| \geq \ell_0$ for some $\ell_0 > 0$. We do further analysis to check if collisions in Table 3.5 indeed lead to instability or not.

3.4.1.5.1 (In)stability analysis for $\Theta = 1$: For periodic perturbations, there are no collisions mentioned for $\Theta = 1$. However, there in one pair of colliding eigenvalues $\{\Omega_{-1,\ell,\mu}, \Omega_{0,\ell,\mu}\}$ for $\beta > 0$ with respect to non-periodic perturbations, colliding for

$$\ell^2 = -\gamma(1 - \mu + \mu^2) + 3\beta k^4\mu^2(1 - \mu)^2 \quad \text{and} \quad k > min\left(\frac{\gamma(1 - \mu + \mu^2)}{3\beta\mu^2(1 - \mu)^2}\right)^{1/4}. \tag{3.4.59}$$

There exists a curve $\ell = \ell_c$ given in (3.4.59) along which

$$\Omega(\ell_c, \mu) := \Omega_{-1,\ell_c,\mu} = \Omega_{0,\ell_c,\mu}.$$

Furthermore,

$$\phi_{0,-1}(z) = e^{-iz} \qquad \textit{and} \qquad \phi_{0,0}(z) = 1, \tag{3.4.60}$$

forms the corresponding eigenspace for $\mathcal{Q}_0(\ell_c, \mu)$ associated with the two eigenvalues. Let

$$i\Omega(\ell_c, \mu) + i\theta_{a,\ell,-1} \qquad \textit{and} \qquad i\Omega(\ell_c, \mu) + i\theta_{a,\ell,0}, \tag{3.4.61}$$

be the eigenvalues of $\mathcal{Q}_a(\ell_c, \mu)$ bifurcating from $i\Omega_{-1,\ell_c,\mu}$ and $i\Omega_{0,\ell_c,\mu}$ respectively for $|a|$ and $|\ell - \ell_c|$ small. Let $\{\phi_{a,\ell,-1}(z), \phi_{a,\ell,0}(z)\}$ be the extended eigenspace associated with two bifurcating eigenvalues. Following [14], we can take,

$$\phi_{a,-1,\ell}(z) = e^{-iz} + a\phi_{-1} + O(a^2), \tag{3.4.62}$$

$$\phi_{a,0,\ell}(z) = 1 + a\phi_0 + O(a^2). \tag{3.4.63}$$

Using orthonormality conditions on the eigenfunctions $\phi_{a,\ell,-1}$ and $\phi_{a,\ell,0}$, we get

$$\phi_{-1} = \phi_0 = 0.$$

In order to find eigenvalues, we compute matrix representations of $\mathcal{Q}_a(\ell_c, \mu)$ and identity operators on $\{\phi_{a,-1,\ell}(z), \phi_{a,0,\ell}(z)\}$ for sufficiently small $|a|$ and $|\ell - \ell_c|$

$$\mathcal{B}_a(\ell, \mu) = \begin{pmatrix} i\Omega(\ell_c, \mu) - i\dfrac{\varsigma}{\mu - 1} & -iak^2\mu \\ -iak^2(\mu - 1) & i\Omega(\ell_c, \mu) - i\dfrac{\varsigma}{\mu} \end{pmatrix} + O(|a|(|\varsigma| + |a|)),$$

where $\varsigma = \ell^2 - \ell_c^2$ and $\mathcal{I}_a$ is the 2×2 identity matrix. Calculating the characteristic equation $\det(\mathcal{B}_a(\ell, \mu) - \lambda\mathcal{I}_a) = 0$ for λ of the form

$$\lambda = i\Omega(\ell_c, \mu) + i\theta,$$

leads to the polynomial equation

$$\theta^2 + \theta\varsigma\left(\frac{1}{\mu-1} + \frac{1}{\mu} + O(a^2)\right) - a^2k^4\mu(\mu-1) + \frac{\varsigma^2}{\mu(\mu-1)} + O(a^2(|\varsigma| + a^2)) = 0.$$

A direct computation shows that the discriminant of this polynomial is

$$\mathrm{disc}_a(\varsigma,\mu) = \frac{\varsigma^2}{\mu^2(\mu-1)^2} + 4k^4\mu(\mu-1)a^2 + O(a^2(|\varsigma| + a^2)).$$

For any a sufficiently small there exists

$$\varsigma_a(\mu) = 2k^2\mu^{3/2}(1-\mu)^{3/2}|a| + O(a^2) > 0,$$

such that the two eigenvalues of $\mathcal{Q}_a(\ell,\mu)$ are purely imaginary when $|\ell^2 - \ell_c^2| \geq \varsigma_a(\mu)$ and complex with opposite nonzero real parts when $|\ell^2 - \ell_c^2| < \varsigma_a(\mu)$, which proves the Theorem 3.4.2.

The instability result in Theorem 3.4.2 is also supported by numerical experiments. We consider the eigenvalue problem (3.4.46) with operator $\mathcal{Q}_a(\ell,\mu)$ defined by (3.4.45). We use the shift-invert technique [67] and consider the following problem,

$$[\mathcal{Q}_a(\ell,\mu)\varphi - i\omega]^{-1} = \frac{1}{\lambda - i\omega}\varphi; \quad \varphi \in L^2(\mathbb{T}) \quad \text{and} \quad \mu \in (0, 1/2]. \tag{3.4.64}$$

Here $\omega \in \mathbb{R}$ and is a guess chosen close to but not equal to λ. We use the MINRES [72] method to invert the operator iteratively. The eigenvalue problem is solved numerically by **eigs** function using MATLAB which is a matrix-free function based on Arnoldi iterations. This allows us to have a $O(N\log N)$ flops scheme instead of $O(N^2)$ operations that would be needed to form an operator matrix.

The spectrum of $\mathcal{Q}_a(\ell,\mu)$ with $\gamma = 1$, $\beta = 1$, and $k = 2 > 4^{1/4}$ is shown in Figure 3.6. High-frequency bubbles appear along the imaginary axis for the solution (3.4.4) with the amplitude of the initial condition chosen to be $\epsilon = 0.01$. In the right panel of Figure 3.6, we zoom into one of the bubbles centered around $0.37916i$ on the imaginary axis. In Figure 3.7, we show the range of parameter μ for two high-frequency bubbles located above the imaginary axis.

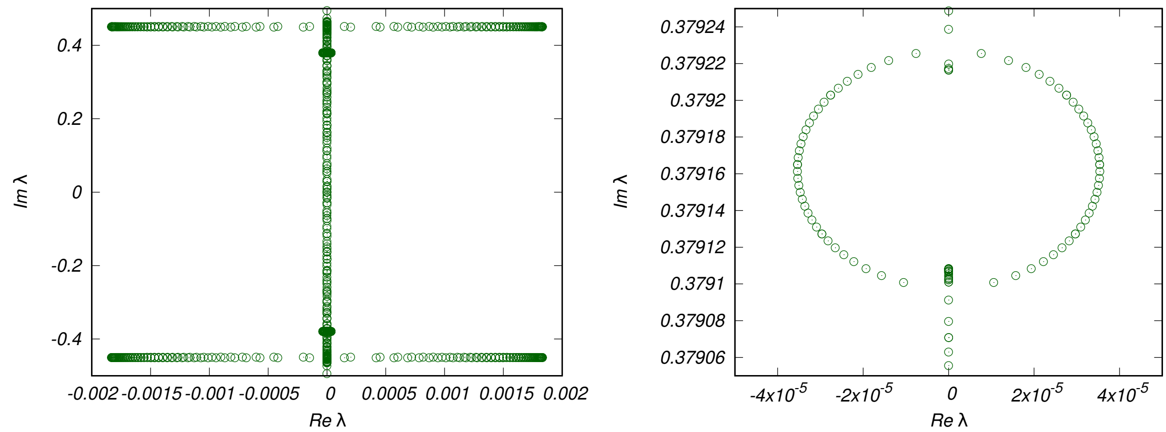


FIGURE 3.6: Left panel: Eigenvalue Spectrum of $\mathcal{Q}_a(\ell, \mu)$ with $\gamma = 1$, $\beta = 1$, $k = 2 > 4^{1/4}$ with the amplitude of the initial condition being $\epsilon = 0.01$.
Right panel: zoom into a high-frequency bubble centered around $0.37916i$ on the imaginary axis.

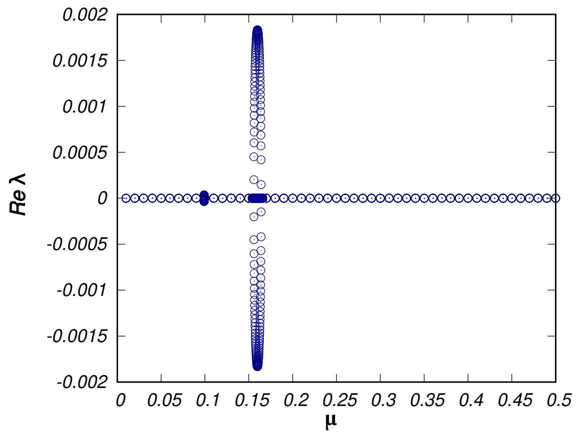


FIGURE 3.7: μ vs. $\Re(\lambda)$ of $\mathcal{Q}_a(\ell, \mu)$ with $\gamma = 1$, $\beta = 1$, $k = 2 > 4^{1/4}$ with the amplitude of the initial condition being $\epsilon = 0.01$.

3.4.1.5.2 <u>(In)stability analysis for $\Theta \geq 2$:</u> For some $n \in \mathbb{Z}$ and a fixed $\Theta \geq 2$, we have

$$i\Omega_{n,\ell_*,\mu} = i\Omega_{n+\Theta,\ell_*,\mu} = i\Omega(\ell_*, \mu), \quad \mu \in (-1/2, 1/2]. \tag{3.4.65}$$

$\mu = 0$ represents the periodic case and $\mu \neq 0$ represents the non-periodic case. $i\Omega$ is an eigenvalue of $\mathcal{Q}_0(\ell_c, \mu)$ of multiplicity two with an orthonormal basis of eigenfunctions $\{e^{inz}, e^{i(n+\Theta)z}\}$. Let $i\Omega + i\nu_{a,n}$ and $i\Omega + i\nu_{a,n+2}$ be the eigenvalues of $\mathcal{Q}_a(\ell, \mu)$ bifurcating from $i\Omega_{n,\ell_c,\mu}$ and $i\Omega_{n+2,\ell_c,\mu}$ respectively, for $|a|$ and $|\ell - \ell_c|$ small. Let $\{\varphi_{a,n}(z), \varphi_{a,n+2}(z)\}$ be a orthonormal basis for the corresponding eigenspace. We suppose the following expansions

$$\varphi_{a,n}(z) = e^{inz} + a\varphi_{n,1} + a^2\varphi_{n,2} + \cdots + a^{\Theta}\varphi_{n,\Theta} + O(a^{\Theta+1}), \tag{3.4.66}$$

$$\varphi_{a,n+\Theta}(z) = e^{i(n+\Theta)z} + a\varphi_{n+\Theta,1} + a^2\varphi_{n+\Theta,2} + \cdots + a^{\Theta}\varphi_{n+\Theta,\Theta} + O(a^{\Theta+1}). \tag{3.4.67}$$

We use orthonormality of $\varphi_{a,n}$ and $\varphi_{a,n+\Theta}$ to find that

$$\varphi_{n,1} = \varphi_{n,2} = \cdots = \varphi_{n,\Theta} = \varphi_{n+\Theta,1} = \varphi_{n+\Theta,2} = \cdots = \varphi_{n+\Theta,\Theta} = 0.$$

Using the expansions of η and c in (3.4.4), we expand $\mathcal{Q}_a(\ell,\mu)$ in a as

$$\mathcal{Q}_a(\ell,\mu) = \mathcal{Q}_0(\ell,\mu) + k^2(\beta_2 a^2 + \beta_4 a^4 + \dots)(\partial_z + i\mu) - 2k^2 a\delta_1(\partial_z + i\mu)(\cos z) \\ - \cdots - 2k^2 a^{\Theta}\delta_{\Theta}(\partial_z + i\mu)(\cos(\Theta z)). \tag{3.4.68}$$

To explicitly obtain the values of all unknown coefficients in the expansion of $\mathcal{Q}_a(\ell,\mu)$, we require coefficients of higher powers of a in the expansion of solution η. Calculating higher coefficients is difficult as the coefficients of the solution do not seem to have any apparent symmetry. Therefore, we pursue the instability analysis without calculating the unknown coefficients explicitly.

Following the same procedure as in the preceding subsection, we arrive at

$$\mathscr{G}_a(\ell,\mu) = \begin{pmatrix} \mathscr{G}_{11} & \mathscr{G}_{12} \\ \mathscr{G}_{21} & \mathscr{G}_{22} \end{pmatrix} + O(a^{\Theta+1}),$$

where

$$\mathscr{G}_{11} = i\Omega(\ell_*,\mu) - i\frac{\varsigma}{n+\mu} + ik^2(n+\mu)(\beta_2 a^2 + \beta_4 a^4 + \dots),$$

$$\mathscr{G}_{12} = -ik^2(n+\mu+\Theta)\delta_{\Theta}a^{\Theta},$$

$$\mathscr{G}_{21} = -ik^2(n+\mu)\delta_{\Theta}a^{\Theta},$$

$$\mathscr{G}_{22} = i\Omega(\ell_*,\mu) - i\frac{\varsigma}{n+\mu+\Theta} + ik^2(n+\mu+\Theta)(\beta_2 a^2 + \beta_4 a^4 + \dots).$$

The resulting discriminant of the characteristic equation $\det(\mathscr{G}_a(\ell,\mu) - (i\Omega(\ell_*,\mu) + i\theta)\mathcal{I}_a) =$

0 is

$$\mathrm{disc}_a(\varsigma) = \frac{\Theta^2\varsigma^2}{(n+\mu)^2(n+\mu+\Theta)^2} + k^4\Theta^2\beta_2^2a^4 + O(a^2(|\varsigma| + |a^3|)).$$

For sufficiently small $|\varsigma|$ and $|a|$ $\mathrm{disc}_a(\varsigma)$ is positive which implies that no eigenvalue of $\mathcal{Q}_a(\ell,\mu)$ is bifurcating from the imaginary axis due to collision. Hence the proof of Theorems 3.4.3 and 3.4.4.

