Topics in the Distribution of Farey Sequences

by

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Submitted
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

to the

Indraprastha Institute of Information Technology Delhi
April 2026

Dedicated to my family

Certificate

This is to certify that the thesis entitled **"Topics in the Distribution of Farey Sequences"** being submitted by **"Mr. Bittu"** to the **Indraprastha Institute of Information Technology Delhi**, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Acknowledgements

First and foremost, I extend my heartfelt gratitude to my advisor, Dr. Sneha Chaubey, for her continuous inspiration and unwavering support. I am grateful for her never-ending encouragement, invaluable guidance, and patience. Her expertise, enthusiastic involvement, and availability for discussions were instrumental in shaping this research endeavor. I also appreciate that I was given the motivation and opportunity to attend several conferences and workshops throughout my Ph.D. journey.

I sincerely thank Prof. Igor Shparlinski for arranging a research visit for me at the University of New South Wales, Sydney. I am grateful to him for his encouragement and for our valuable discussions during my visit.

I am grateful to my review committee members Dr. Debika Banerjee (IIIT Delhi), Dr. Tapas Chatterjee (IIT Ropar), and Dr. Sudhir Kumar Pujahari (NISER Bhubaneswar) for their valuable time, insightful comments and suggestions. I would like to acknowledge all the faculty members of the Department of Mathematics, IIIT Delhi, for their continuous support and cooperation. The research in this thesis was financially supported by University Grants Commission (UGC) and IIIT Delhi.

I also wish to thank all my collaborators who contributed to the manuscripts included in this thesis and to my other research projects.

I would also like to thank my friends Bhavna, Divya Khurana, Leijo Jose, Shivani Goel, and my colleagues, who accompanied me during my time at IIIT Delhi. Especially, I want to acknowledge my good friend Bhavna and Divya Khurana whose positivity and encouragement have been a continuous source of support.

My heartfelt appreciation goes to my family; without their support, this would not have been

possible–particularly my parents, Mr. Omveer Singh and Mrs. Mukesh Devi, whose unwavering support has been my pillar of strength. Their love, encouragement, sacrifices, and faith in me made this achievement possible. I am also grateful to my sister Neeru, my brother-in-law Harbhajan, and my niece Vanika. I also thank my cousins for their love and joy.

I dedicate this work to all of you. Your constant support, love, and faith in me have carried me to this achievement. Thank you for being such a meaningful part of this journey.

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Declaration

I acknowledge that I am fully responsible for the entire content of my thesis, including any sections assisted by online tools, including Artificial Intelligence-based tools. I accept full accountability for any violations of ethical standards in publications arising from the use of such tools.

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Abstract

In this thesis, we study the distribution of Farey sequences. Let Q be a positive integer. The Farey sequence $\mathcal{F}_Q$ of order Q is the set of irreducible fractions in $[0, 1]$ whose denominators do not exceed Q. The first study characterizing the behavior of sequences via equidistribution was carried out in the seminal paper of Weyl. Equidistribution refers to being evenly spaced in a measure space. Equidistributed sequences are particularly useful for performing numerical integration. The notion of equidistribution does not provide information about finer statistics, such as randomness, local clustering, and periodic structure of sequences. To study the fine-scale statistics of a sequence, one can study the nearest neighbor gap distribution, as well as ν-level correlation measure. We study the equidistribution and correlation measure for Farey sequences.

The study of the Farey sequence is of independent interest because of its role in the Diophantine approximation, the circle method, and its connection to the Riemann Hypothesis as established by the classical work of Franel and Landau. The Farey fractions of order Q have a one-to-one correspondence with visible lattice points in the triangle with vertices $(0, 0), (0, Q)$, and (Q, Q) through straight lines passing through the origin. The visible lattice points along polynomials have been introduced and studied by Chaubey et al. Motivated by this, we introduce polynomial Farey fractions as a subset of fractions $a/q \in [0, 1]$ such that the point (a, q) is visible through polynomial curves and examine their distribution. In particular, we study and prove that the lim sup of the pair correlation measure of the polynomial Farey sequence is bounded. For the specific polynomial $P(x) = x(x + 1)$, we show that the pair correlation measure exists and establish an explicit formula for the pair correlation function which is non-Poissonian. Further, when restricting to prime denominators, the pair correlation measure is shown to be Poissonian. A sequence is said to be Poissonian if it behaves like a random uniformly distributed sequence. It is interesting to study the distribution of Farey fractions with denominators in arithmetic progression, as it is closely related to the Generalized Riemann Hypothesis. Moreover, we study an analog of Chebyshev's bias question for polynomial Farey fractions with denominators in an arithmetic progression. Chebyshev's bias question deals with the prime number races and states that there are more primes of the form $4n+3$ than the primes of the form $4n+1$.

Furthermore, we study the distribution of the sequence of Farey fractions with k-free denominators lying in an arithmetic progression, denoted by $\mathscr{F}_{Q,k}^{(m)}$. We prove that the sequence $\left(\mathscr{F}_{Q,k}^{(m)}\right)_Q$ is equidistributed by establishing an estimate for a Weyl sum. Additionally, we establish an equivalent criterion for the Generalized Riemann Hypothesis in terms of the distribution of fractions in $\mathscr{F}_{Q,k}^{(m)}$ analogous to the classical results of Franel and Landau. We also investigate the correlation measure of the sequence $\left(\mathscr{F}_{Q,k}^{(m)}\right)_Q$ and provide an explicit form for the pair correlation measure.

Another effective approach to understanding the distribution of the Farey fractions is to examine their indices. We study the distribution of Farey indices by deriving asymptotic formulas for the moments of the index function of Farey fractions with $\mathcal{B}$-free denominators which lie in a given arithmetic progression.

Contents

List of Figures

List of Symbols

Symbol	**Meaning**
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
μ	The Möbius function
ϕ	The Euler totient function
μ_k	The characteristic function for k-free integers
$\mu_{\mathcal{B}}$	The characteristic function for $\mathcal{B}$-free integers
$\tau(n)$	The divisor function
$\zeta(s)$	The Riemann zeta function
$L(s,\chi)$	The Dirichlet L-function where χ is a Dirichlet character
$\mathcal{L}(s,\chi')$	The Hecke L-function where χ' is a Hecke character
$\#A$	Cardinality of the set A
$\lfloor \cdot \rfloor$	Floor function
$\pi(x;q,a)$	The number of primes $p \leq x$ such that $p \equiv a \pmod q$
$\gcd(a,b)$	The greatest common divisor of a and b
$(a,b)=1$	The integers a and b are coprime

$[a, b]$	The least common multiple of a and b
$\ll$	Vinogradov asymptotic notation
$\mathrm{O}(\cdot)$ and $o(\cdot)$	Big O and little-o asymptotic notation
$f(x) \sim g(x)$	$\lim_{x\to\infty} f(x)/g(x) = 1$
$f(x) \asymp g(x)$	There exist constants C_1 and C_2 such that $C_1 g(x) \le f(x) \le C_2 g(x)$
$f(x) = \Omega_{\pm}(g(x))$	$\limsup_{x\to\infty} f(x)/g(x) > 0$ and $\liminf_{x\to\infty} f(x)/g(x) < 0$

Research Publications

Publications related to the Dissertation

- **B. Chahal**, S. Chaubey, On The Distribution Of Polynomial Farey Points And Chebyshev's Bias Phenomenon, *Math. Z.*, 312(8), 2026. `https://doi.org/10.1007/s00209-025-03886-5`.
- **B. Chahal**, S. Chaubey, Pair Correlation Of Farey Fractions With Square-Free Denominators, *Acta Arith.*, 215(4):289-307, 2024, `DOI:10.4064/aa230329-6-6`.
- **B. Chahal**, S. Chaubey, S. Goel, On The Distribution Of Index Of Farey Sequences, *Res. Number Theory*, 10(2), 2024, `https://doi.org/10.1007/s40993-024-00511-y`.
- **B. Chahal**, S. Chaubey, T. Chatterjee, Distribution Of Farey Fractions With k-Free Denominators, preprint, `https://doi.org/10.48550/arXiv.2507.00228`.

Other Publications

- D. Banerjee, **B. Chahal**, S. Chaubey, and K. Khurana, Distribution Of Values Of General Euler Totient Function, *J. Math. Anal. Appl.*, 530(2), 2024, `https://doi.org/10.1016/j.jmaa.2023.127660`.
- **B. Chahal**, Chebyshev's Bias For Irrational Factor Function, *Ramanujan J*, 69(50), 2026 , `https://doi.org/10.1007/s11139-026-01321-9`.
- **B. Chahal**, E. Elma, N. Fellini, A. Vatwani, T. Vo, On The Second Hardy-Littlewood Conjecture, *Acta Arith.* (accepted). `DOI:10.4064/aa250808-4-11`.

1

Introduction

1.1 Farey sequence

The Farey sequence $\mathcal{F}_Q$ of order Q is an ascending sequence of fractions a/q in the unit interval $(0,1]$ such that $(a,q)=1$ and $0<a\leq q\leq Q$. The cardinality of $\mathcal{F}_Q$ is given by

$$N(Q)=\sum_{q=1}^{Q}\phi(q)=\frac{3Q^2}{\pi^2}+\mathrm{O}\left(Q\log Q\right),$$

where we have used the following estimate for the average order of $\phi(n)$.

Lemma 1.1.1 ([6], Theorem 3.7)**.** *For $x>1$, we have*

$$\sum_{n\leq x}\phi(n)=\frac{3}{\pi^2}x^2+\mathrm{O}\left(x\log x\right).$$

In 1816, Farey, a geologist, published a note [37] stating the "mediant property" of fractions: if $a_1/q_1 < a_2/q_2 < a_3/q_3$ are consecutive fractions in $\mathcal{F}_Q$, then

$$\frac{a_2}{q_2} = \frac{a_1 + a_3}{q_1 + q_3}. \tag{1.1}$$

Cauchy [18] proved this property in 1816 and attributed it to Farey. However, Haros had already [52] constructed $\mathcal{F}_{99}$ using this property in 1802. Due to this, some mathematicians believe that the Farey sequence should be named the Haros-Farey sequence. Farey himself acknowledged his unfamiliarity with Haros's work, remarking that "he is not acquainted, whether this curious property of fractions has been before pointed out; or whether it may admit of any easy or general demonstration".

If $a_1/q_1 < a_2/q_2$ are consecutive Farey fractions in $\mathcal{F}_Q$, then an equivalent property to (1.1) holds:

$$a_2 q_1 - a_1 q_2 = 1. \tag{1.2}$$

Using (1.1) and (1.2), one can recursively construct all the elements of $\mathcal{F}_Q$. Various geometric interpretations of the Farey sequence demonstrate their application in several contexts. Ford circles [39] represent an example of this. One of the important interpretations is via visibility. The Farey fractions are in a one-to-one correspondence with visible lattice points in the triangle with vertices $(0,0), (0,Q)$, and (Q,Q) along straight lines passing through the origin. A point $(a,b) \in \mathbb{Z}^2$ is called visible from the origin if there is no other point of $\mathbb{Z}^2$ on the straight line joining the origin and point (a,b). In various approximation problems, the Farey fractions are often better suited than continued fractions. Hurwitz [55] used the Farey fractions in the Diophantine approximation of real numbers by rationals. In the early 1920s, Hardy, Ramanujan, and Littlewood initiated the circle method, making use of Farey fractions.

It is widely known that the Farey fractions are equidistributed modulo one. In equidistribution modulo one, we examine how the values of a sequence $(x_n)_{n\in\mathbb{N}}$ spread out over the entire interval $[0,1]$. Equidistribution tests the distribution on fixed scales, like intervals of some fixed size.

1.1.1 Equidistribution modulo one

The classical theory of equidistribution modulo one dates back to the early twentieth century, when Weyl [98] laid its foundations in his seminal paper. The study of equidistribution modulo one is concerned with the distribution of fractional parts of real numbers in $[0, 1]$.

Definition 1.1.1 (Equidistribution mod 1)**.** *A sequence* $(x_n)_{n\in\mathbb{N}}$ *of real numbers is said to be equidistributed modulo* 1 *if, for every interval* $I \subseteq [0, 1)$*, we have*

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \leq n \leq N \mid \{x_n\} \in I\} = |I|, \tag{1.3}$$

where $\{x_n\}$ *denotes the fractional part of* x_n*.*

An equivalent form of equidistribution modulo one in (1.3) can also be seen in terms of characteristic function χ_I of I. Therefore, a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is equidistributed modulo one if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_I(x_n) = \int_0^1 \chi_I(x)dx.$$

The preceding statement, combined with the approximation technique, yields a characterization of equidistribution in terms of averages of continuous test functions.

Theorem 1.1.1 ([69], Theorem 1.1)**.** *The sequence* $(x_n)_{n\in\mathbb{N}}$ *of real numbers is equidistributed mod* 1 *if and only if for every real-valued continuous function* f *defined on closed interval* $I = [0, 1]$*, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)dx.$$

Corollary 1.1.1 ([69], Corollary 1.2)**.** *The sequence* $(x_n)_{n\in\mathbb{N}}$ *of real numbers is equidistributed mod* 1 *if and only if for every complex-valued continuous function* f *on* $\mathbb{R}$ *with period* 1*, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)dx.$$

In view of the preceding criteria, determining whether a sequence is equidistributed modulo one amounts to verifying the above condition for all complex-valued continuous functions. Weyl provided the most effective and widely accepted criterion by proving that it suffices to verify Corollary 1.1.1 only for the exponential functions.

Theorem 1.1.2 (Weyl criterion). *The sequence* $(x_n)_{n\in\mathbb{N}}$ *is equidistributed mod* 1 *if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i h x_n}=0 \text{ for all integers } h\neq 0.$$

Proof. See [69, Theorem 2.1, p.7]. □

We next present some examples of sequences to illustrate equidistribution modulo one.

Example 1.1.1. *Let* α *be an irrational number. The sequence* $(n\alpha)_{n\in\mathbb{N}}$ *is equidistributed mod* 1*. Indeed*

$$\left|\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i h n\alpha}\right|=\frac{|e^{2\pi i h N\alpha}-1|}{N|e^{2\pi i h\alpha}-1|}\leq\frac{1}{N|\sin\pi h\alpha|}\to 0 \text{ as } N\to\infty \text{ for all integers } h\neq 0.$$

In the last step, we used the fact that if $\alpha\notin\mathbb{Q}$ *then* $h\alpha\notin\mathbb{Z}$ *for any integer* $h\neq 0$*. Hence, by Weyl criterion, the sequence* $(n\alpha)_{n\in\mathbb{N}}$ *is equidistributed mod* 1 *whenever* $\alpha\in\mathbb{R}\setminus\mathbb{Q}$*. On the other hand, if* $\alpha\in\mathbb{Q}$*, then the sequence* $(n\alpha)_{n\in\mathbb{N}}$ *is not equidistributed mod* 1*. In this case*

$$\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i h n\alpha}\neq 0 \text{ for some integers } h\neq 0;$$

for example, if $\alpha=\frac{a}{h}$ *then*

$$\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i h n\alpha}=1$$

and Weyl's criterion is not fulfilled.

While equidistribution modulo one is a qualitative asymptotic property, it does not provide the speed of convergence in (1.3). It is natural to have a corresponding quantitative concept which applies to finite sequences (or finite truncations of infinite sequences). To quantify the convergence in (1.3), one defines the discrepancy of a sequence as a measure of its deviation from equidistribution, as follows:

Definition 1.1.2 (Discrepancy, star Discrepancy)**.** *For any* $[a,b) \in [0,1]$*, let*

$$A([a,b);N) = \#\{1 \le n \le N \mid a \le \{x_n\} < b\}.$$

Then the discrepancy of the sequence $(x_n)_{n\in\mathbb{N}}$ *is given by*

$$D_N((x_n)_{n\in\mathbb{N}}) = \sup_{0\le a<b\le 1} R_N([a,b)), \tag{1.4}$$

where

$$R_N([a,b)) = \left| \frac{A([a,b);N)}{N} - (b-a) \right|. \tag{1.5}$$

For any $\alpha \in [0,1]$*, let*

$$A([0,\alpha);N) := A(\alpha;N) = \#\{1 \le n \le N \mid \{x_n\} \le \alpha\}.$$

Then the star discrepancy of the sequence $(x_n)_{n\in\mathbb{N}}$ *is given by*

$$D_N^*((x_n)_{n\in\mathbb{N}}) = \sup_{0\le\alpha\le 1} R_N([0,\alpha)), \tag{1.6}$$

where

$$R_N([0,\alpha)) := R_N(\alpha) = \left| \frac{A(\alpha;N)}{N} - \alpha \right|. \tag{1.7}$$

The following result shows that the above two discrepancies are closely related.

Theorem 1.1.3 ([69], Theorem 1.3, p. 91)**.** *For every sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $[0,1)$*, we have*

$$D_N^*((x_n)_{n\in\mathbb{N}}) \le D_N((x_n)_{n\in\mathbb{N}}) \le 2D_N^*((x_n)_{n\in\mathbb{N}}).$$

To establish the connection between equidistribution and discrepancy, we have the following result.

Theorem 1.1.4 ([69], Theorem 1.1, p. 89)**.** *The sequence* $(x_n)_{n\in\mathbb{N}}$ *is equidistributed mod* 1 *if and only if* $\lim_{N\to\infty} D_N^*((x_n)_{n\in\mathbb{N}}) = 0$*, or equivalently* $\lim_{N\to\infty} D_N((x_n)_{n\in\mathbb{N}}) = 0$.

Proof. Suppose that $\lim_{N\to\infty} D_N((x_n)_{n\in\mathbb{N}}) = 0$. This implies that

$$\frac{A(\alpha;N)}{N} \to \alpha \text{ as } N \to \infty.$$

Hence the sequence $(x_n)_{n\in\mathbb{N}}$ is equidistributed mod 1. Conversely, suppose that $(x_n)_{n\in\mathbb{N}}$ is equidistributed mod 1. We choose an integer $m \geq 2$. For $0 \leq k < m-1$, set $I_k = \left[\frac{k}{m}, \frac{k+1}{m}\right)$. Since $(x_n)_{n\in\mathbb{N}}$ is equidistributed mod 1, there exists a positive integer $N_0 = N_0(m)$ such that for all $N \geq N_0$ and for every $0 \leq k \leq m-1$, we have

$$\frac{1}{m}\left(1-\frac{1}{m}\right) \leq \frac{A(I_k;N)}{N} \leq \frac{1}{m}\left(1+\frac{1}{m}\right). \tag{1.8}$$

We now consider an arbitrary subinterval $J = [\beta_1, \beta_2)$ of I. Clearly, there exist intervals J_1 and J_2, each being a finite union of intervals I_k, such that $J_1 \subset J \subset J_2$ and $l(J) - l(J_1) < \frac{2}{m}$ and $l(J_2) - l(J) < \frac{2}{m}$. From (1.8), we obtain for all $N \geq N_0$

$$l(J_1)\left(1-\frac{1}{m}\right) \leq \frac{A(J_1;N)}{N} \leq \frac{A(J;N)}{N} \leq \frac{A(J_2;N)}{N} \leq l(J_2)\left(1+\frac{1}{m}\right).$$

Consequently, we have

$$\left(l(J)-\frac{2}{m}\right)\left(1-\frac{2}{m}\right) < \frac{A(J;N)}{N} < \left(l(J)+\frac{2}{m}\right)\left(1+\frac{1}{m}\right).$$

Since $l(J) \leq 1$, we obtain

$$-\frac{3}{m}+\frac{2}{m^2} < \frac{A(J;N)}{N} - l(J) < \frac{3}{m}+\frac{2}{m^2} \text{ for all } N \geq N_0.$$

Since the bounds in the above inequality are independent of J, it follows that $D_N((x_n)_{n\in\mathbb{N}}) \leq \frac{3}{m}+\frac{2}{m^2}$ for all $N \geq N_0$. Moreover, the quantity $\frac{3}{m}+\frac{2}{m^2}$ can be made arbitrarily small, which completes the proof. □

1.1.2 Farey sequence and Riemann Hypothesis

The primary interest in the distribution of Farey fractions is due to the classical work of Franel [40] and Landau [72], who showed that the Riemann Hypothesis and quantitative statements about the equidistribution of Farey fractions are equivalent. In particular, Franel proved that the supremum of the real parts of the zeros of the Riemann zeta function is the infimum of θ for which the following estimate holds

$$\sum_{i=1}^{N(Q)} R^2_{N(Q)}(\gamma_i) = \mathrm{O}\left(Q^{-2+2\theta}\right),$$

where $\gamma_i \in \mathcal{F}_Q,\ 1 \leq i \leq N(Q)$. Specifically, the Riemann Hypothesis is equivalent to the asymptotic formula

$$\sum_{i=1}^{N(Q)} R^2_{N(Q)}(\gamma_i) = \mathrm{O}\left(Q^{-1+\epsilon}\right), \text{ for all } \epsilon > 0. \tag{1.9}$$

Landau [72] gave a similar version by proving that the Riemann Hypothesis is true if and only if, for all $\epsilon > 0$,

$$\sum_{i=1}^{N(Q)} R_{N(Q)}(\gamma_i) = \mathrm{O}\left(Q^{\frac{1}{2}+\epsilon}\right). \tag{1.10}$$

Much effort has been made to prove the above estimates in terms of discrepancy. The foundational result on the equidistribution of irreducible fractions between 0 and 1, interpreted in terms of frequencies of certain almost periodic functions was given by Erdős et al. [36]. Neville [83] investigated the discrepancy of Farey fractions, proving that $D_{N(Q)}(\mathcal{F}_Q) \asymp \log Q/Q$. Thereafter, it was improved by Niederreiter [84] to $D_{N(Q)}(\mathcal{F}_Q) \asymp 1/Q$ for all $Q \geq 1$. A closed-form formula for the discrepancy was later established by Dress [32] who showed that $D_{N(Q)}(\mathcal{F}_Q) = 1/Q$ holds for every Q. Motivated by a question of Davenport, Huxley [56] studied a generalization of the result of Franel and Landau concerning the zeros of the Dirichlet L-function. He established a connection between Farey fractions with denominators in an arithmetic progression and the Generalized Riemann Hypothesis. In the same direction, the discrepancy of the Farey sequence when denominators are $\mathcal{B}$-free and lie in an arithmetic progression, as well as when the denominators are in a union of arithmetic progressions, was studied by Alkan et al. [1, 2]. Ledoan [73] studied the discrepancy of the Farey sequence with denominators in an arbitrary subinterval of the unit interval $[0, 1]$. He proved upper and lower bound for the discrepancy that demonstrate the sequence is uniformly distributed in all subintervals of $[0, 1]$. In this thesis, we study the discrepancy of Farey sequences by imposing various restrictions on denominators and numerators.

1.1.3 Local statistics

To understand the finer (or local) distribution of a sequence, such as its randomness, local clustering, and periodic patterns, quantities such as nearest-neighbor gap distribution, ν-level correlation measure can be studied. For a sequence of N numbers $x_1, \ldots, x_N$ in $[0, 1)$, these statistics measure the distribution of spacings between x_i at distance of order of the mean spacing $1/N$. As the average distance between neighboring points is $1/N$. The trigonometric functions that determine the spacing statistics of sequences have frequencies of order N , whereas equidistribution is determined by trigonometric functions with constant frequencies. The sequences $(\{ne\})_{n\geq 1}$ and $(\{n\pi\})_{n\geq 1}$ are equidistributed modulo one, but their local spacing statistics exhibit irregularities (see [90]). In this thesis, we study the correlation measure of various sequences.

The motivation to study the correlations of sequences comes from applications in physics, where physicists study the spectra of high energies. For instance, the Berry-Tabor conjecture states that discrete energy spectrum of a quantum system has Poissonian pair correlation except for certain degenerate cases [75]. For some special quantum systems it can be shown that there exists $\alpha \in \mathbb{R}$, and a sequence of positive integers $(a_n)_{n\in\mathbb{N}}$, such that the local distribution properties of the discrete energy spectrum of the system agree with those of the sequence $(\{a_n\alpha\})_{n\in\mathbb{N}}$. Lately, there has been a lot of interest in these notions in applications in number theory, mathematical physics, and probability theory. It has particularly attracted significant interest in number theory following Montgomery [80] and Hejhal's [54] work on the correlations of the zeros of the Riemann zeta function, and Rudnick and Sarnak's [89] work on the correlations of zeros of L-functions.

Correlations measure the distribution between all pairs of elements of the sequence and do not depend on their ordering. Let $\nu \geq 2$ be an integer and let $\mathcal{F}$ be a finite set of N elements in the unit interval $[0,1]$. The ν-level correlation measure $\mathcal{S}_{\mathcal{F}}^{(\nu)}(\mathfrak{B})$ of a box $\mathfrak{B} \subset \mathbb{R}^{\nu-1}$ is defined as follows:

$$\frac{1}{N}\#\left\{(x_1,\ldots,x_\nu) \in \mathcal{F}^\nu : x_i \text{ distinct}, \ (x_1 - x_2, \ldots, x_{\nu-1} - x_\nu) \in \frac{1}{N}\mathfrak{B} + \mathbb{Z}^{\nu-1}\right\}. \tag{1.11}$$

The ν-level correlation measure of a sequence $(\mathcal{F}_n)_n$, for every box $\mathfrak{B} \subset \mathbb{R}^{\nu-1}$, is given (if it exists) by

$$\mathcal{S}^{(\nu)}(\mathfrak{B}) = \lim_{n\to\infty} \mathcal{S}_{\mathcal{F}_n}^{(\nu)}(\mathfrak{B}).$$

The measure $\mathcal{S}^{(2)}$ is called the pair correlation measure. If

$$\mathcal{S}^{(\nu)}(\mathfrak{B}) = \int_{\mathfrak{B}} g_\nu(x_1,\ldots,x_{\nu-1})dx_1\cdots dx_{\nu-1}, \tag{1.12}$$

then g_ν is called the ν-level correlation function of the sequence $(\mathcal{F}_n)_n$, and for $\nu = 2$, it is called the pair correlation function. The ν-level correlation is said to be Poissonian if $g_\nu \equiv 1$. A sequence with a Poissonian distribution behaves like a sequence of randomly chosen numbers in $[0,1)$. We have come up with several examples of explicit sequences which have Poissonian pair correlation.

To study the fine-scale statistics of the Farey sequence, Hall [44] studied the first-level spacing distribution of Farey fractions by estimating the moments of spacings of consecutive Farey fractions. Augustin et al. [7] studied the h-th level spacing distribution of Farey fractions for $h \geq 2$ by showing the convergence of the sequence of probability measures.

Boca and Zaharescu [16] studied the correlations of Farey fractions and proved that the ν-level correlation exists for all $\nu \geq 2$. They also derived an explicit expression for the limiting pair correlation function of $\mathcal{F}_Q$, which is given by

$$g(\lambda) = \frac{6}{\pi^2\lambda^2} \sum_{1\leq k<\frac{\pi^2\lambda}{3}} \phi(k) \log\frac{\pi^2\lambda}{3k}, \tag{1.13}$$

and it shows strong repulsion between the elements of the sequence. The formula in 1.13 appeared in the main term of the second moment of a large sieve matrix [14]. The pair correlation of Farey fractions with prime denominators was studied by Xiong and Zaharescu [100], who showed that it is Poissonian. A more general result on the pair correlation of fractions with prime denominators is contained in [99]. Also, Xiong and Zaharescu [101] studied the pair correlation of Farey fractions with denominators coprime with B_Q, the monotonic increasing sequence of square-free numbers with the condition that $B_{Q_1}|B_{Q_2}$ if $Q_1 < Q_2$. They proved that the pair correlation of the sequence is Poissonian if $\lim_{Q\to\infty}\frac{\phi(B_Q)}{B_Q} = 0$ and showed a strong repulsion if $\lim_{Q\to\infty}\frac{\phi(B_Q)}{B_Q} \neq 0$. Boca and Siskaki [15] reproved (1.13) by studying the pair correlation of Farey fractions with a coprimality condition on denominators, using some different counting arguments. Alkan et al. [5] computed the pair correlation measure of the sum $\mathcal{F}_Q + \mathcal{F}_Q$ modulo one, as $Q \to \infty$.

In this thesis, we study the distribution of polynomial Farey sequence and Farey sequence with k-free denominators lying in an arithmetic progression by establishing estimates for the discrepancy and correlation measure. The Chapter 2 of this thesis consists of several preliminary results that we will be using in proving our main results. In Chapter 3, we analyze the equidistribution and correlation measure of polynomial Farey sequence. We investigate the distribution of the Farey sequence with k-free denominators lying in an arithmetic progression in Chapter 4.

1.1.4 Polynomial Farey sequence

The sequences studied in all the above cases involve restrictions on Farey denominators. It is also natural to consider restrictions on Farey numerators. Such restrictions make the study of the distribution of Farey sequences more challenging, as exponential sums over Farey fractions with restrictions on the numerators become more delicate.

Let $\kappa \geq 1$ be an integer and let $\boldsymbol{c} = (c_\kappa, c_{\kappa-1}, \ldots, c_1) \in \mathbb{Z}^\kappa$ be a fixed non-zero vector. Let $P(x) = c_\kappa x^\kappa + c_{\kappa-1}x^{\kappa-1} + \cdots + c_1 x$ be the associated polynomial. Denote

$$\mathcal{F}_{Q,P} := \left\{ \frac{a}{q} : 1 \leq a \leq q \leq Q,\ (P(a), q) = 1 \right\}. \tag{1.14}$$

If $P(x) = x(x+1)$ then for instance,

$$\mathcal{F}_{5,P} = \left\{ \frac{1}{5}, \frac{1}{3}, \frac{2}{5}, \frac{3}{5}, 1 \right\}.$$

Specifically, $\mathcal{F}_{Q,P}$ reduces to the classical Farey sequence when $P(x) = x$. One may also consider a more general function $F : \mathbb{Z} \to \mathbb{Z}$ instead of a polynomial $P(x)$ in (1.14).

The motivation to define the sequence $(\mathcal{F}_{Q,P})_Q$ comes from a geometric point of view via visibility. Recall that the classical Farey fractions are in a one-to-one correspondence with visible lattice points. Hence, the sequence $\mathcal{F}_Q$ can be viewed in terms of visible lattice points:

$$\mathcal{F}_Q = \{a/q \mid 1 \leq a \leq q \leq Q;\ (a, q) \text{ is visible from origin}\}.$$

The study of visible lattice points in planar and convex domains, along with their generalizations, has attracted considerable attention [8, 12, 58]. Extending this line of investigation, Chaubey et al. [23] recently introduced the concept of polynomial visible lattice points. It is defined as:

Definition 1.1.3. *Let* $\mathbf{c} = (c_n, c_{n-1}, \ldots, c_1) \in \mathbb{Z}^n$ *be a fixed vector with* $c_n \neq 0,\ c_i \geq 0$ *for all* $1 \leq i \leq n$*, and* $\gcd(c_n, c_{n-1}, \ldots, c_1) = 1$*, let* $F(\mathbf{c}) = \{y = rP_{\mathbf{c}}(x) \mid r \in \mathbb{Q}^+\}$*, where* $P_{\mathbf{c}}(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x$*. A point* $(a, b) \in \mathbb{N}^2$ *is called* $F(\mathbf{c})$*-visible if there is no other lattice point on the curve* $y = rP_{\mathbf{c}}(x)$ *joining the origin and the point* (a, b)*.*

The above definition describes the visibility of lattice points through polynomial curves. Let us denote the set of all $F(\boldsymbol{c})$-visible points in $\mathbb{N}^2$ by $V(\boldsymbol{c})$. The natural density of visible lattice points through polynomial curves is known to be one [23]. It is also well known that the natural density of the polynomial visible lattice points for the curve $y = mx^b$, with $b \geq 1$ is equal to $\dfrac{1}{\zeta(b+1)}$ [43]. Denote

$$\mathscr{F}_{Q,V} := \left\{ \frac{a}{q} \mid 1 \leq a \leq q \leq Q,\ (a,q) = 1,\ \text{and } (a,q) \in V(\boldsymbol{c}) \right\}. \tag{1.15}$$

This gives a relation between $\mathscr{F}_{Q,V}$ and the visible lattice points through polynomial curves in the triangle with vertices $(0,0)$, $(0,Q)$, and (Q,Q). It is natural to study the distributional properties of the above sequence but it turns out to be a difficult problem with no arithmetic description of the visibility property in this setting. Instead, we focus on a subset of $\mathscr{F}_{Q,V}$ defined in (1.14). In particular, $\mathcal{F}_{Q,P}$ is a subset of fractions that are in one-to-one correspondence with lattice points visible through polynomial curves. We prove that the lim sup of the pair correlation measure of the sequence $(\mathcal{F}_{Q,P})_Q$ is bounded and provide an explicit formula for the pair correlation function in the case of $P(x) = x(x+1)$. We compare the pair correlation function of $(\mathcal{F}_{Q,P})_Q$ with the Poissonian and GUE distributions by plotting their respective graphs.

We state our main results for correlation measure. Recall that

$$\mathcal{S}_{\mathcal{F}_{Q,P}}(\Lambda) = \frac{1}{\mathcal{N}_{Q,P}} \# \left\{ (\gamma_1, \gamma_2) \in \mathcal{F}_{Q,P}^2 : \gamma_1 \neq \gamma_2, \gamma_1 - \gamma_2 \in \frac{1}{\mathcal{N}_{Q,P}}(0,\Lambda) + \mathbb{Z} \right\},$$

where $\mathcal{N}_{Q,P} = \#\mathcal{F}_{Q,P}$.

Theorem 1.1.5. *Let $\kappa \geq 2$ be an integer and let $P(x) = c_\kappa x^\kappa + c_{\kappa-1}x^{\kappa-1} + \cdots + c_1 x \in \mathbb{Z}[x]$ be a polynomial with non-zero discriminant. Then* $\limsup_{Q\to\infty} \mathcal{S}_{\mathcal{F}_{Q,P}}(\Lambda)$ *is finite. Moreover, the* $\limsup_{Q\to\infty} \mathcal{S}_{\mathcal{F}_{Q,P}}(\Lambda)$ *is bounded by*

$$\limsup_{Q\to\infty} \mathcal{S}_{\mathcal{F}_{Q,P}}(\Lambda) \ll \int_0^\Lambda \frac{1}{\lambda^2} \sum_{1 \leq m < \frac{2\lambda}{\beta_P}} h(m) \log\left(\frac{2\lambda}{m\beta_P}\right) d\lambda \tag{1.16}$$

for any $\Lambda > 0$, where $\beta_P = \prod_p \left(1 - \frac{f_P(p)}{p^2}\right)$, $h(m) = \sum_{\substack{n,\delta,d_1,d_2 \geq 1 \\ n\delta d_1 d_2 = m}} n f_P(d_1) f_P(d_2)$ and

$$f_P(p) := \#\{1 \leq d \leq p : P(d) \equiv 0 \pmod{p}\}. \tag{1.17}$$

We obtain an explicit upper bound on the pair correlation measure. The key point of the above theorem is that, upon imposing the arithmetic restriction $(P(a), q) = 1$, the sequence $(\mathcal{F}_{Q,P})_Q$ does not exhibit extreme clustering at small scales.

For the specific polynomial $P(x) = x(x+1)$, we prove that the limiting pair correlation measure of the sequence $(\mathcal{F}_{Q,P})_{Q\geq 1}$ exists and is non-Poissonian. We also derive an explicit formula for the pair correlation function in this case.