3.4.2 Generalized RMKP

3.4.2.1 Construction of the spectral problem

Linearizing the gRMKP equation (3.4.5) about its one-dimensional periodic traveling wave $\eta_{\mathfrak{g}}$ in (3.4.23) and seeking a solution of the form

$$\eta_{\mathfrak{g}}^*(z,y,t) = e^{\frac{\lambda}{k}t+i\ell y}\varphi(z), \quad \lambda \in \mathbb{C},\ \ell \in \mathbb{R}, \tag{3.4.69}$$

we arrive at

$$\mathcal{G}_a^{\mathfrak{g}}(\lambda,\ell)\varphi := (\lambda\partial_z - k^2\partial_z^2(c - \mathcal{M}_k - 2\alpha_1\eta_{\mathfrak{g}} - 3\alpha_2\eta_{\mathfrak{g}}^2) - \ell^2 - \gamma)\varphi = 0. \tag{3.4.70}$$

3.4.2.2 Periodic perturbations

The invertibility problem

$$\mathcal{G}_a^{\mathfrak{g}}(\lambda,\ell)\varphi = 0; \quad \varphi \in L^2(\mathbb{T}),$$

is transformed into a spectral problem which requires invertibility of ∂_z. Since ∂_z is not invertible in $L^2(\mathbb{T})$, we restrict the problem to mean-zero subspace $L_0^2(\mathbb{T})$, defined in (2.5.11), of $L^2(\mathbb{T})$.

Lemma 3.4.17. *$\mathcal{G}_a^{\mathfrak{g}}(\lambda,\ell)$ acting in $L^2(\mathbb{T})$ with domain $H^{\mathfrak{b}+2}(\mathbb{T})$ is not invertible if and only if λ belongs to $L_0^2(\mathbb{T})$-spectrum of the operator $\mathcal{Q}_a(\ell)$, where*

$$\mathcal{Q}_a^{\mathfrak{g}}(\ell) := k^2\partial_z(c - \mathcal{M}_k - 2\alpha_1\eta_{\mathfrak{g}} - 3\alpha_2\eta_{\mathfrak{g}}^2) + (\gamma + \ell^2)\partial z^{-1}. \tag{3.4.71}$$

Proof. The proof is similar to the proof of Lemma 3.4.10. ☐

We arrive at pseudo-differential spectral problem

$$\mathcal{Q}_a^{\mathfrak{g}}(\ell)\varphi = \lambda\varphi, \tag{3.4.72}$$

where $\varphi \in L_0^2(\mathbb{T})$. With respect to periodic perturbations, we will study the spectrum of the operator $\mathcal{Q}_a^{\mathfrak{g}}(\ell)$ acting in $L_0^2(\mathbb{T})$ with domain $H^{\mathfrak{b}+1}(\mathbb{T})) \cap L_0^2(\mathbb{T})$.

Proposition 3.4.18. *The operator $\mathcal{Q}_a^{\mathfrak{g}}(\ell)$ possess following properties.*

1. *The operator $\mathcal{Q}_a^{\mathfrak{g}}(\ell)$ commutes with the reflection through the real axis.*
2. *The operator $\mathcal{Q}_a^{\mathfrak{g}}(\ell)$ anti-commutes with the reflection through the origin and the imaginary axis.*
3. *The spectrum of $\mathcal{Q}_a^{\mathfrak{g}}(\ell)$ is symmetric with respect to the reflections through origin, real axis and imaginary axis.*

Proof. The proof is similar to the proof of Proposition 3.4.11. □

A straightforward calculation reveals that

$$\mathcal{Q}_0^{\mathfrak{g}}(\ell)e^{inz} = i\Omega_{n,\ell}^{\mathfrak{g}}e^{inz} \quad \text{for all} \quad n \in \mathbb{Z}\setminus\{0\}, \tag{3.4.73}$$

where

$$\Omega_{n,\ell}^{\mathfrak{g}} = \gamma\left(n - \frac{1}{n}\right) + k^2 n(m(k) - m(kn)) - \frac{\ell^2}{n}. \tag{3.4.74}$$

Let eigenvalues $i\Omega_{n,\ell}$ and $i\Omega_{n+\Theta,\ell}$, $n \neq m$, collide at $\ell = \ell_c > 0$. From (3.4.43), Krein signatures χ_n and $\chi_{n+\Theta}$ are opposite at $\ell = \ell_c$ when $n(n+\Theta) < 0$, i.e. n and $n+\Theta$ should be of opposite parity.

Lemma 3.4.19. *For a fixed $\gamma > 0$ and each $\Theta \in \mathbb{N}$, the potentially unstable collisions between the eigenvalues $i\Omega_{n,\ell}^{\mathfrak{g}}$ and $i\Omega_{n+\Theta,\ell}^{\mathfrak{g}}$ are*

1. *$\ell = 0$, $\{n, n+\Theta\} = \{-1, 1\}$ for all m,*
2. *$\ell \neq 0$, $n \in (-\Theta, 0)$, $\Theta \geq 3$ for all m.*

All these collisions occur for k satisfying the relation $-J_{\mathfrak{g}}(n,\Theta)+\dfrac{k^2}{\Theta}Y_{\mathfrak{g}}(n,\Theta)>0$, where $J_{\mathfrak{g}}(n,\Theta)=n(n+\Theta)+1$ and $Y_{\mathfrak{g}}(n,\Theta)=n(n+\Theta)(n(m(k)-m(kn))-(n+\Theta)(m(k)-m(k(n+\Theta)))$. Also, all collisions take place away from the origin in the complex plane except when Θ is even and $n=-\Theta/2$ in which case eigenvalues $\Omega_{n,\ell}$ and $\Omega_{-n,\ell}$ collide at the origin.

Proof. A direct calculation shows that $i\Omega^{\mathfrak{g}}_{n,\ell}$ and $i\Omega^{\mathfrak{g}}_{-t,\ell}$ collide when

$$\ell^2=\gamma J_{\mathfrak{g}}(n,t)+k^2Y_{\mathfrak{g}}(n,t), \tag{3.4.75}$$

where,

$$J_{\mathfrak{g}}(n,t)=nt-1 \quad \text{and} \quad Y_{\mathfrak{g}}(n,t)=\frac{nt}{n+t}(n(m(k)-m(kn))+t(m(k)-m(kt))).$$

When $\ell=0$, (3.4.75) holds only for $n=t=1$. $J_{\mathfrak{g}}(1,1)=0$ and $J_{\mathfrak{g}}(n,t)>0$ for all $n,t\in\mathbb{N}\setminus\{1\}$. $Y_{\mathfrak{g}}(1,1)=0$. $Y_{\mathfrak{g}}(n,t)<0$ when m is monotonically increasing and $Y_{\mathfrak{g}}(n,t)>0$ when m is monotonically decreasing. Therefore, for every m, there exist $\ell\neq 0$ satisfying (3.4.75). If we seek the potentially unstable collisions between the eigenvalues $i\Omega^{\mathfrak{g}}_{n,\ell}$ and $i\Omega^{\mathfrak{g}}_{n+\Theta,\ell}$ for each $\Theta\in\mathbb{N}$, then value of n should be $n\in(-\Theta,0)$ for every m. Observe that $\Omega^{\mathfrak{g}}_{n,\ell_c}=\Omega^{\mathfrak{g}}_{-n,\ell_c}=0$ for $\ell_c^2=|\gamma(n^2-1)+k^2n^2(m(k)-m(kn))|$. Therefore, $\Omega^{\mathfrak{g}}_{n,\ell_c}$ and $\Omega^{\mathfrak{g}}_{n+\Theta,\ell_c}$ collide at the origin when Θ is even and $n=-\Theta/2$. All other collisions are away from the origin.

□

3.4.2.3 Localized or bounded perturbations

In $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$, the operator $\mathcal{G}^{\mathfrak{g}}_a(\lambda,\ell)$ now have continuous spectrum. We can use the Floquet theory such that all solutions of (3.4.70) in $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$ are of the form $\varphi(z)=e^{i}\tilde{\varphi}(z)$ where $\mu\in(-1/2,1/2]$ is the Floquet exponent and $\tilde{\varphi}$ is a 2π-periodic function. We can deduce that the study of the invertibility of $\mathcal{G}^{\mathfrak{g}}_a(\lambda,\ell)$ in $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$ is equivalent to the invertibility of the linear operators $\mathcal{G}^{\mathfrak{g}}_{a,\mu}(\lambda,\ell)$ in $L^2(\mathbb{T})$ with domain $H^{\mathfrak{b}+2}(\mathbb{T})$, for all $\mu\in(-1/2,1/2]$, where

$$\mathcal{G}^{\mathfrak{g}}_{a,\mu}(\lambda,\ell)=\lambda(\partial_z+i\mu)-k^2(\partial_z+i\mu)^2(c-\mathcal{M}_k-2\alpha_1\eta_{\mathfrak{g}}-3\alpha_2\eta_{\mathfrak{g}}^2)-\ell^2-\gamma.$$

Also, $\mathcal{G}^{\mathfrak{g}}_{a,\mu}(\lambda,\ell)$ is not invertible in $L^2(\mathbb{T})$ if and only if zero is an eigenvalue of $\mathcal{G}^{\mathfrak{g}}_{a,\mu}(\lambda,\ell)$. For a $\Psi\in L^2(\mathbb{T})$, $\mathcal{G}_{a,\mu}(\lambda,\ell)\Psi=0$ if and only if $\mathcal{Q}^{\mathfrak{g}}_a(\ell,\mu)\Psi=\lambda\Psi$, where

$$\mathcal{Q}^{\mathfrak{g}}_a(\ell,\mu):=k^2(\partial_z+i\mu)(c-\mathcal{M}_k-2\alpha_1\eta_{\mathfrak{g}}-3\alpha_2\eta_{\mathfrak{g}}^2)+(\gamma+\ell^2)(\partial_z+i\mu)^{-1}. \tag{3.4.76}$$

Therefore, the operator $\mathcal{G}^{\mathfrak{g}}_{a,\mu}(\lambda,\ell)$ is not invertible in $L^2(\mathbb{T})$ for some $\lambda\in\mathbb{C}$ and $\mu\neq 0$ if and only if $\lambda\in\operatorname{spec}_{L^2(\mathbb{T})}(\mathcal{Q}^{\mathfrak{g}}_a(\ell,\mu))$, $L^2(\mathbb{T})$-spectrum of the operator. This is straight forward to observe that $\operatorname{spec}_{L^2(\mathbb{T})}(\mathcal{Q}^{\mathfrak{g}}_a(\ell,\mu))$ is not symmetric with respect to the reflections through the real axis and the origin, rather it exhibit following properties.

Proposition 3.4.20. *The operator $\mathcal{Q}^{\mathfrak{g}}_a(\ell,\mu)$ possess following properties.*

1. *The operator $\mathcal{Q}^{\mathfrak{g}}_a(\ell,-\mu)$ commutes with the reflection through the real axis.*

2. *The operator $\mathcal{Q}^{\mathfrak{g}}_a(\ell,\mu)$ anti-commutes with the reflection through the imaginary axis.*

3. *The operator $\mathcal{Q}^{\mathfrak{g}}_a(\ell,-\mu)$ anti-commutes with the reflection through the origin.*

4. *The spectrum of $\mathcal{Q}^{\mathfrak{g}}_a(\ell,\mu)$ is symmetric with respect to the reflections through the imaginary axis.*

5. *The spectrum of $\mathcal{Q}^{\mathfrak{g}}_a(\ell,-\mu)$ is symmetric with respect to the reflections through the real axis and the origin.*

Proof. The proof is similar to the proof of Proposition 3.4.11. □

A simple calculation yields the following

$$\mathcal{Q}^{\mathfrak{g}}_0(\ell,\mu)e^{inz}=i\Omega^{\mathfrak{g}}_{n,\ell,\mu}e^{inz}\quad\text{for all}\quad n\in\mathbb{Z}, \tag{3.4.77}$$

where

$$\Omega^{\mathfrak{g}}_{n,\ell,\mu}=\gamma\left(n+\mu-\frac{1}{n+\mu}\right)+k^2(n+\mu)(m(k)-m(k(n+\mu)))-\frac{\ell^2}{n+\mu}. \tag{3.4.78}$$

Therefore, the $L^2(\mathbb{T})$-spectrum of $\mathcal{Q}_0(\ell,\mu)$ is given by

$$\operatorname{spec}_{L^2(\mathbb{T})}(\mathcal{Q}_0(\ell,\mu))=\{i\Omega^{\mathfrak{g}}_{n,\ell,\mu};n\in\mathbb{Z},\mu\in(0,1/2]\}. \tag{3.4.79}$$

Let eigenvalues $i\Omega_{n,\ell,\mu}$ and $i\Omega_{n+\Theta,\ell,\mu}$, $\Theta \in \mathbb{N}$, collide at $\ell = \ell_c > 0$. From (3.4.53), Krein signatures χ_n and $\chi_{n+\Theta}$ are opposite at $\ell = \ell_c$ when $(n+\mu)(n+\Theta+\mu) < 0$.

Lemma 3.4.21. *For a fixed $\gamma > 0$ and each $\Theta \in \mathbb{N}$, the potentially unstable collisions between the eigenvalues $i\Omega^{\mathfrak{g}}_{n,\ell,\mu}$ and $i\Omega^{\mathfrak{g}}_{n+\Theta,\ell,\mu}$ are all $n \in [-\Theta, -1]$ for each $\Theta \in \mathbb{N} \setminus \{2\}$ when m is increasing, and all $n \in [-\Theta, -1]$ for each $\Theta \in \mathbb{N} \setminus \{1\}$ when m is decreasing. All these collisions occur for k satisfying the relation $-\gamma R_{\mathfrak{g}}(n,\mu,\Theta) + k^2 S_{\mathfrak{g}}(n,\mu,\Theta) > 0$, where $R_{\mathfrak{g}}(n,\mu,\Theta) = (n+\mu)(n+\mu+\Theta)+1$ and $S_{\mathfrak{g}}(n,\mu,\Theta) = \dfrac{(n+\mu)(n+\mu+\Theta)}{\Theta}((n+\mu)(m(k)-m(k(n+\mu))) - (n+\mu+\Theta)(m(k)-m(k(n+\mu+\Theta))))$. All collisions take place away from the origin in the complex plane except when Θ is odd and $n = -(\Theta+1)/2$ in which case eigenvalues $\Omega_{n,\ell,\Theta}$ and $\Omega_{-n-1,\ell,\Theta}$ collide at the origin for $\ell = \ell_c(1/2)$. Table 3.6 provides range of values of μ for all such collisions.*

m	Θ	n	μ
increasing	$\mathbb{N} \setminus \{2\}$	$(-\Theta, -1]$	$(0, 1/2]$
increasing	1	$-\Theta$	$(0, 1/2]$
increasing	≥ 3	$-\Theta$	$((\Theta - \sqrt{\Theta^2-4})/2, 1/2]$
decreasing	$\mathbb{N} \setminus \{1\}$	$[-\Theta, -1]$	$(0, 1/2]$

Table 3.6: For a given nature of m and value(s) of Θ, each row lists value(s) of n for which collisions takes place along with the value(s) of μ.