Theorem 1.1.6. *Let $P(x) = x(x+1)$ be a fixed polynomial. Then the limiting pair correlation function of the sequence $(\mathcal{F}_{Q,P})_{Q\geq 1}$ exists and is given by*

$$\mathfrak{g}_2(\lambda) = \frac{1}{\zeta(2)\lambda^2} \sum_{d_2,d_4=1}^{\infty} \frac{\mu(d_2)\mu(d_4)}{d_2 d_4 \phi(d_2)\phi(d_4)} \sum_{1\leq m<\frac{2\lambda}{\beta_P}} \mathfrak{h}(m) \log\left(\frac{2\lambda}{m\beta_P}\right),$$

where

$$\mathfrak{h}(m) = \sum_{\substack{n,\delta,d_1,d_3\geq 1\\ n\delta d_1 d_3 = m\\ \left(\frac{d_2}{dG_1},\frac{d_4}{DG_2}\right)=1}} n\mu(d_1)\mu(d_3) dDG_1G_2 \prod_{p\mid \frac{d_2 d_4}{dDG_1G_2}} \left(1+\frac{1}{p}\right)^{-1}, \tag{1.18}$$

and $d = \gcd(d_1, d_2)$, $D = \gcd(d_3, d_4)$, $G_1 = \gcd\left(\delta, \frac{d_2}{d}\right)$, $G_2 = \gcd\left(\delta, \frac{d_4}{D}\right)$.

Pair correlation for Farey fractions is robust under polynomial sieving, but the arithmetic weight in the pair correlation function becomes significantly more complicated. The pair correlation function shows that the sequence $(\mathcal{F}_{Q,P})_{Q\geq 1}$ with $P(x) = x(x+1)$ does not behave as a randomly distributed sequence, as the pair correlation is non-Poissonian. In particular, the deviation of the pair correlation function from the constant function 1 shows that the sequence has strong arithmetic structure. Consequently, the spacings between fractions are correlated, not independent.

Remark 1.1.1. *The key ideas used to prove the above results on the pair correlation measure involve establishing a closed-form formula for the Weyl sum over $\mathcal{F}_{Q,P}$. We also formulate a version of Poisson's summation formula that accounts for the additional weight in the sum of Fourier coefficients.*

Another question of significant interest is determining the values $\tau > 0$ for which there are infinitely many primes p satisfying the Diophantine inequality $||\alpha p|| < p^{-\tau+\epsilon}$ for all $\epsilon > 0$, where $||t||$ denotes the distance from the nearest integer to a real number t. All known results regarding such values of τ are equivalent to quantitative statements about the gaps between the Farey fractions with prime denominators. Motivated by this connection, we study the pair correlation statistics of polynomial Farey sequence whose denominators lie in certain subsets of primes. As a consequence, by taking denominators to be Piatetski-Shapiro primes, Chen primes, prime k-tuples, and primes with restricted digits, we establish explicit sequences that behave like a randomly chosen sequence in $(0,1]$.

For each integer Q, let $\mathcal{B}_Q$ be a fixed subset of prime numbers that are less than or equal to Q. The polynomial Farey sequence of order Q with prime denominators is given by

$$\mathscr{M}_{\mathcal{B}_Q,P} := \left\{ \frac{a}{p} : 1 \leq a \leq p \leq Q,\ (P(a),p) = 1,\ p \in \mathcal{B}_Q \right\}.$$

Theorem 1.1.7. *The limiting pair correlation of the sequence* $(\mathscr{M}_{\mathcal{B}_Q,P})_{Q\geq 1}$ *exists as* $Q \to \infty$ *and is Poissonian if and only if* $\sum_{p\in\mathcal{B}_Q} p^2 = o\big((\#\mathscr{M}_{\mathcal{B}_Q,P})^2\big)$.

This shows that the spacings between any two points of the sequence $(\mathscr{M}_{\mathcal{B}_Q,P})_{Q\geq 1}$ behave like in a randomly distributed sequence in $(0,1)$ when the second moment of the denominators in $\mathcal{B}_Q$ is controlled by the square of number of the fractions in $\mathscr{M}_{\mathcal{B}_Q,P}$.

In 1853, Chebyshev [24] remarked that, for a given modulus, the number of prime quadratic non-residues exceeds that of prime quadratic residues. This is called the Chebyshev's bias phenomenon. We study the Chebyshev's bias question for polynomial Farey sequence with denominators in an arithmetic progression. Let q and l be positive integers; we denote the number of polynomial Farey fractions with denominators in an arithmetic progression as

$$S(Q;q,l) := \#\left\{ \frac{a}{n} \in \mathcal{F}_{Q,P} \mid n \equiv l \pmod{q} \right\}.$$

We ask the following questions for Farey fractions analogous to prime number races.

- Does there exist positive integers Q_0, l_1, and l_2 with $l_1 \not\equiv l_2 \pmod{q}$ such that

$$S(Q; q, l_1) > S(Q; q, l_2) \text{ for all } Q > Q_0?$$

- Are there arbitrarily large values of Q for which $S(Q; q, l_1) < S(Q; q, l_2)$, and arbitrarily large values of Q for which $S(Q; q, l_1) > S(Q; q, l_2)$? In other words, does the function $S(Q; q, l_1) - S(Q; q, l_2)$ change sign infinitely often?

In here, we address the questions listed above. Let $\kappa, J \geq 1$ be integers and let $P(x) = c_\kappa x^\kappa + c_{\kappa-1}x^{\kappa-1} + \cdots + c_1 x \in \mathbb{Z}[x]$ be a polynomial with non-zero discriminant and factorization

$$P(x) = \prod_{i=1}^{J} m_i(x)^{e_i}, \tag{1.19}$$

where $m_i(x) \in \mathbb{Z}[x]$ are irreducible polynomials. To state our result, we need the following condition.

Haselgrove's condition for Hecke L-function mod $\mathfrak{q}$: For all Hecke characters $\chi' \pmod{\mathfrak{q}}$, $\mathcal{L}(s, \chi') \neq 0$ for all $s \in (0, 1)$.

Theorem 1.1.8. *Let $q \geq 2, l_1, l_2$ be positive integers such that $l_1 \not\equiv l_2 \pmod{q}$ and $(q, l_1 l_2) = 1$. Let $P(x) \in \mathbb{Z}[x]$ be as in* (1.19). *Assuming Haselgrove's condition for Hecke L-function $\mathcal{L}_i(s, \chi')$ modulo $\mathfrak{q}_i$, the set of values of Q for which the difference $S(Q; q, l_1) - S(Q; q, l_2)$ is strictly positive and the set of values of Q for which the difference $S(Q; q, l_1) - S(Q; q, l_2)$ is strictly negative are unbounded.*

This chapter appeared in [20].

1.1.5 Farey sequence with k-free denominators

Let $k \geq 2$ be an integer. A number n is said to be k-free if for every prime $p|n$, we have $p^k \nmid n$. It is well known that the density of k-free numbers is $1/\zeta(k)$. Denote

$$\mathscr{F}_{Q,k}^{(m)} := \left\{ \frac{a}{q} \mid 1 \leq a \leq q \leq Q,\ (a, q) = 1,\ q \text{ is } k\text{-free } \&\ q \equiv b \pmod{m} \right\}, \tag{1.20}$$

where $m \in \mathbb{N},\ b \in \mathbb{Z}$ and $(b, m) = 1$.

In Chapter 4, we investigate the distribution of the sequence of Farey fractions with k-free denominators lying in an arithmetic progression, denoted by $\mathscr{F}_{Q,k}^{(m)}$. We prove that the sequence $\left(\mathscr{F}_{Q,k}^{(m)}\right)_Q$ is equidistributed by establishing an estimate for Weyl sum.

Theorem 1.1.9. *The Farey sequence* $\left(\mathscr{F}_{Q,k}^{(m)}\right)_{Q\geq 1}$ *is uniformly distributed modulo one.*

It is natural to establish the quantitative aspect of equidistribution. In order to achieve this, we study discrepancy of the sequence $\left(\mathscr{F}_{Q,k}^{(m)}\right)_{Q\geq 1}$.

Theorem 1.1.10. *For all* $Q \geq 1$, *we have*

$$D_{\mathcal{N}(Q,k,m)}\left(\mathscr{F}_{Q,k}^{(m)}\right) \asymp \frac{1}{Q},$$

where implied constants depend on m.

The coprimality condition $(a, q) = 1$ on fractions in the Farey sequence introduces arithmetic constraints through the Möbius function. By detecting this condition using the following identity

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases}$$

the counting of Farey fractions turns into sums of $\mu(n)$, which are associated to the zeros of the Riemann zeta function. Imposing congruence constraints on Farey denominators invokes Dirichlet characters, leading to twisted sums of $\mu(n)$. Therefore, the distribution of Farey fractions with congruence constraints on denominators is connected to the distribution of the zeros of Dirichlet L-functions.

Motivated by this, we explore a new perspective on the Generalized Riemann Hypothesis. We establish an equivalent criterion for the Generalized Riemann Hypothesis in terms of the distribution of Farey fractions in $\mathscr{F}_{Q,k}^{(m)}$ analogous to the classical results of Franel and Landau. Denote $\mathscr{F}_{Q,k}^{(m)} = \{\gamma_1 < \gamma_2 < \cdots < \gamma_{\mathcal{N}(Q,k,m)}\}$ and $\mathcal{N}(Q, k, m) = \#\mathscr{F}_{Q,k}^{(m)}$.

Theorem 1.1.11. *Let* $R_{\mathcal{N}(Q,k,m)}(\gamma_j) = \left|\gamma_j - \frac{j}{\mathcal{N}(Q,k,m)}\right|$. *The generalized Riemann hypothesis (GRH) holds true if and only if, for all* $\epsilon > 0$,

$$\sum_{j=1}^{\mathcal{N}(Q,k,m)} R_{\mathcal{N}(Q,k,m)}(\gamma_j) = \mathrm{O}_m\left(Q^{\frac{1}{2}+\epsilon}\right).$$

We next prove a closed-form formula for the second moment of the displacement of Farey fractions in $\mathscr{F}_{Q,k}^{(m)}$.

Theorem 1.1.12. *Let* $M_q(x) = \sum_{\substack{n\leq x\\ nq\equiv b \pmod m}} \mu(n)\mu_k(nq)^2$ *and* $R_{\mathcal{N}(Q,k,m)}(\gamma_j) = \left|\gamma_j - \frac{j}{\mathcal{N}(Q,k,m)}\right|$ *for integers* b, m *as in* (4.1). *Then, we have*

$$\begin{aligned}\sum_{j=1}^{\mathcal{N}(Q,k,m)} R^2_{\mathcal{N}(Q,k,m)}(\gamma_j) =& \frac{1}{12\mathcal{N}(Q,k,m)}\left(\sum_{q_1,q_2\leq Q} M_{q_1}\left(\frac{Q}{q_1}\right) M_{q_2}\left(\frac{Q}{q_2}\right)\right.\\ &\left.\times\ \frac{(\gcd(q_1,q_2))^2}{q_1q_2} - 1\right).\end{aligned}$$

Moreover, the right-hand side above is bounded by

$$\ll_m \begin{cases} \exp\left(-c\frac{(\log Q)^{3/5}}{(\log\log Q)^{1/5}}\right), & \textit{unconditionally,}\\ Q^{-1+\epsilon}, & \textit{on the GRH.}\end{cases}$$

We also investigate the ν-level correlations of the sequence $\left(\mathscr{F}_{Q,k}^{(m)}\right)_Q$. In order to investigate correlations measure, we establish a closed-form formula for the exponential sum over Farey fractions whose denominators are k-free and are in an arithmetic progression. We then derive estimates for counting weighted lattice points. As a result, the principal Dirichlet character yields the correlation measure. For the non-principal characters, we provide an estimate for the character sum twisted by a continuously differentiable function and the characteristic function for the k-free numbers. By applying this result, the sum over non-principal characters approaches zero as $Q \to \infty$.

Our next result analyze the pair correlation measure of the sequence $\left(\mathscr{F}_{Q,k}^{(m)}\right)_{Q\geq 1}$.

Theorem 1.1.13. *The pair correlation function of the sequence* $\left(\mathscr{F}_{Q,k}^{(m)}\right)_{Q\geq 1}$ *exists and is given by*

$$\mathfrak{g}_{m,k}(\lambda) = \frac{6}{\lambda^2\pi^2\phi^2(m)} \sum_{1\leq n<\frac{\lambda}{\mathscr{C}(k,m)}} F_k(n)\log\left(\frac{\lambda}{n\mathscr{C}(k,m)}\right) \tag{1.21}$$

for any $\lambda \geq 0$*, where* $\mathscr{C}(k,m)$ *is as in* (4.2)*, and*

$$\begin{aligned} F_k(n) = \sum_{\substack{\delta d_1 d_2 r = n \\ (d_1 d_2 \delta, m)=1}} r\mu_k(\delta)^2\mu(d_1)\mu(d_2) \prod_{\substack{p \\ (p,m)=1}} \left(1 - \frac{\gcd(p^k, d_2\delta)}{p^{k-1}(p+1)}\right) \\ \times \prod_{\substack{p \\ (p,m)=1}} \left(1 - \frac{\gcd(p^k, d_1\delta)}{p^k + p^{k-1} - \gcd(p^k, d_2\delta)}\right). \end{aligned} \tag{1.22}$$

The fact that the pair correlation function in (1.21) is non-Poissonian shows that the sequence $\left(\mathscr{F}_{Q,k}^{(m)}\right)_{Q\geq 1}$ does not behave like a randomly distributed sequence but instead exhibits strong arithmetic correlations.

For the particular case $k = 2$, plotting the graph reveals a strong repulsion near zero. Square-free numbers are closely related to prime numbers but have a positive asymptotic density with a more even distribution. This means one can expect a somewhat less random behavior with the pair correlation function being non-Poissonian, whereas, for prime denominators, it is Poissonian. The results of Chapter 4 appeared in [19] and [21].

1.2 Index of Farey fractions

Another way to investigate the distribution of the Farey sequences is by studying the index of Farey fractions. Let $\gamma' = \frac{a'}{q'} < \gamma = \frac{a}{q} < \gamma'' = \frac{a''}{q''}$ be three consecutive Farey fractions in $\mathcal{F}_Q$. Then, the ratio

$$\nu_Q(\gamma) := \frac{q' + q''}{q} = \frac{a' + a''}{a} \tag{1.23}$$

is an integer and is called the index of the Farey fraction γ in $\mathcal{F}_Q$. In particular, we take $\nu_Q(\gamma_1) = 1$, and $\nu_Q(\gamma_{N_Q}) = 2Q$. The index admits a geometric interpretation in the context of visible lattice points. The index of a visible point (q, a) lying inside or on the boundary of the triangle with vertices $(0, 0), (Q, 0), (Q, Q)$ is defined as the index of the corresponding Farey fraction a/q. In this setting, the index of a visible point (q, a) lying inside or on the boundary of the triangle with vertices $(0, 0), (Q, 0), (Q, Q)$ is defined as the index of the corresponding Farey fraction a/q. When the visible lattice points in the triangle are ordered by increasing slope of the rays emanating from the origin, the index admits a simple area interpretation: for any three consecutive points (q', a'), (q, a), (q'', a''), the index of (q, a) is equal to twice the area of the triangle with vertices $(0, 0)$, (q', a'), and (q'', a'') (see [30, p. 208–209] and [49, p. 23–37]).

The index satisfies the following property:

$$\left\lfloor \frac{2Q+1}{q} \right\rfloor - 1 \leq \nu_Q(\gamma) \leq \left\lfloor \frac{2Q}{q} \right\rfloor$$

which implies that for a given Farey fraction, the corresponding index function can only take one of the two values $\left\lfloor \frac{2Q}{q} \right\rfloor$ or $\left\lfloor \frac{2Q}{q} \right\rfloor - 1$. Using this property, Hall and Shiu [46] proved closed form formulas and asymptotic formulas for the first and second moments of the index function given by

$$\sum_{\gamma \in \mathcal{F}_Q} \nu_Q(\gamma) = 3N(Q) - 1,$$

and

$$\sum_{\gamma \in \mathcal{F}_Q} \nu_Q(\gamma)^2 = \frac{24}{\pi^2} Q^2 \left(\log 2Q - \frac{\zeta'(2)}{\zeta(2)} - \frac{17}{8} + 2\eta \right) + \mathrm{O}\left(Q \log^2 Q\right),$$

where η and ζ are the Euler's constant and the Riemann zeta function, respectively.

Definition 1.2.1 (Deficiency). *The deficiency $\delta(q)$ is the number of fractions $\gamma \in \mathcal{F}_Q$ with denominator q such that $\nu_Q(\gamma)$ takes its lower value.*

A formula for the average of $\delta(\gamma)$ is proved in [46] as

$$\sum_{q=1}^{Q} \delta(q) = Q(2Q+1) - N(2Q) - 2N(Q) + 1.$$

There has been considerable interest in the study of the index function mainly upon imposing extra divisibility constraints on Farey denominators. Partial sums of the Farey indices weighted according to the parity of the denominators were studied by Hall in [45]. Alkan et al. [4] studied the moments of the index of Farey fractions with denominators in a fixed residue class. The index of Farey fractions with square-free denominators lying in an arithmetic progression was investigated by the authors of [3].

In Chapter 5, we extend this line of investigation and study the distribution of the index function. In particular, we establish asymptotic formulas for the moments of Farey indices with $\mathcal{B}$-free Farey denominators twisted by Dirichlet characters.

1.2.1 Statement of the main results

The l-th moment of the Farey indices with $\mathcal{B}$-free Farey denominator in an arithmetic progression is given by

$$\mathcal{M}_{l,\mathcal{B}}(u,m,Q) := \sum_{\substack{\gamma=\frac{a}{q}\in\mathcal{F}_Q \\ q\equiv u \pmod{m} \\ q \text{ is } \mathcal{B}\text{-free}}} \nu_Q(\gamma)^l. \tag{1.24}$$

We establish asymptotic formulas for the first, second, and higher moments of the Farey indices with $\mathcal{B}$-free denominators for the following set. Let $\mathcal{B}$ be a set of primes such that

$$\sum_{p} \frac{1}{p^{\sigma}} < \infty \text{ for some } \sigma < \theta, \text{ where } 1/2 < \theta < 1.$$

In particular

$$\mathcal{B}=\left\{p:\sum_{p}\frac{1}{p^{\sigma}}<\infty \text{ for some } \sigma<\theta, \text{ where } 1/2<\theta<1\right\} \tag{1.25}$$

Theorem 1.2.1. *Let m and u be fixed positive integers with $(m,u)=1$, and let $\mathcal{B}$ be the set defined in* (1.25)*. Then, for all large positive integers Q, we have*

$$\mathcal{M}_{1,\mathcal{B}}(u,m,Q)=\frac{3Q^2}{2\phi(m)\zeta(2)}\prod_{\substack{p\in\mathcal{B}\\ p\nmid m}}\left(1+\frac{1}{p}\right)^{-1}\prod_{p\mid m}\left(1+\frac{1}{p}\right)^{-1}+\mathrm{O}_{m,\mathcal{B}}\left(Q^{1+\theta}(\log Q)^{3/2}\right).$$

Our next objective is to derive an asymptotic formula for the second moment of the Farey indices.

Theorem 1.2.2. *For fixed positive integers m,u such that $(m,u)=1$, and $\mathcal{B}$ is a set of primes defined in* (1.25)*, we have*

$$\begin{aligned}\mathcal{M}_{2,\mathcal{B}}(u,m,Q)&=\frac{4Q^2}{\phi(m)\zeta(2)}\left(2\eta-\frac{\zeta'(2)}{\zeta(2)}-\frac{17}{8}+\sum_{\substack{p\in\mathcal{B}\\ p\nmid m}}\frac{p\log p}{p^2-1}+\sum_{p\mid m}\frac{p\log p}{p^2-1}\right)\prod_{\substack{p\in\mathcal{B}\\ p\nmid m}}\\&\times\left(1+\frac{1}{p}\right)^{-1}\prod_{p\mid m}\left(1+\frac{1}{p}\right)^{-1}+\frac{4Q^2\log 2Q}{\phi(m)\zeta(2)}+\frac{4Q^2}{\phi(m)}\sum_{\substack{\chi \pmod m\\ \chi\neq\chi_0}}\frac{\chi(\bar{u})}{L(2,\chi)}\\&\times L(1,\chi)\prod_{p\in\mathcal{B}}\left(1-\frac{\chi(p)}{p}\right)\left(1-\frac{\chi(p)}{p^2}\right)^{-1}+\mathrm{O}_{m,\mathcal{B}}\left(Q^{1+\theta}(\log Q)^2\right).\end{aligned}$$

An asymptotic formula for the higher moments of the Farey indices is presented in the following result.

Theorem 1.2.3. *For fixed positive integers m,u such that $(m,u)=1$, and set $\mathcal{B}$ defined in* (5.3)*, then for $l\geq 3$, we have*

$$\mathcal{M}_{l,\mathcal{B}}(\chi,Q)=A(Q,l)+E(Q,l),$$

where

$$A(Q,l)=\frac{2^lQ^l}{\phi(m)}\sum_{\chi \pmod m}\frac{\chi(\bar{u})L(l-1,\chi)}{L(l,\chi)}\prod_{p\in\mathcal{B}}\left(\sum_{n=0}^{\infty}\frac{\chi(p)^n}{(p^{l-1})^n}\right)^{-1}\left(\sum_{n=0}^{\infty}\frac{\chi(p)^n\mu(n)}{(p^l)^n}\right)^{-1},$$

and

$$E(Q,l)=\begin{cases}\mathrm{O}_{m,\mathcal{B}}\left(Q^2\log Q\right), & \textit{if } l=3,\\ \mathrm{O}_{m,\mathcal{B}}\left(Q^{l-1}\right), & \textit{if } l\geq 4.\end{cases}$$

The content of this chapter appeared in [22].

1.3 Conclusion and Future Directions

In the final chapter, we conclude this thesis by discussing some ongoing work and outlining future research directions related to the distribution of Farey sequences.

2

Preliminaries

In this chapter, we will recall and prove some important results that we will be using throughout the thesis. We begin by recalling a result proved in [23] on an estimate of the average of the counting function for integer solutions $1 \leq d \leq m$ of the polynomial congruence $P(d) \equiv 0 \pmod{m}$.

Proposition 2.0.1 ([23], Lemma 2.1)**.** *For a fixed non-zero vector* $\mathbf{c} = (c_n, \ldots, c_1) \in \mathbb{Z}^n$*, let* $P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x$ *be a polynomial with non-zero discriminant. If* $f_P(m) = \#\{1 \leq d \leq m : P(d) \equiv 0 \pmod{m}\}$*, then as* $x \to \infty$*, we have*

$$\sum_{m \leq x} f_P(m) \sim Cx(\log x)^{J-1},$$

where $J \geq 2$ *is the number of distinct irreducible factors of the polynomial* $P(x) \in \mathbb{Z}[x]$.

The above proposition can be derived by proving the absolute convergence of the

Dirichlet series associated with $f_P(m)$ in the half plane $\Re(s) > 1$, together with the existence of a pole at $\Re(s) = 1$. A Tauberian theorem [11, Lemma 4.1] then yields the desired conclusion.

We next state a classical result of Landau on the singularities of the Mellin transform of a non-negative function.

Proposition 2.0.2 ([71]). *Let $A(x)$ be a real-valued function in one variable, and $A(x)$ does not change its sign for $x > x_0$, where x_0 is a sufficiently large real number. Suppose also for some real number $\beta < \gamma$, that Mellin transform $g(s) := \int_1^\infty A(x)x^{-s-1}dx$ is analytic for $\mathcal{R}(s) > \gamma$, can be analytically continued to the real segment $(\beta, \gamma]$. Then $g(s)$ represents an analytic function in the half plane $\mathcal{R}(s) > \beta$.*

We also state the following result, which is an analogue of Landau's theorem [81, Theorem 1.7], concerning Dirichlet series with non-negative coefficients.

Proposition 2.0.3 ([81], Lemma 15.1). *Suppose that $A(x)$ is a bounded Riemann integrable function in any finite interval $1 \leq x \leq X$, and that $A(x) \geq 0$ for all $x > X_0$. Let σ_c denote the infimum of those σ for which $\int_{X_0}^\infty A(x)x^{-\sigma}dx < \infty$. Then the function*

$$F(s) = \int_1^\infty A(x)x^{-s}dx$$

is analytic in the half plane $\sigma > \sigma_c$, but not at the point $s = \sigma_c$.

We also state one of the fundamental property of Möbius function that we will use throughout this thesis.

Theorem 2.0.1 ([6], Theorem 2.1). *If $n \geq 1$, we have*

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

We make frequent use of the preceding theorem to handle the coprimality condition $(a, q) = 1$ between the integers a and q.

2.1 Summation formulae

We now state Perron's formula, which represents the partial sums of a Dirichlet series as an integral of its summatory function.

Proposition 2.1.1 ([92], Theorem 2.3, p. 219). *Let* $F(s) := \sum_{n=1}^{\infty} f(n) n^{-s}$ *be the Dirichlet series for the arithmetic function* $f(n)$*, with abscissa of convergence* σ_a*. If* $\alpha > \max(0, \sigma_a)$*,* $T \geq 1$ *and* $x \geq 1$*, then*

$$\sum_{n \leq x} f(n) = \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} F(s) \frac{x^s}{s} ds + R(T),$$

where

$$R(T) \ll x^{\alpha} \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\alpha} |1 + T \log(x/n)|}.$$

We also recall the well known Abel's summation formula.

Proposition 2.1.2 ([6], Theorem 4.2). *For any arithmetical function* $a(n)$ *let* $A(x) = \sum_{n \leq x} a(n)$*, where* $A(x) = 0$ *if* $x < 1$*. Assume* f *has a continuous derivative on the interval* $[y, x]$*, where* $0 < y < x$*. Then we have*

$$\sum_{y < n \leq x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_{y}^{x} A(t) f'(t) dt.$$

Lemma 2.1.1 ([100], Lemma 3). *Let* $H : \mathbb{R} \to \mathbb{R}$ *be a continuously differentiable function with SuppH* $\subset (a, b)$ *for some real numbers* a *and* b*. Then for any* $L > 0$ *one has*

$$\sum_{l \in \mathbb{Z}} H\left(\frac{l}{L}\right) = L \int_{\mathbb{R}} H(x) dx + \mathrm{O}\left(\|DH\|_{\infty} \left(b - a + \frac{2}{L}\right)\right),$$

where

$$\|DH\|_{\infty} = \sup_{x \in \mathbb{R}} |H'(x)|.$$

Let $f \in L^1(\mathbb{R})$ be a real-valued function. The Fourier transform of f is defined by

$$\widehat{f}(x) = \int_{\mathbb{R}} f(y) e(-xy) dy, \ x \in \mathbb{R}.$$

The Poisson summation formula [81, p. 538] is crucial in proving the correlation measure results.

Proposition 2.1.3 (Poisson's summation formula)**.** *Let* $f \in L^1(\mathbb{R})$ *and* $\widehat{f}$ *be the Fourier transform of* f*, then we have*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \widehat{f}(m).$$

2.1.1 Poisson summation formula with smooth weight

We also use a version of the Poisson summation formula as below.

Proposition 2.1.4. *Let* $\alpha, \beta \in \mathbb{R}$*, and let* f *be a smooth function on* $\mathbb{R}$ *such that* $Supp(f) \subset (0, \Lambda)$*. Then we have*

$$\sum_{n\in\mathbb{Z}} f(n+\alpha)e(\beta n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n-\beta)e((n-\beta)\alpha),$$

where $\widehat{f}$ *is the Fourier transform of* f*.*

Proof. Note that $f \in L^1(\mathbb{R})$. We define $g(x) = f(x+\alpha)e(\beta x)$. Clearly, g is a smooth function and $g \in L^1(\mathbb{R})$, since $\int_{\mathbb{R}} |g(x)|dx = \int_{\mathbb{R}} |f(x)|dx < \infty$. The Fourier transform of g is given by

$$\begin{aligned} \widehat{g}(x) &= \int_{\mathbb{R}} g(y)e(-xy)dy = \int_{\mathbb{R}} f(y+\alpha)e(-(x-\beta)y)dy \\ &= \int_{\mathbb{R}} f(z)e(-(x-\beta)(z-\alpha))dz = \widehat{f}(x-\beta)e((x-\beta)\alpha). \end{aligned}$$

We apply Proposition 2.1.3 to $g(x) = f(x+\alpha)e(\beta x)$. Therefore

$$\sum_{n\in\mathbb{Z}} f(n+\alpha)e(\beta n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n-\beta)e((n-\beta)\alpha).$$

This completes the proof of Proposition 2.1.4. □

2.2 Multiplicative functions

Definition 2.2.1. *An arithmetic function $f : \mathbb{N} \to \mathbb{C}$ is said to be multiplicative function if*

$$f(n_1 n_2) = f(n_1) f(n_2) \text{ whenever } (n_1, n_2) = 1.$$

Multiplicative functions play crucial role in forming the Euler product of Dirichlet series.

2.3 Dirichlet characters

Definition 2.3.1. *Let m be a positive integer. A function $\chi : \mathbb{Z} \to \mathbb{C}$ is said to be a Dirichlet character modulo m if*

1. *χ is m-periodic, that is, $\chi(n+m) = \chi(n)$ for all $n \in \mathbb{Z}$;*
2. *χ is completely multiplicative, that is, $\chi(n_1 n_2) = \chi(n_1)\chi(n_2)$ for all $n_1, n_2 \in \mathbb{Z}$;*
3. *$\chi(n) = 0$ if and only if $\gcd(n, m) > 1$.*

We next state the orthogonality relation of Dirichlet characters modulo m.

Proposition 2.3.1 (Orthogonality relation)**.** *Let $m \geq 1$ and a be integers such that $(a, m) = 1$. Then we have*

$$\frac{1}{\phi(m)} \sum_{\chi \pmod{m}} \chi(n\bar{a}) = \begin{cases} 1 & \text{if } n \equiv a \pmod{m}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\bar{a}$ is such that $a\bar{a} \equiv 1 \pmod{m}$.

Proof. See [6, Theorem 6.16]. □

The orthogonality relation plays an important role in the study of numbers in arithmetic progressions. For the arithmetic progression $n \equiv a \pmod{m}$ with $(a, m) = 1$, we make frequent use of the above proposition throughout this thesis.

2.3.1 Weighted character sums

We will also need the following estimate on weighted character sums to deal with the contribution coming from the arithmetic progression $n \equiv a \pmod{m}$ with $(a, m) = 1$ in the computations for correlations measure.

Lemma 2.3.1. *Let $R > 1$, M and Λ be positive real numbers and let δ be a positive integer. Suppose χ is a non-principal Dirichlet character modulo m and f is a continuously differentiable function with Supp$(f) \subset (0, \Lambda)$. Then for any integer $r \geq 1$, we have*

$$\sum_{\substack{a \leq R \\ (a,r)=1}} \mu_k(a\delta)^2 \chi(a) f\left(\frac{M}{a}\right) = \mathrm{O}_{m,\Lambda}\left(\tau(r) R^{\frac{1}{k}} \log R\right).$$

Proof. We have

$$\sum_{\substack{a \leq R \\ (a,r)=1}} \mu_k(a\delta)^2 \chi(a) f\left(\frac{M}{a}\right) = \sum_{\substack{a \leq R \\ (a,r)=1}} \chi(a) f\left(\frac{M}{a}\right) \sum_{d^k \mid a\delta} \mu(d) = \sum_{\substack{d^k \leq R\delta \\ \left(\frac{d^k}{\gcd(d^k,\delta)}, r\right)=1}} \mu(d) \\ \times \chi\left(\frac{d^k}{\gcd(d^k, \delta)}\right) \sum_{\substack{a \leq \frac{R \gcd(d^k,\delta)}{d^k} \\ (a,r)=1}} \chi(a) f\left(\frac{M \gcd(d^k, \delta)}{d^k a}\right).$$

In the last step, we used the fact that $a|bc$ if and only if $\frac{a}{\gcd(a,c)}|b$. Since δ is k-free – otherwise the result would follow trivially – it follows that $\left(\frac{d^k}{\gcd(d^k,\delta)}, r\right) = 1$ if and only if $(d, r) = 1$. Therefore

$$\sum_{\substack{a \leq R \\ (a,r)=1}} \mu_k(a\delta)^2 \chi(a) f\left(\frac{M}{a}\right) \\ = \sum_{\substack{d^k \leq R\delta \\ (d,r)=1}} \mu(d) \chi\left(\frac{d^k}{\gcd(d^k, \delta)}\right) \sum_{\substack{a \leq \frac{R \gcd(d^k,\delta)}{d^k} \\ (a,r)=1}} \chi(a) f\left(\frac{M \gcd(d^k, \delta)}{d^k a}\right)$$

$$= \sum_{s|r} \mu(s)\chi(s) \sum_{\substack{d^k \le R\delta \\ (d,r)=1}} \mu(d)\chi\left(\frac{d^k}{\gcd(d^k,\delta)}\right) \sum_{a \le \frac{R\gcd(d^k,\delta)}{sd^k}} \chi(a) f\left(\frac{M\gcd(d^k,\delta)}{sd^k a}\right). \quad (2.1)$$

To estimate the inner-most sum, we apply Proposition 2.1.2

$$\sum_{a \le \frac{R\gcd(d^k,\delta)}{sd^k}} \chi(a) f\left(\frac{M\gcd(d^k,\delta)}{d^k sa}\right) = f\left(\frac{M}{R}\right) \sum_{a \le \frac{R\gcd(d^k,\delta)}{sd^k}} \chi(a) + \int_1^{\frac{R\gcd(d^k,\delta)}{sd^k}} \sum_{a \le x} \chi(a)$$
$$\times f'\left(\frac{\gcd(d^k,\delta)}{M^{-1}d^k sx}\right) \frac{M\gcd(d^k,\delta)dx}{d^k sx^2} \ll_{m,\Lambda} \log R.$$

The above estimate in conjunction with (2.1) gives the required result. □

2.4 Bounds for Riemann zeta function and Dirichlet L-function

In our applications of Perron's formula, we shall use the following estimates for the Riemann zeta function [93].

$$\zeta(\sigma+it) \ll \begin{cases} t^{\frac{1-\sigma}{2}}\log t, & 0 \le \sigma \le 1, \\ \log t, & 1 \le \sigma \le 2, \\ 1, & \sigma \ge 2. \end{cases} \quad (2.2)$$

and

$$\frac{1}{\zeta(\sigma+it)} \ll \begin{cases} \log t, & 1 \le \sigma \le 2, \\ 1, & \sigma \ge 2. \end{cases} \quad (2.3)$$

We also use Vinogradov-Korobov zero free region (see [95] and [68]). For some fixed constant $c > 0$, we have

$$\zeta(\sigma+it) \neq 0 \text{ for } \sigma \ge 1 - c(\log t)^{-2/3}(\log\log t)^{-1/3}$$

and

$$\frac{1}{\zeta(\sigma+it)} \ll (\log t)^{2/3}(\log\log t)^{1/3},\ 1-c(\log t)^{-2/3}(\log\log t)^{-1/3} \leq \sigma \leq 1. \tag{2.4}$$

For non-principal Dirichlet characters $\chi \pmod{m}$, we use the following bounds of Dirichlet L-function $L(s,\chi)$ (see [67] and [81])

$$L(\sigma+it,\chi) \ll_m \begin{cases} t^{\frac{35(1-\sigma)}{108}}\log^3 t, & 1/2 \leq \sigma \leq 1, \\ \log t, & 1 \leq \sigma \leq 2, \\ 1, & \sigma \geq 2. \end{cases} \tag{2.5}$$

and

$$\frac{1}{L(\sigma+it,\chi)} \ll_m \begin{cases} \log t, & 1 \leq \sigma \leq 2, \\ 1, & \sigma \geq 2. \end{cases} \tag{2.6}$$

We have the following bound for $\frac{1}{L(\sigma+it,\chi)}$ in the Vinogradov-Korobov zero free region [65]. For some fixed constant $c>0$, we have

$$L(\sigma+it) \neq 0 \text{ for } \sigma \geq 1-\frac{c}{\log m+(\log t)^{2/3}(\log\log t)^{1/3}}$$

and

$$\frac{1}{L(\sigma+it,\chi)} \ll (\log t)^{2/3}(\log\log t)^{1/3},\ 1-\frac{c}{\log m+(\log t)^{2/3}(\log\log t)^{1/3}} \leq \sigma \leq 1. \tag{2.7}$$

In our investigation, we shall also make use of results on the mean value estimates for $\zeta(s)/s$ and $L(s,\chi)/s$. To prove these mean value estimates, we will employ the following mean value theorem for Dirichlet series.