Proof. The collision condition for eigenvalues $i\Omega^{\mathfrak{g}}_{n,\ell,\mu}$ and $i\Omega^{\mathfrak{g}}_{n+\Theta,\ell,\mu}$ for $\Theta \in \mathbb{N}$ is

$$\ell^2 = -\gamma R_{\mathfrak{g}}(n,\mu,\Theta) + k^2 S_{\mathfrak{g}}(n+\Theta,\mu,\Theta), \tag{3.4.80}$$

where,

$$R_{\mathfrak{g}}(n,\mu,\Theta) = (n+\mu)(n+\mu+\Theta)+1 \quad \text{and}$$

$$S_{\mathfrak{g}}(n,\mu,\Theta) = \frac{(n+\mu)(n+\mu+\Theta)}{\Theta}((n+\mu)(m(k)-m(k(n+\mu))) - (n+\mu+\Theta)(m(k)-m(k(n+\mu+\Theta)))).$$

Since we intend to find only potentially unstable eigenvalues, therefore $(n+\mu)(n+\mu+\Theta) < 0$ which implies $n \in [-\Theta, -1]$.

The function $R_{\mathfrak{g}}(n,\mu,1), R_{\mathfrak{g}}(n,\mu,2) > 0$ for all $n \in [-\Theta,-1]$; while $R_{\mathfrak{g}}(n,\mu,\geq 3)$ is positive only for $n = -\Theta$ for all $\mu \in \left(0, \dfrac{\Theta - \sqrt{\Theta^2-4}}{2}\right)$; and negative for all

1. $n \in [-\Theta+1, -1]$ for all $\Theta \in (0, 1/2]$,
2. $n = -\Theta$ for all $\mu \in \left(\dfrac{\Theta - \sqrt{\Theta^2-4}}{2}, \dfrac{1}{2}\right]$.

Also, the sign of the function $S_{\mathfrak{g}}(n,\mu,\Theta)$ is

1. When m is monotonically increasing

   (a) The function $S_{\mathfrak{g}}(n,\mu,1) > 0$ for all $n \in [-\Theta,-1]$ and $\mu \in (0,1/2]$.

   (b) The function $S_{\mathfrak{g}}(n,\mu,\geq 2) < 0$ for all $n \in [-\Theta,-1]$ and $\mu \in (0,1/2]$.

2. When m is monotonically decreasing

   (a) The function $S_{\mathfrak{g}}(n,\mu,1) < 0$ for all $n \in [-\Theta,-1]$ and $\mu \in (0,1/2]$.

   (b) The function $S_{\mathfrak{g}}(n,\mu,\geq 2) > 0$ for all $n \in [-\Theta,-1]$ and $\mu \in (0,1/2]$.

The proof follows trivially from here. Note that $\Omega^{\mathfrak{g}}_{n,\ell,\mu} = 0$ at $\ell^2 = |\gamma((n+\mu)^2-1) + k^2(n+\mu)^2(m(k) - m(k(n+\mu)))|$. $\Omega^{\mathfrak{g}}_{n,\ell_c,\mu} = \Omega^{\mathfrak{g}}_{n+\Theta,\ell_c,\mu}{=}0$ for a fixed ℓ_c is possible only for $\Theta = -2n-1$, $\mu = 1/2$. Therefore, $\Omega^{\mathfrak{g}}_{n,\ell_c,\mu}$ and $\Omega^{\mathfrak{g}}_{n+\Theta,\ell_c,\mu}$ collide at the origin for $n = -(\Theta+1)/2$, for all $n \in [-\Theta,-1]\cap\mathbb{Z}$, $\mu = 1/2$ and $\ell_c^2 = |\gamma((n+1/2)^2-1) + k^2(n+1/2)^2(m(k) - m(k(n+1/2)))|$. All other collisions are away from origin.

□

3.4.2.4 Long wavelength transverse perturbations

We start the further analysis with the values of ℓ sufficiently close to origin, that is, $|\ell| \leq \ell_0$ for some $\ell_0 > 0$. Table 3.5 work as it is here too. For a and ℓ sufficiently small, $\{\Omega^{\mathfrak{g}}_{-1,\ell}, \Omega^{\mathfrak{g}}_{1,\ell}\}$ are pair of eigenvalues bifurcating continuously from $\{\Omega^{\mathfrak{g}}_{-1,0}, \Omega^{\mathfrak{g}}_{1,0}\}$. For $a = 0$, $\{\Omega^{\mathfrak{g}}_{-1,0}, \Omega^{\mathfrak{g}}_{1,0}\}$ are equipped with eigenfunctions $\{e^{-iz}, e^{iz}\}$. We choose the real basis $\{\cos z, \sin z\}$. We calculate expansion of a basis $\{\psi_1,\psi_2\}$ for the eigenspace corresponding to the eigenvalues of $\{\Omega^{\mathfrak{g}}_{-1,\ell}, \Omega^{\mathfrak{g}}_{1,\ell}\}$ in $L^2_0(\mathbb{T})$ by using expansions of $\eta_{\mathfrak{g}}$ and $c_{\mathfrak{g}}$ in (3.4.23), for small a and ℓ as

$$\begin{aligned}\psi^{\mathfrak{g}}_1(z) &= \cos z + 2aA_2\cos 2z + O(a^2),\\ \psi^{\mathfrak{g}}_2(z) &= \sin z + 2aA_2\sin 2z + O(a^2).\end{aligned}$$

We have the following expression for $\mathcal{Q}^{\mathfrak{g}}_a(\ell)$ after expanding and using η and c

$$\begin{aligned}\mathcal{Q}^{\mathfrak{g}}_a(\ell) = \gamma(\partial_z + \partial_z^{-1}) + \ell^2\partial_z^{-1} + k^2\partial_z(m(k) - \mathcal{M}_k) - 2a\alpha_1k^2\partial_z(\cos z) + k^2a^2(\alpha_1\eta_{\mathfrak{g}_2} - 3/4\alpha_2)\partial_z\\ - k^2a^2(2\alpha_1\eta_{\mathfrak{g}_2} + 3/2\alpha_2)\partial_z(\cos 2z) + O(a^3).\end{aligned}$$

We use expansion of $\mathcal{Q}_a^{\mathfrak{g}}(\ell)$ to find actions of $\mathcal{Q}_a^{\mathfrak{g}}(\ell)$ and identity operator on $\{\psi_1^{\mathfrak{g}}, \psi_2^{\mathfrak{g}}\}$, and arrive at

$$\mathcal{T}_a^{\mathfrak{g}}(\ell) = \begin{pmatrix} 0 & \ell^2 + \frac{3}{2}\alpha_2 k^2 a^2 + 2a^2\alpha_1 k^2 \eta_{\mathfrak{g}_2} \\ -\ell^2 & 0 \end{pmatrix} + O(a^2(\ell + a)).$$

To locate where these two eigenvalues are bifurcating from the origin, we analyze the characteristic equation $|\mathcal{T}_a^{\mathfrak{g}}(\ell) - \lambda_{\mathfrak{g}}\mathcal{I}_a| = 0$, where $\mathcal{I}_a$ is 2×2 identity matrix. From which we conclude that

$$\lambda_{\mathfrak{g}}^2 + \ell^2(\ell^2 + \frac{3}{2}\alpha_2 k^2 a^2 + 2a^2\alpha_1 k^2\eta_{\mathfrak{g}_2}) = O(|a|\ell^2(\ell^2 + a^2), \qquad (3.4.81)$$

and

$$\lambda_{\mathfrak{g}}^2 = -\ell^2(\ell^2 - \frac{3}{2}|\alpha_2|k^2a^2 + 2a^2|\alpha_1|k^2\eta_{\mathfrak{g}_2}) + O(|a|\ell^2(\ell^2 + a^2). \qquad (3.4.82)$$

For $\ell = a = 0$, we get zero as a double eigenvalue, which agrees with our calculation.

1. $\alpha_1 = 1$, $\alpha_2 = 0$

   (a) m is increasing, then we obtain two real eigenvalues with opposite signs when
   $$|k^2(m(2k) - m(k))| > \frac{3\gamma}{4} \quad \text{and} \quad \ell^2 < 2a^2k^2|\eta_{\mathfrak{g}_2}|,$$

   (b) m is decreasing, we obtain two purely imaginary eigenvalues.

2. $\alpha_1 = 0$, $\alpha_2 = -1$; we obtain two real eigenvalues with opposite signs for all m and $\forall k > 0$ and
   $$|\ell| < \sqrt{\frac{3}{2}}|a|k.$$

3. $\alpha_1 = 1$, $\alpha_2 = -1$, we obtain two real eigenvalues with opposite signs for all m when
   $$\frac{2k^2}{3\gamma + 4k^2(m(k) - m(2k))} < \frac{3}{4} \quad \text{and} \quad \ell^2 < \frac{3}{2}|\alpha_2|k^2a^2 - 2a^2|\alpha_1|k^2|\eta_{\mathfrak{g}_2}|.$$

Hence the Theorem 3.4.6.

3.4.2.5 Finite or short transverse perturbations

Here we work with the values of ℓ away from the origin, $|\ell| \geq \ell_0$ for some $\ell_0 > 0$. We do further analysis to check if collisions in Table 3.5 indeed lead to instability or not.

3.4.2.5.1 (In)stability analysis for $\Theta = 1$: For periodic perturbations, there are no collisions mentioned for $\Theta = 1$. However, there is one pair of colliding eigenvalues $\{\Omega^{\mathfrak{g}}_{-1,\ell,\mu}, \Omega^{\mathfrak{g}}_{0,\ell,\mu}\}$ for $\beta > 0$ with respect to non-periodic perturbations, colliding for

$$\ell^2 = -\gamma(1-\mu+\mu^2) + k^2(\mu(\mu-1)(m(k) - m(k(\mu-1))) - \mu(m(k) - m(k\mu)). \quad (3.4.83)$$

There exists a curve $\ell = \ell_c$ given in (3.4.83) along which

$$\Omega^{\mathfrak{g}}(\ell_c, \mu) := \Omega^{\mathfrak{g}}_{-1,\ell_c,\mu} = \Omega^{\mathfrak{g}}_{0,\ell_c,\mu}.$$

Furthermore,

$$\phi^{\mathfrak{g}}_{0,-1}(z) = e^{-iz} \qquad \textit{and} \qquad \phi^{\mathfrak{g}}_{0,0}(z) = 1, \quad (3.4.84)$$

forms the corresponding eigenspace for $\mathcal{Q}^{\mathfrak{g}}_0(\ell_c, \mu)$ associated with the two eigenvalues. Let

$$i\Omega^{\mathfrak{g}}(\ell_c, \mu) + i\theta^{\mathfrak{g}}_{a,\ell,-1} \qquad \textit{and} \qquad i\Omega^{\mathfrak{g}}(\ell_c, \mu) + i\theta^{\mathfrak{g}}_{a,\ell,0}, \quad (3.4.85)$$

are the eigenvalues of $\mathcal{Q}^{\mathfrak{g}}_a(\ell_c, \mu)$ departing from $i\Omega^{\mathfrak{g}}_{-1,\ell_c,\mu}$ and $i\Omega^{\mathfrak{g}}_{0,\ell_c,\mu}$ respectively for sufficiently small $|a|$ and $|\ell - \ell_c|$. Let $\{\phi^{\mathfrak{g}}_{a,\ell,-1}(z), \phi^{\mathfrak{g}}_{a,\ell,0}(z)\}$ be the extended eigenspace associated with two bifurcating eigenvalues. Following [14], we can take,

$$\phi^{\mathfrak{g}}_{a,-1,\ell}(z) = e^{-iz} + a\phi^{\mathfrak{g}}_{-1} + O(a^2), \quad (3.4.86)$$

$$\phi^{\mathfrak{g}}_{a,0,\ell}(z) = 1 + a\phi^{\mathfrak{g}}_0 + O(a^2). \quad (3.4.87)$$

Using the orthonormality conditions on the eigenfunctions $\phi^{\mathfrak{g}}_{a,\ell,-1}$ and $\phi^{\mathfrak{g}}_{a,\ell,0}$, we get

$$\phi^{\mathfrak{g}}_{-1} = \phi^{\mathfrak{g}}_0 = 0.$$

In order to find the eigenvalues, we compute matrix representations of $\mathcal{Q}_a^{\mathfrak{g}}(\ell_c,\mu)$ and identity operators on $\{\phi^{\mathfrak{g}}_{a,-1,\ell}(z), \phi^{\mathfrak{g}}_{a,0,\ell}(z)\}$ for sufficiently small $|a|$ and $|\ell-\ell_c|$

$$\mathcal{B}_a^{\mathfrak{g}}(\ell,\mu) = \begin{pmatrix} i\Omega^{\mathfrak{g}}(\ell_c,\mu) - i\dfrac{\varsigma}{\mu-1} & -ia\alpha_1 k^2\mu \\ -ia\alpha_1 k^2(\mu-1) & i\Omega^{\mathfrak{g}}(\ell_c,\mu) - i\dfrac{\varsigma}{\mu} \end{pmatrix} + O(|a|(|\varsigma|+|a|)),$$

where $\varsigma = \ell^2 - \ell_c^2$ and $\mathcal{I}_a$ is the 2×2 identity matrix. We solve the characteristic equation $\det(\mathcal{B}_a^{\mathfrak{g}}(\ell,\mu) - \lambda^{\mathfrak{g}}\mathcal{I}_a) = 0$ for λ of the form

$$\lambda^{\mathfrak{g}} = i\Omega^{\mathfrak{g}}(\ell_c,\mu) + i\theta,$$

and obtain the polynomial equation

$$\theta^2 + \theta\varsigma\left(\frac{1}{\mu-1} + \frac{1}{\mu} + O(a^2)\right) - a^2\alpha_1^2k^4\mu(\mu-1) + \frac{\varsigma^2}{\mu(\mu-1)} + O(a^2(|\varsigma|+a^2)) = 0.$$

A direct calculation depicts that the discriminant of this polynomial is

$$\mathrm{disc}_a(\varsigma,\mu) = \frac{\varsigma^2}{\mu^2(\mu-1)^2} + 4\alpha_1^2k^4\mu(\mu-1)a^2 + O(a^2(|\varsigma|+a^2)).$$

For any a sufficiently small, there exists

$$\varsigma_a(\mu) = 2\alpha_1 k^2\mu^{3/2}(1-\mu)^{3/2}|a| + O(a^2) > 0,$$

such that the two eigenvalues of $\mathcal{Q}_a^{\mathfrak{g}}(\ell,\mu)$ are purely imaginary when $|\ell^2-\ell_c^2| \geq \varsigma_a(\mu)$ and complex with opposite nonzero real parts when $|\ell^2-\ell_c^2| < \varsigma_a(\mu)$, which proves the theorem 3.4.7.