Theorem 2.4.1 ([79], Theorem 6.1)**.** *For any real T_0 and T, we have*

$$\int_{T_0}^{T_0+T} |S(it)|^2 dt = \left(T+\theta\frac{4\pi}{\sqrt{3}}N\right)\sum_{n=1}^{N}|a_n|^2,$$

where $S(s)=\sum_{n=1}^{N} a_n n^{-s}$ and $-1 \leq \theta \leq 1$.

Now we state the mean value theorem for the Riemann zeta function which can be derived from the above theorem.

Theorem 2.4.2 ([59], Theorem 1.11, p. 28)**.** *For* $\frac{1}{2} < \sigma < 1$ *fixed*

$$\int_1^T |\zeta(\sigma+it)|^2 dt = \zeta(2\sigma)T + \mathrm{O}\left(T^{2-2\sigma}\log T\right).$$

Mean value estimates of $\zeta(s)/s$

Proposition 2.4.1. *For a complex number* $s = \sigma + it$*, with* $1/2 < \sigma < 1$*, we have*

$$\int_0^T \frac{|\zeta(\sigma+it)|}{|\sigma+it|} dt \ll \log T.$$

Proof. We write

$$\begin{aligned}\int_0^T \frac{|\zeta(\sigma+it)|}{|\sigma+it|} dt &\le \left(\int_0^1 + \sum_{n=0}^{\log T} \int_{2^n}^{2^{n+1}}\right) \frac{|\zeta(\sigma+it)|}{|\sigma+it|} dt \\ &\ll 1 + \log T \max_{0\le n\le \lfloor \log T\rfloor} \int_{2^n}^{2^{n+1}} \frac{|\zeta(\sigma+it)|}{|\sigma+it|} dt \\ &\ll 1 + \log T \max_{0\le n\le \lfloor \log T\rfloor} \frac{1}{2^n} \int_0^{2^{n+1}} |\zeta(\sigma+it)|\, dt. \qquad (2.8)\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we have

$$\int_0^{2^{n+1}} |\zeta(\sigma+it)| dt \le \left(\int_0^{2^{n+1}} 1^2 dt\right)^{\frac{1}{2}} \left(\int_0^{2^{n+1}} |\zeta(\sigma+it)|^2 dt\right)^{\frac{1}{2}}.$$

Using Theorem 2.4.2, we have

$$\int_0^{2^{n+1}} |\zeta(\sigma+it)| dt \ll 2^{n+1}, \qquad (2.9)$$

Substituting (2.9) in (2.8), gives the required result. □

Proposition 2.4.2. *If* $0 < \sigma < 1/2$ *is a real number, then*

$$\int_0^T \frac{|\zeta(\sigma+it)|}{|\sigma+it|} dt \ll T^{\frac{1}{2}-\sigma}\log T.$$

Proof. We begin with the asymptotic formula (2.8):

$$\int_0^T \frac{|\zeta(\sigma+it)|}{|\sigma+it|}dt \ll 1+\log T \max_{0\le n\le\lfloor\log T\rfloor} \frac{1}{2^n}\int_0^{2^n} |\zeta(\sigma+it)|dt. \tag{2.10}$$

The functional equation for Riemann zeta function is given by (see [31], p. 59)

$$\zeta(s) = Y(s)\zeta(1-s),$$

where $Y(s)=\pi^{s-1/2}\Gamma((1-s)/2)/\Gamma(s/2)$. By Stirling's formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\mathrm{O}\left(\frac{1}{n}\right)\right),$$

we have $Y(\sigma+it)\asymp t^{\frac{1}{2}-\sigma}$. Thus, the above identity gives

$$\zeta(\sigma+it) = \mathrm{O}\left(t^{\frac{1}{2}-\sigma}\zeta(1-\sigma+it)\right). \tag{2.11}$$

Employing Cauchy-Schwarz inequality and using Theorem 2.4.2 with (2.11), we obtain

$$\int_0^{2^n} |\zeta(\sigma+it)|dt \ll 2^{\frac{n}{2}+n(1-\sigma)}.$$

Inserting the above estimate into (2.10) gives the required result. ☐

Mean value estimate of $L(s,\chi)/s$

Proposition 2.4.3. *If* $0<\sigma<1/2$ *is a real number, then*

$$\int_0^T \frac{|L(\sigma+it,\chi)|}{|\sigma+it|}dt \ll_m T^{\frac{1}{2}-\sigma}\log T.$$

Proof. The proof follows along the same lines as Proposition 2.4.2, using the functional equation for the Dirichlet L-function modulo m

$$\xi(1-s,\bar{\chi}) = \frac{i^{\mathfrak{a}}m^{\frac{1}{2}}}{\tau(\chi)}\xi(s,\chi),$$

where

$$\xi(s,\chi) = \left(\frac{\pi}{m}\right)^{\frac{-1}{2}(s+\mathfrak{a})} \Gamma\left(\frac{1}{2}(s+\mathfrak{a})\right) L(s,\chi),$$

and where $\mathfrak{a} = 0$ if $\chi(-1) = 1$ and $\mathfrak{a} = 1$ if $\chi(-1) = -1$, together with Theorem 2.4.1. □

2.5 Smooth correlation measure

In this section, we discuss a smooth version of the correlation measure, which is a key point in establishing results on the correlation measure. Let $\nu \geq 2$ be an integer and let $\mathcal{F}$ be a finite set of N elements in the unit interval $[0,1]$. Recall that the ν-level correlation measure $\mathcal{S}_{\mathcal{F}}^{(\nu)}(\mathfrak{B})$ of a box $\mathfrak{B} \subset \mathbb{R}^{\nu-1}$ is defined as follows:

$$\frac{1}{N}\#\left\{(x_1,\ldots,x_\nu) \in \mathcal{F}^\nu : x_i \text{ distinct},\ (x_1-x_2,\ldots,x_{\nu-1}-x_\nu) \in \frac{1}{N}\mathfrak{B} + \mathbb{Z}^{\nu-1}\right\}. \tag{2.12}$$

Let H be a smooth real-valued function on $\mathbb{R}^{\nu-1}$ such that $\mathrm{Supp}H \subset (0,\Lambda_1) \times \cdots \times (0,\Lambda_{\nu-1})$. Define

$$f(y) = \sum_{r\in\mathbb{Z}^{\nu-1}} H(N(y+r)),\ y \in \mathbb{R}^{\nu-1}.$$

Then the smooth ν-level correlation sum is defined as

$$S^\nu = \frac{S}{N}, \tag{2.13}$$

where

$$S = \sum_{\gamma_1,\ldots,\gamma_\nu\in\mathcal{F} \text{ distinct}} f(\gamma_1-\gamma_2,\ldots,\gamma_{\nu-1}-\gamma_\nu).$$

By approximating the characteristic function of $(0,\Lambda_1) \times \cdots \times (0,\Lambda_{\nu-1})$ with H, S^ν becomes the ν-level correlation measure $\mathcal{S}_{\mathcal{F}}^{(\nu)}(\mathfrak{B})$.

It is easy to observe that f is $\mathbb{Z}^{\nu-1}$-periodic function, so we write its Fourier series expansion

$$f(y) = \sum_{r\in\mathbb{Z}^{\nu-1}} c_r e(r\cdot y).$$

The Fourier coefficients of f is given by

$$\begin{aligned} c_r &= \int_{[0,1)^{\nu-1}} f(x)e(-r\cdot x)dx = \sum_{m\in\mathbb{Z}^{\nu-1}} \int_{[0,1)^{\nu-1}} H(N(x+m))e(-n\cdot x)dx \\ &= \int_{\mathbb{R}^{\nu-1}} H(Nv)e(-n\cdot v)dv = \frac{1}{N^{\nu-1}}\int_{\mathbb{R}^{\nu-1}} e\left(-\frac{r\cdot x}{N}\right)H(x)dx = \frac{1}{N^{\nu-1}}\widehat{H}\left(\frac{r}{N}\right), \end{aligned} \tag{2.14}$$

where $\widehat{H}$ is the Fourier transform of H. Using the fact that $\mathrm{Supp}H \subset (0,\Lambda_1)\times\cdots\times(0,\Lambda_{\nu-1})$, the condition that γ_i are distinct can be removed. Thus, by (2.13), we have

$$\begin{aligned} S &= \sum_{\gamma_1,\dots,\gamma_\nu\in\mathcal{F}} f(\gamma_1-\gamma_2,\dots,\gamma_{\nu-1}-\gamma_\nu) \\ &= \sum_{\substack{\gamma_1,\dots,\gamma_\nu\in\mathcal{F}\\ r_1,\dots,r_{\nu-1}\in\mathbb{Z}}} c_r e(r\cdot(\gamma_1-\gamma_2,\dots,\gamma_{\nu-1}-\gamma_\nu)) \\ &= \sum_{\substack{\gamma_1,\dots,\gamma_\nu\in\mathcal{F}\\ r_1,\dots,r_{\nu-1}\in\mathbb{Z}}} c_r e(r_1\gamma_1)e((r_2-r_1)\gamma_2)\dots e((r_{\nu-1}-r_{\nu-2})\gamma_{\nu-1})e(r_{\nu-1}\gamma_\nu). \end{aligned} \tag{2.15}$$

2.6 Lattice points

A Point is called a lattice point if its coordinates are integers. We state a result for counting the lattice points in a bounded domain.

Proposition 2.6.1 ([13], Lemma 1)**.** *Let $R>1$ be a real number. Let $\Omega\subset[1,R]^2$ be a bounded region, and f is a continuously differentiable function on Ω. Then*

$$\begin{aligned} \sum_{(a,b)\in\Omega\cap\mathbb{Z}^2} f(a,b) =& \frac{6}{\pi^2}\iint_\Omega f(x,y)dxdy + \mathrm{O}\left(\left(\left\|\frac{\partial f}{\partial x}\right\|_\infty + \left\|\frac{\partial f}{\partial y}\right\|_\infty\right) \mathit{Area}(\Omega)\right) \\ &+ \mathrm{O}\left(\|f\|_\infty(R+\mathit{length}(\partial\Omega))\right). \end{aligned}$$

We also state a result for counting the lattice points in a bounded domain with some coprimality restriction.

Proposition 2.6.2 ([13], Lemma 2)**.** *Let $R > 1$ be a real number. Let $\Omega \subset [1, R]^2$ be a bounded region, and f is a continuously differentiable function on Ω. Then*

$$\sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2\\(a,b)=1}} f(a,b) = \frac{6}{\pi^2}\iint_\Omega f(x,y)dxdy + \mathrm{O}\left(\left(\left\|\frac{\partial f}{\partial x}\right\|_\infty + \left\|\frac{\partial f}{\partial y}\right\|_\infty\right) \mathit{Area}(\Omega)\log R\right)$$

$$+ \mathrm{O}\left(\|f\|_\infty(R + \mathit{length}(\partial\Omega)\log R)\right).$$

2.6.1 Weighted lattice points with additional constraints

The results in this section are essential in the derivation of the ν-level correlation function and the explicit expression for the pair correlation function. We first prove the following result on the lattice point counting, involving weight and coprimality conditions.

Lemma 2.6.1. *Let $\Omega \subset [1, R]^2$ be a bounded region, and f be a continuously differentiable function on Ω. For any positive integers r_1 and r_2, we have*

$$\sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2\\(a,r_1)=1=(b,r_2)\\(a,b)=1}} f(a,b) = \frac{M(r_1,r_2)}{\zeta(2)}\iint_\Omega f(x,y)dxdy + E, \tag{2.16}$$

where

$$M(r_1,r_2) = \prod_{p|r_1r_2}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{p|r_1}\left(1-\frac{1}{p}\right)\prod_{p|r_2}\left(1-\frac{1}{p}\right)$$

and

$$E \ll \left(\tau(r_1)\left\|\frac{\partial f}{\partial x}\right\|_\infty + \tau(r_2)\left\|\frac{\partial f}{\partial y}\right\|_\infty\right)\mathit{Area}(\Omega)\log^2 R + \|f\|_\infty(\tau(r_1)+\tau(r_2))R\log^2 R.$$

Proof. We begin by removing the coprimality conditions on the left-hand side of (2.16) using Möbius summation (Theorem 2.0.1). Therefore, we obtain

$$L := \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2\\(a,r_1)=1=(b,r_2)\\(a,b)=1}} f(a,b) = \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2\\(a,r_1)=1=(b,r_2)}} f(a,b)\sum_{\substack{d|a\\d|b}}\mu(d)$$

$$
\begin{aligned}
&= \sum_{d\le R} \mu(d) \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2\\ (a,r_1)=1=(b,r_2)\\ d|a,d|b}} f(a,b)\\
&= \sum_{\substack{d\le R\\ (d,r_1r_2)=1}} \mu(d) \sum_{\substack{(a_1,b_1)\in\frac{1}{d}\Omega\cap\mathbb{Z}^2\\ (a_1,r_1)=1=(b_1,r_2)}} f(da_1,db_1)\\
&= \sum_{\substack{d\le R\\ (d,r_1r_2)=1}} \mu(d) \sum_{(a_1,b_1)\in\frac{1}{d}\Omega\cap\mathbb{Z}^2} f(da_1,db_1) \sum_{\substack{d_1|a_1\\ d_1|r_1}} \mu(d_1) \sum_{\substack{d_2|b_1\\ d_2|r_2}} \mu(d_2)\\
&= \sum_{\substack{d\le R\\ (d,r_1r_2)=1}} \mu(d) \sum_{d_1|r_1} \mu(d_1) \sum_{d_2|r_2} \mu(d_2) \sum_{(a',b')\in\Delta\cap\mathbb{Z}^2} \mathfrak{f}(a',b'), \qquad (2.17)
\end{aligned}
$$

where $\mathfrak{f}(a',b') = f(dd_1a', dd_2b')$ and $\Delta = \Big\{(x,y) : x \in \frac{1}{dd_1}[1,R], y \in \frac{1}{dd_2}[1,R]\Big\}$. We use proposition 2.6.1 to estimate the inner-most sum in the above identity and obtain

$$
\begin{aligned}
\sum_{(a',b')\in\Delta\cap\mathbb{Z}^2} \mathfrak{f}(a',b') = &\iint_\Delta \mathfrak{f}(x,y)dxdy\\
&+ \mathrm{O}\left(\left(\left\|\frac{\partial\mathfrak{f}}{\partial x}\right\|_\infty + \left\|\frac{\partial\mathfrak{f}}{\partial y}\right\|_\infty\right)\mathrm{Area}(\Delta) + \|\mathfrak{f}\|_\infty(1+\mathrm{length}(\partial\Delta))\right)\\
=&\frac{1}{d^2d_1d_2}\iint_\Omega f(x,y)dxdy + \mathrm{O}\left(\|f\|_\infty \frac{R}{d}\left(\frac{1}{d_1}+\frac{1}{d_2}\right)\right)\\
&+ \mathrm{O}\left(\left(\frac{1}{dd_2}\left\|\frac{\partial f}{\partial x}\right\|_\infty + \frac{1}{dd_1}\left\|\frac{\partial f}{\partial y}\right\|_\infty\right)\mathrm{Area}(\Omega)\right).
\end{aligned}
$$

Inserting the above estimate into (2.17), we obtain

$$
\begin{aligned}
L = &\sum_{\substack{d\le R\\ (d,r_1r_2)=1}} \frac{\mu(d)}{d^2} \sum_{d_1|r_1} \frac{\mu(d_1)}{d_1} \sum_{d_2|r_2} \frac{\mu(d_2)}{d_2} \iint_\Omega f(x,y)dxdy\\
&+ \mathrm{O}\left(\left(\tau(r_1)\left\|\frac{\partial f}{\partial x}\right\|_\infty + \tau(r_2))\left\|\frac{\partial f}{\partial y}\right\|_\infty\right)\mathrm{Area}(\Omega)\log^2 R\right)\\
&+ \mathrm{O}\left(\|f\|_\infty(\tau(r_1)+\tau(r_2))R\log^2 R\right)\\
=&\frac{1}{\zeta(2)}\prod_{p|r_1r_2}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{p|r_1}\left(1-\frac{1}{p}\right)\prod_{p|r_2}\left(1-\frac{1}{p}\right)\iint_\Omega f(x,y)dxdy\\
&+ \mathrm{O}\left(\left(\tau(r_1)\left\|\frac{\partial f}{\partial x}\right\|_\infty + \tau(r_2)\left\|\frac{\partial f}{\partial y}\right\|_\infty\right)\mathrm{Area}(\Omega)\log^2 R\right)
\end{aligned}
$$

$$+ \mathrm{O}\left(\|f\|_\infty(\tau(r_1)+\tau(r_2))R\log^2 R\right).$$

This completes the proof of Lemma 2.6.1. □

We next prove weighted versions of the Proposition 2.6.2 consisting of coprimality constraints and twisted by characteristic functions of k-free numbers. This involves several significant modifications. In particular, we need to deal with the extra Möbius twists in the sums and handle the extra coprimality conditions by carefully reducing the regions using several change of variables.

Lemma 2.6.2. *Let $R > 1$ be a real number and let δ_1 and δ_2 be k-free numbers. Let $\Omega \subset [1,R]^2$ be a bounded region and assume that f is a continuously differentiable function on Ω. For any positive integers r_1 and r_2, we have*

$$\sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2\\(a,r_1)=(b,r_2)=(a,b)=1}} \mu_k(a\delta_1)^2\mu_k(b\delta_2)^2 f(a,b) = \frac{6P^k_{r_1,r_2}(\delta_1,\delta_2)}{\pi^2}\iint_\Omega f(x,y)dxdy + E(r_1,r_2), \tag{2.18}$$

where

$$\begin{aligned}P^k_{r_1,r_2}(\delta_1,\delta_2) = &\frac{\phi(r_1)\phi(r_2)}{r_1r_2}\prod_{p\mid r_1r_2}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{\substack{p\mid r_1\\(p,r_2)=1}}\left(1-\frac{\gcd(p^k,\delta_2)}{p^k}\right)\\ &\times\prod_{\substack{p\mid r_2\\(p,r_1)=1}}\left(1-\frac{\gcd(p^k,\delta_1)}{p^k}\right)\prod_{\substack{p\\(p,r_1r_2)=1}}\left(1-\frac{\gcd(p^k,\delta_2)}{p^{k-1}(p+1)}\right)\\ &\times\left(1-\frac{\gcd(p^k,\delta_1)}{p^{k-1}(p+1)}\left(1-\frac{\gcd(p^k,\delta_2)}{p^{k-1}(p+1)}\right)^{-1}\right)\end{aligned}$$

and

$$\begin{aligned}E(r_1,r_2) \ll_k &\left(\tau(r_1)\left\|\frac{\partial f}{\partial x}\right\|_\infty + \tau(r_2)\left\|\frac{\partial f}{\partial y}\right\|_\infty\right) Area(\Omega)R^{\frac{1}{k}}\log^2 R\\ &+ R^{1+\frac{1}{k}}\log^2 R\,\|f\|_\infty(\tau(r_1)+\tau(r_2)).\end{aligned}$$

Proof. We have

$$M := \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2\\(a,r_1)=(b,r_2)=(a,b)=1}} \mu_k(a\delta_1)^2\mu_k(b\delta_2)^2 f(a,b).$$

In view of the identity

$$\mu_k(n)^2 = \sum_{d^k|n}\mu(d), \tag{2.19}$$

the above sum can be expressed as

$$\begin{aligned}
M &= \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2\\(a,r_1)=(b,r_2)=(a,b)=1}} f(a,b)\sum_{d_1^k|a\delta_1}\mu(d_1)\sum_{d_2^k|b\delta_2}\mu(d_2)\\
&= \sum_{\substack{d_1^k\le R\delta_1\\d_2^k\le R\delta_2}}\mu(d_1)\mu(d_2)\sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2\\(a,r_1)=(b,r_2)=(a,b)=1\\d_1^k|a\delta_1,d_2^k|b\delta_2}} f(a,b).
\end{aligned}$$

Using the fact that $a|bc$ if and only if $\frac{a}{\gcd(a,c)}|b$, the above identity can be expressed as

$$\begin{aligned}
M &= \sum_{\substack{d_1^k\le R\delta_1\\d_2^k\le R\delta_2}}\mu(d_1)\mu(d_2)\sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2\\(a,r_1)=(b,r_2)=(a,b)=1\\\frac{d_1^k}{\gcd(d_1^k,\delta_1)}|a,\frac{d_2^k}{\gcd(d_2^k,\delta_2)}|b}} f(a,b)\\
&= \sum_{\substack{d_1^k\le R\delta_1\\d_2^k\le R\delta_2}}\mu(d_1)\mu(d_2)\sum_{\substack{(a,b)\in\Omega_{(d_1,d_2)}\cap\mathbb{Z}^2\\\left(\frac{d_1^ka_1}{\gcd(d_1^k,\delta_1)},r_1\right)=\left(\frac{d_2^kb_1}{\gcd(d_2^k,\delta_2)},r_2\right)=1\\\left(\frac{d_1^ka_1}{\gcd(d_1^k,\delta_1)},\frac{d_2^kb_1}{\gcd(d_2^k,\delta_2)}\right)=1}} g(a_1,b_1)\\
&= \sum_{\substack{d_1^k\le R\delta_1,d_2^k\le R\delta_2\\\left(\frac{d_1^k}{G_1},r_1\right)=\left(\frac{d_2^k}{G_2},r_2\right)=1\\\left(\frac{d_1^k}{G_1},\frac{d_2^k}{G_2}\right)=1}}\mu(d_1)\mu(d_2)\sum_{\substack{(a_1,b_1)\in\Omega_{(d_1,d_2)}\cap\mathbb{Z}^2\\\left(a_1,\frac{r_1d_2^k}{G_2}\right)=1=\left(b_1,\frac{r_2d_1^k}{G_1}\right)\\(a_1,b_1)=1}} g(a_1,b_1),
\end{aligned} \tag{2.20}$$

where $G_1 = \gcd(d_1^k, \delta_1),\ G_2 = \gcd(d_2^k, \delta_2),\ g(a_1, b_1) = f\left(\frac{d_1^k a_1}{G_1}, \frac{d_2^k b_1}{G_2}\right)$ and

$$\Omega_{(d_1,d_2)} = \left\{(x, y)\ :\ x \in \frac{G_1}{d_1^k}[1, R],\ y \in \frac{G_2}{d_2^k}[1, R]\right\}.$$

We first estimate the inner sum in the above identity. In doing so, we use Theorem 2.0.1. Therefore

$$\begin{aligned}
M^{(1)} :=& \sum_{\substack{(a_1,b_1)\in\Omega_{(d_1,d_2)}\cap\mathbb{Z}^2\\ \left(a_1,\frac{r_1 d_2^k}{G_2}\right)=1=\left(b_1,\frac{r_2 d_1^k}{G_1}\right)\\ (a_1,b_1)=1}} g(a_1, b_1) = \sum_{\substack{(a_1,b_1)\in\Omega_{(d_1,d_2)}\cap\mathbb{Z}^2\\ \left(a_1,\frac{r_1 d_2^k}{G_2}\right)=1=\left(b_1,\frac{r_2 d_1^k}{G_1}\right)}} g(a_1, b_1) \sum_{d\mid \gcd(a_1,b_1)} \mu(d)\\
=& \sum_{\substack{d\le R\min\left(\frac{G_1}{d_1^k},\frac{G_2}{d_2^k}\right)\\ \left(d,\frac{r_1 r_2 d_1^k d_2^k}{G_1 G_2}\right)=1}} \mu(d) \sum_{\substack{(a_2,b_2)\in\frac{1}{d}\Omega_{(d_1,d_2)}\cap\mathbb{Z}^2\\ \left(a_2,\frac{r_1 d_2^k}{G_2}\right)=1=\left(b_2,\frac{r_2 d_1^k}{G_1}\right)}} g(da_2, db_2)\\
=& \sum_{\substack{d\le R\min\left(\frac{G_1}{d_1^k},\frac{G_2}{d_2^k}\right)\\ \left(d,\frac{r_1 r_2 d_1^k d_2^k}{G_1 G_2}\right)=1}} \mu(d) \sum_{(a_2,b_2)\in\frac{1}{d}\Omega_{(d_1,d_2)}\cap\mathbb{Z}^2} g(da_2, db_2) \sum_{\substack{s\mid a_2\\ s\mid\frac{r_1 d_2^k}{G_2}}} \mu(s) \sum_{\substack{t\mid b_2\\ t\mid\frac{r_2 d_1^k}{G_1}}} \mu(t)\\
=& \sum_{\substack{d\le R\min\left(\frac{G_1}{d_1^k},\frac{G_2}{d_2^k}\right)\\ \left(d,\frac{r_1 r_2 d_1^k d_2^k}{G_1 G_2}\right)=1}} \mu(d) \sum_{s\mid\frac{r_1 d_2^k}{G_2}} \mu(s) \sum_{t\mid\frac{r_2 d_1^k}{G_1}} \mu(t) \sum_{(a_3,b_3)\in\Gamma\cap\mathbb{Z}^2} h(a_3, b_3), \qquad (2.21)
\end{aligned}$$

where $h(a_3, b_3) = f\left(\frac{dsd_1^k a_3}{G_1}, \frac{dtd_2^k b_3}{G_2}\right)$ and

$$\Gamma = \left\{(x, y)\ :\ x \in \frac{G_1}{dsd_1^k}[1, R],\ y \in \frac{G_2}{dtd_2^k}[1, R]\right\}.$$

We use Proposition 2.6.1 to estimate the innermost sum in (2.21)

$$\begin{aligned}
\sum_{(a_3,b_3)\in\Gamma\cap\mathbb{Z}^2} h(a_3, b_3) =& \iint_\Gamma h(x, y)dxdy + \mathrm{O}\left(\left(\left\|\frac{\partial h}{\partial x}\right\|_\infty + \left\|\frac{\partial h}{\partial y}\right\|_\infty\right) \mathrm{Area}(\Gamma)\right)\\
&+ \mathrm{O}\left(\|h\|_\infty \left(1 + \mathrm{length}(\partial\Gamma)\right)\right)\\
=& \frac{G_1 G_2}{std^2 d_1^k d_2^k} \iint_\Omega f(x, y)dxdy + \mathrm{O}\left(\|f\|_\infty \frac{R}{d}\left(\frac{G_1}{sd_1^k} + \frac{G_2}{td_2^k}\right)\right)
\end{aligned}$$

$$+\,\mathrm{O}\left(\left(\frac{G_2}{dtd_2^k}\left\|\frac{\partial f}{\partial x}\right\|_\infty+\frac{G_1}{dsd_1^k}\left\|\frac{\partial f}{\partial y}\right\|_\infty\right)\mathrm{Area}(\Omega)\right).$$

By invoking the above estimate into (2.21), we obtain

$$\begin{aligned}M^{(1)}=&\frac{G_1G_2}{d_1^kd_2^k}\sum_{\substack{d\le R\min\left(\frac{G_1}{d_1^k},\frac{G_2}{d_2^k}\right)\\ \left(d,\frac{r_1r_2d_1^kd_2^k}{G_1G_2}\right)=1}}\frac{\mu(d)}{d^2}\sum_{s\mid\frac{r_1d_2^k}{G_2},\,t\mid\frac{r_2d_1^k}{G_1}}\frac{\mu(s)\mu(t)}{st}\iint_\Omega f(x,y)dxdy\\ &+\mathrm{O}\left(\left(\tau\left(\frac{r_1d_2^k}{G_2}\right)\frac{G_2}{d_2^k}\left\|\frac{\partial f}{\partial x}\right\|_\infty+\tau\left(\frac{r_2d_1^k}{G_1}\right)\frac{G_1}{d_1^k}\left\|\frac{\partial f}{\partial y}\right\|_\infty\right)\mathrm{Area}(\Omega)\log^2 R\right)\\ &+\mathrm{O}\left(R\log^2 R\,\|f\|_\infty\left(\tau\left(\frac{r_2d_1^k}{G_1}\right)\frac{G_1}{d_1^k}+\tau\left(\frac{r_1d_2^k}{G_2}\right)\frac{G_2}{d_2^k}\right)\right).\end{aligned}\tag{2.22}$$

We next estimate the summation in (2.22)

$$\begin{aligned}M^{(11)}:=&\sum_{\substack{d\le R\min\left(\frac{G_1}{d_1^k},\frac{G_2}{d_2^k}\right)\\ \left(d,\frac{r_1r_2d_1^kd_2^k}{G_1G_2}\right)=1}}\frac{\mu(d)}{d^2}\sum_{s\mid\frac{r_1d_2^k}{G_2},\,t\mid\frac{r_2d_1^k}{G_1}}\frac{\mu(s)\mu(t)}{st}\\ =&\sum_{\substack{d=1\\ \left(d,\frac{r_1r_2d_1^kd_2^k}{G_1G_2}\right)=1}}^{\infty}\frac{\mu(d)}{d^2}\prod_{p\mid\frac{r_1d_2^k}{G_2}}\left(1-\frac{1}{p}\right)\prod_{p\mid\frac{r_2d_1^k}{G_1}}\left(1-\frac{1}{p}\right)+\mathrm{O}\left(\frac{\max(d_1^k,d_2^k)}{R}\right)\\ =&\prod_{p\mid\frac{r_1d_2^k}{G_2}}\left(1-\frac{1}{p}\right)\prod_{p\mid\frac{r_2d_1^k}{G_1}}\left(1-\frac{1}{p}\right)\prod_{\substack{p\\ \left(p,\frac{r_1r_2d_1^kd_2^k}{G_1G_2}\right)=1}}\left(1-\frac{1}{p^2}\right)+\mathrm{O}\left(\frac{\max(d_1^k,d_2^k)}{R}\right).\end{aligned}\tag{2.23}$$

The above estimate in conjunction with (2.20) and (2.22) gives

$$M=\frac{1}{\zeta(2)}\iint_\Omega f(x,y)dxdy\sum_{\substack{d_1^k\le R\delta_1,\,d_2^k\le R\delta_2\\ \left(\frac{d_1^k}{G_1},r_1\right)=\left(\frac{d_2^k}{G_2},r_2\right)=1\\ \left(\frac{d_1^k}{G_1},\frac{d_2^k}{G_2}\right)=1}}\frac{\mu(d_1)\mu(d_2)G_1G_2}{d_1^kd_2^k}\prod_{p\mid\frac{r_1r_2d_1^kd_2^k}{G_1G_2}}\left(1-\frac{1}{p^2}\right)^{-1}$$

$$\times \prod_{p|\frac{r_1 d_2^k}{G_2}} \left(1-\frac{1}{p}\right) \prod_{p|\frac{r_2 d_1^k}{G_1}} \left(1-\frac{1}{p}\right) + \mathrm{O}\left(R^{1+\frac{1}{k}} \log^2 R \left\|f\right\|_\infty \left(\tau(r_1)+\tau(r_2)\right)\right)$$

$$+\mathrm{O}\left(\left(\tau\left(r_1\right)\left\|\frac{\partial f}{\partial x}\right\|_\infty + \tau\left(r_2\right)\left\|\frac{\partial f}{\partial y}\right\|_\infty\right)\mathrm{Area}(\Omega) R^{\frac{1}{k}}\log^2 R\right)$$

$$=\frac{1}{\zeta(2)}\prod_{p|r_1r_2}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{p|r_1}\left(1-\frac{1}{p}\right)\prod_{p|r_2}\left(1-\frac{1}{p}\right)\iint_\Omega f(x,y)dxdy$$

$$\times \sum_{\substack{d_1,d_2=1\\ \left(\frac{d_1^k}{G_1},r_1\right)=\left(\frac{d_2^k}{G_2},r_2\right)=1\\ \left(\frac{d_1^k}{G_1},\frac{d_2^k}{G_2}\right)=1}}^{\infty} \frac{\mu(d_1)\mu(d_2)G_1G_2}{d_1^k d_2^k}\prod_{\substack{p|\frac{d_1^k}{G_1}\\(p,r_2)=1}}\left(1-\frac{1}{p}\right)\prod_{\substack{p|\frac{d_2^k}{G_2}\\(p,r_1)=1}}\left(1-\frac{1}{p}\right)$$

$$\times\prod_{\substack{p|\frac{d_1^k d_2^k}{G_1G_2}\\(p,r_1r_2)=1}}\left(1-\frac{1}{p^2}\right)^{-1}+E(r_1,r_2). \tag{2.24}$$

We next estimate the summation in (2.24); let us denote it by M_{r_1,r_2}. Since δ_1 and δ_2 are k-free, it follows that $\left(\frac{d_1^k}{G_1},\frac{d_2^k}{G_2}\right)=1$ if and only if $(d_1,d_2)=1$; that $p|\frac{d_1^k}{G_1}$ if and only if $p|d_1$; and that $p|\frac{d_2^k}{G_2}$ if and only if $p|d_2$. Therefore the sum in (2.24) becomes

$$M_{r_1,r_2} := \sum_{\substack{d_1,d_2=1\\ \left(\frac{d_1^k}{G_1},r_1\right)=\left(\frac{d_2^k}{G_2},r_2\right)=1\\ \left(\frac{d_1^k}{G_1},\frac{d_2^k}{G_2}\right)=1}}^{\infty} \frac{\mu(d_1)\mu(d_2)G_1G_2}{d_1^k d_2^k}\prod_{\substack{p|\frac{d_1^k}{G_1}\\(p,r_2)=1}}\left(1-\frac{1}{p}\right)$$

$$\times\prod_{\substack{p|\frac{d_2^k}{G_2}\\(p,r_1)=1}}\left(1-\frac{1}{p}\right)\prod_{\substack{p|\frac{d_1^k d_2^k}{G_1G_2}\\(p,r_1r_2)=1}}\left(1-\frac{1}{p^2}\right)^{-1}$$

$$=\sum_{\substack{d_1=1\\(d_1,r_1)=1}}^{\infty}\frac{\mu(d_1)G_1}{d_1^k}\prod_{\substack{p|d_1\\(p,r_2)=1}}\left(1-\frac{1}{p}\right)\prod_{\substack{p|d_1\\(p,r_1r_2)=1}}\left(1-\frac{1}{p^2}\right)^{-1}\sum_{\substack{d_2=1\\(d_2,r_2d_1)=1}}^{\infty}\frac{\mu(d_2)G_2}{d_2^k}$$

$$\times\prod_{\substack{p|d_2\\(p,r_1)=1}}\left(1-\frac{1}{p}\right)\prod_{\substack{p|d_2\\(p,r_1r_2d_1)=1}}\left(1-\frac{1}{p^2}\right)^{-1}+\mathrm{O}\left(\frac{1}{R^{1-\frac{1}{k}}}\right)$$

$$\begin{aligned}
&= \sum_{\substack{d_1=1\\(d_1,r_1)=1}}^{\infty} \frac{\mu(d_1)G_1}{d_1^k} \prod_{\substack{p|d_1\\(p,r_2)=1}} \left(1-\frac{1}{p}\right) \prod_{\substack{p|d_1\\(p,r_1r_2)=1}} \left(1-\frac{1}{p^2}\right)^{-1} \\
&\quad\times \prod_{\substack{p\\(p,r_1r_2d_1)=1}} \left(1-\frac{\gcd(p^k,\delta_2)}{p^{k-1}(p+1)}\right) \prod_{\substack{p|r_1\\(p,r_2d_1)=1}} \left(1-\frac{\gcd(p^k,\delta_2)}{p^k}\right) + \mathrm{O}\left(\frac{1}{R^{1-\frac{1}{k}}}\right) \\
&= \prod_{\substack{p\\(p,r_1r_2)=1}} \left(1-\frac{\gcd(p^k,\delta_2)}{p^{k-1}(p+1)}\right) \prod_{\substack{p|r_1\\(p,r_2)=1}} \left(1-\frac{\gcd(p^k,\delta_2)}{p^k}\right) \sum_{\substack{d_1=1\\(d_1,r_1)=1}}^{\infty} \frac{\mu(d_1)G_1}{d_1^k} \\
&\quad\times \prod_{\substack{p|d_1\\(p,r_2)=1}} \left(1-\frac{1}{p}\right) \prod_{\substack{p|d_1\\(p,r_1r_2)=1}} \left(1-\frac{1}{p^2}\right)^{-1} \prod_{\substack{p|d_1\\(p,r_2)=1}} \left(1-\frac{\gcd(p^k,\delta_2)}{p^{k-1}(p+1)}\right)^{-1} \\
&\quad\times \prod_{\substack{p|r_1,p|d_1\\(p,r_2)=1}} \left(1-\frac{\gcd(p^k,\delta_2)}{p^k}\right)^{-1} + \mathrm{O}\left(\frac{1}{R^{1-\frac{1}{k}}}\right) \\
&= \prod_{\substack{p\\(p,r_1r_2)=1}} \left(1-\frac{\gcd(p^k,\delta_2)}{p^{k-1}(p+1)}\right) \left(1-\frac{\gcd(p^k,\delta_1)}{p^{k-1}(p+1)} \left(1-\frac{\gcd(p^k,\delta_2)}{p^{k-1}(p+1)}\right)^{-1}\right) \\
&\quad\times \prod_{\substack{p|r_2\\(p,r_1)=1}} \left(1-\frac{\gcd(p^k,\delta_1)}{p^k}\right) \prod_{\substack{p|r_1\\(p,r_2)=1}} \left(1-\frac{\gcd(p^k,\delta_2)}{p^k}\right) + \mathrm{O}\left(\frac{1}{R^{1-\frac{1}{k}}}\right).
\end{aligned}$$

Inserting the above estimate into (2.24) completes the proof of Lemma 2.6.2. □

Lemma 2.6.3. *Let* $\Omega \subset [1,R]^2$ *be a bounded region and let* f *be a continuously differentiable function on* Ω*. Then, we have*

$$\sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2\\(am,b)=1}} \mu_k(b)^2 f(a,b) = \frac{6\phi(m)P_k(m)}{\pi^2} \iint_{\Omega} f(x,y)\,dxdy + E, \tag{2.25}$$

where

$$P_k(m) = \frac{1}{m} \prod_{p|m} \left(1-\frac{1}{p^2}\right)^{-1} \prod_{\substack{p\\(p,m)=1}} \left(1-\frac{1}{p^{k-1}(p+1)}\right),$$

and

$$E \ll_{k,m} \left(\left\| \frac{\partial f}{\partial x} \right\|_\infty + \left\| \frac{\partial f}{\partial y} \right\|_\infty \right) Area(\Omega) R^{\frac{1}{k}} \log^2 R + \|f\|_\infty R^{1+\frac{1}{k}} \log^2 R.$$

Proof. We begin by considering the left-hand side of (2.25) and use (2.19). We have

$$\begin{aligned}
\sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2 \\ (am,b)=1}} \mu_k(b)^2 f(a,b) &= \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2 \\ (am,b)=1}} f(a,b) \sum_{d^k|b} \mu(d) \\
&= \sum_{d^k \le R} \mu(d) \sum_{\substack{(a,d^k b_1)\in\Omega\cap\mathbb{Z}^2 \\ (am,d^k b_1)=1}} f(a,d^k b_1) \\
&= \sum_{\substack{d^k \le R \\ (d,m)=1}} \mu(d) \sum_{\substack{(a,b_1)\in\Omega_1\cap\mathbb{Z}^2 \\ (a,db_1)=1=(b_1,m)}} f(a,d^k b_1), \qquad (2.26)
\end{aligned}$$

where $\Omega_1 = \{(x,y) \mid x \in [1,R],\ y \in \frac{1}{d^k}[1,R]\}$. We now use Theorem 2.0.1 and obtain

$$\begin{aligned}
\sum_{\substack{(a,b_1)\in\Omega_1\cap\mathbb{Z}^2 \\ (a,db_1)=1=(b_1,m)}} f(a,d^k b_1) &= \sum_{\substack{(a,b_1)\in\Omega_1\cap\mathbb{Z}^2 \\ (a,d)=1=(b_1,m)}} f(a,d^k b_1) \sum_{\substack{d'|a \\ d'|b_1}} \mu(d') \\
&= \sum_{d' \le \frac{R}{d^k}} \mu(d') \sum_{\substack{(a_2,b_2)\in\frac{1}{d'}\Omega_1\cap\mathbb{Z}^2 \\ (d'a_2,d)=1=(d'b_2,m)}} f(d'a_2, d'd^k b_2) \\
&= \sum_{\substack{d' \le \frac{R}{d^k} \\ (d',dm)=1}} \mu(d') \sum_{\substack{(a_2,b_2)\in\frac{1}{d'}\Omega_1\cap\mathbb{Z}^2 \\ (a_2,d)=1=(b_2,m)}} f(d'a_2, d'd^k b_2) \\
&= \sum_{\substack{d' \le \frac{R}{d^k} \\ (d',dm)=1}} \mu(d') \sum_{(a_2,b_2)\in\frac{1}{d'}\Omega_1\cap\mathbb{Z}^2} f(d'a_2, d'd^k b_2) \sum_{\substack{s|a_2 \\ s|d}} \mu(s) \sum_{\substack{t|b_2 \\ t|m}} \mu(t) \\
&= \sum_{\substack{d' \le \frac{R}{d^k} \\ (d',dm)=1}} \mu(d') \sum_{s|d} \mu(s) \sum_{t|m} \mu(t) \sum_{(a_3,b_3)\in\Omega_2\cap\mathbb{Z}^2} g(a_3,b_3),
\end{aligned}$$

(2.27)

where $g(a_3, b_3) = f(d'sa_3, d'd^ktb_3)$ and

$$\Omega_2 = \left\{(x, y) \mid x \in \frac{1}{d's}[1, R],\ y \in \frac{1}{d'd^kt}[1, R]\right\}.$$

To estimate the inner-most sum in (2.27), we use Proposition 2.6.1.

$$\begin{aligned}
\sum_{(a_3,b_3)\in\Omega_2\cap\mathbb{Z}^2} g(a_3, b_3) = &\iint_{\Omega_2} g(x, y)dxdy + \mathrm{O}\left(\left(\left\|\frac{\partial g}{\partial x}\right\|_\infty + \left\|\frac{\partial g}{\partial y}\right\|_\infty\right)\mathrm{Area}(\Omega_2)\right)\\
&+ \mathrm{O}\left(\|g\|_\infty \left(1 + \mathrm{length}(\partial\Omega)\right)\right)\\
=&\frac{1}{(d')^2d^kst}\iint_\Omega f(x, y)dxdy + \mathrm{O}\left(\|f\|_\infty \frac{R}{d'}\left(\frac{1}{s} + \frac{1}{td^k}\right)\right)\\
&+ \mathrm{O}\left(\left(\frac{1}{d'd^kt}\left\|\frac{\partial f}{\partial x}\right\|_\infty + \frac{1}{d's}\left\|\frac{\partial f}{\partial y}\right\|_\infty\right)\mathrm{Area}(\Omega_2)\right).
\end{aligned}$$

Invoking the above estimate into (2.27) and then into (2.26), we obtain

$$\begin{aligned}
\sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2\\(am,b)=1}} \mu_k(b)^2 f(a, b) = &\iint_\Omega dxdy \sum_{\substack{d^k\le R\\(d,m)=1}} \frac{\mu(d)}{d^k} \sum_{\substack{d'\le\frac{R}{d^k}\\(d',dm)=1}} \frac{\mu(d')}{(d')^2} \sum_{s|d} \frac{\mu(s)}{s} \sum_{t|m} \frac{\mu(t)}{t}\\
&\times \iint_\Omega f(x, y)dxdy + \mathrm{O}\left(R^{1+\frac{1}{k}} \log^2 R\|f\|_\infty\right)\\
&+ \mathrm{O}\left(\left(\left\|\frac{\partial f}{\partial x}\right\|_\infty + \left\|\frac{\partial f}{\partial y}\right\|_\infty\right) R^{1/k} \log R\ \mathrm{Area}(\Omega)\right)\\
=&\frac{6\phi(m)P_k(m)}{\pi^2}\iint_\Omega f(x, y)dxdy + \mathrm{O}\left(R^{1+\frac{1}{k}} \log^2 R\|f\|_\infty\right)\\
&+ \mathrm{O}\left(\left(\left\|\frac{\partial f}{\partial x}\right\|_\infty + \left\|\frac{\partial f}{\partial y}\right\|_\infty\right) R^{1/k} \log R\ \mathrm{Area}(\Omega)\right).
\end{aligned}$$

□

Our next result is proved along the same lines as the proof of Proposition 2.6.1.

Lemma 2.6.4. *Let $\Omega \subset [1, R]^2$ be a bounded region and let f be a continuously differentiable function on Ω. For any positive integers $n_j, e_j, \Delta_j,\ 1 \le j \le \nu - 1$, we* *have*

$$\sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2\\ (a,b)=1\\ \mu_k(n_j(be_j-a\Delta_j))^2=1\\ 1\le j\le \nu-1}} \mu_k(b)^2 f(a,b) = \Xi(k,n_j,e_j,\Delta_j)\iint_{\Omega} f(x,y)dxdy + E,$$

where $\Xi(k,n_j,e_j,\Delta_j) > 0$ *is a constant depending on* k, n_j, e_j, Δ_j *and*

$$E \ll_k \left(\left\|\frac{\partial f}{\partial x}\right\|_\infty + \left\|\frac{\partial f}{\partial y}\right\|_\infty\right) Area(\Omega) R^{\frac{1}{k}+\epsilon}\log^2 R + \|f\|_\infty R^{1+\frac{1}{k}+\epsilon}\log^2 R.$$

Proof. The proof is similar to Lemma 2.6.2. We use

$$\mu_k(n)^2 = \sum_{d^k|n}\mu(d)$$

to detect the k-free numbers. □

Remark 2.6.1. *The constant* $\Xi(k,n_j,e_j,\Delta_j)$ *can be viewed as natural density of the set* $\{(a,b)\in\Omega\cap\mathbb{Z}^2 \ :\ (a,b)=1, \mu_k(b)^2=1, \mu_k(n_j(be_j-a\Delta_j))^2=1; 1\le j\le \nu-1\}$. *We assume it to be positive.*

3

Distribution of polynomial Farey fractions

3.1 Introduction

Let $\kappa \geq 1$ be an integer and let $\boldsymbol{c} = (c_\kappa, c_{\kappa-1}, \ldots, c_1) \in \mathbb{Z}^\kappa$ be a fixed non-zero vector and $P(x) = c_\kappa x^\kappa + c_{\kappa-1}x^{\kappa-1} + \cdots + c_1 x$ be a polynomial. Denote

$$\mathcal{F}_{Q,P} := \left\{ \frac{a}{q} : 1 \leq a \leq q \leq Q,\ (P(a), q) = 1 \right\}. \tag{3.1}$$

Recall that

$$\mathscr{F}_{Q,V} := \left\{ \frac{a}{q} \mid 1 \leq a \leq q \leq Q,\ (a,q) = 1,\ \text{and } (a,q) \in V(\boldsymbol{c}) \right\}.$$

To see $\mathcal{F}_{Q,P} \subset \mathscr{F}_{Q,V}$, we prove the following lemma.

Lemma 3.1.1. *Let* $\mathcal{F}_{Q,P}$ *be as in* (3.1)*, and let* $\mathcal{F}_{Q,V}$ *be as in* (1.15)*. We have*

$$\mathcal{F}_{Q,P} \subset \mathscr{F}_{Q,V}$$

Proof. We take $\frac{a}{q} \in \mathcal{F}_{Q,P}$ and choose $r = \frac{q}{P(a)} \in \mathbb{Q}^+$, so that $q = rP(a)$. This implies that the point (a, q) lies on the curve $y = rP(x)$. Let $(a', q') \in \mathbb{N}^2$ be such that $q' = rP(a')$ and $a' < a$. This in turn implies that $P(a)|qP(a')$, but since $(P(a), q) = 1$, it follows that $P(a)|P(a')$ which is not true since $a' < a$. Therefore, $(a, q) \in V(\boldsymbol{c})$, which implies that $\frac{a}{q} \in \mathscr{F}_{Q,V}$. The inclusion is strict because for example, if $P(x) = x(x + 1)$, then clearly $\frac{1}{2} \in \mathscr{F}_{Q,V}$ as $(1, 2) \in V((1, 1))$ since $2 = 1 \cdot P(1)$. However, $\frac{1}{2} \notin \mathcal{F}_{Q,P}$ since $\gcd(P(1), 2) \neq 1$. □

For certain specific polynomials and fixed Q, we explicitly write the set of polynomial Farey fractions. For instance, if $P(x) = x(x^2 + 1)$, then

$$\mathcal{F}_{5,P} = \left\{ \frac{1}{5}, \frac{1}{3}, \frac{2}{3}, \frac{4}{5}, 1 \right\}.$$

For $P(x) = x(x + 1)(x + 2)$, we have

$$\mathcal{F}_{5,P} = \left\{ \frac{1}{5}, \frac{2}{5}, 1 \right\}.$$

The cardinality of the set $\mathcal{F}_{Q,P}$ is given by

$$\mathcal{N}_{Q,P} = \#\mathcal{F}_{Q,P} = \frac{Q^2}{2} \prod_p \left(1 - \frac{f_P(p)}{p^2}\right) + \mathrm{O}\left(Q^{\frac{3}{2}+\epsilon}\right),$$

where

$$f_P(p) := \#\{1 \leq d \leq p : P(d) \equiv 0 \pmod{p}\}.$$

Further, the density of $\mathcal{F}_{Q,P}$ is given by

$$\operatorname{dens}(\mathcal{F}_{Q,P}) = \lim_{Q\to\infty} \frac{\#\mathcal{F}_{Q,P}}{\#\{a/b \mid 1 \leq a \leq b \leq Q\}} = \prod_p \left(1 - \frac{f_P(p)}{p^2}\right).$$

Hence, the set $\mathcal{F}_{Q,P}$ is dense in $[0,1]$. We now explore the quantitative equidistribution and correlation measures of the sequence $(\mathcal{F}_{Q,P})_{Q\geq 1}$. Furthermore, we construct sequences that behave like a sequence of randomly chosen numbers from $[0,1]$ by restricting the denominators to a subset of prime. To do so, we require a result on the cardinality of $\mathcal{F}_{Q,P}$. Therefore, we begin by counting polynomial Farey fractions.

3.2 Counting polynomial Farey fractions

In this section, we prove an asymptotic formula for the number of polynomial Farey fractions with denominators in an arithmetic progression. Let q and l be positive integers; we denote the number of polynomial Farey fractions with denominators in an arithmetic progression as

$$S(Q;q,l) := \#\left\{\frac{a}{n} \in \mathcal{F}_{Q,P} \mid n \equiv l \pmod{q}\right\}. \tag{3.2}$$

Proposition 3.2.1. *Let $\kappa \geq 1$ be an integer and let $P(x) = c_\kappa x^\kappa + c_{\kappa-1}x^{\kappa-1} + \cdots + c_1 x \in \mathbb{Z}[x]$ be a polynomial with non-zero discriminant. Let q and l be positive integers with $(q,l)=1$. If $S(Q;q,l)$ is as in (3.2), then*

$$S(Q;q,l) = \frac{Q^2}{2q}\prod_{p\nmid q}\left(1-\frac{f_P(p)}{p^2}\right) + \mathrm{O}_{q,P}\left(Q^{\frac{3}{2}+\epsilon}\right),$$

where

$$f_P(p) := \#\{1 \leq d \leq p : P(d) \equiv 0 \pmod{p}\}.$$

Proof. We begin with the sum

$$S(Q;q,l) = \sum_{\substack{n\leq Q\\ n\equiv l \pmod{q}}} \sum_{\substack{a\leq n\\ (P(a),n)=1}} 1. \tag{3.3}$$

For fixed positive integers q,l with $(q,l)=1$ using Proposition 2.3.1, the sum in

(3.3) can be written as

$$S(Q;q,l) = \frac{1}{\phi(q)} \sum_{\chi \pmod q} \bar{\chi}(l) \sum_{n \le Q} \chi(n) \sum_{\substack{a \le n \\ (P(a),n)=1}} 1.$$

For $(P(a),n)=1$, we apply Theorem 2.0.1 and obtain

$$\begin{aligned}
S(Q;q,l) &= \frac{1}{\phi(q)} \sum_{\chi \pmod q} \bar{\chi}(l) \sum_{n \le Q} \chi(n) \sum_{a \le n} \sum_{\substack{d|P(a) \\ d|n}} \mu(d) \\
&= \frac{1}{\phi(q)} \sum_{\chi \pmod q} \bar{\chi}(l) \sum_{n \le Q} \chi(n) \sum_{d|n} \mu(d) \sum_{\substack{a \le n \\ P(a)\equiv 0 \pmod d}} 1 \\
&= \frac{1}{\phi(q)} \sum_{\chi \pmod q} \bar{\chi}(l) \sum_{n \le Q} n\chi(n) \sum_{d|n} \frac{\mu(d) f_P(d)}{d} \\
&= \frac{1}{\phi(q)} \sum_{\chi \pmod q} \bar{\chi}(l) \sum_{n \le Q} n\chi(n) K(n), \qquad (3.4)
\end{aligned}$$

where $K(n) = \sum_{d|n} \frac{\mu(d) f_P(d)}{d}$. Note that the arithmetic function $n\chi(n)K(n)$ is multiplicative. The Dirichlet series of $n\chi(n)K(n)$ is given by

$$\begin{aligned}
F(s) = \sum_{n=1}^{\infty} \frac{\chi(n)K(n)}{n^{s-1}} &= \prod_p \left(1 + \left(1 - \frac{f_P(p)}{p}\right)\left(\frac{\chi(p)}{p^{s-1}} + \frac{\chi(p^2)}{p^{2s-2}} + \cdots\right)\right) \\
= \prod_p \left(1 + \left(1 - \frac{f_P(p)}{p}\right) \frac{\chi(p)p^{-(s-1)}}{\left(1 - \frac{\chi(p)}{p^{s-1}}\right)}\right) &= L(s-1,\chi) \prod_p \left(1 - \frac{\chi(p) f_P(p)}{p^s}\right),
\end{aligned} \qquad (3.5)$$

which is absolutely convergent for $\Re(s) > 2$. Moreover, the product term on the far right side is absolutely convergent for $\Re(s) > 1$. Thus, the Dirichlet series $F(s)$ has an analytic continuation to the half plane $\Re(s) > 1$ except for a simple pole at $s = 2$ in the case of principal Dirichlet character χ_0. We use Proposition 2.1.1 for the Dirichlet series $F(s)$ and $x = Q + \frac{1}{2}$ with some fixed $\alpha = 2 + 1/\log Q$ to obtain

$$\sum_{n \le Q} n\chi(n)K(n) = \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{F(s)(Q + \frac{1}{2})^s}{s} ds + R(T), \qquad (3.6)$$

where

$$R(T) \ll \frac{Q^\alpha}{T}\sum_{n=1}^{\infty}\frac{K(n)}{n^\alpha|\log\frac{Q+\frac{1}{2}}{n}|}.$$

We use the fact that $K(n) \ll n^\epsilon$ and first estimate the error term $R(T)$. We divide the sum into three subsums over the following ranges of n: $n \le \frac{Q}{2}$, $\frac{Q}{2} < n < \frac{3Q}{2}$, and $n \ge \frac{3Q}{2}$. For $n \le \frac{Q}{2}$ and $n \ge \frac{3Q}{2}$, it is clear that $\left|\log\frac{Q+\frac{1}{2}}{n}\right| \ge \log\frac{3}{2}$. Hence the first and last subsums are $\mathrm{O}\left(Q^\epsilon/(\alpha-2)\right)$. For values of n satisfying $\frac{Q}{2} < n < \frac{3Q}{2}$ the middle subsum is

$$\ll Q^{-\alpha+\epsilon}\sum_{\frac{-Q}{2}<n\le\frac{Q}{2}}\frac{1}{\left|\log\frac{Q+\frac{1}{2}}{Q+n}\right|} \ll Q^{1-\alpha+\epsilon}\sum_{\frac{-Q}{2}<n\le\frac{Q}{2}}\frac{1}{|n-\frac{1}{2}|} \ll Q^{1-\alpha+\epsilon}\log Q.$$

Therefore,

$$R(T) \ll \frac{Q^{2+\epsilon}\log Q}{T}.$$

To estimate the integral in (3.6), we shift the path of integration into a rectangular contour with line segments connecting the points $\alpha - iT, \alpha + iT, 3/2+\epsilon+iT$, and $3/2+\epsilon-iT$. We first consider the principal character $\chi_0 \pmod q$. Applying Cauchy's residue theorem, we have

$$\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\frac{F(s)(Q+\frac{1}{2})^s}{s}ds = \frac{(Q+\frac{1}{2})^2}{2}\prod_{p|q}\left(1-\frac{1}{p}\right)\prod_{p\nmid q}\left(1-\frac{f_P(p)}{p^2}\right)+\sum_{j=1}^{3}I_j, \tag{3.7}$$

where I_1 and I_3 are the integrals along the horizontal line segments connecting the points $3/2+\epsilon+iT, \alpha+iT$ and $3/2+\epsilon-iT, \alpha-iT$, respectively, and I_2 is integral along vertical line $[3/2+\epsilon-iT, 3/2+\epsilon+iT]$. The first term in the above identity is due to the simple pole of the integrand at $s = 2$. To estimate the integrals I_1 and I_3, we use the standard bounds for $\zeta(s)$ (see 2.2). Thus,

$$I_1, I_3 \ll_q \frac{\log T}{T^{3/4}}\int_{3/2+\epsilon}^{2}Q^\sigma d\sigma + \frac{\log T}{T}\int_{2}^{\alpha}Q^\sigma d\sigma \ll_q \frac{Q^2\log T}{T^{3/4}\log Q}+\frac{Q^\alpha\log T}{T\log Q}.$$

We use Proposition 2.4.1 to estimate the integral I_2 and obtain

$$I_2 \ll_q Q^{3/2+\epsilon}\int_0^T\frac{|\zeta(\frac{1}{2}+\epsilon+it)|}{|\frac{1}{2}+\epsilon+it|}dt \ll_q Q^{3/2+\epsilon}\log T.$$

We next consider the case for the non-principal character $\chi \neq \chi_0$. We continue with the same contour defined above and use the bounds in 2.5 for $L(s-1,\chi)$. Therefore

$$I_1, I_3 \ll_q \frac{\log T}{T^{181/216}} \int_{3/2+\epsilon}^{2} Q^{\sigma} d\sigma + \frac{\log T}{T} \int_{2}^{\alpha} Q^{\sigma} d\sigma \ll_q \frac{Q^2 \log T}{T^{181/216} \log Q} + \frac{Q^{\alpha} \log T}{T \log Q},$$

and

$$I_2 \ll_q Q^{3/2+\epsilon} \int_0^T \frac{|L(\frac{1}{2}+\epsilon+it)|}{|\frac{1}{2}+\epsilon+it|} dt \ll_q Q^{3/2+\epsilon} \log T.$$

Collecting all the above estimates in (3.6) and choosing $T = Q$, for $\chi = \chi_0$, we have

$$\sum_{n \leq Q} n\chi_0(n) K(n) = \frac{Q^2}{2} \prod_{p|q} \left(1 - \frac{1}{p}\right) \prod_{p \nmid q} \left(1 - \frac{f_P(p)}{p^2}\right) + \mathrm{O}_{q,P}\left(Q^{\frac{3}{2}+\epsilon}\right), \tag{3.8}$$

and for $\chi \neq \chi_0$, we have

$$\sum_{n \leq Q} n\chi(n) K(n) = \mathrm{O}_{q,P}\left(Q^{\frac{3}{2}+\epsilon}\right). \tag{3.9}$$

Inserting (3.8) and (3.9) into (3.4) gives the required result. □

An immediate consequence of Proposition 3.2.1 is the following corollary which is obtained upon taking $q = 1$.

Corollary 3.2.1. *Let $\kappa \geq 1$ be an integer and let $P(x) = c_\kappa x^\kappa + c_{\kappa-1} x^{\kappa-1} + \cdots + c_1 x \in \mathbb{Z}[x]$ be a polynomial with non-zero discriminant. If $\mathcal{F}_{Q,P}$ is as in* (3.1)*, then*

$$\mathcal{N}_{Q,P} = \#\mathcal{F}_{Q,P} = \frac{Q^2}{2} \prod_{p} \left(1 - \frac{f_P(p)}{p^2}\right) + \mathrm{O}\left(Q^{\frac{3}{2}+\epsilon}\right), \tag{3.10}$$

3.3 Equidistribution

Our first main result concerns the quantitative equidistribution of the polynomial Farey sequence $(\mathcal{F}_{Q,P})_{Q \geq 1}$, for which we establish bounds on its discrepancy. We denote by $D_{\mathcal{N}_{Q,P}}(\mathcal{F}_{Q,P})$ the discrepancy of this sequence.

Theorem 3.3.1. *Let $\kappa \geq 2$ be an integer and let $\mathbf{c} = (c_\kappa, c_{\kappa-1}, \ldots, c_1) \in \mathbb{Z}^\kappa$ be a fixed non-zero vector and $P(x) = c_\kappa x^\kappa + c_{\kappa-1} x^{\kappa-1} + \cdots + c_1 x$ be a polynomial with* *non-zero discriminant. For all* $Q \geq 1$, *we have*

$$\frac{1}{Q} \ll D_{\mathcal{N}_{Q,P}}(\mathcal{F}_{Q,P}) \ll \frac{(\log Q)^J}{Q},$$

where J *is the number of distinct irreducible factors of the polynomial* $P(x) \in \mathbb{Z}[x]$ *and the implied constants depend on the polynomial* $P(x)$.

From Theorem 3.3.1, it follows that $D_{\mathcal{N}_{Q,P}}(\mathcal{F}_{Q,P}) \to 0$ as $Q \to \infty$ for all $\alpha \in [0,1]$. By Theorem 1.1.4, we obtain the following corollary.

Corollary 3.3.1. *Let* $\kappa \geq 2$ *be an integer and let* $\mathbf{c} = (c_\kappa, c_{\kappa-1}, \dots, c_1) \in \mathbb{Z}^\kappa$ *be a fixed non-zero vector and* $P(x) = c_\kappa x^\kappa + c_{\kappa-1}x^{\kappa-1} + \cdots + c_1 x$ *be a polynomial with non-zero discriminant. Then the Farey sequence* $(\mathcal{F}_{Q,P})_{Q\geq 1}$ *is uniformly distributed modulo one.*

3.3.1 Proof of Theorem 3.3.1

Let $\alpha \in [0,1]$ be a real number. Recall that

$$D^*_{\mathcal{N}_{Q,P}}(\mathcal{F}_{Q,P}) = \sup_{0\leq\alpha\leq 1} R_{\mathcal{N}_{Q,P}}([0,\alpha)), \tag{3.11}$$

where

$$R_{\mathcal{N}_{Q,P}}([0,\alpha)) := R_{\mathcal{N}_{Q,P}}(\alpha) = \left|\frac{A(\alpha;\mathcal{N}_{Q,P})}{\mathcal{N}_{Q,P}} - \alpha\right| = \frac{1}{\mathcal{N}_{Q,P}}|A(\alpha;\mathcal{N}_{Q,P}) - \alpha\mathcal{N}_{Q,P}|, \tag{3.12}$$

and

$$A(\alpha;\mathcal{N}_{Q,P}) = \sum_{\gamma\in\mathcal{F}_{Q,P}\cap[0,\alpha]} 1.$$

We next consider the far-right side of (3.12) to obtain

$$A(\alpha;\mathcal{N}_{Q,P}) - \alpha\mathcal{N}_{Q,P} = \sum_{q\leq Q}\left(\sum_{\substack{a\leq\alpha q\\(P(a),q)=1}} 1 - \alpha\sum_{\substack{a\leq q\\(P(a),q)=1}} 1\right)$$

$$
\begin{aligned}
&= \sum_{q\le Q}\left(\sum_{a\le \alpha q}\sum_{\substack{d\mid P(a)\\ d\mid q}}\mu(d) - \alpha\sum_{a\le q}\sum_{\substack{d\mid P(a)\\ d\mid q}}\mu(d)\right)\\
&= \sum_{q\le Q}\left(\sum_{d\mid q}\mu(d)\sum_{\substack{a\le \alpha q\\ d\mid P(a)}}1 - \sum_{d\mid q}\mu(d)\sum_{\substack{a\le q\\ d\mid P(a)}}1\right)\\
&= \sum_{q\le Q}\sum_{d\mid q}\mu(d)\left(\sum_{\substack{a\le \alpha q\\ d\mid P(a)}}1 - \alpha\sum_{\substack{a\le q\\ d\mid P(a)}}1\right).
\end{aligned}
$$

Since $f_P(d)$ counts the number of solutions of the polynomial congruence $P(a) \equiv 0 \pmod{d}$, so using this fact in the above estimate, we have

$$
\sum_{\substack{a\le \alpha q\\ d\mid P(a)}}1 = \left\lfloor \frac{\alpha q}{d}\right\rfloor f_P(d) + \mathrm{O}\left(f_P(d)\right) \text{ and } \sum_{\substack{a\le q\\ d\mid P(a)}}1 = \frac{q}{d}f_P(d).
$$

Therefore, we have

$$
\begin{aligned}
|A(\alpha;\mathcal{N}_{Q,P}) - \alpha\mathcal{N}_{Q,P}| &= \left|\sum_{q\le Q}\sum_{d\mid q}\mu(d)\left(\left\lfloor \frac{\alpha q}{d}\right\rfloor f_P(d) + \mathrm{O}\left(f_P(d)\right) - \frac{\alpha q}{d}f_P(d)\right)\right|\\
&= \left|\sum_{q\le Q}\sum_{d\mid q}\mu(d)\left(\left(\frac{\alpha q}{d} + \mathrm{O}\left(1\right)\right)f_P(d) - \frac{\alpha q}{d}f_P(d)\right)\right|\\
&\ll \sum_{q\le Q}\sum_{d\mid q}f_P(d) \ll \sum_{q\le Q}\sum_{d\le \frac{Q}{q}}f_P(d). \qquad (3.13)
\end{aligned}
$$

By Proposition 2.0.1, we have, as $Q \to \infty$

$$
\sum_{d\le \frac{Q}{q}}f_P(d) \sim \frac{Q}{q}\left(\log\frac{Q}{q}\right)^{J-1},
$$

where J is the number of distinct irreducible factors of $P(x)$. Thus, the above estimate with (3.13) yields

$$|A(\alpha;\mathcal{N}_{Q,P}) - \alpha\mathcal{N}_{Q,P}| \ll \sum_{q\leq Q} \frac{Q}{q}\left(\log\frac{Q}{q}\right)^{J-1} \ll Q\left(\log Q\right)^{J-1}\sum_{q\leq Q}\frac{1}{q} \ll Q\left(\log Q\right)^{J}.$$

Therefore

$$R_{\mathcal{N}_{Q,P}}(\alpha) = \frac{1}{\mathcal{N}_{Q,P}}\left|A(\alpha;\mathcal{N}_{Q,P}) - \alpha\mathcal{N}_{Q,P}\right| \ll \frac{(\log Q)^J}{Q} \tag{3.14}$$

uniformly in $\alpha \in [0,1]$. Hence,

$$D^*_{\mathcal{N}_{Q,P}}(\mathcal{F}_{Q,P}) \ll \frac{(\log Q)^J}{Q}. \tag{3.15}$$

Next, let $\epsilon > 0$ be arbitrarily small and we take $\alpha = 1/Q - \epsilon$ to obtain a lower bound for $D^*_{\mathcal{N}_{Q,P}}(\mathcal{F}_{Q,P})$. Since $\gamma \geq \frac{1}{Q}$ for all $\gamma \in \mathcal{F}_{Q,P}$, it follows from the definition of $A(\alpha;\mathcal{N}_{Q,P})$ that $A(1/Q-\epsilon;\mathcal{N}_{Q,P}) = 0$. Therefore, using (3.11) and (3.12), we obtain

$$D^*_{\mathcal{N}_{Q,P}}(\mathcal{F}_{Q,P}) \geq R_{\mathcal{N}_{Q,P}}(\alpha) = R_{\mathcal{N}_{Q,P}}\left(\frac{1}{Q}-\epsilon\right) = \frac{1}{Q}-\epsilon$$

for all $\epsilon > 0$. Since $\epsilon > 0$ is arbitrary, we have

$$D^*_{\mathcal{N}_{Q,P}}(\mathcal{F}_{Q,P}) \geq \frac{1}{Q}. \tag{3.16}$$

Therefore, using (3.15), (3.16), and Theorem 1.1.3, we obtain

$$\frac{1}{Q} \ll D_{\mathcal{N}_{Q,P}}(\mathcal{F}_{Q,P}) \ll \frac{(\log Q)^J}{Q}.$$

This completes the proof of Theorem 3.3.1.

3.4 Pair correlation measure

In this section, we study the finer distribution of the sequence $(\mathcal{F}_{Q,P})_{Q\geq 1}$ by analyzing their pair correlation. In particular, we will prove Theorems 1.1.5 and 1.1.6.

Recall that the pair correlation function of the sequence $(\mathcal{F}_{Q,P})_{Q\geq 1}$ is given by

$$\mathfrak{g}_2(\lambda) = \frac{1}{\zeta(2)\lambda^2} \sum_{d_2,d_4=1}^{\infty} \frac{\mu(d_2)\mu(d_4)}{d_2 d_4 \phi(d_2)\phi(d_4)} \sum_{1\leq m<\frac{2\lambda}{\beta_P}} \mathfrak{h}(m) \log\left(\frac{2\lambda}{m\beta_P}\right),$$

where $\mathfrak{h}(m)$ is as in (1.18).