3.4.2.5.2 <u>(In)stability analysis for $\Theta \geq 2$:</u> For some $n \in \mathbb{Z}$ and a fixed $\Theta \geq 2$, we have

$$i\Omega^{\mathfrak{g}}_{n,\ell_*,\mu} = i\Omega^{\mathfrak{g}}_{n+\Theta,\ell_*,\mu} = i\Omega(\ell_*,\mu), \quad \mu \in (-1/2, 1/2]. \tag{3.4.88}$$

$\mu = 0$ represents the periodic case and $\mu \neq 0$ represents the non-periodic case. $i\Omega^{\mathfrak{g}}$ is an eigenvalue of $\mathcal{Q}_0^{\mathfrak{g}}(\ell_c,\mu)$ of multiplicity two with an orthonormal basis of eigenfunctions $\{e^{inz}, e^{i(n+\Theta)z}\}$. Let $i\Omega^{\mathfrak{g}} + i\nu_{a,n}$ and $i\Omega^{\mathfrak{g}} + i\nu_{a,n+2}$ be the eigenvalues of $\mathcal{Q}_a^{\mathfrak{g}}(\ell,\mu)$ bifurcating from $i\Omega^{\mathfrak{g}}_{n,\ell_c,\mu}$ and $i\Omega^{\mathfrak{g}}_{n+2,\ell_c,\mu}$ respectively, for $|a|$ and $|\ell-\ell_c|$ small. For the corresponding eigenspace, suppose $\{\varphi^{\mathfrak{g}}_{a,n}(z), \varphi^{\mathfrak{g}}_{a,n+2}(z)\}$ be an orthonormal basis. We suppose the following expansions

$$\varphi^{\mathfrak{g}}_{a,n}(z) = e^{inz} + a\varphi_{n,1} + a^2\varphi_{n,2} + \cdots + a^{\Theta}\varphi_{n,\Theta} + O(a^{\Theta+1}), \tag{3.4.89}$$

$$\varphi^{\mathfrak{g}}_{a,n+\Theta}(z) = e^{i(n+\Theta)z} + a\varphi_{n+\Theta,1} + a^2\varphi_{n+\Theta,2} + \cdots + a^{\Theta}\varphi_{n+\Theta,\Theta} + O(a^{\Theta+1}). \tag{3.4.90}$$

We use orthonormality of $\varphi^{\mathfrak{g}}_{a,n}$ and $\varphi^{\mathfrak{g}}_{a,n+\Theta}$ to find that

$$\varphi^{\mathfrak{g}}_{n,1} = \varphi^{\mathfrak{g}}_{n,2} = \cdots = \varphi^{\mathfrak{g}}_{n,\Theta} = \varphi^{\mathfrak{g}}_{n+\Theta,1} = \varphi^{\mathfrak{g}}_{n+\Theta,2} = \cdots = \varphi^{\mathfrak{g}}_{n+\Theta,\Theta} = 0.$$

Using the expansions of η and c in (3.4.4), we expand $\mathcal{Q}^{\mathfrak{g}}_a(\ell,\mu)$ in a as

$$\mathcal{Q}^{\mathfrak{g}}_a(\ell,\mu) = \mathcal{Q}_0(\ell,\mu) + k^2(\beta_2 a^2 + \beta_4 a^4 + \dots)(\partial_z + i\mu) - 2k^2 a\delta_1(\partial_z + i\mu)(\cos z) - \cdots - 2k^2 a^{\Theta}\delta_{\Theta}(\partial_z + i\mu)(\cos(\Theta z)). \tag{3.4.91}$$

Following the same procedures as in the preceding subsection, we arrive at

$$\mathscr{G}^{\mathfrak{g}}_a(\ell,\mu) = \begin{pmatrix} \mathscr{G}^{\mathfrak{g}}_{11} & \mathscr{G}^{\mathfrak{g}}_{12} \\ \mathscr{G}^{\mathfrak{g}}_{21} & \mathscr{G}^{\mathfrak{g}}_{22} \end{pmatrix} + O(a^{\Theta+1}),$$

where

$$\begin{aligned}
\mathscr{G}^{\mathfrak{g}}_{11} &= i\Omega(\ell_*,\mu) - i\frac{\varsigma}{n+\mu} + ik^2(n+\mu)(\beta_2 a^2 + \beta_4 a^4 + \dots),\\
\mathscr{G}^{\mathfrak{g}}_{12} &= -ik^2(n+\mu+\Theta)\delta^{\Theta},\\
\mathscr{G}^{\mathfrak{g}}_{21} &= -ik^2(n+\mu)\delta^{\Theta},\\
\mathscr{G}^{\mathfrak{g}}_{22} &= i\Omega(\ell_*,\mu) - i\frac{\varsigma}{n+\mu+\Theta} + ik^2(n+\mu+\Theta)(\beta_2 a^2 + \beta_4 a^4 + \dots).
\end{aligned}$$

The resulting discriminant of the characteristic equation $\det(\mathscr{G}^{\mathfrak{g}}_a(\ell,\mu) - (i\Omega(\ell_*,\mu) + i\theta)\mathcal{I}_a) = 0$ is

$$\mathrm{disc}^{\mathfrak{g}}_a(\varsigma) = \frac{\Theta^2\varsigma^2}{(n+\mu)^2(n+\mu+\Theta)^2} + k^4\Theta^2\beta_2^2 a^4 + O(a^2(|\varsigma| + |a^3|)).$$

For sufficiently small $|\varsigma|$ and $|a|$, $\mathrm{disc}_a^{\mathfrak{g}}(\varsigma)$ is positive which implies that no eigenvalue of $\mathcal{Q}_a^{\mathfrak{g}}(\ell,\mu)$ is bifurcating from the imaginary axis due to collision. Hence the proof of Theorems 3.4.6 and 3.4.7.

3.4.3 Applications

3.4.3.1 RM-fKdV-KP Equation

The RM-fKdV-KP equation can be derived from (3.4.5) by choosing

$$m(k) = 1 + |k|^{\alpha}, \qquad \alpha > 1/2$$

with $\alpha_1 = 1$, $\alpha_2 = 0$. Clearly, the symbol $m(k)$ meets the hypotheses 2.2.1 $J1$, $J2$ ($\mathfrak{b} = \alpha$, $A_1 = 1$ and $A_2 = 2$), and $J3$ (m is strictly increasing for $k > 0$). We can obtain the periodic solutions of rotation-modified fKdV equation from (3.4.23) by replacing $m(k)$ with $1 + |k|^{\alpha}$. Transverse stability and instability of these solutions can be derived using Theorems 3.4.6, 3.4.7, 3.4.8 and 3.4.9.

Corollary 3.4.22 (Transverse stability vs. instability of RM-fKdV-KP)**.** *For any a sufficiently small and $\gamma > 0$,*

1. *(a) For all $k > 0$ and $\beta < 0$, periodic traveling waves (3.4.23) of (3.4.11) are transversely stable with respect to either periodic or non-periodic (localized or bounded) perturbations in the direction of propagation and periodic perturbations in the transverse direction.*

   (b) For all $k > 0$ and $\beta > 0$, periodic traveling waves (3.4.23) of (3.4.11) are transversely stable with respect to periodic perturbations in the direction of propagation and, periodic with finite wavelength perturbations in the transverse direction.

2. *(a) $\beta > 0$, periodic traveling waves (3.4.23) of (3.4.11) are transversely unstable with respect to periodic perturbations in both directions and long wavelength perturbations in the transverse direction if*

$$k > \left| \frac{3\gamma}{4\beta(2^{\alpha} - 1)} \right|^{1/\alpha+2}.$$

(b) $\beta > 0$, *periodic traveling waves* (3.4.23) *of* (3.4.11) *are transversely unstable with respect to non-periodic (localized or bounded) perturbations in the direction of propagation of the wave and, periodic with finite wavelength perturbations in the transverse direction if*

$$k > \left(\frac{3(2^\alpha)\gamma}{(2^\alpha - 1)\beta} \right)^{1/\alpha+2}.$$

Note that, for $\gamma = 0$, all the results mentioned in Corollary 3.4.22 agree with the findings in [5] for KP-fKdV equation.

3.4.3.2 RMBO-KP Equation

RMBO-KP equation analogous to RM-fKdV-KP equation with $\alpha = 1$. We have derived transverse stability and instability of these solutions in Corollary 3.4.23 using Theorems 3.4.6, 3.4.7, 3.4.8 and 3.4.9.

Corollary 3.4.23 (Transverse stability vs. instability of RMKP-BO)**.** *For any a sufficiently small and* $\gamma > 0$*,*

1. (a) *For all* $k > 0$ *and* $\beta < 0$*, periodic traveling waves* (3.4.23) *of* (3.4.10) *are transversely stable with respect to either periodic or non-periodic (localized or bounded) perturbations in the direction of propagation and periodic perturbations in the transverse direction.*

   (b) *For all* $k > 0$ *and* $\beta > 0$*, periodic traveling waves* (3.4.23) *of* (3.4.10) *are transversely stable with respect to periodic perturbations in the direction of propagation and, periodic with finite wavelength perturbations in the transverse direction.*

2. (a) $\beta > 0$*, periodic traveling waves* (3.4.23) *of* (3.4.10) *are transversely unstable with respect to periodic perturbations in both directions and long wavelength perturbations in the transverse direction if*

$$k > \left| \frac{3\gamma}{4\beta} \right|^{1/3}.$$

   (b) $\beta > 0$*, periodic traveling waves* (3.4.23) *of* (3.4.10) *are transversely unstable* *with respect to non-periodic (localized or bounded) perturbations in the direction of propagation of the wave and, periodic with finite wavelength perturbations in the transverse direction if*

$$k > \left(\frac{6\gamma}{\beta}\right)^{1/3}.$$

For $\gamma = 0$, all the results mentioned in Corollary 3.4.23 agree with the findings in [5] for KP-BO equation.

3.4.3.3 RMG-KP Equation

The RMG-KP equation can be derived from (3.4.5) by choosing

$$m(k) = 1 + |k|^2,$$

with $\alpha_1 = 1$, $\alpha_2 = -1$. Clearly, the symbol $m(k)$ meets the hypotheses 2.2.1 $J1$, $J2$ ($A_1 = 1$, and $A_2 = 2$), and $J3$ (m is strictly increasing for $k > 0$). We can obtain the periodic solutions of rotation-modified Gardner equation from (3.4.23) by replacing $m(k)$ with $1 + |k|^2$. We have derived transverse stability and instability of these solutions in Corollary 3.4.24 using Theorems 3.4.6, 3.4.7, 3.4.8 and 3.4.9.

Corollary 3.4.24 (Transverse stability vs. instability of RMG-KP)**.** *For any a sufficiently small and $\gamma > 0$,*

1. (a) *For all $k > 0$ and $\beta < 0$, periodic traveling waves (3.4.23) of (3.4.16) are transversely stable with respect to non-periodic (localized or bounded) perturbations in the direction of propagation and periodic perturbations in the transverse direction.*

   (b) *For all $k > 0$ and $\beta \neq 0$, periodic traveling waves (3.4.4) of (3.4.16) are transversely stable with respect to periodic perturbations in the direction of propagation and, periodic with finite wavelength perturbations in the transverse direction.*

2. (a) *periodic traveling waves (3.4.23) of (3.4.16) are transversely unstable with respect to periodic perturbations cin both directions and long wavelength perturbations in the transverse direction if*

i. $\beta > 0$,

$$36|\beta|k^4 + 8k^2 < 9\gamma,$$

ii. $\beta < 0$,

$$-36|\beta|k^4 + 8k^2 < 9\gamma.$$

(b) $\beta > 0$, *periodic traveling waves* (3.4.23) *of* (3.4.16) *are transversely unstable with respect to non-periodic (localized or bounded) perturbations in the direction of propagation of the wave and, periodic with finite wavelength perturbations in the transverse direction if*

$$k > \left(\frac{4\gamma}{\beta}\right)^{1/4}.$$

3.4.3.4 Reduced RMKP Equation

The Reduced RMKP equation can be derived from (3.4.5) by choosing

$$m(k) = 1 + |k|^2,$$

with $\beta = 0$, $\alpha_1 = 1$ and $\alpha_2 = 0$. Clearly, the symbol $m(k)$ meets the hypotheses 2.2.1 $J1$, $J2$ ($A_1 = 1$, and $A_2 = 2$), and $J3$ (m is strictly increasing for $k > 0$). We can obtain the periodic solutions of reduced Ostrovsky equation from (3.4.23) by replacing $m(k)$ with $1 + |k|^2$. We can obtain transverse stability and instability of these solutions using Theorems 3.4.6, 3.4.7, 3.4.8 and 3.4.9.

Corollary 3.4.25. *Assume that small-amplitude periodic traveling waves of the reduced Ostrovsky equation are spectrally stable in* $L^2(\mathbb{T})$ *with respect to one-dimensional perturbations. Then for any* a *sufficiently small,* $\gamma > 0$, $k > 0$ *and* $\beta = 0$, *periodic traveling waves* (3.4.23) *of* (3.4.1) *are transversely stable with respect to either periodic or non-periodic (localized or bounded) perturbations in the direction of propagation and periodic perturbations in the transverse direction.*

3.4.3.5 RM-mKdV-KP Equation

The RM-mKdV-KP equation can be derived from (3.4.5) by choosing

$$m(k) = 1 + |k|^2,$$

with $\alpha_1 = 0$ and $\alpha_2 = -1$. Clearly, the symbol $m(k)$ meets the hypotheses 2.2.1 $J1$, $J2$ ($A_1 = 1$, and $A_2 = 2$), and $J3$ (m is strictly increasing for $k > 0$). We can obtain the periodic solutions of rotation-modified mKdV equation from (3.4.23) by replacing $m(k)$ with $1 + |k|^2$. We can obtain transverse stability and instability of these solutions using Theorems 3.4.6, 3.4.7, 3.4.8 and 3.4.9.

Corollary 3.4.26. *For any a sufficiently small and $\gamma > 0$,*

1. (a) *For all $k > 0$ and $\beta < 0$, periodic traveling waves (3.4.23) of (3.4.17) are transversely stable with respect to non-periodic (localized or bounded) perturbations in the direction of propagation and periodic perturbations in the transverse direction.*

   (b) *For all $k > 0$ and $\beta \neq 0$, periodic traveling waves (3.4.4) of (3.4.17) are transversely stable with respect to periodic perturbations in the direction of propagation and, periodic with finite wavelength perturbations in the transverse direction.*

2. (a) *For all $k > 0$ and $\beta \neq 0$, periodic traveling waves (3.4.23) of (3.4.17) are transversely unstable with respect to periodic perturbations in both directions and long wavelength perturbations in the transverse direction.*

   (b) *$\beta > 0$, periodic traveling waves (3.4.23) of (3.4.17) are transversely unstable with respect to non-periodic (localized or bounded) perturbations in the direction of propagation of the wave and, periodic with finite wavelength perturbations in the transverse direction if*

$$k > \left(\frac{4\gamma}{\beta}\right)^{1/4}.$$

For $\gamma = 0$, all the results mentioned in Corollary 3.4.26 are in accordance with the findings in [6, 47] for mKP-II equation.

3.4.3.6 RMILW-KP Equation

The RMILW-KP equation can be derived from (3.4.5) by choosing,

$$m(k) = k \coth k.$$

The symbol $m(k)$ satisfies Hypotheses 2.2.1 $J1$, $J2$ ($\mathfrak{b} = 2$, $A_1 = 1$, and $A_2 = 2$), and $J3$ (m is strictly increasing for $k > 0$). We can obtain the periodic solutions of rotation-modified ILW equation from (3.4.23) by replacing $m(k)$ with $k \coth k$. We can obtain transverse stability and instability of these solutions using Theorems 3.4.6, 3.4.7, 3.4.8 and 3.4.9.