Using Huxley's [57] bound for $f_P(n)$, which states that if $\deg P(x) = \kappa \geq 2$ and $P(x)$ has no multiple roots, then $f_P(l) \ll \kappa^{\omega(l)}$, where $\omega(l)$ counts the number of distinct prime divisors of l, we obtain $f_P(n) \ll n^{\epsilon}$. Thus by (1.16), we have

$$g_P(\lambda) := \frac{1}{22\lambda^2} \sum_{1\leq m<\frac{2\lambda}{\beta_P}} \left(\sum_{\substack{n,\delta,d_1,d_2\geq 1\\ n\delta d_1 d_2 = m}} n(d_1 d_2)^{0.01} \right) \log\left(\frac{2\lambda}{m\beta_P}\right). \tag{3.17}$$

In Figure 1, we compare the plots of the pair correlation function $\mathfrak{g}_2(\lambda)$, the above bound of the pair correlation function for the polynomial $P(x) = x(x^2+1)$, and the pair correlation function $g(\lambda)$ of the classical Farey sequence, against various distributions: Poissonian distribution ($g_{P_o} = 1$), and GUE distribution (g_{GUE}). Note that for $P(x) = x(x^2+1)$, $f_P(2) = 2$; for other values, $f_P(p) = 3$ if $p \equiv 1 \pmod 4$, and $f_P(p) = 1$ if $p \equiv 3 \pmod 4$.

Outline of proof: To establish the pair correlation measure of the sequence $(\mathcal{F}_{Q,P})_{Q\geq 1}$, we need to count tuples, for any positive real number Λ, in the following set:

$$\left\{ (\gamma_1,\gamma_2) \in \mathcal{F}_{Q,P}^2 : \gamma_1 \neq \gamma_2, \gamma_1 - \gamma_2 \in \frac{1}{\mathcal{N}_{Q,P}}(0,\Lambda) + \mathbb{Z} \right\}.$$

Using Fourier series techniques for a smooth real-valued function H with support contained in $(0,\Lambda)$, we reduce the problem of counting tuples in the above set to estimating an exponential sum over $\mathcal{F}_{Q,P}$. We derive a closed-form formula for this

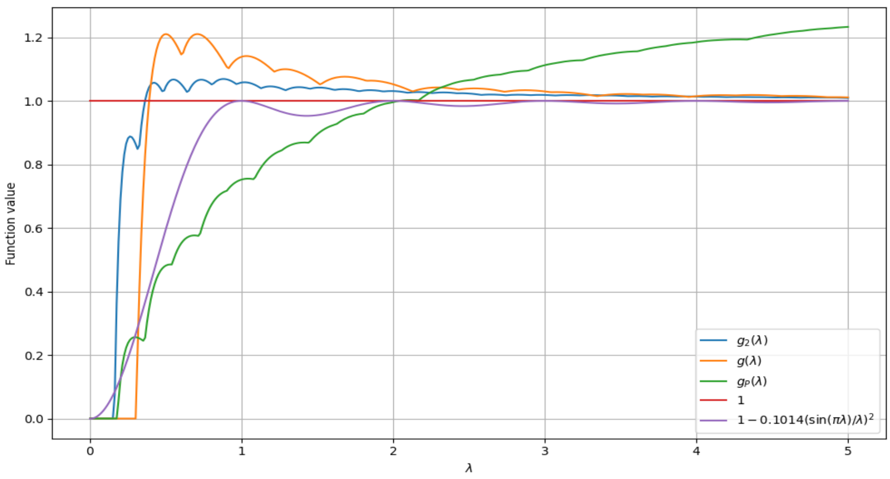

FIGURE 3.1: The graphs of pair correlation functions $\mathfrak{g}_2(\lambda)$, $g_P(\lambda)$, $g(\lambda)$, $g_{Po}(\lambda) \equiv 1$ and $g_{GUE}(\lambda) = 1 - \left(\frac{\sin \pi\lambda}{\pi\lambda}\right)^2$.

exponential sum. For the specific polynomial $P(x) = x(x+1)$, we expand the Weyl sum using Dirichlet characters, which provides a crucial estimate for obtaining an explicit pair correlation formula for this polynomial. Further, we use a weighted Poisson summation formula to handle the Fourier coefficients. This weight arises because we have a less explicit closed-form formula for exponential sums for a general polynomial. This analysis leads us to establish a result for a lattice point counting problem involving weight and coprimality conditions. All this, together with some further simplifications and standard approximation arguments gives the required result.

We first establish results that will be essential in the proofs of pair correlation measure.

3.4.1 Exponential sum over polynomial Farey fractions

We begin by establishing results for the exponential sum over the Farey fractions in $\mathcal{F}_{Q,P}$.

Lemma 3.4.1. *Let r and $\kappa \geq 2$ be integers and let $P(x) = c_\kappa x^\kappa + c_{\kappa-1}x^{\kappa-1} + \cdots +$*

$c_1x \in \mathbb{Z}[x]$ be a polynomial with non-zero discriminant. Then for any $\epsilon > 0$, we have

$$\sum_{\gamma\in\mathcal{F}_{Q,P}} e(r\gamma) = \sum_{\substack{q\le Q\\ q|r}} q \sum_{d\le\frac{Q}{q}} \mu(d) \sum_{\substack{1\le a\le d\\ P(a)\equiv 0 \pmod d}} e\left(\frac{ra}{qd}\right),$$

where $e(x) = \exp(2\pi i x)$.

Proof. We have

$$\sum_{\gamma\in\mathcal{F}_{Q,P}} e(r\gamma) = \sum_{q\le Q} \sum_{\substack{1\le a\le q\\ (P(a),q)=1}} e\left(\frac{ar}{q}\right).$$

We apply Theorem 2.0.1 to obtain

$$\begin{aligned}\sum_{\gamma\in\mathcal{F}_{Q,P}} e(r\gamma) &= \sum_{q\le Q}\sum_{1\le a\le q} e\left(\frac{ar}{q}\right)\sum_{\substack{d|P(a)\\ d|q}}\mu(d)\\ &= \sum_{d\le Q}\mu(d)\sum_{\substack{q\le Q\\ d|q}}\sum_{\substack{1\le a\le q\\ d|P(a)}} e\left(\frac{ar}{q}\right) = \sum_{d\le Q}\mu(d)\sum_{q\le\frac{Q}{d}}\sum_{\substack{1\le a\le qd\\ P(a)\equiv 0 \pmod d}} e\left(\frac{ar}{qd}\right).\end{aligned} \tag{3.18}$$

We first consider the innermost sum in the above equation

$$\begin{aligned}&\sum_{\substack{1\le a\le qd\\ P(a)\equiv 0 \pmod d}} e\left(\frac{ar}{qd}\right)\\ &= \left(\sum_{\substack{1\le a\le d\\ P(a)\equiv 0 \pmod d}} + \sum_{\substack{d<a\le 2d\\ P(a)\equiv 0 \pmod d}} + \cdots + \sum_{\substack{(q-1)d<a\le qd\\ P(a)\equiv 0 \pmod d}}\right) e\left(\frac{ar}{qd}\right)\\ &= \sum_{j=0}^{q-1}\sum_{\substack{1\le a\le d\\ P(a)\equiv 0 \pmod d}} e\left(\frac{r(jd+a)}{qd}\right) = \sum_{j=0}^{q-1} e\left(\frac{rj}{q}\right)\sum_{\substack{1\le a\le d\\ P(a)\equiv 0 \pmod d}} e\left(\frac{ra}{qd}\right).\end{aligned} \tag{3.19}$$

In view of the identity

$$\sum_{n=1}^{m} e(nl/m) = \begin{cases} m, & \text{if } m|l,\\ 0, & \text{otherwise,}\end{cases} \tag{3.20}$$

the first sum on the right-hand side of (3.19) is q if $q|r$ and 0 otherwise. So (3.19) in conjunction with (3.18) gives

$$\begin{aligned}\sum_{\gamma\in\mathcal{F}_{Q,P}} e(r\gamma) &= \sum_{d\le Q}\mu(d)\sum_{\substack{q\le\frac{Q}{d}\\ q|r}} q \sum_{\substack{1\le a\le d\\ P(a)\equiv 0 \pmod d}} e\left(\frac{ra}{qd}\right)\\ &= \sum_{\substack{q\le Q\\ q|r}} q \sum_{d\le\frac{Q}{q}}\mu(d) \sum_{\substack{1\le a\le d\\ P(a)\equiv 0 \pmod d}} e\left(\frac{ra}{qd}\right).\end{aligned}$$

This gives the required result. □

For the specific polynomial $P(x)=x(x+1)$, we can expand the Weyl sum using Dirichlet characters.

Lemma 3.4.2. *For the polynomial $P(x)=x(x+1)$, and $r\in\mathbb{Z}$, we have*

$$\begin{aligned}\sum_{\gamma\in\mathcal{F}_{Q,P}} e(r\gamma) &= \sum_{d_1\le Q}\mu(d_1)\sum_{d_2\le Q}\frac{\mu(d_2)d_2}{\phi(d_2)\gcd(d_1,d_2)}\sum_{\substack{q\le\frac{Q\gcd(d_1,d_2)}{d_1d_2}\\ \frac{qd_2}{\gcd(d_1,d_2)}|r}} q\\ &\quad+\sum_{q\le Q}\sum_{d|q}\frac{\mu(d)}{\phi(d)}\sum_{\substack{\chi \pmod d\\ \chi\neq\chi_0}}\bar{\chi}(-1)\sum_{\substack{a\le q\\ (a,q)=1}}\chi(a)e\left(\frac{ar}{q}\right),\end{aligned}$$

where $e(x)=\exp(2\pi i x)$.

Proof. We have

$$\sum_{\gamma\in\mathcal{F}_{Q,P}} e(r\gamma) = \sum_{q\le Q}\sum_{\substack{a\le q\\ (a(a+1),q)=1}} e\left(\frac{ar}{q}\right).$$

To deal with the condition $(a+1,q)=1$, we apply Theorem 2.0.1 and obtain

$$\begin{aligned}\sum_{\gamma\in\mathcal{F}_{Q,P}} e(r\gamma) &= \sum_{q\le Q}\sum_{\substack{a\le q\\ (a(a+1),q)=1}} e\left(\frac{ar}{q}\right) = \sum_{q\le Q}\sum_{\substack{a\le q\\ (a,q)=1}} e\left(\frac{ar}{q}\right)\sum_{\substack{d_2|a+1\\ d_2|q}}\mu(d_2)\\ &= \sum_{q\le Q}\sum_{d_2|q}\mu(d_2)\sum_{\substack{a\le q\\ (a,q)=1\\ a\equiv -1 \pmod{d_2}}} e\left(\frac{ar}{q}\right)\end{aligned}$$

$$= \sum_{q\le Q}\sum_{d_2|q}\frac{\mu(d_2)}{\phi(d_2)}\sum_{\chi \pmod{d_2}}\bar{\chi}(-1)\sum_{\substack{a\le q\\(a,q)=1}}\chi(a)e\left(\frac{ar}{q}\right) = S(\chi_0)+S(\chi).$$

For $\chi=\chi_0$, we have

$$\begin{aligned}
S(\chi_0) &= \sum_{q\le Q}\sum_{d_2|q}\frac{\mu(d_2)}{\phi(d_2)}\sum_{\substack{a\le q\\(a,q)=1}}\chi_0(a)e\left(\frac{ar}{q}\right)\\
&= \sum_{q\le Q}\sum_{d_2|q}\frac{\mu(d_2)}{\phi(d_2)}\sum_{\substack{a\le q\\(a,q)=1}}e\left(\frac{ar}{q}\right) = \sum_{q\le Q}\sum_{d_2|q}\frac{\mu(d_2)}{\phi(d_2)}\sum_{a\le q}e\left(\frac{ar}{q}\right)\sum_{\substack{d_1|a\\d_1|q}}\mu(d_1)\\
&= \sum_{q\le Q}\sum_{d_2|q}\frac{\mu(d_2)}{\phi(d_2)}\sum_{d_1|q}\mu(d_1)\sum_{\substack{a\le q\\d_1|a}}e\left(\frac{ar}{q}\right)\\
&= \sum_{d_1\le Q}\mu(d_1)\sum_{q\le \frac{Q}{d_1}}\sum_{d_2|qd_1}\frac{\mu(d_2)}{\phi(d_2)}\sum_{a\le q}e\left(\frac{ar}{q}\right).
\end{aligned}$$

We use (3.20) to estimate the inner-most sum in the above identity. Therefore

$$\begin{aligned}
S(\chi_0) &= \sum_{d_1\le Q}\mu(d_1)\sum_{\substack{q\le\frac{Q}{d_1}\\q|r}}q\sum_{d_2|qd_1}\frac{\mu(d_2)}{\phi(d_2)} = \sum_{d_1\le Q}\mu(d_1)\sum_{d_2\le Q}\frac{\mu(d_2)}{\phi(d_2)}\sum_{\substack{q\le\frac{Q}{d_1}\\q|r,\ d_2|qd_1}}q\\
&= \sum_{d_1\le Q}\mu(d_1)\sum_{d_2\le Q}\frac{\mu(d_2)}{\phi(d_2)}\sum_{\substack{q\le\frac{Q}{d_1}\\q|r,\ \frac{d_2}{\gcd(d_1,d_2)}|q}}q\\
&= \sum_{d_1\le Q}\mu(d_1)\sum_{d_2\le Q}\frac{\mu(d_2)d_2}{\phi(d_2)\gcd(d_1,d_2)}\sum_{\substack{q\le\frac{Q\gcd(d_1,d_2)}{d_1d_2}\\\frac{qd_2}{\gcd(d_1,d_2)}|r}}q.
\end{aligned}$$

In the second last step, we used the fact that $n_1|n_2n_3$ if and only if $\frac{n_1}{\gcd(n_1,n_2)}|n_3$.

For $\chi\ne\chi_0$, we have

$$S(\chi) = \sum_{q\le Q}\sum_{d|q}\frac{\mu(d)}{\phi(d)}\sum_{\substack{\chi \pmod{d}\\\chi\ne\chi_0}}\bar{\chi}(-1)\sum_{\substack{a\le q\\(a,q)=1}}\chi(a)e\left(\frac{ar}{q}\right).$$

This completes the proof of Lemma 3.4.2. $\square$

We now turn to the proofs of Theorems 1.1.5 and 1.1.6.

3.4.2 Proof of Theorem 1.1.5

To prove Theorem 1.1.5, we need to estimate, for any positive real number Λ, the quantity

$$\mathcal{S}_{\mathcal{F}_{Q,P}}(\Lambda)=\frac{1}{\mathcal{N}_{Q,P}}\#\left\{(\gamma_1,\gamma_2)\in\mathcal{F}_{Q,P}^2:\gamma_1\neq\gamma_2,\gamma_1-\gamma_2\in\frac{1}{\mathcal{N}_{Q,P}}(0,\Lambda)+\mathbb{Z}\right\},\tag{3.21}$$

as $Q\to\infty$. Let H be any continuously differentiable function with Supp $H\subset(0,\Lambda)$. For $\nu=2$ and $\mathcal{F}=\mathcal{F}_{Q,P}$, we use smooth analogue of correlation measure defined in (2.15). Recall that

$$f(y)=\sum_{n\in\mathbb{Z}}H(\mathcal{N}_{Q,P}(y+n)),\ y\in\mathbb{R},$$

and

$$\begin{aligned}S_{Q,P}&=\sum_{\substack{\gamma_1,\gamma_2\in\mathcal{F}_{Q,P}\\ \gamma_1\neq\gamma_2}}f(\gamma_1-\gamma_2)\\ &=\sum_{\substack{\gamma_1,\gamma_2\in\mathcal{F}_{Q,P}\\ r\in\mathbb{Z}}}c_r e(r(\gamma_1-\gamma_2))=\sum_{r\in\mathbb{Z}}c_r\left|\sum_{\gamma\in\mathcal{F}_{Q,P}}e(r\gamma)\right|^2,\end{aligned}\tag{3.22}$$

where

$$f(y)=\sum_{r\in\mathbb{Z}}c_r e(ry)\tag{3.23}$$

is the Fourier series expansion of f with the Fourier coefficients

$$\begin{aligned}c_r&=\int_{[0,1)}f(x)e(-rx)dx=\sum_{m\in\mathbb{Z}}\int_{[0,1)}H(\mathcal{N}_{Q,P}(x+m))e(-nx)dx\\ &=\int_{\mathbb{R}}H(\mathcal{N}_{Q,P}v)e(-n\cdot v)dv=\frac{1}{\mathcal{N}_{Q,P}}\int_{\mathbb{R}}e\left(-\frac{rx}{\mathcal{N}_{Q,P}}\right)H(x)dx=\frac{1}{\mathcal{N}_{Q,P}}\widehat{H}\left(\frac{r}{\mathcal{N}_{Q,P}}\right),\end{aligned}$$

where $\widehat{H}$ is the Fourier transform of H. By approximating the characteristic function of $(0,\Lambda)$ with H, $\frac{1}{\mathcal{N}_{Q,P}}S_{Q,P}$ becomes $\mathcal{S}_{\mathcal{F}_{Q,P}}(\Lambda)$. We employ Lemma 3.4.1 in (3.22) to obtain

$$\begin{aligned} S_{Q,P} =& \sum_{r\in\mathbb{Z}} c_r \sum_{\substack{q_1\le Q\\ q_1|r}} q_1 \sum_{d_1\le\frac{Q}{q_1}} \mu(d_1) \sum_{\substack{1\le a_1\le d_1\\ P(a_1)\equiv 0 \pmod{d_1}}} e\left(\frac{ra_1}{q_1d_1}\right)\\ &\times \sum_{\substack{q_2\le Q\\ q_2|r}} q_2 \sum_{d_2\le\frac{Q}{q_2}} \mu(d_2) \sum_{\substack{1\le a_2\le d_2\\ P(a_2)\equiv 0 \pmod{d_2}}} e\left(-\frac{ra_2}{q_2d_2}\right)\\ =& \sum_{d_1,d_2\le Q} \mu(d_1)\mu(d_2) \sum_{\substack{q_1\le\frac{Q}{d_1}\\ q_2\le\frac{Q}{d_2}}} q_1q_2 \sum_{\substack{1\le a_1\le d_1\\ 1\le a_2\le d_2\\ P(a_1)\equiv 0 \pmod{d_1}\\ P(a_2)\equiv 0 \pmod{d_2}}}\\ &\times \sum_{n\in\mathbb{Z}} c_{n[q_1,q_2]} e\left(n[q_1,q_2]\left(\frac{a_1}{q_1d_1}-\frac{a_2}{q_2d_2}\right)\right), \end{aligned} \tag{3.24}$$

where $[q_1,q_2]$ is the least common multiple of q_1 and q_2. In order to estimate the inner-most sum, for each $y>0$, we consider the function

$$H_y(x) = \frac{1}{y}H\left(\frac{x\mathcal{N}_{Q,P}}{y}\right),\ x\in\mathbb{R}. \tag{3.25}$$

Then

$$\widehat{H}_y(z) = \frac{1}{\mathcal{N}_{Q,P}}\widehat{H}\left(\frac{yz}{\mathcal{N}_{Q,P}}\right). \tag{3.26}$$

Since $c_r = \frac{1}{\mathcal{N}_{Q,P}}\widehat{H}\left(\frac{r}{\mathcal{N}_{Q,P}}\right)$, using (3.26) the inner-most sum in (3.24) can be expressed as

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} c_{n[q_1,q_2]} e\left(n[q_1,q_2]\left(\frac{a_1}{q_1d_1}-\frac{a_2}{q_2d_2}\right)\right)\\ &\qquad= \sum_{n\in\mathbb{Z}} \frac{1}{\mathcal{N}_{Q,P}}\widehat{H}\left(\frac{n[q_1,q_2]}{\mathcal{N}_{Q,P}}\right) e\left(n[q_1,q_2]\left(\frac{a_1}{q_1d_1}-\frac{a_2}{q_2d_2}\right)\right)\\ &\qquad= \sum_{n\in\mathbb{Z}} \widehat{H}_{[q_1,q_2]}(n) e\left(n[q_1,q_2]\left(\frac{a_1}{q_1d_1}-\frac{a_2}{q_2d_2}\right)\right). \end{aligned} \tag{3.27}$$

Next, we apply Proposition 2.1.4 to the right-hand side of the above identity. We obtain

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \widehat{H}_{[q_1,q_2]}(n) e\left(n[q_1,q_2]\left(\frac{a_1}{q_1d_1}-\frac{a_2}{q_2d_2}\right)\right) \\
&= \sum_{n\in\mathbb{Z}} H_{[q_1,q_2]}\left(n+\frac{a_1[q_1,q_2]}{q_1d_1}-\frac{a_2[q_1,q_2]}{q_2d_2}\right) \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{[q_1,q_2]} H\left(\frac{\mathcal{N}_{Q,P}}{[q_1,q_2]}\left(n+\frac{a_1[q_1,q_2]}{q_1d_1}-\frac{a_2[q_1,q_2]}{q_2d_2}\right)\right). \qquad (3.28)
\end{aligned}$$

The above identity, in conjunction with (3.27) and (3.24), yields

$$\begin{aligned}
S_{Q,P} &= \sum_{d_1,d_2\le Q} \mu(d_1)\mu(d_2) \sum_{\substack{q_1\le \frac{Q}{d_1}\\ q_2\le \frac{Q}{d_2}}} q_1q_2 \sum_{\substack{1\le a_1\le d_1\\ 1\le a_2\le d_2\\ P(a_1)\equiv 0 \pmod{d_1}\\ P(a_2)\equiv 0 \pmod{d_2}}} \\
&\quad\times \sum_{n\in\mathbb{Z}} \frac{1}{[q_1,q_2]} H\left(\frac{\mathcal{N}_{Q,P}}{[q_1,q_2]}\left(n+\frac{a_1[q_1,q_2]}{q_1d_1}-\frac{a_2[q_1,q_2]}{q_2d_2}\right)\right) \\
&= \sum_{d_1,d_2\le Q} \mu(d_1)\mu(d_2) \sum_{\substack{1\le a_1\le d_1\\ 1\le a_2\le d_2\\ P(a_1)\equiv 0 \pmod{d_1}\\ P(a_2)\equiv 0 \pmod{d_2}}} \sum_{\substack{q_1\le \frac{Q}{d_1}\\ q_2\le \frac{Q}{d_2}}} \gcd(q_1,q_2) \\
&\quad\times \sum_{n\in\mathbb{Z}} H\left(\frac{\mathcal{N}_{Q,P}}{[q_1,q_2]}\left(n+\frac{a_1[q_1,q_2]}{q_1d_1}-\frac{a_2[q_1,q_2]}{q_2d_2}\right)\right).
\end{aligned}$$

Take $\gcd(q_1,q_2)=\delta$, so that $q_1=q_1'\delta$ and $q_2=q_2'\delta$ with $(q_1',q_2')=1$. Substituting this into above equation, we arrive at the following expression

$$\begin{aligned}
S_{Q,P} &= \sum_{d_1,d_2\le Q} \mu(d_1)\mu(d_2) \sum_{\substack{1\le a_1\le d_1\\ 1\le a_2\le d_2\\ P(a_1)\equiv 0 \pmod{d_1}\\ P(a_2)\equiv 0 \pmod{d_2}}} \sum_{\delta\le \frac{Q}{\max(d_1,d_2)}} \delta \sum_{\substack{q_1'\le \frac{Q}{\delta d_1}\\ q_2'\le \frac{Q}{\delta d_2}\\ (q_1',q_2')=1}} \\
&\quad\times \sum_{n\in\mathbb{Z}} H\left(\mathcal{N}_{Q,P}\left(\frac{n}{q_1'q_2'\delta}+\frac{a_1}{q_1'\delta d_1}-\frac{a_2}{q_2'\delta d_2}\right)\right)
\end{aligned}$$

$$= \sum_{d_1,d_2\le Q} \mu(d_1)\mu(d_2) \sum_{\substack{1\le a_1\le d_1\\ 1\le a_2\le d_2\\ P(a_1)\equiv 0 \pmod{d_1}\\ P(a_2)\equiv 0 \pmod{d_2}}} \sum_{\delta\le \frac{Q}{\max(d_1,d_2)}} \delta \sum_{\substack{q_1'\le \frac{Q}{\delta d_1},\ q_2'\le \frac{Q}{\delta d_2}\\ \frac{a_1}{q_1'd_1}=\frac{a_2}{q_2'd_2}\\ (q_1',q_2')=1}} \sum_{n\in\mathbb{Z}} H\left(\frac{n\mathcal{N}_{Q,P}}{q_1'q_2'\delta}\right)$$

$$\ll \sum_{d_1,d_2\le Q} \sum_{\substack{1\le a_1\le d_1\\ 1\le a_2\le d_2\\ P(a_1)\equiv 0 \pmod{d_1}\\ P(a_2)\equiv 0 \pmod{d_2}}} \sum_{\delta\le \frac{Q}{\max(d_1,d_2)}} \delta \sum_{\substack{q_1'\le \frac{Q}{\delta d_1},\ q_2'\le \frac{Q}{\delta d_2}\\ (q_1',q_2')=1}} \sum_{n\in\mathbb{Z}} H\left(\frac{n\mathcal{N}_{Q,P}}{q_1'q_2'\delta}\right). \tag{3.29}$$

In the second last step, we used the fact that Supp $H \subset (0,\Lambda)$, since for large Q, either $\mathcal{N}_{Q,P}\left(\frac{a_1}{q_1'\delta d_1}-\frac{a_2}{q_2'\delta d_2}\right) > \Lambda$ or $\left(\frac{a_1}{q_1'\delta d_1}-\frac{a_2}{q_2'\delta d_2}\right) \le 0$. Further, for a non-zero contribution from H, one must have $0 < \frac{n\mathcal{N}_{Q,P}}{q_1'q_2'\delta} < \Lambda$ which implies $\delta n d_1 d_2 < \frac{2\Lambda}{\beta_P} =: \mathscr{C}_\Lambda$, where $\beta_P = \prod_p \left(1-\frac{f_P(p)}{p^2}\right)$. By utilizing the aforementioned inequality and taking into account the observation that

$$H\left(\frac{n\mathcal{N}_{Q,P}}{q_1'q_2'\delta}\right) = H\left(\frac{n\beta_P Q^2}{2q_1'q_2'\delta}\right) + \mathrm{O}\left(\frac{n}{q_1'q_2'\delta}Q^{\frac{3}{2}+\epsilon}\right), \tag{3.30}$$

the sum in (3.29) can be expressed as

$$S_{Q,P} \ll \sum_{\substack{d_1,d_2,\delta,n\ge 1\\ n\delta d_1 d_2<\mathscr{C}_\Lambda}} \delta f_P(d_1) f_P(d_2) \sum_{\substack{q_1'\le \frac{Q}{\delta d_1},\ q_2'\le \frac{Q}{\delta d_2}\\ (q_1',q_2')=1}} H\left(\frac{n\beta_P Q^2}{2q_1'q_2'\delta}\right). \tag{3.31}$$

In order to estimate the inner sum in (3.31), we employ Proposition 2.6.2, which gives an asymptotic result for counting the number of lattice points with some weight within a bounded region. We obtain

$$\begin{aligned} \sum_{\substack{q_1'\le \frac{Q}{\delta d_1}, q_2'\le \frac{Q}{\delta d_2}\\ \gcd(q_1',q_2')=1}} H\left(\frac{n\beta_P Q^2}{2q_1'q_2'\delta}\right) &= \frac{6}{\pi^2}\int_0^{\frac{Q}{\delta d_2}}\int_0^{\frac{Q}{\delta d_1}} H\left(\frac{n\beta_P Q^2}{2xy\delta}\right) dxdy + \mathrm{O}\left(Q\log Q\right)\\ &= \frac{6Q^2}{\pi^2}\int_0^{\frac{1}{\delta d_2}}\int_0^{\frac{1}{\delta d_1}} H\left(\frac{n\beta_P}{2xy\delta}\right) dxdy + \mathrm{O}\left(Q\log Q\right). \end{aligned} \tag{3.32}$$

Further, we put $\lambda = \frac{n\beta_P}{2xy\delta}$, then the double integral transforms into the following expression

$$\begin{aligned}\int_0^{\frac{1}{\delta d_2}} \int_0^{\frac{1}{\delta d_1}} H\left(\frac{n\beta_P}{2xy\delta}\right) dx dy &= \frac{n\beta_P}{2\delta} \int_0^{\frac{1}{\delta d_1}} \int_{\frac{nd_2\beta_P}{2x}}^{\Lambda} \frac{H(\lambda)}{x\lambda^2} d\lambda dx \\ &= \frac{n\beta_P}{2\delta} \int_{\frac{\delta n d_1 d_2 \beta_P}{2}}^{\Lambda} \int_{\frac{nd_2\beta_P}{2\lambda}}^{\frac{1}{\delta d_1}} \frac{H(\lambda)}{x\lambda^2} dx d\lambda \\ &= \frac{n\beta_P}{2\delta} \int_{\frac{\delta n d_1 d_2 \beta_P}{2}}^{\Lambda} \frac{H(\lambda)}{\lambda^2} \log\left(\frac{2\lambda}{\delta n d_1 d_2 \beta_P}\right) d\lambda. \qquad (3.33)\end{aligned}$$

Thus, (3.32) and (3.33) in conjunction with (3.31) gives

$$\begin{aligned}S_{Q,P} &\ll Q^2\beta_P \sum_{\substack{d_1,d_2,\delta,n\geq 1\\ n\delta d_1 d_2 < \mathscr{C}_\Lambda}} n f_P(d_1) f_P(d_2) \int_{\frac{\delta n d_1 d_2 \beta_P}{2}}^{\Lambda} \frac{H(\lambda)}{\lambda^2} \log\left(\frac{2\lambda}{\delta n d_1 d_2 \beta_P}\right) d\lambda \\ &\ll Q^2\beta_P \sum_{1\leq m < \mathscr{C}_\Lambda} \int_{\frac{m\beta_P}{2}}^{\Lambda} \frac{H(\lambda)}{\lambda^2} \log\left(\frac{2\lambda}{m\beta_P}\right) d\lambda \sum_{\substack{n,\delta,d_1,d_2\geq 1\\ n\delta d_1 d_2 = m}} n f_P(d_1) f_P(d_2) \\ &\ll Q^2\beta_P \int_0^{\Lambda} \frac{H(\lambda)}{\lambda^2} \sum_{1\leq m < \frac{2\lambda}{\beta_P}} h(m) \log\left(\frac{2\lambda}{m\beta_P}\right) d\lambda, \qquad (3.34)\end{aligned}$$

where

$$h(m) := \sum_{\substack{n,\delta,d_1,d_2\geq 1\\ n\delta d_1 d_2 = m}} n f_P(d_1) f_P(d_2).$$

Using (3.10) and (3.34), we conclude that

$$\frac{S_{Q,P}}{\#\mathcal{F}_{Q,P}} \ll \int_0^{\Lambda} \frac{H(\lambda)}{\lambda^2} \sum_{1\leq m < \frac{2\lambda}{\beta_P}} h(m) \log\left(\frac{2\lambda}{m\beta_P}\right) d\lambda.$$

Next, we use the standard approximation argument to approximate the characteristic function of the interval $(0, \Lambda)$ from below and above by the smooth functions with compact support in $(0, \Lambda)$ and obtain

$$\mathcal{S}_{\mathcal{F}_{Q,P}}(\Lambda) \ll \int_0^{\Lambda} \frac{1}{\lambda^2} \sum_{1\leq m < \frac{2\lambda}{\beta_P}} h(m) \log\left(\frac{2\lambda}{m\beta_P}\right) d\lambda.$$

Therefore,

$$\limsup_{Q\to\infty} \mathcal{S}_{\mathcal{F}_{Q,P}}(\Lambda) \ll \int_0^\Lambda \frac{1}{\lambda^2} \sum_{1\le m<\frac{2\lambda}{\beta_P}} h(m)\log\left(\frac{2\lambda}{m\beta_P}\right) d\lambda.$$

Note that $h(m) < \infty$ for every m. Since $1 \le m < \frac{2\lambda}{\beta_P}$, it follows that for every $\Lambda > 0$, the sum on the right hand side in the above inequality has only finitely many terms and so is finite. Therefore, $\limsup_{Q\to\infty} \mathcal{S}_{\mathcal{F}_{Q,P}}(\Lambda)$ is finite. This completes the proof of Theorem 1.1.5.

3.4.3 Proof of Theorem 1.1.6

To prove Theorem 1.1.6, we need to estimate, for any positive real number Λ and polynomial $P(x) = x(x+1)$, the quantity

$$\mathcal{S}_{\mathcal{F}_{Q,P}}(\Lambda) = \frac{1}{\mathcal{N}_{Q,P}}\#\left\{(\gamma_1,\gamma_2)\in\mathcal{F}_{Q,P}^2 : \gamma_1\neq\gamma_2, \gamma_1-\gamma_2\in\frac{1}{\mathcal{N}_{Q,P}}(0,\Lambda)+\mathbb{Z}\right\},$$

as $Q\to\infty$. As in Theorem 1.1.5, we change the problem of counting the tuples in (3.21) into estimating the exponential sum over polynomial Farey fractions. We have

$$S_{Q,P} = \sum_{r\in\mathbb{Z}} c_r \left|\sum_{\gamma\in\mathcal{F}_{Q,P}} e(r\gamma)\right|^2, \tag{3.35}$$

where $c_r = \frac{1}{\mathcal{N}_{Q,P}}\widehat{H}\left(\frac{r}{\mathcal{N}_{Q,P}}\right)$ is Fourier coefficient of the Fourier series

$$f(y) = \sum_{r\in\mathbb{Z}} c_r e(ry)$$

and $\widehat{H}$ is the Fourier transform of H. We employ Lemma 3.4.2 in the above identity to obtain

$$
\begin{aligned}
S_{Q,P} = &\sum_{r\in\mathbb{Z}} c_r \left(\sum_{d_1\le Q} \mu(d_1) \sum_{d_2\le Q} \frac{\mu(d_2)d_2}{d\phi(d_2)} \sum_{\substack{q_1\le \frac{dQ}{d_1 d_2} \\ \frac{q_1 d_2}{d} \mid r}} q_1 + \sum_{\substack{d_1'\mid q_1' \\ q_1'\le Q}} \frac{\mu(d_1')}{\phi(d_1')} \sum_{\substack{\chi \pmod{d_1'} \\ \chi\ne\chi_0}} \bar{\chi}(-1) \right. \\
&\left. \times \sum_{\substack{a_1\le q_1' \\ (a_1,q_1')=1}} \chi(a_1) e\left(\frac{a_1 r}{q_1'}\right) \right) \left(\sum_{d_3\le Q} \mu(d_3) \sum_{d_4\le Q} \frac{\mu(d_4)d_4}{D\phi(d_4)} \sum_{\substack{q_2\le \frac{QD}{d_3 d_4} \\ \frac{q_2 d_4}{D} \mid r}} q_2 \right. \\
&\left. + \sum_{\substack{d_2'\mid q_2' \\ q_2'\le Q}} \frac{\mu(d_2')}{\phi(d_2')} \sum_{\substack{\chi \pmod{d_2'} \\ \chi\ne\chi_0}} \bar{\chi}(-1) \sum_{\substack{a_2\le q_2' \\ (a_2,q_2')=1}} \chi(a_2) e\left(\frac{-a_2 r}{q_2'}\right) \right),
\end{aligned}
$$

where $d=\gcd(d_1,d_2)$ and $D=\gcd(d_3,d_4)$.

$$
\begin{aligned}
S_{Q,P} = &\sum_{d_1,d_3\le Q} \mu(d_1)\mu(d_3) \sum_{d_2,d_4\le Q} \frac{\mu(d_2)\mu(d_4)d_2 d_4}{\phi(d_2)\phi(d_4)dD} \sum_{\substack{q_1\le \frac{Qd}{d_1 d_2} \\ q_2\le \frac{QD}{d_3 d_4}}} q_1 q_2 \sum_{n\in\mathbb{Z}} c_{\left[\frac{q_1 d_2}{d}, \frac{q_2 d_4}{D}\right] n} \\
&+ \sum_{d_1\le Q} \mu(d_1) \sum_{d_2\le Q} \frac{\mu(d_2)d_2}{d\phi(d_2)} \sum_{q_1\le \frac{Qd}{d_1 d_2}} q_1 \sum_{q_2'\le Q} \sum_{d_2'\mid q_2'} \frac{\mu(d_2')}{\phi(d_2')} \sum_{\substack{\chi \pmod{d_2'} \\ \chi\ne\chi_0}} \bar{\chi}(-1) \\
&\times \sum_{\substack{a_2\le q_2' \\ (a_2,q_2')=1}} \chi(a_2) \sum_{n\in\mathbb{Z}} c_{\frac{q_1 d_2 n}{d}} e\left(-\frac{a_2 q_1 d_2 n}{q_2' d}\right) + \sum_{d_3\le Q} \mu(d_3) \sum_{d_4\le Q} \frac{\mu(d_4)d_4}{\phi(d_4)D} \\
&\times \sum_{q_2\le \frac{QD}{d_3 d_4}} q_2 \sum_{q_1'\le Q} \sum_{d_1'\mid q_1'} \frac{\mu(d_1')}{\phi(d_1')} \sum_{\substack{\chi \pmod{d_1'} \\ \chi\ne\chi_0}} \bar{\chi}(-1) \sum_{\substack{a_1\le q_1' \\ (a_1,q_1')=1}} \chi(a_1) \\
&\times \sum_{n\in\mathbb{Z}} c_{\frac{q_2 d_4 n}{D}} e\left(\frac{a_1 q_2 d_4 n}{q_1' D}\right) + \sum_{q_1'\le Q} \sum_{d_1'\mid q_1'} \frac{\mu(d_1')}{\phi(d_1')} \sum_{\substack{\chi \pmod{d_1'} \\ \chi\ne\chi_0}} \bar{\chi}(-1) \sum_{\substack{a_1\le q_1' \\ \gcd(a_1,q_1')=1}} \chi(a_1)
\end{aligned}
$$

$$\times \sum_{q_2' \le Q} \sum_{d_2' | q_2'} \frac{\mu(d_2')}{\phi(d_2')} \sum_{\substack{\chi \pmod{d_2'} \\ \chi \ne \chi_0}} \bar{\chi}(-1) \sum_{\substack{a_2 \le q_2' \\ (a_2, q_2')=1}} \chi(a_2) \sum_{r \in \mathbb{Z}} c_r e\left(\frac{a_1 r}{q_1'} - \frac{a_2 r}{q_2'}\right), \tag{3.36}$$

where $[a,b]$ is the least common multiple of a and b. We estimate the sum of Fourier coefficients in the first term on the right-hand side of the above identity using (3.25), (3.26), and Proposition 2.1.3. Therefore, we have

$$\sum_{n \in \mathbb{Z}} c_{\left[\frac{q_1 d_2}{d}, \frac{q_2 d_4}{D}\right] n} = \sum_{n \in \mathbb{Z}} \frac{1}{\left[\frac{q_1 d_2}{d}, \frac{q_2 d_4}{D}\right]} H\left(\frac{n \mathcal{N}_{Q,P}}{\left[\frac{q_1 d_2}{d}, \frac{q_2 d_4}{D}\right]}\right). \tag{3.37}$$

In order to estimate the other inner-most sums containing the Fourier coefficients, we use (3.25), (3.26) and Proposition 2.1.4 to obtain

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} c_{\frac{q_1 d_2 n}{d}} e\left(\frac{-a_2 q_1 d_2 n}{q_2' d}\right) &= \sum_{n \in \mathbb{Z}} \frac{1}{\mathcal{N}_{Q,P}} \widehat{H}\left(\frac{q_1 d_2 n}{d \mathcal{N}_{Q,P}}\right) e\left(-\frac{a_2 q_1 d_2 n}{q_2' d}\right) \\
&= \sum_{n \in \mathbb{Z}} \widehat{H}_{\frac{q_1 d_2}{d}}(n)\, e\left(\frac{-a_2 q_1 d_2 n}{q_2' d}\right) = \sum_{n \in \mathbb{Z}} H_{\frac{q_1 d_2}{d}}\left(n - \frac{a_2 q_1 d_2}{q_2' d}\right) \\
&= \sum_{n \in \mathbb{Z}} \frac{d}{q_1 d_2} H\left(\frac{d \mathcal{N}_{Q,P}}{q_1 d_2}\left(n - \frac{a_2 q_1 d_2}{q_2' d}\right)\right).
\end{aligned} \tag{3.38}$$

Similarly, we have

$$\sum_{n \in \mathbb{Z}} c_{\frac{q_2 d_4 n}{D}} e\left(\frac{a_1 q_2 d_4 n}{q_1' D}\right) = \sum_{n \in \mathbb{Z}} \frac{D}{q_2 d_4} H\left(\frac{D \mathcal{N}_{Q,P}}{q_2 d_4}\left(n + \frac{a_1 q_2 d_4}{q_1' D}\right)\right) \tag{3.39}$$

and

$$\sum_{r \in \mathbb{Z}} c_r e\left(\frac{a_1 r}{q_1'} - \frac{a_2 r}{q_2'}\right) = \sum_{r \in \mathbb{Z}} H\left(\mathcal{N}_{Q,P}\left(r + \frac{a_1}{q_1'} - \frac{a_2}{q_2'}\right)\right). \tag{3.40}$$

Note that $q_1 \le \frac{dQ}{d_1 d_2}$, $q_2 \le \frac{DQ}{d_3 d_4}$ and $\mathcal{N}_{Q,P} \sim cQ^2$. Given that Supp $H \subset (0, \Lambda)$ such that $Q > \Lambda$, for sufficiently large Q, we have $H\left(\frac{d\mathcal{N}_{Q,P}}{q_1 d_2}\left(n - \frac{a_2 q_1 d_2}{d q_2'}\right)\right) = 0$, $H\left(\frac{D\mathcal{N}_{Q,P}}{q_2 d_4}\left(n + \frac{a_1 q_2 d_4}{D q_1'}\right)\right) = 0$, and $H\left(\mathcal{N}_{Q,P}\left(r + \frac{a_1}{q_1'} - \frac{a_2}{q_2'}\right)\right) = 0$. This leads to the vanishing of the sums in (3.38), (3.39), and (3.40) reducing (3.36) to the following identity:

$$
\begin{aligned}
&S_{Q,P}\\
&= \sum_{d_1,d_3\le Q}\mu(d_1)\mu(d_3)\sum_{d_2,d_4\le Q}\frac{\mu(d_2)\mu(d_4)d_2d_4}{\phi(d_2)\phi(d_4)dD}\sum_{\substack{q_1\le\frac{Qd}{d_1d_2}\\ q_2\le\frac{DQ}{d_3d_4}}}\frac{q_1q_2}{\left[\frac{q_1d_2}{d},\frac{q_2d_4}{D}\right]}\sum_{n\in\mathbb{Z}}H\left(\frac{n\mathcal{N}_{Q,P}}{\left[\frac{q_1d_2}{d},\frac{q_2d_4}{D}\right]}\right)\\
&= \sum_{d_1,d_2,d_3,d_4\le Q}\frac{\mu(d_1)\mu(d_2)\mu(d_3)\mu(d_4)}{\phi(d_2)\phi(d_4)}\sum_{\substack{q_1\le\frac{dQ}{d_1d_2}\\ q_2\le\frac{DQ}{d_3d_4}}}\gcd\left(\frac{q_1d_2}{d},\frac{q_2d_4}{D}\right)\sum_{n\in\mathbb{Z}}H\left(\frac{n\mathcal{N}_{Q,P}}{\left[\frac{q_1d_2}{d},\frac{q_2d_4}{D}\right]}\right).
\end{aligned}
$$

Denote $\gcd\left(\frac{q_1d_2}{d},\frac{q_2d_4}{D}\right)=\delta$. Note that $\gcd\left(\frac{q_1d_2}{d},\frac{q_2d_4}{D}\right)=\delta$ if and only if $\left(\frac{q_1d_2}{d\delta},\frac{q_2d_4}{D\delta}\right)=1$. Furthermore, $\delta|\frac{q_1d_2}{d}$ if and only if $\frac{\delta}{G_1}|q_1$ and $\delta|\frac{q_2d_4}{D}$ if and only if $\frac{\delta}{G_2}|q_2$, where $G_1:=\gcd\left(\delta,\frac{d_2}{d}\right)$ and $G_2:=\gcd\left(\delta,\frac{d_4}{D}\right)$. That is, we have $q_1=\frac{q_1'\delta}{G_1}$ and $q_2=\frac{q_2'\delta}{G_2}$ for some $q_1',q_2'\in\mathbb{Z}$. With this reduction, we have

$$
S_{Q,P}=\sum_{\substack{d_1,d_2,d_3,d_4\le Q\\ \delta\le\frac{Q}{\max(d_1,d_3)}}}\frac{\delta\mu(d_1)\mu(d_2)\mu(d_3)\mu(d_4)}{\phi(d_2)\phi(d_4)}\sum_{\substack{n\in\mathbb{Z}\\ (q_1',q_2')\in\mathfrak{S}}}H\left(\frac{ndDG_1G_2\mathcal{N}_{Q,P}}{q_1'q_2'\delta d_2d_4}\right), \tag{3.41}
$$

where $\mathfrak{S}:=\left\{q_1'\le\frac{QdG_1}{\delta d_1d_2},q_2'\le\frac{DQG_2}{\delta d_3d_4}:\left(\frac{q_1'd_2}{dG_1},\frac{q_2'd_4}{DG_2}\right)=1\right\}$. We use the fact that Supp $H\subset(0,\Lambda)$ and (3.10). For a non-zero contribution from H, one must have $0<\frac{ndDG_1G_2\mathcal{N}_{Q,P}}{q_1'q_2'\delta d_2d_4}<\Lambda$ which implies $\delta nd_1d_3<\frac{2\Lambda}{\beta_P}=:\mathscr{C}_\Lambda$, where $\beta_P=\prod_p\left(1-\frac{f_P(p)}{p^2}\right)$. By utilizing the aforementioned inequality and taking into account Corollary 3.2.1, the sum in (3.41) becomes

$$
\begin{aligned}
S_{Q,P}=&\sum_{\delta nd_1d_3<\frac{2\Lambda}{\beta_P}}\delta\mu(d_1)\mu(d_3)\sum_{\substack{d_2,d_4\le Q\\ \left(\frac{d_2}{dG_1},\frac{d_4}{DG_2}\right)=1}}\frac{\mu(d_2)\mu(d_4)}{\phi(d_2)\phi(d_4)}\\
&\times\sum_{(q_1',q_2')\in\mathfrak{S}_1}H\left(\frac{ndDG_1G_2\beta_PQ^2}{2q_1'q_2'\delta d_2d_4}\right)+\mathrm{O}\left(Q^{\frac{3}{2}+\epsilon}\right),
\end{aligned}\tag{3.42}
$$

where $\mathfrak{S}_1:=\left\{q_1'\le\frac{dQG_1}{\delta d_1d_2},q_2'\le\frac{DQG_2}{\delta d_3d_4}\ :\ \left(q_1',\frac{d_4}{DG_2}\right)=(q_1',q_2')=\left(q_2',\frac{d_2}{dG_1}\right)=1\right\}$. We next employ Lemma 2.6.1 to estimate the innermost sum on the right-hand side of

(3.42). Denoting this sum by $S^{(1)}$, we have

$$\begin{aligned} S^{(1)} =& \frac{1}{\zeta(2)} \prod_{p\mid\frac{d_2d_4}{dDG_1G_2}} \left(1-\frac{1}{p^2}\right)^{-1} \prod_{p\mid\frac{d_2}{dG_1}} \left(1-\frac{1}{p}\right) \prod_{p\mid\frac{d_4}{DG_2}} \left(1-\frac{1}{p}\right) \\ &\times \int_0^{\frac{DQG_2}{\delta d_3 d_4}} \int_0^{\frac{dQG_1}{\delta d_1 d_2}} H\left(\frac{ndDG_1G_2\beta_P Q^2}{2xy\delta d_2 d_4}\right) dxdy + \mathrm{O}\left((\tau(d_2)+\tau(d_4))Q\log^2 Q\right). \end{aligned} \tag{3.43}$$

We denote the double integral in the above estimate by $\mathcal{I}$. By suitable change of variables, we obtain

$$\mathcal{I} = \frac{Q^2}{d_2 d_4} dDG_1G_2 \int_0^{\frac{1}{\delta d_3}} \int_0^{\frac{1}{\delta d_1}} H\left(\frac{n\beta_P}{2\delta xy}\right) dxdy.$$

We further put $\frac{n\beta_P}{2\delta xy} = \lambda$ to obtain

$$\begin{aligned} \mathcal{I} &= \frac{Q^2 n\beta_P}{2\delta d_2 d_4} dDG_1G_2 \int_{\frac{n\delta d_1 d_3 \beta_P}{2}}^{\Lambda} \int_{\frac{nd_3\beta_P}{2\lambda}}^{\frac{1}{\delta d_1}} \frac{H(\lambda)}{\lambda^2 x} dxd\lambda \\ &= \frac{Q^2 n\beta_P}{2\delta d_2 d_4} dDG_1G_2 \int_{\frac{n\delta d_1 d_3 \beta_P}{2}}^{\Lambda} \frac{H(\lambda)}{\lambda^2} \log\left(\frac{2\lambda}{n\delta d_1 d_3 \beta_P}\right) d\lambda. \end{aligned}$$

The above estimate in conjunction with (3.42) and (3.43) gives

$$\begin{aligned} S_{Q,P} =& \frac{Q^2\beta_P}{2\zeta(2)} \sum_{d_2,d_4\le Q} \frac{\mu(d_2)\mu(d_4)}{d_2 d_4 \phi(d_2)\phi(d_4)} \sum_{\substack{\delta,n,d_1,d_3\ge 1 \\ \delta n d_1 d_3 < \frac{2\Lambda}{\beta_P} \\ \left(\frac{d_2}{dG_1},\frac{d_4}{DG_2}\right)=1}} n\mu(d_1)\mu(d_3) dDG_1G_2 \\ &\times \prod_{p\mid\frac{d_2}{dG_1}} \left(1-\frac{1}{p}\right) \prod_{p\mid\frac{d_4}{DG_2}} \left(1-\frac{1}{p}\right) \prod_{p\mid\frac{d_2d_4}{dDG_1G_2}} \left(1-\frac{1}{p^2}\right)^{-1} \\ &\times \int_{\frac{n\delta d_1 d_3 \beta_P}{2}}^{\Lambda} \frac{H(\lambda)}{\lambda^2} \log\left(\frac{2\lambda}{n\delta d_1 d_3 \beta_P}\right) d\lambda + \mathrm{O}\left(Q^{\frac{3}{2}+\epsilon}\right) \\ =& \frac{Q^2\beta_P}{2\zeta(2)} \sum_{d_2,d_4=1}^{\infty} \frac{\mu(d_2)\mu(d_4)}{d_2 d_4 \phi(d_2)\phi(d_4)} \int_0^{\Lambda} \frac{H(\lambda)}{\lambda^2} \sum_{m<\frac{2\lambda}{\beta_P}} \mathfrak{h}(m) \log\left(\frac{2\lambda}{m\beta_P}\right) d\lambda \\ &+ \mathrm{O}\left(Q^{\frac{3}{2}+\epsilon}\right) \end{aligned}$$

$$=\frac{Q^2\beta_P}{2}\int_0^{\Lambda}\mathfrak{g}_2(\lambda)H(\lambda)d\lambda+\mathrm{O}\left(Q^{\frac{3}{2}+\epsilon}\right),$$

where $\mathfrak{h}(m)$ and $\mathfrak{g}_2(\lambda)$ are as defined in the statement of Theorem 1.1.6. The above estimate with (3.10) yields

$$\frac{S_{Q,P}}{\mathcal{N}_{Q,P}}=\int_0^{\Lambda}H(\lambda)\mathfrak{g}_2(\lambda)d\lambda+\mathrm{O}\left(Q^{-\frac{1}{2}+\epsilon}\right).$$

Using an appropriate approximation argument to approximate the characteristic function of the interval $(0,\Lambda)$ from below and from above by the smooth functions with compact support in $(0,\Lambda)$, we have

$$\lim_{Q\to\infty}\mathcal{S}_{\mathcal{F}_{Q,P}}(\Lambda)=\int_0^{\Lambda}\mathfrak{g}_2(\lambda)d\lambda.$$

This completes the proof of Theorem 1.1.6.

3.5 Polynomial Farey fractions with prime denominators

Another question of significant interest is determining the values $\tau>0$ for which there are infinitely many primes that satisfy the Diophantine inequality

$$||\alpha p||<p^{-\tau+\epsilon} \tag{3.44}$$

for all $\epsilon>0$, where $||t||$ denotes the distance from the nearest integer to a real number t. Vinogradov [96] was the first to determine that $\tau=1/5$ is admissible and subsequently, his result was improved by various authors [50, 51, 53, 60–63, 94]. The inequality in (3.44) is equivalent to the existence of a fraction with prime denominator a/p satisfying $|\alpha-\frac{a}{p}|<p^{-1-\tau+\epsilon}$. All the above results on τ are equivalent to quantitative statements on the gap between the Farey fractions with prime denominators. Motivated by this connection, we study the pair correlation statistics of polynomial Farey sequence with prime denominators. For each integer

Q, let $\mathcal{B}_Q$ be a fixed subset of prime numbers that are less than or equal to Q. The polynomial Farey sequence of order Q with prime denominators is given by

$$\mathscr{M}_{\mathcal{B}_Q,P} := \left\{ \frac{a}{p} : 1 \leq a \leq p \leq Q,\ (P(a),p) = 1,\ p \in \mathcal{B}_Q \right\}. \tag{3.45}$$

A similar set as $\mathcal{B}_Q$ was considered by Xiao [99], where author studied the pair correlation of the usual Farey fractions with denominators in that set.

3.5.1 Counting polynomial Farey fractions with prime denominators

We prove an asymptotic result to count the number of polynomial Farey fractions of order Q with denominators in $\mathcal{B}_Q$.

Proposition 3.5.1. *Let $\kappa \geq 1$ be an integer and $P(x) = c_\kappa x^\kappa + c_{\kappa-1}x^{\kappa-1} + \cdots + c_1 x \in \mathbb{Z}[x]$ be a polynomial with non-zero discriminant. If $\mathscr{M}_{\mathcal{B}_Q,P}$ is as in* (3.45)*, then*

$$\mathscr{N}_{\mathcal{B}_Q,P} = \#\mathscr{M}_{\mathcal{B}_Q,P} = \sum_{p \in \mathcal{B}_Q} p + \mathrm{O}\left(\#\mathcal{B}_Q\right).$$

Proof. We begin by expressing the number of fractions in $\mathscr{M}_{\mathcal{B}_Q,P}$ as a summation and then use the Möbius sum for the coprimality condition:

$$\begin{aligned}\mathscr{N}_{\mathcal{B}_Q,P} &= \sum_{p \in \mathcal{B}_Q} \sum_{\substack{a \leq p \\ (P(a),p)=1}} 1 = \sum_{p \in \mathcal{B}_Q} \sum_{a \leq p} \sum_{\substack{d \mid P(a) \\ d \mid p}} \mu(d) = \sum_{p \in \mathcal{B}_Q} \sum_{a \leq p} 1 - \sum_{p \in \mathcal{B}_Q} \sum_{\substack{a \leq p \\ P(a) \equiv 0 \pmod{p}}} 1 \\ &= \sum_{p \in \mathcal{B}_Q} p - \sum_{p \in \mathcal{B}_Q} f_P(p) = \sum_{p \in \mathcal{B}_Q} p + \mathrm{O}\left(\#\mathcal{B}_Q\right).\end{aligned}$$

In the last step, we applied Lagrange's theorem [49, Chapter 7] which implies that $P(x)$ has at most $\deg(P)$ roots modulo p, thus $f_P(p) \leq \deg(P)$. $\square$

3.5.2 Main results

Let us recall the main result of polynomial Farey sequence with prime denominators, namely Theorem 1.1.7.

Theorem 3.5.1. *The limiting pair correlation of the sequence* $(\mathscr{M}_{\mathcal{B}_Q,P})_{Q\geq 1}$ *exists as* $Q \to \infty$ *and is Poissonian if and only if* $\sum_{p\in\mathcal{B}_Q} p^2 = o\left((\#\mathscr{M}_{\mathcal{B}_Q,P})^2\right)$.

If $P(x) = x$ and for each Q, $\mathcal{B}_Q$ is the set of prime numbers that are less than or equal to Q, and $\mathscr{M}_Q$ is the set of Farey fractions with prime denominators not exceeding Q, then, by the prime number theorem and a partial summation formula using Theorem 3.5.1, we immediately deduce the following result of Xiong et al. [100] on the pair correlation of Farey fractions with prime denominators.

Corollary 3.5.1 ([100], Theorem 1)**.** *The limiting pair correlation of the sequence* $(\mathscr{M}_Q)_{Q\geq 1}$ *exists as* $Q \to \infty$ *and is constant equal to* 1.

Using Theorem 3.5.1, we next investigate the distribution of the fractions in $\mathscr{M}_{\mathcal{B}_Q,P}$ for different choices of the set $\mathcal{B}_Q$.

Piatetski-Shapiro primes

Definition 3.5.1 (Piatetski-Shapiro primes)**.** *Let* $1 < c < 2$ *be a fixed real number. A prime* p *is said to be Piatetski-Shapiro prime if it is of the form* $\lfloor n^c \rfloor$.

In order to count the Piatetski-Shapiro primes, we have the following asymptotic result.

Theorem 3.5.2 ([86])**.** *Let* $\pi_c(x)$ *denote the number of Piatetski-Shapiro primes* $p \leq x$. *Then for fixed* $c \in (1, 1.16)$, *we have*

$$\pi_c(x) \sim \frac{x^{1/c}}{\log x} \text{ as } x \to \infty.$$

For more details on the Piatetski-Shapiro primes, one may refer to the well-known article of [85]. Employing Theorems 3.5.1 and 3.5.2, we derive the following corollary.

Corollary 3.5.2. *Let* $c \in (1, 1.16)$ *be a fixed real number. If* $\mathcal{B}_Q^{(1)}$ *is the set of Piatetski-Shapiro primes not exceeding* Q *then the limiting pair correlation of the sequence* $\left(\mathscr{M}_{\mathcal{B}_Q^{(1)},P}\right)_{Q\geq 1}$ *exists as* $Q \to \infty$ *and it is Poissonian.*

Chen primes

Definition 3.5.2 (Chen primes). *A prime* p *is said to be a Chen prime if* $p+2$ *is a product of at most two primes.*

Theorem 3.5.3 ([28], Theorem II). *There exist infinitely many primes* p *such that* $p+h$ *is a product of at most* 2 *primes,* h *being any even integer, and*

$$\#\{p \leq x \mid p+h \textit{ is a product of at most two primes}\} \geq \frac{0.67xC_h}{(\log x)^2},$$

where C_h *is a constant depending on* h.

Corollary 3.5.3. *If* $\mathcal{B}_Q^{(2)}$ *is the set of Chen primes less than or equal to* Q *then the limiting pair correlation of the sequences* $\left(\mathscr{M}_{\mathcal{B}_Q^{(2)},P}\right)_{Q\geq 1}$ *exists as* $Q \to \infty$ *and it is Poissonian.*

Prime k-tuple

Let $a_1, \ldots, a_k$ be distinct positive even integers. A prime p is said to be a prime k-tuple if $p+a_1, \ldots, p+a_k$ are all prime. In 1922, Hardy and Littlewood [48] formulated the prime k-tuple conjecture, which is the following:

Conjecture 3.5.1 (Hardy-Littlewood prime k-tuple conjecture). *Let* $a_1, \ldots, a_k$ *be distinct integers, and let* $b(p)$ *be the number of distinct residue classes* $\pmod p$ *represented by* a_i, $1 \leq i \leq k$. *If* $b(p) < p$ *for every prime* p, *the prime* k*-tuple conjecture asserts that the number of* $n \leq x$ *such that all* k *numbers* $n + a_i$ *are prime for* $1 \leq i \leq k$ *is asymptotic to*

$$\mathfrak{S}(a_1, \ldots, a_k)\frac{x}{(\log x)^k},$$

where

$$\mathfrak{S}(a_1, \ldots, a_k) = \prod_p \left(1 - \frac{b(p)}{p}\right)\left(1 - \frac{1}{k}\right)^{-k}$$

Assuming the Hardy-Littlewood conjecture [48], there are infinitely many prime k-tuples. Using Theorem 3.5.1 and assuming the Hardy-Littlewood conjecture, we obtain the following corollary.

Corollary 3.5.4. *If $\mathcal{B}_Q^{(3)}$ is the set of prime k-tuples less than or equal to Q, then under the Hardy-Littlewood prime k-tuple conjecture, the limiting pair correlation of the sequence $\left(\mathscr{M}_{\mathcal{B}_Q^{(3)},P}\right)_{Q\geq 1}$ exists as $Q \to \infty$ and is Poissonian.*

Primes with restricted digits

Let q be a positive integer. For any choice of $\mathcal{D} \subseteq \{0,1,\ldots,q-1\}$, let

$$\boldsymbol{A} = \left\{ \sum_{0\leq i\leq k} n_i q^i \ : \ n_i \in \{0,1,\ldots,q-1\} \setminus \mathcal{D}, \ k \geq 0 \right\}$$

be a set of integers with no digit in base q in the set $\mathcal{D}$. A prime $p \in \boldsymbol{A}$ is said to be a prime with restricted digits. It is natural to ask whether the above set $\boldsymbol{A}$ contains infinitely many primes. In his well-known article [78], Maynard showed that there are infinitely many primes in the set $\boldsymbol{A}$. In particular, he proved the following result:

Theorem 3.5.4 ([78], Theorem 1.2)**.** *Let q be sufficiently large, and let $X \geq q$. For any choice $B \subset \{0,\ldots,q-1\}$ with $\#B = s \leq q^{23/80}$, let*

$$A = \left\{ \sum_{0\leq i\leq k} n_i q^i \ : \ n_i \in \{0,1,\ldots,q-1\} \setminus B, \ k \geq 0 \right\}$$

be the set of integers less than X with no digit in base q in the set B. Then we have

$$\#\{p \leq X \mid p \in A\} \asymp \frac{X^{\log(q-s)/\log q}}{\log X}.$$

Corollary 3.5.5. *Let q be a sufficiently large positive integer. For any choice of $\mathcal{D} \subseteq \{0,1,\ldots,q-1\}$ with $|\mathcal{D}| \leq q^{23/80}$, let $\mathcal{B}_Q^{(4)} \subset \mathbf{A}$ be the set of primes that are less than or equal to Q with restricted digits. Then the limiting pair correlation of the sequence $\left(\mathscr{M}_{\mathcal{B}_Q^{(4)},P}\right)_{Q\geq 1}$ exists as $Q \to \infty$ and is Poissonian.*

3.5.3 Proof of Theorem 3.5.1

In order to prove Theorem 3.5.1, we intent to estimate, for any positive real number Λ, subset of primes $\mathcal{B}_Q$, and polynomial $P(x) = c_\kappa x^\kappa + c_{\kappa-1}x^{\kappa-1} + \cdots + c_1 x$, the quantity

$$\mathcal{S}_{\mathscr{M}_{\mathcal{B}_Q,P}}(\Lambda) = \frac{1}{\mathscr{N}_{\mathcal{B}_Q,P}} \# \left\{ (\gamma_1, \gamma_2) \in \mathscr{M}^2_{\mathcal{B}_Q,P} : \gamma_1 \neq \gamma_2, \gamma_1 - \gamma_2 \in \frac{1}{\mathscr{N}_{\mathcal{B}_Q,P}}(0,\Lambda) + \mathbb{Z} \right\}, \tag{3.46}$$

as $Q \to \infty$. Let H be any continuously differentiable function with Supp $H \subset (0,\Lambda)$. Define $f(y) = \sum_{n\in\mathbb{Z}} H(\mathscr{N}_{\mathcal{B}_Q,P}(y+n))$, $y \in \mathbb{R}$, and $\mathcal{S} = \sum_{\substack{\gamma_1,\gamma_2 \in \mathscr{M}_{\mathcal{B}_Q,P} \\ \gamma_1 \neq \gamma_2}} f(\gamma_1 - \gamma_2)$. Thus, for $\nu = 2$ and $\mathcal{F} = \mathscr{M}_{\mathcal{B}_Q,P}$, using (2.15), we transform the problem of estimating (3.46) into the exponential sum over $\mathscr{M}_{\mathcal{B}_Q,P}$. We obtain

$$\mathcal{S} = \sum_{r\in\mathbb{Z}} c_r \left| \sum_{\gamma \in \mathscr{M}_{\mathcal{B}_Q,P}} e(r\gamma) \right|^2 . \tag{3.47}$$

Next, we estimate the exponential sum over polynomial Farey fractions with denominators in $\mathcal{B}_Q$. In doing so, we use Theorem 2.0.1. We have

$$\begin{aligned}
\sum_{\gamma \in \mathscr{M}_{\mathcal{B}_Q,P}} e(r\gamma) &= \sum_{p \in \mathcal{B}_Q} \sum_{\substack{a \leq p \\ (P(a),p)=1}} e\left(\frac{ar}{p}\right) = \sum_{p \in \mathcal{B}_Q} \sum_{a \leq p} e\left(\frac{ar}{p}\right) \sum_{\substack{d \mid P(a) \\ d \mid p}} \mu(d) \\
&= \sum_{p \in \mathcal{B}_Q} \sum_{a \leq p} e\left(\frac{ar}{p}\right) - \sum_{p \in \mathcal{B}_Q} \sum_{\substack{a \leq p \\ P(a) \equiv 0 \pmod p}} e\left(\frac{ar}{p}\right) \\
&= \sum_{\substack{p \in \mathcal{B}_Q \\ p \mid r}} p - \sum_{p \in \mathcal{B}_Q} \sum_{\substack{a \leq p \\ P(a) \equiv 0 \pmod p}} e\left(\frac{ar}{p}\right).
\end{aligned} \tag{3.48}$$

In the last step, we have used the following identity

$$\sum_{n=1}^{m} e(nl/m) = \begin{cases} m, & \text{if } m \mid l, \\ 0, & \text{otherwise.} \end{cases}$$

Using (3.47) and (3.48), we obtain

$$
\begin{aligned}
\mathcal{S} =& \sum_{r\in\mathbb{Z}} c_r \sum_{\substack{p_1\in\mathcal{B}_Q\\ p_1\mid r}} p_1 \sum_{\substack{p_2\in\mathcal{B}_Q\\ p_2\mid r}} p_2 - \sum_{r\in\mathbb{Z}} c_r \sum_{\substack{p_1\in\mathcal{B}_Q\\ p_1\mid r}} p_1 \sum_{p_2\in\mathcal{B}_Q} \sum_{\substack{a\le p_2\\ P(a)\equiv 0 \pmod{p_2}}} e\left(-\frac{ar}{p_2}\right) - \sum_{r\in\mathbb{Z}} c_r \\
&\times \sum_{\substack{p_1\in\mathcal{B}_Q\\ p_1\mid r}} p_1 \sum_{p_2\in\mathcal{B}_Q} \sum_{\substack{a\le p_2\\ P(a)\equiv 0 \pmod{p_2}}} e\left(\frac{ar}{p_2}\right) + \sum_{r\in\mathbb{Z}} c_r \left| \sum_{p\in\mathcal{B}_Q} \sum_{\substack{a\le p\\ P(a)\equiv 0 \pmod{p}}} e\left(\frac{ar}{p}\right)\right|^2 \\
=& \sum_{p_1,p_2\in\mathcal{B}_Q} p_1p_2 \sum_{r\in\mathbb{Z}} c_{[p_1,p_2]r} - \sum_{p_1\in\mathcal{B}_Q} p_1 \sum_{p_2\in\mathcal{B}_Q} \sum_{\substack{a\le p_2\\ P(a)\equiv 0 \pmod{p_2}}} \sum_{r\in\mathbb{Z}} c_{p_1r} e\left(-\frac{ap_1r}{p_2}\right) \\
&- \sum_{p_1\in\mathcal{B}_Q} p_1 \sum_{p_2\in\mathcal{B}_Q} \sum_{\substack{a\le p_2\\ P(a)\equiv 0 \pmod{p_2}}} \sum_{r\in\mathbb{Z}} c_{p_1r} e\left(\frac{ap_1r}{p_2}\right) \\
&+ \sum_{p_1,p_2\in\mathcal{B}_Q} \sum_{\substack{a_1\le p_1\\ a_2\le p_2\\ P(a_1)\equiv 0 \pmod{p_1}\\ P(a_2)\equiv 0 \pmod{p_2}}} \sum_{r\in\mathbb{Z}} c_r e\left(\frac{a_1r}{p_1}-\frac{a_2r}{p_2}\right). \qquad (3.49)
\end{aligned}
$$

Since c_r is a Fourier coefficient, we estimate the sum of Fourier coefficients in the first term on the right-hand side of the above identity using Proposition 2.1.3 as in (3.37). Therefore,

$$
\sum_{r\in\mathbb{Z}} c_{[p_1,p_2]r} = \sum_{r\in\mathbb{Z}} \frac{1}{[p_1,p_2]} H\left(\frac{r\mathscr{N}_{\mathcal{B}_Q,P}}{[p_1,p_2]}\right),
$$

The other sums of Fourier coefficients are estimated using Proposition 2.1.4 as demonstrated in (3.28)

$$
\sum_{r\in\mathbb{Z}} c_{p_1r} e\left(\pm\frac{ap_1r}{p_2}\right) = \sum_{r\in\mathbb{Z}} \frac{1}{p_1} H\left(\frac{\mathscr{N}_{\mathcal{B}_Q,P}}{p_1}\left(n \pm \frac{ap_1}{p_2}\right)\right)
$$

and

$$
\sum_{r\in\mathbb{Z}} c_r e\left(\frac{a_1r}{p_1}-\frac{a_2r}{p_2}\right) = \sum_{r\in\mathbb{Z}} H\left(\mathscr{N}_{\mathcal{B}_Q,P}\left(r+\frac{a_1}{p_1}-\frac{a_2}{p_2}\right)\right).
$$

Inserting the above estimates into (3.49) gives

$$\begin{aligned}\mathcal{S} = &\sum_{p_1,p_2\in\mathcal{B}_Q} \frac{p_1p_2}{[p_1,p_2]}\sum_{r\in\mathbb{Z}} H\left(\frac{r\mathscr{N}_{\mathcal{B}_Q,P}}{[p_1,p_2]}\right) - \sum_{p_1\in\mathcal{B}_Q}\sum_{p_2\in\mathcal{B}_Q}\sum_{\substack{a\le p_2\\ P(a)\equiv 0 \pmod{p_2}}}\\ &\times\sum_{r\in\mathbb{Z}}\left(H\left(\frac{\mathscr{N}_{\mathcal{B}_Q,P}}{p_1}\left(n-\frac{ap_1}{p_2}\right)\right)H\left(\frac{\mathscr{N}_{\mathcal{B}_Q,P}}{p_1}\left(n+\frac{ap_1}{p_2}\right)\right)\right)\\ &+\sum_{p_1,p_2\in\mathcal{B}_Q}\sum_{\substack{a_1\le p_1\\ a_2\le p_2\\ P(a_1)\equiv 0 \pmod{p_1}\\ P(a_2)\equiv 0 \pmod{p_2}}}\sum_{r\in\mathbb{Z}} H\left(\mathscr{N}_{\mathcal{B}_Q,P}\left(r+\frac{a_1}{p_1}-\frac{a_2}{p_2}\right)\right). \end{aligned}\tag{3.50}$$

Since Supp $H \subset (0,\Lambda)$ for some $\Lambda > 0$. Thus, for sufficiently large Q, we have $H\left(\mathscr{N}_{\mathcal{B}_Q,P}\left(r+\frac{a_1}{p_1}-\frac{a_2}{p_2}\right)\right) = 0$. If $\sum_{p\in\mathcal{B}_Q} p^2 = o\left((\mathscr{N}_{\mathcal{B}_Q,P})^2\right)$, then $p^2 = o\left((\mathscr{N}_{\mathcal{B}_Q,P})^2\right)$ for all $p \in \mathcal{B}_Q$, this yields $H\left(\frac{\mathscr{N}_{\mathcal{B}_Q,P}}{p_1}\left(n \pm \frac{ap_1}{p_2}\right)\right) = 0$ for large Q. Hence

$$\mathcal{S} = \sum_{p_1,p_2\in\mathcal{B}_Q} \frac{p_1p_2}{[p_1,p_2]}\sum_{r\in\mathbb{Z}} H\left(\frac{r\mathscr{N}_{\mathcal{B}_Q,P}}{[p_1,p_2]}\right) = \sum_{\substack{p_1,p_2\in\mathcal{B}_Q\\ p_1\ne p_2\\ p_1p_2>\mathscr{N}_{\mathcal{B}_Q,P}/\Lambda}}\sum_{r\in\mathbb{Z}} H\left(\frac{r\mathscr{N}_{\mathcal{B}_Q,P}}{p_1p_2}\right).$$

In the last step, we used the fact that Supp $H \subset (0,\Lambda)$. We employ Lemma 2.1.1 to estimate the inner sum and use the fact that derivative of a smooth, compactly supported function is compactly supported to obtain

$$\begin{aligned}\mathcal{S} &= \frac{\int_{\mathbb{R}} H(x)dx}{\mathscr{N}_{\mathcal{B}_Q,P}}\sum_{\substack{p_1,p_2\in\mathcal{B}_Q\\ p_1\ne p_2}} p_1p_2 + \mathrm{O}_H\left(\sum_{p_1,p_2\in\mathcal{B}_Q} 1\right)\\ &= \frac{\int_{\mathbb{R}} H(x)dx}{\mathscr{N}_{\mathcal{B}_Q,P}}\left(\sum_{p_1,p_2\in\mathcal{B}_Q} p_1p_2 - \sum_{p\in\mathcal{B}_Q} p^2\right) + \mathrm{O}_H\left((\#\mathcal{B}_Q)^2\right)\\ &= \mathscr{N}_{\mathcal{B}_Q,P}(1+o(1))\int_{\mathbb{R}} H(x)dx + \mathrm{O}_H\left((\#\mathcal{B}_Q)^2\right).\end{aligned}$$

Therefore,

$$\frac{\mathcal{S}}{\mathscr{N}_{\mathcal{B}_Q,P}} = (1+o(1))\int_{\mathbb{R}} H(x)dx + \mathrm{O}_H\left(\frac{(\#\mathcal{B}_Q)^2}{\mathscr{N}_{\mathcal{B}_Q,P}}\right). \tag{3.51}$$

Suppose $\#\mathcal{B}_Q = m$. Then, by Proposition 3.5.1, we have

$$\mathscr{N}_{\mathcal{B}_Q,P} \gg \sum_{k=1}^{m} p_k \sim \sum_{k=1}^{m} k \log k \gg m^2 \log m,$$

where p_k is the kth prime. Thus $(\#\mathcal{B}_Q)^2 = o(\mathscr{N}_{\mathcal{B}_Q,P})$. By (3.51), we obtain

$$\lim_{Q\to\infty} \frac{\mathcal{S}}{\mathscr{N}_{\mathcal{B}_Q,P}} = \int_{\mathbb{R}} H(x)dx.$$

Using the standard approximation argument, we approximate the characteristic function of the interval $(0,\Lambda)$ from above and below by smooth functions with compact support in $(0,\Lambda)$, we deduce that the pair correlation function of the sequence $(\mathscr{M}_{\mathcal{B}_Q,P})_{Q\geq 1}$ is constant equal to 1 as $Q \to \infty$.