Corollary 3.4.27 (Transverse stability vs. instability of RMILW-KP)**.** *For any a sufficiently small and $\gamma > 0$,*

1. *(a) For all $k > 0$ and $\beta < 0$, periodic traveling waves (3.4.23) of (3.4.14) are transversely stable with respect to either periodic or non-periodic (localized or bounded) perturbations in the direction of propagation and periodic perturbations in the transverse direction.*

   (b) For all $k > 0$ and $\beta > 0$, periodic traveling waves (3.4.23) of (3.4.5) are transversely stable with respect to periodic perturbations in the direction of propagation and, periodic with finite wavelength perturbations in the transverse direction.

2. *(a) $\beta > 0$, periodic traveling waves (3.4.23) of (3.4.14) are transversely unstable with respect to periodic perturbations in both directions and long wavelength perturbations in the transverse direction if*

$$k^2(2k \coth 2k - k \coth k) > \frac{3\gamma}{4\beta}.$$

   (b) $\beta > 0$, periodic traveling waves (3.4.23) of (3.4.14) are transversely unstable with respect to non-periodic (localized or bounded) perturbations in the direction of propagation of the wave and, periodic with finite wavelength in the transverse direction if

$$k^2(k \coth k - k/2 \coth (k/2)) > \frac{3\gamma}{\beta}.$$

Note that, for $\gamma = 0$, all the results mentioned in Corollary 3.4.27 agree with the results in [5] for KP-ILW equation.

3.4.3.7 RM-Whitham-KP Equation

The RM-Whitham-KP equation can be derived from (3.4.5) by choosing,

$$m(k) = \sqrt{\frac{\tanh k}{k}}.$$

The symbol $m(k)$ satisfies Hypotheses 2.2.1 $J1$, $J2$ with $\mathfrak{b} = -\frac{1}{2}$, $A_1 = 1$, and $A_2 = 2$, and $J3$ as m is strictly decreasing for $k > 0$. We can obtain the periodic solutions of rotation-modified Whitham equation from (3.4.23) by replacing $m(k)$ with $\sqrt{\frac{\tanh k}{k}}$. We can obtain transverse stability and instability of these solutions using Theorems 3.4.6, 3.4.7, 3.4.8 and 3.4.9.

Corollary 3.4.28 (Transverse stability vs. instability of RM-Whitham-KP)**.** *For any a sufficiently small and $\gamma > 0$,*

1. *(a) For all $k > 0$ and $\beta > 0$, periodic traveling waves* (3.4.23) *of* (3.4.12) *are transversely stable with respect to either periodic or non-periodic (localized or bounded) perturbations in the direction of propagation and periodic perturbations in the transverse direction.*

   (b) For all $k > 0$ and $\beta < 0$, periodic traveling waves (3.4.23) *of* (3.4.12) *are transversely stable with respect to periodic perturbations in the direction of propagation and, periodic with finite wavelength perturbations in the transverse direction.*

2. *(a) $\beta < 0$, periodic traveling waves* (3.4.23) *of* (3.4.12) *are transversely unstable with respect to periodic perturbations in both directions and long wavelength perturbations in the transverse direction if*

$$k^2\left(\sqrt{\frac{\tanh k}{k}} - \sqrt{\frac{\tanh 2k}{2k}}\right) > \frac{3\gamma}{4|\beta|}.$$

   (b) $\beta < 0$, periodic traveling waves (3.4.23) *of* (3.4.12) *are transversely unstable with respect to non-periodic (localized or bounded) perturbations in the direction*

of propagation of the wave and, periodic with finite wavelength perturbations in the transverse direction if

$$k^2\left(\sqrt{\frac{\tanh k/2}{k/2}}-\sqrt{\frac{\tanh k}{k}}\right)>\frac{3\gamma}{|\beta|}.$$

For $\gamma=0$, all the results mentioned in Corollary 3.4.28 agree with the findings in [5] for KP-Whitham equation.

4

High-frequency Instabilities

4.1 Introduction

This chapter is devoted to the study of high-frequency spectral instabilities of small-amplitude periodic traveling wave solutions in the Ostrovsky equation and its generalizations. While modulational and transverse instabilities have been extensively studied in previous chapters, high-frequency instability, associated with finite and short-wavelength perturbations, remains a distinct and important mechanism of spectral destabilization, especially in rotationally influenced dispersive systems.

The Ostrovsky equation models the evolution of weakly nonlinear, weakly dispersive long internal waves under the influence of Earth's rotation and serves as a rotational extension of the classical Korteweg–de Vries (KdV) equation. In the rotational regime, the dispersive properties of the equation are significantly altered, and the introduction of high-frequency perturbations leads to rich and nontrivial spectral behavior.

This chapter is based on the results presented in our work: *"High-frequency instability in the Ostrovsky equation and related models"* with Atul Kumar and Ashish Kumar Pandey. We rigorously establish the existence of a one-parameter family of small-amplitude periodic traveling wave solutions to the Ostrovsky equation using Lyapunov–Schmidt reduction. We then perform a detailed spectral analysis of the linearized operator about these periodic waves and identify conditions under which high-frequency instabilities arise.

4.2 The Model

The Ostrovsky equation

$$(u_t - \beta u_{xxx} + (u^2)_x)_x = \gamma u, \quad x \in \mathbb{R}, \tag{4.2.1}$$

was derived by Ostrovsky (see [62]) as a model for the unidirectional propagation of weakly nonlinear long surface and internal waves of small amplitude in rotating liquid. The liquid is assumed to be incompressible and inviscid. Here, $u(x,t)$ represents the free surface displacement of the fluid. The constant γ measures the effect of rotation and is rather small for the real conditions of the Earth rotation [24]. The parameter β determines the type of dispersion, namely $\beta < 0$ (negative dispersion) for surface and internal waves in the ocean and surface waves in a shallow channel with an uneven bottom and $\beta > 0$ (positive dispersion) for capillary waves on the surface of a liquid or for oblique magneto-acoustic waves in plasma [25]. Note that, if $u(x,t)$ satisfies (4.2.1) for a particular choice of β and γ, then $-u(x,-t)$ satisfies (4.2.1) with β and γ replaced by $-\beta$ and $-\gamma$ respectively. Due to this symmetry, we restrict $\gamma > 0$.

Setting $\gamma = 0$ in (4.2.1) and integrating with respect to $x \in \mathbb{R}$ and assuming the solution $u(x,t)$ and all the derivatives are vanishing at infinity, one obtains the well-known Korteweg-de Vries (KdV) equation

$$u_t - \beta u_{xxx} + (u^2)_x = 0. \tag{4.2.2}$$

The Ostrovsky equation is non-local and dispersive with linear dispersion as

$$\omega(k) = \frac{\gamma}{k} + \beta k^3.$$

It is also Hamiltonian

$$u_t = \frac{\partial \mathcal{H}(u)}{\partial u},$$

where

$$\mathcal{H}(u) = \int_{\mathbb{R}} \left(\frac{\beta}{2}|u_x|^2 + \frac{\gamma}{2}|D_x^{-1}u|^2 + \frac{1}{3}u^3 \right) dx,$$

and for $k \in \mathbb{N}$, the operator D_x^{-k} is defined by

$$(\widehat{D_x^{-k} f})(\mu) = (i\mu)^{-k}\hat{f}(\mu).$$

The Ostrovsky equation, unlike KdV, is nonintegrable by the method of the inverse scattering transform. The local and global-well posedness of the Ostrovsky equation are known in some weighted Sobolev spaces [56, 73].

In this chapter, we investigate the spectral stability of small-amplitude periodic traveling waves of the Ostrovsky equation. We use a standard argument based on implicit function theorem and Lyapunov-Schmidt reduction to establish the existence of a family of periodic traveling waves. As a consequence, we obtain a small-amplitude expansion of these periodic traveling waves. We linearize (4.2.1) about the obtained periodic traveling wave and examine the $L^2(\mathbb{R})$-spectrum of the linearized operator. In the case of periodic perturbations, one needs to restrict to the mean zero space because of the presence of ∂_z^{-1} in the linearized operator. But for square-integrable perturbations on the whole real line, we use Floquet-Bloch theory which transforms ∂_z^{-1} to $(\partial_z + i\mu)^{-1}$, where μ is the Floquet exponent. As a result, for $\mu \neq 0$, we need not restrict to mean zero space. In terms of perturbations, $\mu \neq 0$ corresponds to non-modulational perturbations and the resulting spectral instability is termed as *high-frequency instability* [17]. We show that the obtained small-amplitude periodic traveling waves of (4.2.1) exhibit high-frequency instability.

In Section 4.3, we obtain periodic traveling waves of the Ostrovsky equation bifurcating from the trivial solution. Existence of periodic traveling wave solutions We set up the spectral stability problem in Section 4.4 and prove the existence of high-frequency instabilities in Section 4.5.

4.3 Sufficiently small and periodic traveling waves

A traveling wave of (4.2.1) is a solution which propagates at a constant velocity without change of form. That is, $u(x,t) = U(x - ct)$ for some $c \in \mathbb{R}$. Substituting this in (4.2.1) leads to

$$cU'' + \beta U'''' - (U^2)'' + \gamma U = 0.$$

We seek a *periodic traveling wave* of (4.2.1). That is, U is a $2\pi/k$-periodic function of its argument where $k > 0$ is the wave number. Taking $z := kx$, the function $\eta(z) := U(x)$ is 2π-periodic in z and satisfies

$$ck^2\eta'' + \beta k^4 \eta'''' - k^2(\eta^2)'' + \gamma\eta = 0. \tag{4.3.1}$$

Note that (4.3.1) is invariant under $z \mapsto z + z_0$ and $z \mapsto -z$ and therefore, we may assume that η is even. Also, note that (4.3.1) does not possess scaling invariance. Hence, we may not a priori assume that $k = 1$. In fact, the stability result reported in Theorem 4.5.1 depends on k. To compare, the KdV equation (4.2.2) for periodic traveling waves possesses scaling invariance, and stability results are independent of the carrier wave number, see [10], for instance.

In what follows, we seek a non-trivial 2π-periodic solution η of (4.3.1). For fixed β and γ, let $F : H^4(\mathbb{T}) \times \mathbb{R} \times \mathbb{R}^+ \to L^2(\mathbb{T})$ be defined as

$$F(\eta, c; k) = ck^2\eta'' + \beta k^4 \eta'''' - k^2(\eta^2)'' + \gamma\eta. \tag{4.3.2}$$

It is well defined by a Sobolev inequality. We seek a solution $\eta \in H^4(\mathbb{T})$, $c \in \mathbb{R}$ and $k > 0$ of

$$F(\eta, c; k) = 0.$$

Note that if $\eta \in H^4(\mathbb{T})$, then from (4.3.1), $\eta'''' \in H^2(\mathbb{T})$ by a Sobolev inequality. Therefore, $\eta \in H^6(\mathbb{T})$. By a bootstrap argument, we obtain that $\eta \in H^\infty(\mathbb{T})$.

The operator F in (4.3.2) is a polynomial in parameters c and k. Its Fréchet derivatives with respect to η are all continuous from $H^4(\mathbb{T})$ to $L^2(\mathbb{T})$. Therefore, F is a real analytic operator.

Clearly, $F(0,c;k)=0$ for all $c\in\mathbb{R}$ and $k>0$. If non-trivial solutions of $F(\eta,c;k)=0$ bifurcate from $\eta\equiv 0$ for some c then

$$L_0 := \partial_\eta F(0,c;k) = ck^2\partial_z^2 + \beta k^4\partial_z^4 + \gamma,$$

from $H^4(\mathbb{T})$ to $L^2(\mathbb{T})$, is not an isomorphism. From a straightforward calculation,

$$L_0 e^{inz} = (-ck^2n^2 + \beta k^4 n^4 + \gamma)e^{inz} = 0, \quad n\in\mathbb{Z},$$

if and only if

$$c = \frac{\gamma}{k^2n^2} + \beta k^2 n^2, \quad n\in\mathbb{Z}.$$

Without loss of generality, we take $n=1$ viz.

$$c_0 = \frac{\gamma}{k^2} + \beta k^2. \tag{4.3.3}$$

Note that for $\beta>0$, wavenumbers, $k=\left(\frac{\gamma}{\beta n^2}\right)^{1/4}$, $2\le n\in\mathbb{N}$, satisfy resonance condition

$$\frac{\gamma}{k^2} + \beta k^2 = \frac{\gamma}{k^2n^2} + \beta k^2 n^2, \tag{4.3.4}$$

of the fundamental mode and nth harmonic, and hence the kernel of L_0 is four-dimensional. For all other values of k, L_0 is a Fredholm operator of index zero with both kernel and co-kernel spanned by $e^{\pm iz}$.

Next, we employ a Lyapunov-Schmidt procedure to establish the existence of a one-parameter family of non-trivial solutions of $F(\eta,c;k)=0$ bifurcating from $\eta\equiv 0$ and $c=c_0$. The proof follows along the same lines as the arguments in [38, 41] and we provide it in Appendix A.1. We summarize the existence result for periodic traveling waves of (4.2.1) and their small-amplitude expansion below.

Theorem 4.3.1. *Consider the following wavenumbers depending on the sign of β,*

1. *for $\beta<0$, all $k>0$, and*

2. *for $\beta>0$, all $k>0$ but $\left(\frac{\gamma}{\beta n^2}\right)^{1/4}$, $2\le n\in\mathbb{N}$.*

Then, for all such wavenumbers k, a one parameter family of solutions of (4.3.1) exists, given by $u(x,t) = \eta(a;k)(k(x-c(a;k)t))$ for $a \in \mathbb{R}$ and $|a|$ sufficiently small; $\eta(a;k)(\cdot)$ is 2π-periodic, even and smooth in its argument, and $c(a;k)$ is even in a; $\eta(a;k)$ and $c(a;k)$ depend analytically on a and k. Moreover,

$$\eta(a;k)(z) = a\cos(z) + a^2 A_2 \cos 2z + a^3 A_3 \cos 3z + a^4(A_{42}\cos 2z + A_{44}\cos 4z) + O(a^5),$$

and

$$c(a;k) = c_0 + a^2 c_2 + a^4 c_4 + O(a^6),$$

as $a \to 0$, where c_0 is in (4.3.3),

$$A_2 = \frac{2k^2}{3\gamma - 12\beta k^4}, \quad A_3 = \frac{9k^2 A_2}{8\gamma - 72\beta k^4}, \quad A_{42} = 2A_2A_3 - 2A_2^3, \quad A_{44} = \frac{8k^2(A_2^2 + 2A_3)}{15\gamma - 240\beta k^4},$$

$$c_2 = A_2, \quad \textit{and} \quad c_4 = 3A_2A_3 - 2A_2^3.$$

Remark 4.3.2. *The Lyapunov-Schmidt procedure described above guarantees the existence of periodic traveling waves of (4.2.1) for only small amplitude. It would be interesting to explore if the bifurcation curves for the periodic traveling waves of large amplitudes exist.*

4.4 Linearization and the spectral problem

We linearize (4.2.1) about the solution η in Theorem 4.3.1 in the coordinate frame moving at the speed c. The result becomes

$$k(v_t - ckv_z - \beta k^3 v_{zzz} + 2k(\eta v)_z)_z = \gamma v.$$

We seek a solution of the form $v(z,t) = e^{\frac{\lambda}{k}t}\tilde{v}(z)$, $\lambda \in \mathbb{C}$, to arrive at

$$\mathcal{T}_{k,a}^{\lambda}\tilde{v} := (\lambda\partial_z - k^2\partial_z^2(c + \beta k^2\partial_z^2 - 2\eta) - \gamma)\tilde{v} = 0. \tag{4.4.1}$$

The operator $\mathcal{T}_{k,a}^{\lambda}$ is defined on $L^2(\mathbb{R})$ with dense domain $H^4(\mathbb{R})$.