For the other direction, suppose that the pair correlation of $\mathscr{M}_{\mathcal{B}_Q,P}$ is Poissonian and on the contrary $\sum_{p\in\mathcal{B}_Q} p^2$ is not $o\left((\#\mathscr{M}_{\mathcal{B}_Q,P})^2\right)$. In (3.50), the characteristic function of the interval $(0,\Lambda)$ is approximated from both above and below by the smooth functions with compact support in $(0,\Lambda)$, we obtain

$$\begin{aligned}
\mathscr{N}_{\mathcal{B}_Q,P}\mathcal{S}_{\mathscr{M}_{\mathcal{B}_Q,P}}(\Lambda) = & \sum_{\substack{p_1,p_2\in\mathcal{B}_Q\\ p_1\neq p_2}} \left[\frac{p_1p_2\Lambda}{\mathscr{N}_{\mathcal{B}_Q,P}}\right] + \sum_{p\in\mathcal{B}_Q} p\left[\frac{p\Lambda}{\mathscr{N}_{\mathcal{B}_Q,P}}\right] - \sum_{p_1,p_2\in\mathcal{B}_Q} \sum_{\substack{a\leq p_2\\ P(a)\equiv 0 \pmod{p_2}}} \\
& \times \left[\frac{p_1\Lambda}{\mathscr{N}_{\mathcal{B}_Q,P}} + \frac{ap_1}{p_2}\right] - \sum_{p_1,p_2\in\mathcal{B}_Q} \sum_{\substack{a\leq p_2\\ P(a)\equiv 0 \pmod{p_2}}} \left[\frac{p_1\Lambda}{\mathscr{N}_{\mathcal{B}_Q,P}} - \frac{ap_1}{p_2}\right] \\
= & \frac{\Lambda}{\mathscr{N}_{\mathcal{B}_Q,P}} \sum_{p_1,p_2\in\mathcal{B}_Q} p_1p_2 - \sum_{p\in\mathcal{B}_Q} p\left\{\frac{p\Lambda}{\mathscr{N}_{\mathcal{B}_Q,P}}\right\} \\
& - 2\sum_{p_1,p_2\in\mathcal{B}_Q} \sum_{\substack{a\leq p_2\\ P(a)\equiv 0 \pmod{p_2}}} \frac{p_1\Lambda}{\mathscr{N}_{\mathcal{B}_Q,P}} + \mathrm{O}\left((\#\mathcal{B}_Q)^2\right),
\end{aligned}$$

where $\{x\}$ is the fractional part of x. If $p = o(\mathscr{N}_{\mathcal{B}_Q,P})$ for every $p \in \mathcal{B}_Q$, then $\lim_{Q\to\infty} \frac{1}{\mathscr{N}_{\mathcal{B}_Q,P}} \sum_{p\in\mathcal{B}_Q} p\left\{\frac{p\Lambda}{\mathscr{N}_{\mathcal{B}_Q,P}}\right\} = \lim_{Q\to\infty} \frac{\Lambda}{(\mathscr{N}_{\mathcal{B}_Q,P})^2} \sum_{p\in\mathcal{B}_Q} p^2 \neq 0$. If there exists a $p \in \mathcal{B}_Q$ such that p is not $o(\mathscr{N}_{\mathcal{B}_Q,P})$, then $\lim_{Q\to\infty} \frac{1}{\mathscr{N}_{\mathcal{B}_Q,P}} \times \sum_{p\in\mathcal{B}_Q} p\left\{\frac{p\Lambda}{\mathscr{N}_{\mathcal{B}_Q,P}}\right\} \geq \lim_{Q\to\infty} \frac{1}{\mathscr{N}_{\mathcal{B}_Q,P}} p\left\{\frac{p\Lambda}{\mathscr{N}_{\mathcal{B}_Q,P}}\right\} \neq 0$. Therefore, $\lim_{Q\to\infty} \mathcal{S}_{\mathscr{M}_{\mathcal{B}_Q,P}}(\Lambda) \neq \Lambda$, thus the pair correlation is not Poissonian, which is a contradiction. This completes the proof of

Theorem 3.5.1.

3.5.4 Piatetski-Shapiro primes

In this section, we derive Corollary 3.5.2, which concerns the distribution of polynomial Farey fractions with denominators in the set of Piatetski-Shapiro primes. We use the following counting estimate to verify the condition

$$\sum_{p\in\mathcal{B}_Q} p^2 = o\left((\#\mathscr{M}_{\mathcal{B}_Q,P})^2\right), \tag{3.52}$$

in Theorem 3.5.1 for the set of Piatetski-Shapiro primes.

Let $c \in (1, 1.16)$ be a fixed real number. Suppose $\mathcal{B}_Q^{(1)}$ is the set of Piatetski-Shapiro primes less than or equal to Q. We use Proposition 2.1.2 and Theorem 3.5.2 to estimate the sum on the left-hand side of (3.52):

$$\begin{aligned}\sum_{p\in\mathcal{B}_Q^{(1)}} p^2 &= \sum_{\substack{p\le Q\\ p=\lfloor n^c\rfloor}} p^2 = Q^2 \sum_{\substack{p\le Q\\ p=\lfloor n^c\rfloor}} 1 - \int_2^Q 2t\left(\sum_{\substack{p\le t\\ p=\lfloor n^c\rfloor}} 1\right)dt\\ &= Q^2\left(\frac{Q^{1/c}}{\log Q} + \mathrm{O}\left(\frac{Q^{1/c}}{\log^2 Q}\right)\right) - 2\int_2^Q \left(\frac{t^{1+\frac{1}{c}}}{\log t} + \mathrm{O}\left(\frac{t^{1+\frac{1}{c}}}{\log^2 t}\right)\right)dt\\ &= \frac{Q^{2+\frac{1}{c}}}{(2c+1)\log Q} + \mathrm{O}\left(\frac{Q^{2+\frac{1}{c}}}{\log^2 Q}\right).\end{aligned} \tag{3.53}$$

To estimate the right-hand side of (3.52), we use Propositions 3.5.1, 2.1.2, and Theorem 3.5.2. This yields

$$\begin{aligned}\#\mathscr{M}_{\mathcal{B}_Q^{(1)},P} &= \sum_{p\in\mathcal{B}_Q^{(1)}} p + \mathrm{O}\left(\#\mathcal{B}_Q^{(1)}\right) = \sum_{\substack{p\le Q\\ p=\lfloor n^c\rfloor}} p + \mathrm{O}\left(\#\mathcal{B}_Q^{(1)}\right)\\ &= Q\left(\frac{Q^{1/c}}{\log Q} + \mathrm{O}\left(\frac{Q^{1/c}}{\log^2 Q}\right)\right) - \int_2^Q \left(\frac{t^{\frac{1}{c}}}{\log t} + \mathrm{O}\left(\frac{t^{\frac{1}{c}}}{\log^2 t}\right)\right)dt\\ &= \frac{Q^{1+\frac{1}{c}}}{(c+1)\log Q} + \mathrm{O}\left(\frac{Q^{1+\frac{1}{c}}}{\log^2 Q}\right).\end{aligned}$$

Thus the above estimate with (3.53) gives the required result.

3.5.5 Chen primes

The goal of this section is to prove Corollary 3.5.3. To do so, using Theorem 3.5.1, we need to verify the condition for $\mathcal{B}_Q^{(2)}$, the set of Chen primes

$$\sum_{p\in\mathcal{B}_Q^{(2)}} p^2 = o\left((\#\mathscr{M}_{\mathcal{B}_Q^{(2)},P})^2\right). \tag{3.54}$$

Suppose $\mathcal{B}_Q^{(2)}$ is the set of Chen primes not exceeding Q. To prove the asymptotic formula in (3.54) for Chen primes, we apply Propositions 3.5.1, 2.1.2, and Theorem 3.5.3 to obtain

$$\begin{aligned}\sum_{p\in\mathcal{B}_Q^{(2)}} p^2 &= \sum_{\substack{p\le Q\\ p\text{ Chen prime}}} p^2 = Q^2 \sum_{\substack{p\le Q\\ p\text{ Chen prime}}} 1 - \int_2^Q 2t\left(\sum_{\substack{p\le t\\ p\text{ Chen prime}}} 1\right)dt\\ &\ll Q^2\cdot\frac{Q}{\log Q} - 2\int_2^Q \frac{t^2}{(\log t)^2}dt \ll \frac{Q^3}{\log Q},\end{aligned} \tag{3.55}$$

and

$$\begin{aligned}\#\mathscr{M}_{\mathcal{B}_Q^{(2)},P} &= \sum_{p\in\mathcal{B}_Q^{(2)}} p + \mathrm{O}\left(\#\mathcal{B}_Q^{(2)}\right) = \sum_{\substack{p\le Q\\ p\text{ Chen prime}}} p + \mathrm{O}\left(\#\mathcal{B}_Q^{(2)}\right)\\ &= Q\sum_{\substack{p\le Q\\ p\text{ Chen prime}}} 1 - \int_2^Q\left(\sum_{\substack{p\le t\\ p\text{ Chen prime}}} 1\right)dt + \mathrm{O}\left(\#\mathcal{B}_Q^{(2)}\right) \gg \frac{Q^2}{\log^2 Q}.\end{aligned}$$

The above estimate with (3.55) and (3.54) completes the proof of Corollary 3.5.3.

3.5.6 Prime k-tuples

This section devoted to establish Corollary 3.5.4. Let $\mathcal{B}_Q^{(3)}$ be a set of prime k-tuples that are less than or equal to Q. We use Proposition 3.5.1 and 2.1.2 with the Hardy-Littlewood conjecture to derive the asymptotic formulas

$$\begin{aligned}\sum_{p\in\mathcal{B}_Q^{(3)}} p^2 &= \sum_{\substack{p\le Q\\ p\ \text{prime } k\text{-tuple}}} p^2 = Q^2 \sum_{\substack{p\le Q\\ p\ \text{prime } k\text{-tuple}}} 1 - \int_2^Q 2t\left(\sum_{\substack{p\le t\\ p\ \text{prime } k\text{-tuple}}} 1\right)dt\\ &= (1+o(1))\frac{c_kQ^3}{(\log Q)^k} - 2\int_2^Q (1+o(1))\frac{c_kt^2}{(\log Q)^k}dt = (1+o(1))\frac{c_kQ^3}{3(\log Q)^k},\end{aligned}$$

and

$$\begin{aligned}\#\mathscr{M}_{\mathcal{B}_Q^{(3)},P} &= \sum_{p\in\mathcal{B}_Q^{(3)}} p + \mathrm{O}\left(\#\mathcal{B}_Q^{(3)}\right) = (1+o(1))\frac{c_kQ^2}{(\log Q)^k} - \int_2^Q (1+o(1))\frac{c_kt}{(\log Q)^k}dt\\ &= (1+o(1))\frac{c_kQ^2}{2(\log Q)^k},\end{aligned}$$

where c_k is a positive constant depending on k. Collecting the above estimates establishes the following estimate

$$\sum_{p\in\mathcal{B}_Q^{(2)}} p^2 = o\left((\#\mathscr{M}_{\mathcal{B}_Q^{(2)},P})^2\right).$$

Thus, applying Theorem 3.5.1 completes the proof of Corollary 3.5.4.

3.5.7 Primes with restricted digits

In this section, we prove Corollary 3.5.5. Suppose $\mathcal{B}_Q^{(4)}$ is the set of primes that are less than or equal to Q with restricted digits. Now, employing Proposition 2.1.2 and Theorem 3.5.4, we have

$$\begin{aligned}\sum_{p\in\mathcal{B}_Q^{(4)}} p^2 &= \sum_{\substack{p\le Q\\ p\in\mathbf{A}}} p^2 = Q^2\sum_{\substack{p\le Q\\ p\in\mathbf{A}}} 1 - \int_2^Q 2t\left(\sum_{\substack{p\le t\\ p\in\mathbf{A}}} 1\right)dt \asymp Q^2\cdot\frac{Q^{\frac{\log(q-\#\mathcal{D})}{\log q}}}{\log Q}\\ &\quad - 2\int_2^Q \frac{t^{1+\frac{\log(q-\#\mathcal{D})}{\log q}}}{\log t}dt \asymp \frac{Q^{2+\frac{\log(q-\#\mathcal{D})}{\log q}}}{\log Q}.\end{aligned}$$

The right-hand side of (3.52) is estimated using Propositions 3.5.1, 2.1.2, and Theorem 3.5.4

$$\begin{aligned}\#\mathscr{M}_{\mathcal{B}_Q^{(4)},P} &= \sum_{p\in\mathcal{B}_Q^{(4)}} p + \mathrm{O}\left(\#\mathcal{B}_Q^{(4)}\right) = \sum_{\substack{p\le Q\\ p\in\mathbf{A}}} p + \mathrm{O}\left(\#\mathcal{B}_Q^{(4)}\right) = Q\sum_{\substack{p\le Q\\ p\in\mathbf{A}}} 1 - \int_2^Q \left(\sum_{\substack{p\le t\\ p\in\mathbf{A}}} 1\right) dt \\ &\asymp Q\cdot\frac{Q^{\frac{\log(q-\#\mathcal{D})}{\log q}}}{\log Q} - \int_2^Q \frac{t^{\frac{\log(q-\#\mathcal{D})}{\log q}}}{\log t} dt \asymp \frac{Q^{1+\frac{\log(q-\#\mathcal{D})}{\log q}}}{\log Q}.\end{aligned}$$

Therefore, the above two estimates together with Theorem 3.5.1 give the required result.

3.6 Chebyshev's bias for Farey fractions

In this section, we study the problem of the distribution of polynomial Farey sequence along arithmetic progressions and bias over one from another. As a consequence of Dirichlet's prime number theorem, the prime numbers in arithmetic progressions $a + nq$, with a relatively prime to q, are evenly distributed. In particular, primes are asymptotically same among the residue classes modulo q that are coprime to q. It was asserted by Chebyshev that there are more primes $\equiv 3 \pmod 4$ than $\equiv 1 \pmod 4$. Littlewood [74] disproved Chebyshev's assertion by proving that the set of values of x for which the difference $\pi(x; 4, 3) - \pi(x; 4, 1)$ is positive is unbounded, also there exists an unbounded set of values of x for which the difference $\pi(x; 4, 3) - \pi(x; 4, 1)$ is negative. Here, we study Chebyshev's bias question for classical and polynomial Farey fractions. Let q and l be positive integers; we denote the number of polynomial Farey fractions with denominators in an arithmetic progression as

$$S(Q; q, l) := \#\left\{\frac{a}{n} \in \mathcal{F}_{Q,P} \mid n \equiv l \pmod q\right\}. \tag{3.56}$$

We ask the following questions for Farey fractions analogous to prime number races.

- Does there exist positive integers Q_0, l_1, and l_2 with $l_1 \not\equiv l_2 \pmod{q}$ such that

$$S(Q; q, l_1) > S(Q; q, l_2) \text{ for all } Q > Q_0?$$

- Are there arbitrarily large values of Q for which $S(Q; q, l_1) < S(Q; q, l_2)$, and arbitrarily large values of Q for which $S(Q; q, l_1) > S(Q; q, l_2)$? In other words, does the function $S(Q; q, l_1) - S(Q; q, l_2)$ change sign infinitely often?

In here, we address the questions listed above. To state our results, we need the following conditions.

Haselgrove's condition for modulus q [66, p. 309]: For all Dirichlet characters $\chi \pmod{q}$, $L(s, \chi) \neq 0$ for all $s \in (0, 1)$.

Note that J. B. Rosser [87, 88] showed that no Dirichlet L-function attached to a real character modulo $q \leq 1000$ has a real zero in the strip $0 < \Re(s) < 1$, and Watkins [97] proved that one can take $q \leq 300000000$ if we restrict the Dirichlet L-functions to odd Dirichlet characters. For $q|24$, there are only real characters modulo q. Therefore, Haselgrove's condition is known to hold when modulus q divides 24.

Let $\kappa, J \geq 1$ be integers and let $P(x) = c_\kappa x^\kappa + c_{\kappa-1} x^{\kappa-1} + \cdots + c_1 x \in \mathbb{Z}[x]$ be a polynomial with non-zero discriminant and factorization

$$P(x) = \prod_{i=1}^{J} m_i(x)^{e_i}, \tag{3.57}$$

where $m_i(x) \in \mathbb{Z}[x]$ are irreducible polynomials. Let $K_i = \mathbb{Q}[x]/(m_i(x))$ be number fields with ring of integers $\mathcal{O}_{K_i}$. Let $\mathfrak{q}_i = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r} \subset \mathcal{O}_{K_i}$ be an ideal with the unique prime factorization such that $p \in \mathfrak{p}_j$ for some j and $p|q$. For $1 \leq i \leq J$, let $\mathcal{L}_i(s, \chi')$ be the Hecke L-functions associated with Hecke characters $\chi' \pmod{\mathfrak{q}_i}$ of $\mathcal{O}_{K_i}$.

Our main result for races of classical and polynomial Farey fractions is as follows.

Theorem 3.6.1. *Let* $q \geq 2, l_1, l_2$ *be positive integers such that* $l_1 \not\equiv l_2 \pmod{q}$ *and* $(q, l_1 l_2) = 1$*. Let* $P(x) \in \mathbb{Z}[x]$ *be as in* (3.57)*. Assuming Haselgrove's condition for Hecke L-function* $\mathcal{L}_i(s, \chi')$ *modulo* $\mathfrak{q}_i$*, the set of values of* Q *for which the difference*

$S(Q;q,l_1)-S(Q;q,l_2)$ is strictly positive and the set of values of Q for which the difference $S(Q;q,l_1)-S(Q;q,l_2)$ is strictly negative are unbounded.

Note that Theorem 3.6.1 does not give any information on the frequency of sign changes of the function $S(Q;q,l_1)-S(Q;q,l_2)$, in particular the number of sign changes within an interval. To obtain information about the frequency of sign changes, we use the following result of Kaczorowski and Wiertelak [64, Lemma 3.1].

Lemma 3.6.1. *Let $f(x)$ be real for $x>0$ and suppose that the integral*

$$F(s)=\int_0^\infty f(x)x^{-s-1}dx$$

converges absolutely for $\sigma>\sigma_0$ and has meromorphic continuation to a half-plane $\sigma>\theta$ for certain $\theta<\sigma_0$. Suppose that $F(s)$ is not holomorphic for $\sigma>\theta$ but is holomorphic on the segment $(\theta,\sigma_0]$ of the real axis. Then for sufficiently large T, $f(x)$ has in the interval $(0,T]$ at least $\ll \log T$ oscillations of size x^θ.

Remark 3.6.1. *We denote*

$$\mathscr{A}(Q):=S(Q;q,l_1)-S(Q;q,l_2)\pm cQ^{\frac{1}{2}-\epsilon},$$

where c is some positive constant and $\epsilon>0$ is arbitrarily small real number. In the definition of $\mathscr{A}(Q)$, we take the exponent of Q to be $1/2-\epsilon$ in order to include the line $\Re(s)=1/2$ in our region enabling us to apply Lemma 3.6.1. Assuming Haselgrove's condition for Hecke L-function $\mathcal{L}_i(s,\chi')$ modulo $\mathfrak{q}_i$, we apply Lemma 3.6.1 to the function $\mathscr{A}(Q)$, and obtain a sequence $\{Q_i\}_{i=1}^{[\log T]}$ in the interval $(1,T]$ of length $\log T$ such that sgn$\mathscr{A}(Q_i)\neq$ sgn$\mathscr{A}(Q_{i+1})$ and $|\mathscr{A}(Q_i)|>Q_i^{1/2-\epsilon}$. Hence, $\mathscr{A}(Q)$ has at least $\gg \log T$ oscillations of size $Q^{1/2-\epsilon}$, in the interval $(1,T]$.

Moreover, we prove an Ω-result for the error term in the asymptotic formula of $S(Q;q,l)$ in Proposition 3.2.1.

Theorem 3.6.2. *Let $q\geq 2$ and l be positive integers with $(q,l)=1$. Let $P(x)\in\mathbb{Z}[x]$ be as in (3.57). If Θ denotes the supremum of real parts of the zeros of the Hecke L-functions $\mathcal{L}_i(s,\chi)$ modulo $\mathfrak{q}_i$, then for any $\epsilon>0$, assuming the Haselgrove's* *condition, we have*

$$S(Q;q,l)-\frac{Q^2}{2\phi(q)}\prod_{p|q}\left(1-\frac{1}{p}\right)\prod_{p\nmid q}\left(1-\frac{f_P(p)}{p^2}\right)=\Omega_{\pm}(Q^{\Theta-\epsilon}),$$

where $f_P(p)$ is as in (1.17).

3.6.1 Proof of Theorem 3.6.1

We use the approach of Knapowski and Turán [66] in proving Theorem 3.6.1. We begin with the integral

$$\int_1^{\infty}(S(Q;q,l_1)-S(Q;q,l_2)\pm cQ^{\frac{1}{2}-\epsilon})Q^{-s-1}dQ,\ \text{for } \epsilon>0.$$

Now, the difference can be estimated using Theorem 2.3.1 as

$$\begin{aligned}S(Q;q,l_1)-S(Q;q,l_2)=&\left(\sum_{\substack{n\leq Q\\ n\equiv l_1 \pmod q}}1-\sum_{\substack{n\leq Q\\ n\equiv l_2 \pmod q}}1\right)\sum_{\substack{a\leq n\\ (P(a),n)=1}}1\\ =&\frac{1}{\phi(q)}\left(\sum_{n\leq Q}\sum_{\chi \pmod q}\chi(n\bar{l}_1)-\sum_{n\leq Q}\sum_{\chi \pmod q}\chi(n\bar{l}_2)\right)\sum_{\substack{a\leq n\\ (P(a),n)=1}}1\\ =&\frac{1}{\phi(q)}\left(\sum_{\chi \pmod q}(\bar{\chi}(l_1)-\bar{\chi}(l_2))\right)\sum_{n\leq Q}\chi(n)\\ &\times\sum_{\substack{a\leq n\\ (P(a),n)=1}}1.\end{aligned}$$

We next use Theorem 2.0.1 and obtain

$$S(Q;q,l_1)-S(Q;q,l_2)=\frac{1}{\phi(q)}\left(\sum_{\chi \pmod q}(\bar{\chi}(l_1)-\bar{\chi}(l_2))\right)\sum_{n\leq Q}\chi(n)\sum_{a\leq n}\sum_{\substack{d|P(a)\\ d|n}}\mu(d)$$

$$= \frac{1}{\phi(q)} \left(\sum_{\chi \pmod q} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \right) \sum_{n \le Q} n\chi(n)K(n),$$

where $K(n) = \sum_{d|n} \frac{\mu(d) f_P(d)}{d}$. Therefore,

$$\begin{aligned} &\int_1^\infty (S(Q;q,l_1) - S(Q;q,l_2) \pm cQ^{\frac{1}{2}-\epsilon}) Q^{-s-1} dQ \\ &= \frac{1}{\phi(q)} \sum_{\chi \pmod q} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \int_1^\infty \frac{\sum_{n \le Q} n\chi(n)K(n)}{Q^{s+1}} dQ \pm \frac{c}{s - \frac{1}{2} + \epsilon}. \end{aligned} \tag{3.58}$$

The integral on the right-hand side represents the Dirichlet series $\sum_{n=1}^\infty \frac{\chi(n)K(n)}{n^{s-1}}$ as a Mellin transform. Using (3.5), we have

$$\begin{aligned} \int_1^\infty \frac{\sum_{n \le Q} n\chi(n)K(n)}{Q^{s+1}} dQ &= \frac{1}{s} \sum_{n=1}^\infty \frac{\chi(n)K(n)}{n^{s-1}} \\ &= \frac{1}{s} \prod_p \left(1 + \left(1 + \frac{f_P(p)}{p} \right) \left(\frac{\chi(p)}{p} + \frac{\chi(p^2)}{p^2} + \cdots \right) \right) \\ &= \frac{L(s-1,\chi)}{s} \prod_p \left(1 - \frac{\chi(p) f_P(p)}{p^s} \right), \text{ for } \sigma > 2. \end{aligned} \tag{3.59}$$

This along with (3.58) gives

$$\begin{aligned} &\int_1^\infty (S(Q;q,l_1) - S(Q;q,l_2) \pm cQ^{\frac{1}{2}-\epsilon}) Q^{-s-1} dQ \\ &= \frac{1}{\phi(q)} \sum_{\chi \pmod q} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \frac{1}{s} L(s-1,\chi) \prod_p \left(1 - \frac{\chi(p) f_P(p)}{p^s} \right) \pm \frac{c}{s - \frac{1}{2} + \epsilon} \\ &= \frac{1}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \frac{1}{s} L(s-1,\chi) \prod_p \left(1 - \frac{\chi(p) f_P(p)}{p^s} \right) \pm \frac{c}{s - \frac{1}{2} + \epsilon}. \end{aligned} \tag{3.60}$$

In the last step, we used the fact that $\bar{\chi}_0(l_1) = \bar{\chi}_0(l_2) = 1$ for $(l_1 l_2, q) = 1$. We will apply Proposition 2.0.2 with

$$\mathscr{A}(Q) = S(Q;q,l_1) - S(Q;q,l_2) \pm cQ^{\frac{1}{2}-\epsilon}.$$

Thus by (3.60), we have

$$\begin{aligned} g(s) &= \int_1^{\infty} \mathscr{A}(Q)Q^{-s-1}dQ \\ &= \frac{1}{s\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi\neq\chi_0}} (\bar{\chi}(l_1)-\bar{\chi}(l_2))L(s-1,\chi)\prod_p\left(1-\frac{\chi(p)f_P(p)}{p^s}\right) \pm \frac{c}{s-\frac{1}{2}+\epsilon}. \end{aligned}$$

Since $L(s-1,\chi)$ is entire for non-principal Dirichlet character $\chi \pmod q$ and $\prod_p\left(1-\frac{\chi(p)f_P(p)}{p^s}\right)$ is absolutely convergent for $\sigma > 1$ so, $g(s)$ is analytic in the half plane $\sigma > 1$. Let $\Delta \neq 0$ be discriminant of $P(x)$. The product term in the last equation can be written as

$$\prod_p\left(1-\frac{\chi(p)f_P(p)}{p^s}\right) = \prod_{p\mid\Delta}\left(1-\frac{\chi(p)f_P(p)}{p^s}\right)\prod_{p\nmid\Delta}\left(1-\frac{\chi(p)f_P(p)}{p^s}\right) \tag{3.61}$$

Note that $f_P(p)$ is non-negative integer for all prime p. Therefore, by the binomial theorem, we obtain

$$\begin{aligned} \prod_{p\nmid\Delta}\left(1-\frac{\chi(p)}{p^s}\right)^{f_P(p)} =& \prod_{p\nmid\Delta}\left(1-\frac{f_P(p)\chi(p)}{p^s}+\frac{f_P(p)(f_P(p)-1)\chi(p^2)}{2p^{2s}}+\cdots\right. \\ &\left. + \frac{(-1)^{f_P(p)}\chi(p^{f_P(p)})}{p^{f_P(p)s}}\right) = \prod_{p\nmid\Delta}\left(1-\frac{\chi(p)f_P(p)}{p^s}\right)\mathscr{P}_1(\Delta). \end{aligned}$$

where

$$\begin{aligned} \mathscr{P}_1(\Delta) =& \prod_{p\nmid\Delta}\left(1+\left(1-\frac{\chi(p)f_P(p)}{p^s}\right)^{-1}\right. \\ &\left.\times\left(\frac{\chi(p^2)f_P(p)(f_P(p)-1)}{2p^{2s}}+\cdots+\frac{(-1)^{f_P(p)}\chi(p^{f_P(p)})}{p^{sf_P(p)}}\right)\right). \end{aligned}$$

Inserting the above identity into (3.61) gives

$$\prod_p\left(1-\frac{\chi(p)f_P(p)}{p^s}\right) = \prod_{p\mid\Delta}\left(1-\frac{\chi(p)f_P(p)}{p^s}\right)\prod_{p\nmid\Delta}\left(1-\frac{\chi(p)}{p^s}\right)^{f_P(p)}\mathscr{P}_1(\Delta)^{-1} \tag{3.62}$$

Let us assume that $P(x)$ is irreducible. An application of Dedekind's theorem (see Theorem 5.5.1, [82]) implies that for primes $p \nmid \Delta$, $f_P(p)$ is the number of prime ideals of a ring of integers, $\mathcal{O}_K$, where $K = \mathbb{Q}[x]/(P(x))$. This yields

$$\begin{aligned}
\prod_{p \nmid \Delta} \left(1 - \frac{\chi(p)}{p^s}\right)^{f_P(p)} &= \prod_{p \nmid \Delta} \prod_{\substack{p \in \mathfrak{P} \subset \mathcal{O}_K \\ \|\mathfrak{P}\| = p}} \left(1 - \frac{\chi'(\mathfrak{P})}{\| \mathfrak{P} \|^s}\right) \\
&= \prod_{p \nmid \Delta} \prod_{p \in \mathfrak{P} \subset \mathcal{O}_K} \left(1 - \frac{\chi'(\mathfrak{P})}{\| \mathfrak{P} \|^s}\right) \prod_{p \nmid \Delta} \prod_{\substack{p \in \mathfrak{P} \subset \mathcal{O}_K \\ \|\mathfrak{P}\| \neq p}} \left(1 - \frac{\chi'(\mathfrak{P})}{\| \mathfrak{P} \|^s}\right)^{-1} \\
&= \prod_{p} \prod_{p \in \mathfrak{P} \subset \mathcal{O}_K} \left(1 - \frac{\chi'(\mathfrak{P})}{\| \mathfrak{P} \|^s}\right) \prod_{p \mid \Delta} \prod_{p \in \mathfrak{P} \subset \mathcal{O}_K} \left(1 - \frac{\chi'(\mathfrak{P})}{\| \mathfrak{P} \|^s}\right)^{-1} \mathscr{P}_2(\Delta) \\
&= \frac{1}{\mathcal{L}(s, \chi')} \prod_{p \mid \Delta} \left(1 + \mathrm{O}\left(|p^{-s}|\right)\right)^{-1} \mathscr{P}_2(\Delta), \qquad (3.63)
\end{aligned}$$

where $\mathscr{P}_2(\Delta) = \prod_{p \nmid \Delta} \prod_{\substack{p \in \mathfrak{P} \subset \mathcal{O}_K \\ \|\mathfrak{P}\| \neq p}} \left(1 - \frac{\chi'(\mathfrak{P})}{\|\mathfrak{P}\|^s}\right)^{-1}$, $\| \mathfrak{P} \|$ denotes the norm of a prime ideal $\mathfrak{P}$, and $\mathcal{L}$ is the Hecke L-function modulo $\mathfrak{q}$. So, from (3.62) and (3.63), we have

$$\begin{aligned}
\prod_{p} \left(1 - \frac{\chi(p) f_P(p)}{p^s}\right) &= \frac{1}{\mathcal{L}(s, \chi')} \prod_{p \mid \Delta} \left(1 - \frac{\chi(p) f_P(p)}{p^s}\right) \\
&\qquad \times \prod_{p \mid \Delta} \left(1 + \mathrm{O}\left(|p^{-s}|\right)\right)^{-1} \mathscr{P}_1(\Delta)^{-1} \mathscr{P}_2(\Delta).
\end{aligned}$$

This yields

$$\begin{aligned}
g(s) &= \frac{1}{s\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \frac{L(s-1, \chi)}{\mathcal{L}(s, \chi')} \prod_{p \mid \Delta} \left(1 - \frac{\chi(p) f_P(p)}{p^s}\right) \\
&\quad \times \prod_{p \mid \Delta} \left(1 + \mathrm{O}\left(|p^{-s}|\right)\right)^{-1} \mathscr{P}_1(\Delta) \mathscr{P}_2(\Delta) \pm \frac{c}{s - \frac{1}{2} + \epsilon}. \qquad (3.64)
\end{aligned}$$

Clearly $1/\mathcal{L}(s, \chi')$ is analytic in the half plane $\sigma \geq 1$ for all character χ' as none of the denominators $\mathcal{L}(s, \chi')$ have zeros in the half-plane $\sigma \geq 1$. The products on the right-hand side run over the prime divisors of $\Delta \neq 0$, so they are finite. The product terms $\mathscr{P}_1(\Delta)$ and $\mathscr{P}_2(\Delta)$ are convergent in the half plane $\sigma > \frac{1}{2}$. Thus, the product $\prod_p \left(1 - \frac{\chi(p) f_P(p)}{p^s}\right)$ is absolutely convergent for $\sigma \geq 1$. Hence, the function $g(s)$ is analytic in the half plane $\sigma \geq 1$.