Definition 4.4.1. *The periodic traveling wave solution η is said to be spectrally stable if*

$\mathcal{T}_{k,a}^{\lambda}$ *is invertible for any* $\lambda \in \mathbb{C}$ *with* $\Re(\lambda) > 0$*, otherwise, it is deemed to be spectrally unstable.*

The operator $\mathcal{T}_{k,a}^{\lambda}$ has continuous spectrum in $L^2(\mathbb{R})$. By Floquet theory, since coefficients of $\mathcal{T}_{k,a}^{\lambda}$ are periodic functions, all solutions of (4.4.1) in $L^2(\mathbb{R})$ are of the form $\tilde{v}(z) = e^{i\mu z}V(z)$ where $\mu \in (-1/2, 1/2]$ is the Floquet exponent and V is a 2π-periodic function, see [31] for a similar situation. This helps to break the invertibility problem of $\mathcal{T}_{k,a}^{\lambda}$ in $L^2(\mathbb{R})$ into a family of invertibility problems in $L^2(\mathbb{T})$.

Lemma 4.4.2. *The linear operator* $\mathcal{T}_{k,a}^{\lambda}$ *is invertible in* $L^2(\mathbb{R})$ *if and only if linear operators*

$$\mathcal{T}_{k,a,\mu}^{\lambda} = \lambda(\partial_z + i\mu) - k^2(\partial_z + i\mu)^2(c + \beta k^2(\partial_z + i\mu)^2 - 2\eta) - \gamma,$$

acting in $L^2(\mathbb{T})$ *with dense domain* $H^4(\mathbb{T})$ *are invertible, for any* $\mu \in (-1/2, 1/2]$.

We refer to [31, Proposition A.1] for detailed proof in a similar situation.

Remark 4.4.3. *The* $L^2(\mathbb{T})$*-spectra of operators* $T_{k,a,\mu}^{\lambda}$ *consist of eigenvalues of finite multiplicity. Therefore,* $T_{k,a,\mu}^{\lambda}$ *is invertible in* $L^2(\mathbb{T})$ *if zero is not an eigenvalue of* $T_{k,a,\mu}^{\lambda}$.

Using Remark 4.4.3, we have the following result.

Lemma 4.4.4. *The operator* $\mathcal{T}_{k,a,\mu}^{\lambda}$ *is not invertible in* $L^2(\mathbb{T})$ *for some* $\lambda \in \mathbb{C}$ *and* $\mu \neq 0$ *if and only if* $\lambda \in \sigma(\mathcal{A}_{k,a,\mu})$*,* $L^2(\mathbb{T})$*-spectrum of the operator,*

$$\mathcal{A}_{k,a,\mu} := k^2(\partial_z + i\mu)(c + \beta k^2(\partial_z + i\mu)^2 - 2\eta) + \gamma(\partial_z + i\mu)^{-1}.$$

Proof. The operator $\mathcal{T}_{k,a,\mu}^{\lambda}$ is not invertible in $L^2(\mathbb{T})$ for some $\lambda \in \mathbb{C}$ and $\mu \neq 0$ if and only if zero is an eigenvalue of $\mathcal{T}_{k,a,\mu}^{\lambda}$. Moreover, for a $V \in L^2(\mathbb{T})$, $\mathcal{T}_{k,a,\mu}^{\lambda}V = 0$ if and only if $\mathcal{A}_{k,a,\mu}V = \lambda V$. The proof follows trivially. □

Note that $\mu \neq 0$ is important in Lemma 4.4.4. For $\mu = 0$, $\mathcal{A}_{k,a,0}$ is not well-defined on $L^2(\mathbb{T})$ since ∂_z^{-1} is not well-defined on $L^2(\mathbb{T})$. In what follows, we restrict μ to be non-zero and examine the $L^2(\mathbb{T})$-spectrum of $\mathcal{A}_{k,a,\mu}$. To ease the notation, we will drop k from subscript in $\mathcal{A}_{k,a,\mu}$. We observe that if $\lambda \in \sigma(\mathcal{A}_{a,\mu})$ then $\bar{\lambda} \in \sigma(\mathcal{A}_{a,-\mu})$, therefore, it is enough to consider $\mu \in (0, 1/2]$. Also, since $\eta(z)$ is even in z, we have

$$\sigma(\mathcal{A}_{a,\mu}) = \sigma(-\mathcal{A}_{a,-\mu}).$$

Consequently, we obtain spectral instability of η if $\sigma(\mathcal{A}_{a,\mu})$ is not contained in the imaginary axis for some $\mu \in (0, 1/2]$.

A straightforward calculation shows that

$$\mathcal{A}_{0,\mu} e^{inz} = i\omega_{n,\mu} e^{inz}, \quad n \in \mathbb{Z}, \tag{4.4.2}$$

where

$$\omega_{n,\mu} = k^2(n+\mu)(c_0 - \beta k^2(n+\mu)^2) - \frac{\gamma}{n+\mu} = (\gamma - \beta k^4(n+\mu)^2)\frac{((n+\mu)^2-1)}{n+\mu}. \tag{4.4.3}$$

We have $\sigma(\mathcal{A}_{0,\mu}) \subset i\mathbb{R}$, which should be the case since $a = 0$ corresponds to the zero solution, which is trivially stable. As $|a|$ increases, the eigenvalues in (4.4.2) move around and may leave the imaginary axis to give spectral instability. Because of the symmetry of the spectrum around the real and imaginary axes, spectral instability takes place only if a pair of imaginary eigenvalues collide on the imaginary axis. If the spectral instability arises from a collision away from the origin on an imaginary axis, it is termed as *High-frequency instability* [17].

Let $n \neq m \in \mathbb{Z}$, and $\mu_{n,m} \in (0, 1/2]$ be such that

$$\omega_{n,\mu_{n,m}} = \omega_{m,\mu_{n,m}}. \tag{4.4.4}$$

For a fixed value of $\gamma > 0$ and $\beta \neq 0$, collisions at the origin take place only when

$$(n+\mu)^2 = (m+\mu)^2,$$

with $k^4 = \frac{\gamma}{\beta(n+\mu)^2}$. This reduces to $2\mu = -(n+m)$, which is an integer and therefore forces $\mu = 1/2$ and $m = -n-1$. Hence, collisions at the origin take place only for $\beta > 0$, all $n \in \mathbb{Z}$, $m = -n-1$, $\mu_{n,m} = 1/2$, and $k = \left(\frac{\gamma}{\beta(n+1/2)^2}\right)^{1/4}$. There is no collision at the origin if $\beta < 0$. The collision at the origin for non-zero Floquet exponent is an interesting characteristic of the Ostrovsky equation. In similar studies on other various water wave models such collisions have not been observed, see [38, 40, 41], for example.

Here, we seek to find high-frequency instabilities and, therefore, list all collisions away from the origin below.

Lemma 4.4.5. *For each $n \in \mathbb{Z}$ and $\Delta n \in \mathbb{N}$, the eigenvalues ω_{n,μ_c} and $\omega_{n+\Delta n,\mu_c}$ of the operator $\mathcal{A}_{0,\mu}$ would collide away from the origin for*

1. *all pairs $\{n, n+\Delta n\}$ except $\{-1,1\}$ and $\{-2,0\}$ when $\beta > 0$, and*

2. *pairs $\{-1,1\}$ and $\{-n,0\} : n \geq 2$, when $\beta < 0$.*

Moreover, all collisions take place in an interval $k \in (k^{\min}_{n,\Delta n}, k^{\max}_{n,\Delta n})$, where $k^{\min}_{n,\Delta n} > 0$ and $k^{\max}_{n,\Delta n} \in \mathbb{R}^+ \cup \{\infty\}$ can be calculated explicitly depending on n and Δn.

Proof. In (4.4.4), without loss of generality, we can assume that $n < m$ and $m = n + \Delta n$ with $\Delta n \in \mathbb{N}$. Then the collision condition (4.4.4) can be written as $\omega_{n,\mu_c} = \omega_{n+\Delta n,\mu_c}$, that is

$$(\gamma - \beta k^4 (n+\mu)^2)\frac{((n+\mu)^2-1)}{n+\mu} = (\gamma - \beta k^4 (n+\Delta n+\mu)^2)\frac{((n+\Delta n+\mu)^2-1)}{n+\Delta n+\mu}.$$

We can rearrange the above equation to obtain

$$k^4 = \frac{\gamma \Delta n}{\beta} K(x,\Delta n) := \frac{\gamma \Delta n}{\beta}\frac{1 + x(x+\Delta n)}{x(x+\Delta n)((x+\Delta n)^3 - x^3 - \Delta n)},$$

where $x = n+\mu$. In terms of the function $K(x,\Delta n)$, since $\gamma > 0$, collision between n and $n+\Delta n$ takes place

1. for $\beta > 0$ if $K(n+\mu,\Delta n) > 0$ for some $\mu \in (0,1/2]$, and

2. for $\beta < 0$ if $K(n+\mu,\Delta n) < 0$ for some $\mu \in (0,1/2]$.

Hence, it is important to know the sign of $K(x,\Delta n)$ for a fixed Δn and $x \in \mathbb{R}$. We examine this case by case.

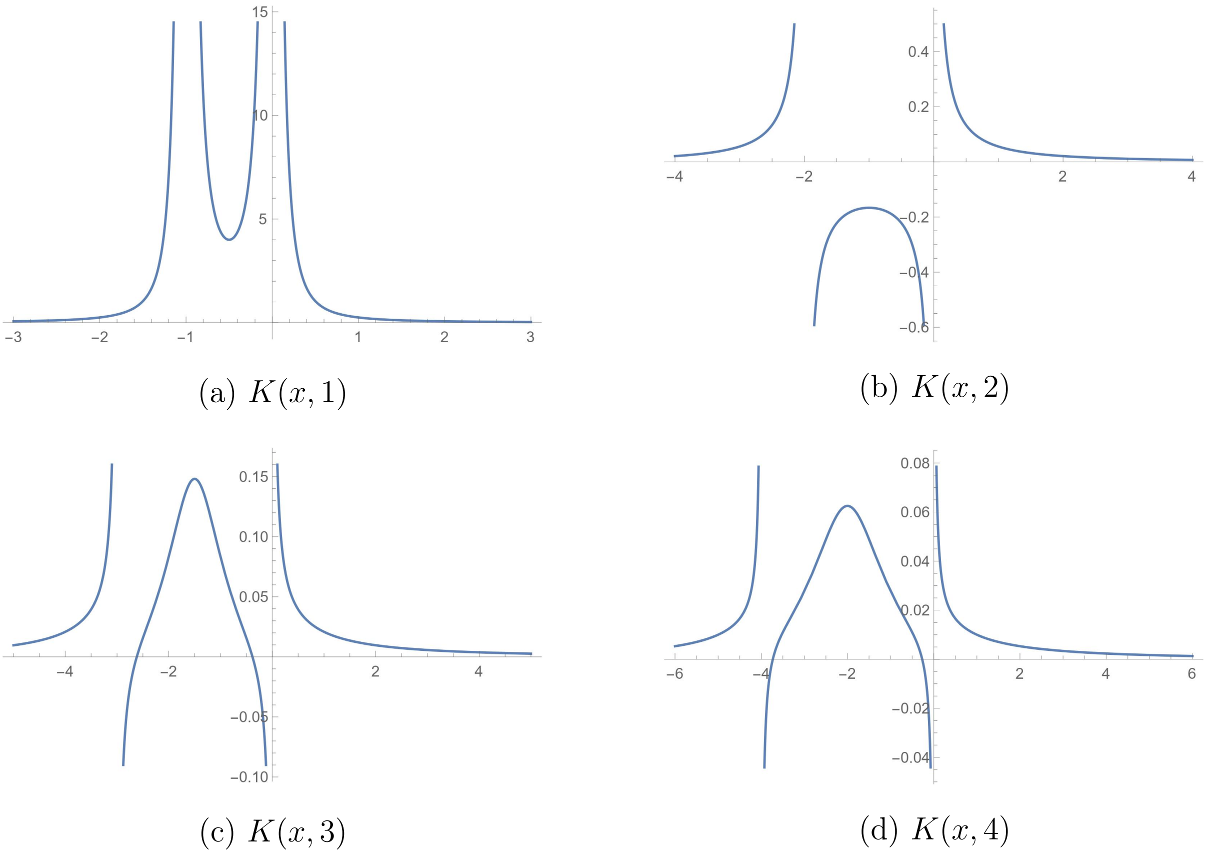
1. **Case 1 ($\Delta n = 1$):** The function $K(x,1)$, see Figure 4.1(a), is always positive except at singularities -1 and 0. Therefore, there is a collision between n and $n+1$ for all $n \in \mathbb{Z}$ when $\beta > 0$ while there is no collision between n and $n+1$ for any $n \in \mathbb{Z}$ when $\beta < 0$.

2. **Case 2 ($\Delta n = 2$):** The function $K(x,2)$, see Figure 4.1(b), is positive for $x \in (-\infty,-2) \cup (0,\infty)$ and negative for $x \in (-2,0)$. Since $\mu \in \left(0,\frac{1}{2}\right]$, there is a collision between n and $n+2$ for all $n \in \mathbb{Z}\backslash\{-1,-2\}$ when $\beta > 0$ while there is a collision between n and $n+2$ only for $n=-2$, and -1 when $\beta < 0$.

3. **Case 3 ($\Delta n \geq 3$):** The function $K(x,\Delta n)$, $\Delta n \geq 3$, see Figures 4.1(c) and 4.1(d) for example, is positive in
$$(-\infty,-\Delta n) \cup \left(-\frac{\Delta n + \sqrt{\Delta n^2-4}}{2}, -\frac{\Delta n - \sqrt{\Delta n^2-4}}{2}\right) \cup (0,\infty),$$
and negative in
$$\left(-\Delta n, -\frac{\Delta n + \sqrt{\Delta n^2-4}}{2}\right) \cup \left(-\frac{\Delta n - \sqrt{\Delta n^2-4}}{2}, 0\right).$$
For $\Delta n \geq 3$, we have
$$-\Delta n < -\frac{\Delta n + \sqrt{\Delta n^2-4}}{2} < -\Delta n + \frac{1}{2}, \text{ and } -\frac{1}{2} < -\frac{\Delta n - \sqrt{\Delta n^2-4}}{2} < 0.$$
Therefore, there is a collision between n and $n+\Delta n$ for all $n \in \mathbb{Z}$ when $\beta > 0$ while there is a collision between n and $n+\Delta n$ only for $n = -\Delta n$ when $\beta < 0$.

This proves the existence of all pairs satisfying collision condition (4.4.4) away from the origin.

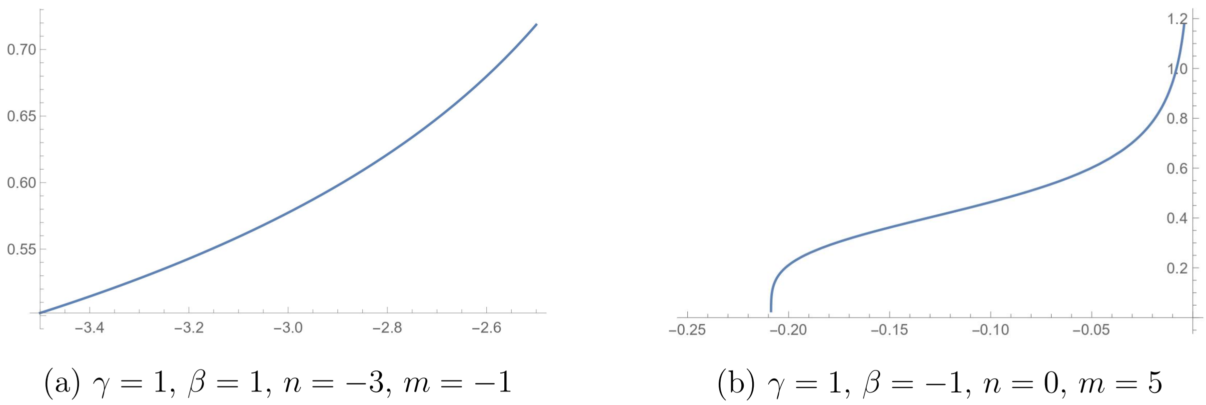
Now, for a fixed Δn, if $K(n,\Delta n) > 0$ for some $n \neq -\Delta n, 0$ then it continues to be positive in $[n, n+1/2]$ and therefore, for $\beta > 0$, collision takes place between n and $n+\Delta n$ for all $\mu \in (0,1/2]$. Since $K(x,\Delta n)$ restricted to $x \in (n, n+1/2]$ is a continuous function, it attains a maximum and minimum in $[n, n+1/2]$ and therefore, collision takes place in a bounded interval of wavenumbers $k \in (k_{n,\Delta n}^{\min}, k_{n,\Delta n}^{\max}) \subset (0,\infty)$, see Figure 4.2(a) for an example. Similarly, for $\beta > 0$ and $n = -\Delta n$, the collision takes place in a bounded interval of wavenumbers. For $n = 0$, $K(x,\Delta n)$ is positive and unbounded either in $(n, n+1/2]$ and therefore $K(x,\Delta n)$ is bounded below but unbounded above. In these cases, collision takes place in an interval of wavenumbers $k \in (k_{n,\Delta n}^{\min}, \infty) \subset (0,\infty)$, see Figure 4.2(b) for an example. Similarly, for $\beta < 0$ and $n = -\Delta n$, collision takes place in an unbounded interval of wavenumbers. This completes the proof. □



(a) $K(x, 1)$ (b) $K(x, 2)$

(c) $K(x, 3)$ (d) $K(x, 4)$

FIGURE 4.1: Graph of function $K(x, \Delta n)$ vs. x for $\Delta n = 1, 2, 3$, and 4.



(a) $\gamma = 1,\ \beta = 1,\ n = -3,\ m = -1$ (b) $\gamma = 1,\ \beta = -1,\ n = 0,\ m = 5$

FIGURE 4.2: Graph of wavenumbers vs. $n+\mu$ for two collisions. The range of wavenumbers for which collision is taking place is approximately $(0.5, 0.73)$ for the left plot and $(0, \infty)$ for the right plot.

A necessary condition for collisions in Lemma 4.4.5 to provide high-frequency instability is that their *Krein signatures* at collision should be opposite. Since the Ostrovsky equation possesses a Hamiltonian structure, the linear operator $\mathcal{A}_{a,\mu}$ can be decomposed as

$$\mathcal{A}_{a,\mu} = J_\mu \mathcal{L}_{a,\mu},$$

$\boldsymbol{\Delta n}$	$\boldsymbol{\beta > 0}$	$\boldsymbol{\beta < 0}$
1	$\{-1, 0\}$	none
2	none	$\{-2, 0\}, \{-1, 1\}$
≥ 3	$\{-1, \Delta n - 1\}, \{-2, \Delta n - 2\}, \ldots, \{-\Delta n + 1, 1\}, \{-\Delta n, 0\}$	$\{-\Delta n, 0\}$

Table 4.1: Collisions with opposite Krein signatures for a given Δn for $\beta > 0$ and $\beta < 0$.

where $J_\mu = \partial_z + i\mu$ is skew-adjoint and

$$\mathcal{L}_{a,\mu} = k^2(c + \beta k^2(\partial_z + i\mu)^2 - 2\eta) + \gamma(\partial_z + i\mu)^{-2},$$

is self-adjoint. With this decomposition, the Krein signature $\kappa_{n,\mu}$ of eigenvalues $i\omega_{n,\mu}$ in (4.4.3) of $\mathcal{A}_{0,\mu}$ is given by

$$\kappa_{n,\mu} = \operatorname{sgn}(\langle_{0,\mu} e^{inz}, e^{inz}\rangle) = \operatorname{sgn}\left(\frac{1}{n+\mu}\omega_{n,\mu}\right), \tag{4.4.5}$$

where sgn is the signum function which determines the sign of a real number. If the collision condition (4.4.4) is satisfied for some $n, m \in \mathbb{Z}$ and $\mu_{n,m} \in (0, 1/2]$ then (4.4.5) provides that eigenvalues $i\omega_{n,\mu}$ and $i\omega_{m,\mu}$ have opposite Krein signatures at the collision if

$$(n + \mu_{n,m})(m + \mu_{n,m}) < 0, \tag{4.4.6}$$

otherwise they have same Krein signatures at the collision. Using (4.4.6), we can rule out some collisions in Lemma 4.4.5 which will not lead to high-frequency instability.

Lemma 4.4.6. *1. For $\beta > 0$, out of all collisions mentioned in Lemma 4.4.5, $\{n, m\}$ with $n \leq -1$ and $m \geq 0$, have opposite Krein signatures.*

2. For $\beta < 0$, all collisions mentioned in Lemma 4.4.5 have opposite Krein signatures.

Proof. If $n \neq 0$ and $m \neq 0$ then from (4.4.6), n and m must be of opposite signs for (4.4.6) to hold. If one of n or m is zero then the other needs to be negative in order for (4.4.6) to hold. Then the proof follows. □

4.5 High-frequency instabilities

Table 4.1 summarizes all the collisions with opposite Krein signatures based on Lemma 4.4.6 for a given Δn for both $\beta > 0$ and $\beta < 0$. In what follows, we do further analysis to check if collisions in Table 4.1 corresponding to $\Delta n = 1$, and 2, lead to high-frequency instability.

4.5.1 $\Delta n = 1$ calculation and conclusion

For a fixed $n \in \mathbb{Z}$, let $\mu_0 \in (0, 1/2]$ be such that

$$0 \neq \omega_{n,\mu_0} = \omega_{n+1,\mu_0} =: \omega.$$

Therefore, $i\omega$ is an eigenvalue of $\mathcal{A}_{0,\mu_0}$ of multiplicity two with an orthonormal basis of eigenfunctions $\{e^{inz}, e^{i(n+1)z}\}$. For $|a|$ small, let λ_{n,a,μ_0} and λ_{n+1,a,μ_0} be eigenvalues of $\mathcal{A}_{a,\mu_0}$ bifurcating from $i\omega$ with an orthonormal basis of eigenfunctions $\{\phi_{n,a,\mu_0}(z), \phi_{n+1,a,\mu_0}(z)\}$. Note that $\lambda_{n,0,\mu_0} = \lambda_{n+1,0,\mu_0} = i\omega$ with $\phi_{n,0,\mu_0}(z) = e^{inz}$ and $\phi_{n+1,0,\mu_0}(z) = e^{i(n+1)z}$. Let

$$\lambda_{n,a,\mu_0} = i\omega + i\mu_{n,a,\mu_0} \qquad \text{and} \qquad \lambda_{n+1,a,\mu_0} = i\omega + i\mu_{n+1,a,\mu_0}. \tag{4.5.1}$$

We are interested in the location of $\mu_{n,a,\mu}$ and $\mu_{n+1,a,\mu}$ for $|a|$ small as if they have non-zero imaginary parts then we obtain high-frequency instability.

We start with the following expansions of eigenfunctions

$$\phi_{n,a,\mu_0} = e^{inz} + a\phi_{n,1} + a^2\phi_{n,2} + O(a^3), \tag{4.5.2}$$
$$\phi_{n+1,a,\mu_0} = e^{i(n+1)z} + a\phi_{n+1,1} + a^2\phi_{n+1,2} + O(a^3). \tag{4.5.3}$$

We use orthonormality of ϕ_{n,a,μ_0} and ϕ_{n+1,a,μ_0} to find that

$$\phi_{n,1} = \phi_{n,2} = \phi_{n+1,1} = \phi_{n+1,2} = 0.$$

To trace the bifurcation of the eigenvalues from the point of the collision on the imaginary axis for $|a|$ sufficiently small, we compute the actions of $\mathcal{A}_{a,\mu_0}$ and identity operators on the extended eigenspace $\{\phi_{n,a,\mu_0}(z), \phi_{n+1,a,\mu_0}(z)\}$ viz.

$$\mathcal{B}_{a,\mu_0} = \left[\frac{\langle \mathcal{A}_a(\mu_0)\phi_{i,a,\mu_0}(z), \phi_{j,a,\mu_0}(z)\rangle}{\langle \phi_{i,a,\mu_0}(z), \phi_{i,a,\mu_0}(z)\rangle}\right]_{i,j=n,n+1} \quad \text{and } \mathcal{I}_a = \left[\frac{\langle \phi_{i,a,\mu_0}(z), \phi_{j,a,\mu_0}(z)\rangle}{\langle \phi_{i,a,\mu_0}(z), \phi_{i,a,\mu_0}(z)\rangle}\right]_{i,j=n,n+1}. \tag{4.5.4}$$

Here $\langle \cdot, \cdot \rangle$ denotes the $L^2(\mathbb{T})$- inner product as defined in (2.5.11).

Using the expansions of η and c in Theorem 4.3.1, we expand $\mathcal{A}_{a,\mu_0}$ in a as

$$\mathcal{A}_{a,\mu_0} = \mathcal{A}_{0,\mu_0} - 2ak^2(\partial_z + i\mu_0)\cos z + a^2k^2(\partial_z + i\mu_0)(c_2 - 2A_2\cos 2z) + O(a^3),$$

and use the expansion of eigenfunctions in (4.5.2)-(4.5.3) to find the matrices in (4.5.4) as

$$\mathcal{B}_{a,\mu_0} = \begin{bmatrix} i\omega + ik^2a^2(n+\mu_0)c_2 & -ik^2a(n+1+\mu_0) \\ \\ -ik^2a(n+\mu_0) & i\omega + ik^2a^2(n+1+\mu_0)c_2 \end{bmatrix} + O(a^3),$$

and $\mathcal{I}_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + O(a^3)$. Note that $\mathcal{B}_{0,\mu_0} = \text{diag}(i\omega, i\omega)$ which should be the case as $i\omega$ is an eigenvalue of $\mathcal{A}_{0,\mu_0}$ of multiplicity two. The two values of μ solving the equation

$$\det(\mathcal{B}_{a,\mu_0} - (i\omega + i\mu)\mathcal{I}_a) = 0, \tag{4.5.5}$$

would coincide with μ_{n,a,μ_0} and μ_{n+1,a,μ_0} in (4.5.1) in leading order of a. Plugging the values in (4.5.5) and calculating the discriminant of the quadratic in μ, we arrive at

$$\mathbb{D}_{a,\mu_0} = 4k^4a^2(n+\mu_0)(n+1+\mu_0) + O(a^3).$$

Therefore, for sufficiently small $|a|$, if $(n+\mu_0)(n+1+\mu_0)$ is negative then we would obtain high-frequency instability.

From Table 4.1, the only collision for $\Delta n = 1$ is when $\beta > 0$ and $n = -1$. This collision takes place for all values of $\mu \in (0, 1/2]$, see Figure 4.3. Analyzing the function $K(x, \Delta n)$ for $x = \mu - 1$ and $\Delta n = 1$, we easily deduce that this collision takes place for wavenumbers $k \in ((4\gamma/\beta)^{1/4}, \infty)$. We summarize the result in the following theorem.

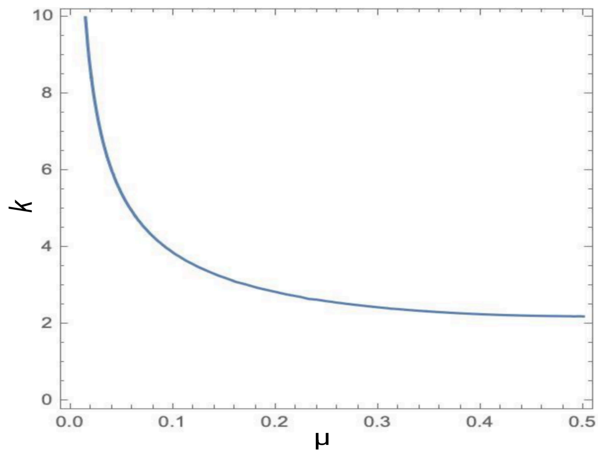


Figure 4.3: Collision contour describing collision between eigenvalues $i\omega_{-1,\mu}$ and $i\omega_{0,\mu}$ for different values of k and μ for $\gamma = 6$ and $\beta = 1$.

Theorem 4.5.1. *For a fixed $\gamma > 0$ and $\beta > 0$, a $2\pi/k$-periodic traveling wave of* (4.2.1) *given by $u(x,t) = \eta(k(x-ct))$ where η and c are given in Theorem 4.3.1 suffers high-frequency instability if*

$$k > \sqrt[4]{\frac{4\gamma}{\beta}}.$$

4.5.2 $\Delta n = 2$ calculation and conclusion

We proceed as in the previous section. For a fixed $n \in \mathbb{Z}$, let $\mu_0 \in (0, 1/2]$ be such that

$$0 \neq \omega_{n,\mu_0} = \omega_{n+2,\mu_0} =: \omega.$$

That is, $i\omega$ is an eigenvalue of $\mathcal{A}_{0,\mu_0}$ of multiplicity two with an orthonormal basis of eigenfunctions $\{e^{inz}, e^{i(n+2)z}\}$. As before, for $|a|$ small, let λ_{n,a,μ_0} and λ_{n+2,a,μ_0} be eigenvalues of $\mathcal{A}_{a,\mu_0}$ bifurcating from $i\omega$ with an orthonormal basis of eigenfunctions $\{\phi_{n,a,\mu_0}(z), \phi_{n+2,a,\mu_0}(z)\}$. Let

$$\lambda_{n,a,\mu_0} = i\omega + i\mu_{n,a,\mu_0} \qquad \text{and} \qquad \lambda_{n+2,a,\mu_0} = i\omega + i\mu_{n+2,a,\mu_0}, \tag{4.5.6}$$

and we are interested in the location of $\mu_{n,a,\mu}$ and $\mu_{n+2,a,\mu}$ for $|a|$ small. Again, using orthonormality of ϕ_{n,a,μ_0} and ϕ_{n+2,a,μ_0} we find that

$$\phi_{n,a,\mu_0} = e^{inz} + O(a^5) \qquad \text{and} \qquad \phi_{n+2,a,\mu_0} = e^{i(n+2)z} + O(a^5). \tag{4.5.7}$$

As before, we compute the action matrices of $\mathcal{A}_{a,\mu_0}$ and identity operators on the extended eigenspace $\{\phi_{n,a,\mu_0}(z), \phi_{n+2,a,\mu_0}(z)\}$. We use the expansions of η and c in Theorem 4.3.1 to expand $\mathcal{A}_{a,\mu_0}$ in a as

$$\begin{aligned}\mathcal{A}_{a,\mu_0} =& \mathcal{A}_{0,\mu_0} - 2ak^2(\partial_z + i\mu_0)\cos z + a^2k^2(\partial_z + i\mu_0)(c_2 - 2A_2\cos 2z) - 2a^3k^2A_3(\partial_z + i\mu_0)\cos 3z \\ &+ a^4k^2(\partial_z + i\mu_0)(c_4 - 2(A_{42}\cos 2z + A_{44}\cos 4z)) + O(a^5).\end{aligned}$$

Using the expansion of eigenfunctions in (4.5.7), the matrices in (4.5.4) turn out to be

$$\mathcal{B}_{a,\mu_0} = \begin{bmatrix} i\omega + ik^2(a^2A_2 + a^4c_4)(n+\mu_0) & -ik^2(a^2A_2 + a^4A_{42})(n+2+\mu_0) \\ \\ -ik^2(a^2A_2 + a^4A_{42})(n+\mu_0) & i\omega + ik^2(a^2A_2 + a^4c_4)(n+2+\mu_0) \end{bmatrix} + O(a^5),$$

and $\mathcal{I}_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + O(a^5)$. Again, we solve the equation

$$\det(\mathcal{B}_{a,\mu_0} - (i\omega + i\mu)\mathcal{I}_a) = 0,$$

to obtain a quadratic in μ whose discriminant is given by

$$\mathbb{D}_{a,\mu_0} = 4k^4a^4A_2^2(n + \mu_0 + 1)^2 + O(a^5).$$

Note that, irrespective of the values of n and μ_0, the leading term in the discriminant is always positive. Therefore, we do not observe any high-frequency instability for the $\Delta n = 2$ case by performing the perturbation calculation up to the fourth power of the amplitude parameter a.

4.5.3 $\Delta n \geq 3$ discussion

It is evident from calculations for $\Delta n = 1$ and 2 that higher is the value of Δn, higher powers of a are needed in the expansion of the operator $\mathcal{A}_{a,\mu}$. That is, we need to calculate more terms in the expansion of solution η in the Theorem 4.3.1. Since we do not see any obvious symmetry in the coefficients of the solution, calculating higher coefficients is challenging, and so is the high-frequency instability analysis for higher Δn.

Therefore, we only present the analysis for $\Delta n = 1$ and 2. In some recent works [14], high-frequency instability have been explored numerically for higher values of Δn. We believe that numerical methods in [14], can be applied for the Ostrovsky equation, too, and we will pursue it as a future project. Moreover, if the general term in the expansion of the solution η can be calculated explicitly, then the high-frequency instability can be explored for higher values of Δn analytically.

5

Ongoing and Future Work

Building upon the analytical foundations laid in this thesis, several promising directions remain for further investigation. Some of these have already been initiated and are under active development, while others are proposed as future extensions of the current research.

5.1 Ongoing work

5.1.1 The Benjamin-Feir instability in KdV-like equations with general dispersion and monomial nonlinearity

In collaboration with Bernard Deconinck and Ashish Kumar Pandey, this work examines a class of scalar equations with general dispersion and monomial nonlinearity, including various KdV-like equations. For small-amplitude traveling wave solutions, the spectrum near the origin of the linear operator is characterized by linearizing around periodic traveling waves. Building on the work of Berti, Maspero, and Ventura [1, 2], the analysis uses Kato's similarity transformations and considers the reversibility of the Hamiltonian system. Symplectic methods reduce the problem to finding eigenvalues of a 3x3 complex Hamiltonian matrix, which is then block-diagonalized into a 2x2 unstable spectrum and a purely imaginary stable element. The primary result is that the unstable spectrum of generalized KdV equations may form figure-8 curves and more complex patterns with a modulational instability X crossing at the origin. This phenomenon, previously observed numerically, is now rigorously proven for a wide class of equations under certain non-degeneracy conditions on the dispersion relation, with the Whitham-Benjamin coefficient playing a crucial role.

5.1.2 Transverse Instabilities in the full-dispersion Kadomtsev-Petviashvili equation

This work is in collaboration with Mathew A. Johnson and Ashish Kumar Pandey.

The full-dispersion KP (FDKP) equation

$$u_t + \boldsymbol{\phi}(D_x, D_y)u + (u^2)_x = 0,$$

where $u(x, y, t)$ is typically the height of the wave from equilibrium position at spatial variables x and y and time variable t, $D_x = -i\partial_x$, $D_y = -i\partial_y$ and $\boldsymbol{\phi}(D_x, D_y)$ is a Fourier multiplier defined as

$$\boldsymbol{\phi}(D_x, D_y) = i\sqrt{D_x^2 + D_y^2}\, c\Big(\sqrt{D_x^2 + D_y^2}\Big)\frac{D_x}{|D_x|},$$

where

$$c(\kappa) := \sqrt{(1 + T\kappa^2)\frac{\tanh\kappa}{\kappa}}.$$

Here $T > 0$ measures the effect of surface tension. The symbol $p(\kappa_1, \kappa_2)$ of $\boldsymbol{\phi}$ is given as

$$p(\kappa_1, \kappa_2) = i\,\mathrm{sgn}(\kappa_1)\sqrt{\kappa_1^2 + \kappa_2^2}\, c\Big(\sqrt{\kappa_1^2 + \kappa_2^2}\Big).$$

We investigate the transverse stability and instability of one-dimensional small-amplitude periodic traveling waves in the aforementioned FDKP equation.

5.2 Future work

5.2.1 Benjamin-Feir Instability in KdV-Like Equations with General Dispersion and Polynomial Nonlinearity

The objective of this work is to generalize the analysis of Benjamin-Feir instability in KdV-like equations by incorporating polynomial nonlinearity, as opposed to the simpler monomial nonlinearity used in previous studies. The analysis investigates the conditions under which the Benjamin-Feir instability occurs in this generalized setting, examining how the interplay between general dispersion and polynomial nonlinearity affects wave stability. A comprehensive spectral analysis will be performed to determine the stability of wave solutions in the generalized model. This involves identifying the full spectrum of perturbation growth rates as a function of system parameters, providing insights into both stable and unstable regimes. Understanding the spectrum is crucial for predicting the onset of Benjamin-Feir instability and the behavior of perturbed solutions over time.

5.2.2 Determining the Pinch-Off Steepness of the Benjamin-Feir Figure-Eight for Stokes Waves

The objective of this project is to identify the steepness threshold at which the figure-eight instability curve of the Benjamin-Feir spectrum pinches off in Stokes waves. This steepness represents a critical point for the stability of high-amplitude waves. The methodology involves spectral analysis and perturbation methods, with a focus on determining the steepness values at which the onset of instability occurs.

5.2.3 High-Frequency Instabilities for Near-Extreme Stokes Waves

The objective of this project is to investigate the behavior of high-frequency instabilities in near-extreme Stokes waves, extending the analysis of the stability spectrum beyond the vicinity of the origin and across a range of steepnesses. High-frequency instabilities in near-extreme conditions are not fully understood, and this work will enhance our ability to model and predict wave behaviors in extreme physical settings. Leveraging insights from Prof. Deconinck's recent work with Sergey Dyachenko and Anastassiya Semenova

[15], this project will apply spectral techniques to explore the evolution of instability spectra in high-amplitude conditions.

Conclusion

The research directions outlined in this chapter build naturally on the analytical and spectral foundations developed throughout this thesis. From ongoing investigations of Benjamin–Feir and transverse instabilities in generalized dispersive models to ambitious future projects involving extreme wave behavior, well-posedness, and numerical validation, these efforts aim to broaden the mathematical understanding of nonlinear water wave dynamics. Together, they reflect a sustained commitment to exploring the intricate relationship between wave structure, dispersion, nonlinearity, and stability in both theoretical and applied settings.

A

Lyapunov-Schmidt Procedure

A.1 Lyapunov-Schmidt Procedure

Since

$$\partial_k F(\eta, c; k) := (2ck\eta'' + 4\beta k^3\eta'''' - 2k(\eta^2)'' + \gamma\eta) \quad \text{and} \quad \partial_c F(\eta, c; k) := k^2\eta''$$

are continuous, $F : H^4(\mathbb{T}) \times \mathbb{R} \times \mathbb{R}^+ \to L^2(\mathbb{T})$ is C^1. Note that

$$L_0 e^{\pm iz} = 0.$$

For arbitrary $k > 0$, we seek a non-trivial solution $\eta \in H^4(\mathbb{T})$ near trivial solution $\eta \equiv 0$ of

$$F(\eta, c; k) = 0, \tag{A.1.1}$$

for some c near c_0, where c_0 is defined in (4.3.3). Let

$$\eta(z) = 0 + \frac{1}{2}ae^{iz} + \frac{1}{2}\bar{a}e^{-iz} + v(z) \quad \text{and} \quad c = c_0 + r,$$

where $a \in \mathbb{C}$ and $v \in H^4(\mathbb{T})$ satisfying that

$$\int_{\mathbb{T}} v(z)\, e^{\pm iz}\, dz = 0,$$

and $r \in \mathbb{R}$. Plugging these into (A.1.1) and using the fact that $L_0 e^{\pm iz} = 0$, we arrive at

$$L_0 v = g(a, \bar{a}, v, r) = -rk^2\eta'' + k^2(\eta^2)'', \tag{A.1.2}$$

where g is analytic in its argument and $g(0,0,0,r) = 0$ for all $r \in \mathbb{R}$. We define the projection operator $\Pi : L^2(\mathbb{T}) \to \ker L_0$ as

$$\Pi f(z) = \widehat{f}(1)e^{iz} + \widehat{f}(-1)e^{-iz}.$$

Since $\Pi v = 0$, we may write (A.1.2) as

$$L_0 v = (I - \Pi)g(a, \bar{a}, v, r) \quad \text{and} \quad 0 = \Pi g(a, \bar{a}, v, r). \tag{A.1.3}$$

Note that, for wavenumbers not satisfying the resonance condition (2.4), we can write

$$f(z) = L_{0_{|(I-\Pi)H^4(\mathbb{T})}} \left(\sum_{n \neq \pm 1} \frac{\widehat{f}(n)}{(-c_0k^2n^2 + \beta k^4 n^4 + \gamma)} e^{inz} \right),$$

or

$$(L_{0_{|(I-\Pi)H^4(\mathbb{T})}})^{-1} f(z) = \sum_{n \neq \pm 1} \frac{\widehat{f}(n)}{(-c_0k^2n^2 + \beta k^4 n^4 + \gamma)} e^{inz}.$$

Consequently, we can rewrite (A.1.3) as

$$v = L_0^{-1}(I - \Pi)g(a, \bar{a}, v, r) \quad \text{and} \quad 0 = \Pi g(a, \bar{a}, v, r). \tag{A.1.4}$$

Note that $(L_{0_{|(I-\Pi)H^4(\mathbb{T})}})^{-1}$ depends analytically on its arguments.

In parallel with the proof in [38], it follows from the repeated use of implicit function theorem and symmetries of the equation that a unique solution

$$(v, r) = (V(a, \bar{a}, r), r(|a|)),$$

exists to equations in (A.1.4) in the vicinity of $(a, \bar{a}, r) = (0, 0, 0)$, which depends analytically on its argument. Moreover,

$$\eta(z) = 0 + a\cos z + V(a, a, r(|a|))(z) \quad \text{and} \quad c = c_0 + r(|a|),$$

solve (A.1.1) for $|a|$ sufficiently small. Since details are exactly similar, we omit it here.

Since η and c depend analytically on a for $|a|$ sufficiently small and since c is even in a, we write that

$$\eta(k; a)(z) = 0 + a\cos z + a^2\eta_2(z) + a^3\eta_3(z) + a^4\eta_4(z) + O(a^5),$$

and

$$c(k; a) = c_0(k) + a^2 c_2 + a^4 c_4 + O(a^6),$$

as $a \to 0$, where η_2, η_3 and η_4 are even and 2π-periodic in z. Substituting these into (A.1.1), we see that the coefficients of a both sides are equal. Equating the coefficients of a^2, we arrive at

$$\beta k^4 \eta_2'''' + c_0 k^2 \eta_2'' + \gamma\eta_2 = -2k^2\cos(2z),$$

which has a solution

$$\eta_2(z) = A_2\cos(2z) := \frac{2k^2}{3\gamma - 12\beta k^4}\cos(2z). \tag{A.1.5}$$

For $O(a^3)$, we get

$$\beta k^4 \eta_3'''' + c_0 k^2 \eta_3'' + \gamma\eta_3 = k^2(c_2 - A_2)\cos(z) - 9k^2 A_2\cos(3z),$$

which has solution

$$c_2 = A_2, \tag{A.1.6}$$

and

$$\eta_3(z) = A_3 \cos(3z) := \frac{9k^2 A_2}{8\gamma - 72\beta k^4} \cos(3z). \tag{A.1.7}$$

For $O(a^4)$, we get

$$c_4 = 3A_2A_3 - 2A_2^3, \tag{A.1.8}$$

and

$$\eta_4(z) = A_{42}\cos 2z + A_{44}\cos 4z := \big(2A_2A_3 - 2A_2^3\big)\cos 2z + \left(\frac{8k^2(A_2^2 + 2A_3)}{15\gamma - 240\beta k^4}\right)\cos 4z. \tag{A.1.9}$$

This completes the proof.

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