By the Haselgrove's condition, $\mathcal{L}(s, \chi') \neq 0$ on the real segment $0 < \sigma < 1$ so, $1/\mathcal{L}(s, \chi')$ defines an analytic function on the real segment $0 < \sigma < 1$. Hence, $g(s)$ is regular on the real segment $\frac{1}{2} - \epsilon < \sigma < 1$.

Suppose there exists a positive constant Q_0 such that $\mathscr{A}(Q)$ does not change its sign for $Q > Q_0$. Then by Proposition 2.0.2, $g(s)$ is an analytic function in the half plane $\Re(s) > \frac{1}{2} - \epsilon$. Therefore, each zero of the denominator $\mathcal{L}(s, \chi')$ must also be a zero of the numerator $L(s-1, \chi)$. However, it is known that all nontrivial zeros of $\mathcal{L}(s, \chi')$ are in the critical strip $0 < \sigma < 1$. Similarly, all nontrivial zeros of $L(s-1, \chi)$ are in the critical strip $1 < \sigma < 2$. Thus, the nontrivial zeros of the denominator $\mathcal{L}(s, \chi')$ in (3.64) cannot be canceled by the zeros of numerator $L(s-1, \chi)$. Since the zeros of $\mathcal{L}(s, \chi')$ has a symmetry along critical line $\sigma = 1/2$ (see [70, Section 5]). Hence, there exist zeros of the denominator $\mathcal{L}(s, \chi')$ in (3.64) in the strip $1/2 \leq \sigma < 1$. Any such zero is a pole of $g(s)$ and this by Proposition 2.0.2, contradicts the assumption that there exists Q_0 such that $\mathscr{A}(Q)$ does not change sign for $Q > Q_0$ and thus we conclude the result for irreducible polynomials. We next consider that $P(x)$ is reducible with the factorization

$$P(x) = \prod_{i=1}^{J} m_i(x)^{e_i},$$

where $m_i(x) \in \mathbb{Z}[x]$ are irreducible polynomials. Let $\Delta = \prod_{i=1}^{J} \Delta_i$ be the discriminant of $P(x)$, where $\Delta_i \neq 0$ is discriminant of $m_i(x)$. For primes $p \nmid \Delta$,

$$f_P(p) = \sum_{i=1}^{J} f_{m_i}(p),$$

where $f_{m_i}(p)$ is the number of prime ideals of a ring of integers, $\mathcal{O}_{K_i}$ and $K_i = \mathbb{Q}[x]/(m_i(x))$, by an application of Dedekind's Theorem (see Theorem 5.5.1, [82]).

We follow the proof of irreducible case and the above identity to obtain

$$\prod_{p\nmid\Delta}\left(1-\frac{\chi(p)}{p^s}\right)^{f_P(p)}=\prod_{i=1}^{J}\mathcal{L}_i(s,\chi')^{-1}\prod_{p|\Delta}\left(1+\mathrm{O}\left(|p^{-s}|\right)\right)^{-1}\mathscr{P}_2(\Delta)^J,$$

and

$$\begin{aligned}\prod_{p}\left(1-\frac{\chi(p)f_P(p)}{p^s}\right)=&\prod_{i=1}^{J}\mathcal{L}_i(s,\chi')^{-1}\mathscr{P}_{1,i}(\Delta)\prod_{p|\Delta}\left(1-\frac{\chi(p)f_P(p)}{p^s}\right)\\&\times\prod_{p|\Delta}\left(1+\mathrm{O}\left(|p^{-s}|\right)\right)^{-1}\mathscr{P}_2(\Delta)^J,\end{aligned}\tag{3.65}$$

where

$$\begin{aligned}\mathscr{P}_{1,i}(\Delta)=&\prod_{p\nmid\Delta}\left(1+\left(1-\frac{\chi(p)f_{m_i}(p)}{p^s}\right)^{-1}\right.\\&\left.\times\left(\frac{\chi(p^2)f_{m_i}(p)(f_{m_i}(p)-1)}{2p^{2s}}+\cdots+\frac{(-1)^{f_{m_i}(p)}\chi(p^{f_{m_i}(p)})}{p^{sf_{m_i}(p)}}\right)\right).\end{aligned}$$

By the same argument made for irreducible case, $g(s)$ is analytic in the half-plane $\Re(s)\geq 1$ and can be analytically continued on the real segment $\frac{1}{2}-\epsilon<\sigma<1$. Suppose there exists a positive constant Q_0 such that $\mathscr{A}(Q)$ does not change its sign for $Q>Q_0$. In a similar manner as for the irreducible case in view of Proposition 2.0.2, we get a contradiction on the assumption that $\mathscr{A}(Q)$ does not change the sign for $Q>Q_0$. This completes the proof of Theorem 3.6.1.

3.6.2 Proof of Theorem 3.6.2

In order to prove Theorem 3.6.2, we use Proposition 2.0.3. We denote

$$\mathcal{A}(Q)=-S(Q;q,l)+\frac{Q^2}{2\phi(q)}\prod_{p|q}\left(1-\frac{1}{p}\right)\prod_{p\nmid q}\left(1-\frac{f_P(p)}{p^2}\right)+Q^{\Theta-\epsilon},$$

where Θ denotes the supremum of the real parts of zeros of the Hecke L-functions $\mathcal{L}_i(s,\chi')$ modulo $\mathfrak{q}_i$. The second term on the right-hand side of the above identity is the main term in the asymptotic formula of $S(Q;q,l)$ (see Proposition 3.2.1). For $\sigma > 2$, we consider

$$
\begin{aligned}
&\int_1^{\infty} \mathcal{A}(Q) Q^{-s-1} dQ \\
=&\int_1^{\infty} \left(-S(Q;q,l) + \frac{Q^2}{2\phi(q)} \prod_{p|q} \left(1 - \frac{1}{p}\right) \prod_{p\nmid q} \left(1 - \frac{f_P(p)}{p^2}\right) + Q^{\Theta-\epsilon} \right) Q^{-s-1} dQ \\
=&\frac{-1}{\phi(q)} \sum_{\chi \pmod q} \bar{\chi}(l) \frac{1}{s} L(s-1,\chi) \prod_{p} \left(1 - \frac{\chi(p) f_P(p)}{p^s}\right) + \frac{1}{2\phi(q)} \prod_{p|q} \left(1 - \frac{1}{p}\right) \\
&\times \prod_{p\nmid q} \left(1 - \frac{f_P(p)}{p^2}\right) \frac{1}{s-2} + \frac{1}{s-\Theta+\epsilon} \\
=&\frac{-1}{s\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \bar{\chi}(l_1) L(s-1,\chi) \prod_{p} \left(1 - \frac{\chi(p) f_P(p)}{p^s}\right) - \frac{L(s-1,\chi_0)}{s\phi(q)} \\
&\times \prod_{p} \left(1 - \frac{\chi_0(p) f_P(p)}{p^s}\right) - \frac{2^{-1}}{\phi(q)} \prod_{p|q} \left(1 - \frac{1}{p}\right) \prod_{p\nmid q} \left(1 - \frac{f_P(p)}{p}\right) \frac{1}{s-2} + \frac{1}{s-\Theta+\epsilon} \\
=&\frac{-1}{s\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \bar{\chi}(l) L(s-1,\chi) \prod_{p} \left(1 - \frac{\chi(p) f_P(p)}{p^s}\right) - \frac{\zeta(s-1)}{s\phi(q)} \prod_{p|q} \left(1 - \frac{1}{p^{s-1}}\right) \\
&\times \prod_{p} \left(1 - \frac{\chi_0(p) f_P(p)}{p^s}\right) + \frac{1}{2\phi(q)} \prod_{p|q} \left(1 - \frac{1}{p}\right) \frac{1}{s-2} \prod_{p\nmid q} \left(1 - \frac{f_P(p)}{p^2}\right) \\
&+ \frac{1}{s-\Theta+\epsilon}.
\end{aligned} \tag{3.66}
$$

By (3.65), we have

$$
\begin{aligned}
\prod_{p} \left(1 - \frac{\chi(p) f_P(p)}{p^s}\right) =& \prod_{i=1}^{J} \mathcal{L}_i(s,\chi')^{-1} \mathscr{P}_{1,i}(\Delta) \prod_{p|\Delta} \left(1 - \frac{\chi(p) f_P(p)}{p^s}\right) \\
&\times \prod_{p|\Delta} \left(1 + \mathrm{O}\left(|p^{-s}|\right)\right)^{-1} \mathscr{P}_2(\Delta)^J.
\end{aligned}
$$

Note that the second term in equation (3.66) has a pole at $s = 2$ due to $\zeta(s-1)$, which will be canceled by the pole of the third term at $s = 2$. Thus, the right-hand side of (3.66) is analytic in the half-plane $\Re(s) > 1$ and using the Haselgrove's condition for Hecke L-function, it can be analytically continued on the real segment

$(\Theta-\epsilon, 1]$ with a simple pole at $s=\Theta-\epsilon$. This yields, $\int_1^\infty \mathcal{A}(Q)Q^{-\sigma-1}dQ < \infty$ for $\sigma > \Theta-\epsilon$. Let us assume that

$$S(Q;q,l) - \frac{Q^2}{2\phi(q)} \prod_{p|q}\left(1-\frac{1}{p}\right)\prod_{p\nmid q}\left(1-\frac{f_P(p)}{p^2}\right) < Q^{\Theta-\epsilon} \text{ for all } Q > Q_0(\epsilon). \quad (3.67)$$

Employing Proposition 2.0.3, we deduce that the integral $\int_1^\infty \mathcal{A}(Q)Q^{-s-1}dQ$ is analytic in the half plane $\Re(s) > \Theta-\epsilon$. In view of the definition of Θ, the product $\prod_p \left(1-\frac{\chi(p)f_P(p)}{p^s}\right)$ has poles with $\Re(s) > \Theta-\epsilon$, due to the zeros of $\mathcal{L}_i(s,\chi')$. This leads to a contradiction, and one can deduce that the assertion (3.67) is not true. Hence

$$S(Q;q,l) - \frac{Q^2}{2\phi(q)} \prod_{p|q}\left(1-\frac{1}{p}\right)\prod_{p\nmid q}\left(1-\frac{f_P(p)}{p^2}\right) = \Omega_+(Q^{\Theta-\epsilon}).$$

To derive the corresponding estimate for Ω_-, we proceed in a similar manner with

$$\mathcal{A}(Q) = S(Q;q,l) - \frac{Q^2}{2\phi(q)} \prod_{p|q}\left(1-\frac{1}{p}\right)\prod_{p\nmid q}\left(1-\frac{f_P(p)}{p^2}\right) + Q^{\Theta-\epsilon},$$

and obtain

$$S(Q;q,l) - \frac{Q^2}{2\phi(q)} \prod_{p|q}\left(1-\frac{1}{p}\right)\prod_{p\nmid q}\left(1-\frac{f_P(p)}{p^2}\right) = \Omega_-(Q^{\Theta-\epsilon}).$$

This completes the proof of Theorem 3.6.2.

4

Distribution of Farey fractions with k-free denominators

Let $k \geq 2$ be an integer. A number n is said to be k-free if for every prime $p|n$, we have $p^k \nmid n$. It is well known that the density of k-free numbers is $1/\zeta(k)$. Denote

$$\mathscr{F}_{Q,k}^{(m)} := \left\{ \frac{a}{q} \mid 1 \leq a \leq q \leq Q,\ (a,q)=1,\ q \text{ is } k\text{-free } \&\ q \equiv b \pmod{m} \right\}, \quad (4.1)$$

where $m \in \mathbb{N},\ b \in \mathbb{Z}$ and $(b,m)=1$.

In this chapter, we prove Theorems 1.1.11, 1.1.12, 1.1.9, 1.1.10, 4.4.1, and 1.1.13 by analyzing Weyl sums, quantitative equidistribution, and correlation measures associated with the sequence $\left(\mathscr{F}_{Q,k}^{(m)}\right)_{Q\geq 1}$. We also examine a criterion for the generalized Riemann hypothesis that is equivalent to the quantitative equidistribution of fractions in $\mathscr{F}_{Q,k}^{(m)}$.

4.1 Key lemmas

In this section, we state and prove several results that will be used throughout the chapter.

4.1.1 Counting Farey fractions with k-free denominators

In this section, we establish asymptotic formula for the number of fractions in $\mathscr{F}_{Q,k}^{(m)}$.

Proposition 4.1.1. *Let m and b be positive integers. Then, we have*

$$\mathcal{N}(Q,k,m) = Q^2\mathscr{C}(k,m) + \mathrm{O}_m\left(Q^{\frac{2(2k-1)}{3k-2}}\exp\left(-c\sqrt{\log Q}\right)\right),$$

where $c > 0$ is some constant and

$$\mathscr{C}(k,m) = \frac{1}{2\phi(m)L(k,\chi_0)}\prod_{p\mid m}\left(1-\frac{1}{p}\right)\prod_{\substack{p\\(p,m)=1}}\left(1-\frac{p^{k-1}-1}{p(p^k-1)}\right). \tag{4.2}$$

Proof. We begin with expressing $\mathcal{N}(Q,k,m)$ in summation form

$$\mathcal{N}(Q,k,m) = \sum_{\substack{n\le Q\\ n\equiv l \pmod m}}\mu_k(n)^2\sum_{\substack{a\le n\\(a,n)=1}}1 = \sum_{\substack{n\le Q\\ n\equiv l \pmod m}}\mu_k(n)^2\phi(n). \tag{4.3}$$

Using Theorem 2.3.1, the above identity can be expressed as

$$\mathcal{N}(Q,k,m) = \frac{1}{\phi(m)}\sum_{\chi \pmod m}\bar{\chi}(b)\sum_{n\le Q}\chi(n)\phi(n)\mu_k(n)^2. \tag{4.4}$$

The Dirichlet series of $\chi(n)\phi(n)\mu_k(n)^2$ is given by

$$\begin{aligned}F(s) &= \sum_{n=1}^{\infty}\frac{\chi(n)\phi(n)\mu_k(n)^2}{n^s}\\ &= \prod_p\left(1+\frac{\chi(p)(p-1)}{p^s}+\frac{\chi(p^2)p(p-1)}{p^{2s}}+\cdots+\frac{\chi(p^{k-1})p^{k-1}(p-1)}{p^{(k-1)s}}\right)\end{aligned}$$

$$= \prod_p \left(1 + \frac{\chi(p)(p-1)}{p^s} \cdot \frac{1 - \frac{\chi(p^{k-1})}{p^{(k-1)(s-1)}}}{1 - \frac{\chi(p)}{p^{(s-1)}}} \right)$$
$$= \frac{L(s-1,\chi)}{L(ks-k,\chi^k)} \prod_p \left(1 + \frac{\chi(p^k) - \chi(p)p^{(k-1)(s-1)}}{p(p^{k(s-1)} - \chi(p^k))} \right), \tag{4.5}$$

which is absolutely convergent for $\Re(s) > 2$ and has an analytic continuation to the half-plane $\Re(s) > 1$ except for a simple pole at $s = 2$ when $\chi = \chi_0$. For some fixed $\alpha = 2 + 1/\log Q$ and the Dirichlet series $F(s)$, we apply Proposition 2.1.1

$$\sum_{n \le Q} \chi(n)\phi(n)\mu_k(n)^2 = \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} F(s) \frac{(Q + \frac{1}{2})^s}{s} ds + \mathrm{O}\left(R(T)\right), \tag{4.6}$$

where

$$R(T) \ll \frac{Q^\alpha}{T} \sum_{n=1}^{\infty} \frac{\phi(n)}{n^\alpha \left| \log \frac{Q+\frac{1}{2}}{n} \right|} \ll \frac{Q^\alpha}{T} \sum_{n=1}^{\infty} \frac{1}{n^{\alpha-1} \left| \log \frac{Q+\frac{1}{2}}{n} \right|}.$$

We divide the sum into three subsums over the following ranges of n: $n \le \frac{Q}{2}$, $\frac{Q}{2} < n < \frac{3Q}{2}$, and $n \ge \frac{3Q}{2}$. For $n \le \frac{Q}{2}$ and $n \ge \frac{3Q}{2}$, it is clear that $\left| \log \frac{Q+\frac{1}{2}}{n} \right| \ge \log \frac{3}{2}$. For values of n satisfying $\frac{Q}{2} < n < \frac{3Q}{2}$ the middle subsum is

$$\ll Q^{1-\alpha} \sum_{\frac{-Q}{2} < n \le \frac{Q}{2}} \frac{1}{\left| \log \frac{Q+\frac{1}{2}}{Q+n} \right|} \ll Q^{2-\alpha} \sum_{\frac{-Q}{2} < n \le \frac{Q}{2}} \frac{1}{|n - \frac{1}{2}|} \ll Q^{2-\alpha} \log Q.$$

Therefore,

$$R(T) \ll \frac{Q^2 \log Q}{T}. \tag{4.7}$$

We use the zero-free region for the Dirichlet L-functions modulo m to estimate the integral in (4.6). We shift the line integral to the left of the line $\Re(s) = \alpha$, thereby replacing it by a rectangular contour with vertices $\alpha \pm iT$ and $\beta \pm iT$, where $\beta = 1 + 1/k - c/k\sqrt{\log T}$.

Principal Dirichlet character: We first consider the principal Dirichlet character $\chi = \chi_0$. Since the integrand in (4.6) is holomorphic on and within this contour except for a pole at $s=2$. Thus, by Cauchy's residue theorem, we have

$$\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} F(s)\frac{(Q+\frac{1}{2})^s}{s}ds = \frac{(Q+\frac{1}{2})^2}{2L(k,\chi_0)}\prod_{p|m}\left(1-\frac{1}{p}\right)\prod_{\substack{p\\(p,m)=1}}\left(1-\frac{p^{k-1}-1}{p(p^k-1)}\right)+\sum_{j=1}^{3} I_j,$$

where I_1 and I_3 are integrals along the horizontal segments $[\alpha-iT,\beta-iT]$ and $[\beta+iT,\alpha+iT]$, respectively and I_2 is defined as the integral along the vertical segment $[\beta-iT,\beta+iT]$. In order to estimate the integrals I_j's, we use the standard bounds for $\zeta(s)$ provided in (2.2), modulo multiplication by constant depending on m. Therefore,

$$\begin{aligned} I_1, I_3 &\ll_m \int_\beta^\alpha \frac{Q^\sigma|\zeta(\sigma-1+iT)|d\sigma}{|\sigma+iT||\zeta(k\sigma-k+ikT)|} \ll_m \frac{(\log T)^2}{T}\left(\int_\beta^2 Q^\sigma T^{1-\frac{\sigma}{2}}d\sigma+\int_2^\alpha Q^\sigma d\sigma\right)\\ &\ll_m \frac{Q^2(\log T)^2}{T}. \end{aligned}$$

Next, we estimate the integral I_2 using the Proposition 2.4.2.

$$\begin{aligned} I_2 &\ll_m Q^\beta \int_0^T \frac{|\zeta(\beta-1+it)|}{|\beta+it||\zeta(k\beta-k+ikt)|}dt \ll_m Q^\beta \log T\int_0^T \frac{|\zeta(\beta-1+it)|}{|\beta+it|}dt\\ &\ll_m Q^\beta T^{\frac{3}{2}-\beta}(\log T)^2. \end{aligned}$$

Non-principal Dirichlet character: We next consider the case for non-principal character $\chi \pmod m$. We continue with the contour defined above and use the bounds for Dirichlet L-function provided in (2.5). Therefore

$$\begin{aligned} I_1, I_3 &\ll_m \int_\beta^\alpha \frac{Q^\sigma|L(\sigma-1+iT,\chi)|}{|\sigma+iT||L(k\sigma-k+ikT,\chi^k)|}d\sigma \ll_m \log T\left((\log T)^3\int_\beta^{\frac{3}{2}}\frac{Q^\sigma T^{\frac{127-73\sigma}{108}}}{T}\right.\\ &\qquad \left.\times\, d\sigma+(\log T)^3\int_{\frac{3}{2}}^2 \frac{Q^\sigma T^{\frac{35(2-\sigma)}{108}}}{T}d\sigma+\int_2^\alpha \frac{Q^\sigma\log T}{T}d\sigma\right) \ll_m \frac{Q^\alpha(\log T)^2}{T\log Q}, \end{aligned}$$

and using the bound $|L(k\sigma-k+ikt,\chi^k)| \gg_m 1/\log T$ (see (2.7)) and Proposition 2.4.3, we obtain

$$I_2 \ll_m \int_{-T}^{T}\frac{|Q^{\beta+it}||L(\beta-1+it,\chi)|}{|\beta+it||L(k\beta-k+ikt,\chi^k)|}dt \ll_m Q^\beta\log T\int_0^T\frac{|L(\beta-1+it,\chi)|}{|\beta-1+it|}dt$$

$$\ll_m Q^{\beta} T^{\frac{3}{2}-\beta} (\log T)^2.$$

We choose optimally

$$T = Q^{\frac{2(k-1)}{3k-2}} \exp\left(c\sqrt{\log T}\right).$$

By collecting all the above estimates, we obtain the required result. □

4.1.2 Averages of weighted Möbius function

Lemma 4.1.1. *Let* $b \in \mathbb{Z}$ *and* d, l, m *be positive integers. If* $\xi_{d,k}(n) = \mu_k(nd)^2$ *then for* $x \geq 1$*, we have*

$$\sum_{\substack{n \leq x \\ (n,\ell)=1 \\ n \equiv b \pmod{m}}} \mu(n)\xi_{d,k}(n)$$

$$\ll_m \begin{cases} x \exp\left(-c\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right) \prod_{p|d} \left(\frac{\sqrt{p}}{\sqrt{p}-1}\right) \prod_{p|\ell} \left(1-\frac{1}{\sqrt{p}}\right)^{-1}, & \textit{unconditionally,} \\ x^{\frac{1}{2}+\epsilon} \prod_{p|d} \left(1+\frac{1}{\sqrt{p}-1}\right) \prod_{p|\ell} \left(1-\frac{1}{\sqrt{p}}\right)^{-1}, & \textit{on the GRH.} \end{cases}$$

Proof. It is easy to observe that if d is not k-free, then the result follows trivially. Thus, we assume that d is k-free. Using Proposition (2.3.1), we have

$$\sum_{\substack{n \leq x \\ (n,\ell)=1 \\ n \equiv b \pmod{m}}} \mu(n)\xi_{d,k}(n) = \frac{1}{\phi(m)} \sum_{\chi} \chi(\bar{b}) \sum_{\substack{n \leq x \\ (n,\ell)=1}} \chi(n)\mu(n)\xi_{d,k}(n).$$

Note that $\xi_{d,k}(n)$ is a multiplicative function of n. Let $(n_1, n_2) = 1$. If $n_1 n_2 d$ is k-free, then it is easy to observe that $n_1 d$ and $n_2 d$ are k-free. Conversely, suppose that $n_1 d$ and $n_2 d$ are k-free. We need to show that $n_1 n_2 d$ is also k-free. Suppose, for contradiction, that $n_1 n_2 d$ is not k-free; that is, there exists a prime p such that $p^k | n_1 n_2 d$. Since $(n_1, n_2) = 1$, it follows that either $p^k | n_1 d$ or $p^k | n_2 d$, which is a contradiction. This proves that $\xi_{d,k}(n)$ is a multiplicative function of n. The

Dirichlet series of $\chi(n)\mu(n)\xi_{d,k}(n)$ is given by

$$\begin{aligned}
F(s) &= \sum_{\substack{n=1\\(n,\ell)=1}}^{\infty} \frac{\chi(n)\mu(n)\xi_{d,k}(n)}{n^s} = \prod_{\substack{p\\(p,\ell)=1}} \left(1-\frac{\chi(p)\xi_{d,k}(p)}{p^s}\right)\\
&= \frac{1}{L(s,\chi)} \prod_{p} \left(1+\frac{\chi(p)(1-\xi_{d,k}(p))}{p^s}\left(1-\frac{\chi(p)}{p^s}\right)^{-1}\right) \prod_{p\mid \ell}\left(1-\frac{\chi(p)\xi_{d,k}(p)}{p^s}\right)^{-1}\\
&= \frac{1}{L(s,\chi)} \prod_{p\mid d} \left(1+\frac{\chi(p)(1-\xi_{d,k}(p))}{p^s}\left(1-\frac{\chi(p)}{p^s}\right)^{-1}\right) \prod_{p\mid \ell}\left(1-\frac{\chi(p)\xi_{d,k}(p)}{p^s}\right)^{-1}.
\end{aligned}$$

In the last step, we used the fact that $\xi_{d,k}(p)=1$ if $(p,d)=1$. The Dirichlet series $F(s)$ is absolutely convergent for $\Re(s)\geq \beta$, where $\beta = 1-c/(\log T)^{2/3}(\log\log T)^{1/3}$. Employing Proposition 2.1.1 for the Dirichlet series $F(s)$ with $\alpha = 1+\frac{1}{\log x}$, we have

$$\sum_{\substack{n\leq x\\(n,\ell)=1}} \chi(n)\mu(n)\xi_{d,k}(n) = \frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} F(s)\frac{x^s}{s}ds + \mathrm{O}\left(R(T)\right),$$

where

$$R(T) \ll \frac{x^{\alpha}}{T}\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}|\log x/n|} \ll \frac{x\log x}{T}. \tag{4.8}$$

In here, we bound the error term $R(T)$ as in (4.7). We next move the path of integration into a rectangular contour with line segments $[\alpha-iT,\alpha+iT]$, $[\alpha+iT,\beta+iT]$, $[\beta+iT,\beta-iT]$, and $[\beta-iT,\alpha-iT]$. For $\beta\leq\sigma\leq\alpha$, we have

$$\left|\prod_{p\mid\ell}\left(1-\frac{\chi(p)\xi_{d,k}(p)}{p^s}\right)^{-1}\right| \leq \prod_{p\mid\ell}\left(1-\frac{1}{\sqrt{p}}\right)^{-1}$$

and

$$\left|\prod_{p\mid d}\left(1+\frac{\chi(p)(1-\xi_{d,k}(p))}{p^s-\chi(p)}\right)\right| \leq \prod_{p\mid d}\left(1+\frac{1}{\sqrt{p}-1}\right).$$

By Cauchy's theorem, we have

$$\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} F(s)\frac{x^s}{s}ds = \frac{1}{2\pi i}\left(\int_{\alpha-iT}^{\beta-iT}+\int_{\beta-iT}^{\beta+iT}+\int_{\beta+iT}^{\alpha+iT}\right)F(s)\frac{x^s}{s}ds := I_1+I_2+I_3.$$

We first estimate the integrals I_1 and I_3:

$$I_1, I_3 \ll_m \frac{x\log T}{T\log x}\prod_{p\mid\ell}\left(1-\frac{1}{\sqrt{p}}\right)^{-1}\prod_{p\mid d}\left(1+\frac{1}{\sqrt{p}-1}\right).$$

The integral I_2 is estimated as

$$I_2 \ll_m x^{\beta}(\log T)^2\prod_{p\mid d}\left(1+\frac{1}{\sqrt{p}-1}\right)\prod_{p\mid\ell}\left(1-\frac{1}{\sqrt{p}}\right)^{-1}.$$

We collect all the above estimate and take $T=\exp\left(\frac{c(\log x)^{\frac{3}{5}}}{(\log\log x)^{\frac{1}{5}}}\right)$. This completes the proof unconditionally. Assuming GRH, the Dirichlet series $F(s)$ is absolutely convergent for $\Re(s)>1/2$. By using Perron's formula with $\alpha=1+\frac{1}{\log x}$ and $\beta=\frac{1}{2}+\epsilon$, and proceeding in a similar manner as in the unconditional case, we obtain the proof under GRH. This completes the proof of Lemma 4.1.1. $\square$

Lemma 4.1.2. *Let $b\in\mathbb{Z}$, and let d, l, m be positive integers. Suppose d is k-free and $\xi_{d,k}(n)=\mu_k(nd)^2$. For $x\geq 2$, we have*

$$\sum_{\substack{n\leq x\\ (n,\ell)=1\\ n\equiv b \pmod{m}}}\frac{\xi_{d,k}(n)}{n} = \mathcal{M}_{m,d,l}(x)+\mathrm{O}_{k,m,l,d}\left(x^{\frac{-2(k-1)}{3k-2}}\exp\left(-c\sqrt{\log x}\right)\right),$$

where c is some positive constant and

$$\begin{aligned}\mathcal{M}_{m,d,l}(x)=&\left(\sum_{\substack{p\mid d\\(p,m)=1}}\left(\frac{-k\log p}{p^k-1}+\frac{\log p}{p-1}-\log p\sum_{j=1}^{k-1}\frac{j\xi_{d,k}(p^j)}{p^j}\left(1+\sum_{j=1}^{k-1}\frac{\xi_{d,k}(p^j)}{p^j}\right)^{-1}\right)\right.\\ &+\log p\sum_{\substack{p\mid\ell\\(p,m)=1}}\sum_{j=1}^{k-1}\frac{j\xi_{d,k}(p^j)}{p^j}\left(1+\sum_{j=1}^{k-1}\frac{\xi_{d,k}(p^j)}{p^j}\right)^{-1}-k\frac{L'(k,\chi_0)}{L(k,\chi_0)}+\log x\\ &\left.+\gamma+\sum_{p\mid m}\frac{\log p}{p-1}\right)\frac{1}{L(k,\chi_0)}\prod_{p\mid m}\left(1-\frac{1}{p}\right)\prod_{\substack{p\mid d\\(p,m)=1}}\left(1+\sum_{j=1}^{k-1}\frac{\xi_{d,k}(p^j)}{p^j}\right)\end{aligned}$$

$$\times \prod_{\substack{p\mid d\\(p,m)=1}} \left(1-\frac{1}{p^k}\right)^{-1} \left(1-\frac{1}{p}\right) \prod_{\substack{p\mid \ell\\(p,m)=1}} \left(1+\sum_{j=1}^{k-1} \frac{\xi_{d,k}(p^j)}{p^j}\right)^{-1}.$$

Proof. We take into account Proposition 2.3.1 to obtain

$$\sum_{\substack{n\le x\\(n,\ell)=1\\n\equiv b \pmod m}} \frac{\xi_{d,k}(n)}{n} = \frac{1}{\phi(m)} \sum_{\chi} \chi(\bar{b}) \sum_{\substack{n\le x\\(n,\ell)=1}} \frac{\chi(n)\xi_{d,k}(n)}{n}.$$

The Dirichlet series of $\frac{\chi(n)\xi_{d,k}(n)}{n}$ is as follows:

$$\begin{aligned}
F(s) = \sum_{\substack{n=1\\(n,l)=1}}^{\infty} \frac{\xi_{d,k}(n)\chi(n)}{n^{s+1}} &= \prod_{\substack{p\\(p,l)=1}} \left(1+\frac{\chi(p)\xi_{d,k}(p)}{p}+\cdots+\frac{\chi(p^{k-1})\xi_{d,k}(p^{k-1})}{p^{k-1}}\right)\\
&= \frac{L(s+1,\chi)}{L(ks+k,\chi^k)} \prod_{p\mid d} \left(1-\frac{\chi(p)}{p^{s+1}}\right)\left(1-\frac{\chi(p^k)}{p^{k(s+1)}}\right)^{-1}\\
&\times \prod_{p\mid d}\left(1+\sum_{j=1}^{k-1} \frac{\xi_{d,k}(p^j)\chi(p^j)}{p^{j(s+1)}}\right) \prod_{p\mid \ell}\left(1+\sum_{j=1}^{k-1} \frac{\xi_{d,k}(p^j)\chi(p^j)}{p^{j(s+1)}}\right)^{-1}.
\end{aligned}$$

Note that $F(s)$ is absolutely convergent for $\Re(s) > 0$ and it can be analytically continued to the half-plane $\Re(s) = \beta > -1 + \frac{1}{k} - \frac{c}{k\sqrt{\log T}}$ except for a pole at $s = 0$ when $\chi = \chi_0$. For some fixed $\alpha = 1/\log x$ and the Dirichlet series $F(s)$, we apply Proposition 2.1.1

$$\sum_{\substack{n\le x\\(n,\ell)=1}} \frac{\xi_{d,k}(n)\chi(n)}{n} = \frac{1}{2\pi i} \int_{\alpha-iT}^{\alpha+iT} F(s)\frac{x^s}{s} ds + \mathrm{O}\left(R(T)\right), \tag{4.9}$$

where

$$R(T) \ll \frac{x^\alpha}{T} \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}|\log x/n|} \ll \frac{\log x}{T}.$$

To estimate the integral on the right hand side of (4.9), we shift the line of integral into a rectangular contour with vertices $\alpha \pm iT$ and $\beta \pm iT$. We first consider the principal character. In this case, the integrand in (4.9) has a pole of order 2 at

$s = 0$. Denote

$$Z(s) = \frac{x^s L(s+1,\chi)}{sL(ks+k,\chi^k)} \prod_{p|d}\left(1-\frac{\chi(p)}{p^{s+1}}\right)\left(1-\frac{\chi(p^k)}{p^{k(s+1)}}\right)^{-1}\left(1+\sum_{j=1}^{k-1}\frac{\xi_{d,k}(p^j)\chi(p^j)}{p^{j(s+1)}}\right)$$
$$\times \prod_{p|\ell}\left(1+\sum_{j=1}^{k-1}\frac{\xi_{d,k}(p^j)\chi(p^j)}{p^{j(s+1)}}\right)^{-1}.$$

By Cauchy's residue theorem, we have

$$\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} F(s)\frac{x^s}{s}ds = \mathrm{Res}_{s=0}Z(s) + \sum_{i=1}^{3} I_i,$$

where I_1 and I_3 are integrals along horizontal segments $[\alpha - iT, \beta - iT]$ and $[\beta + iT, \alpha + iT]$, respectively and I_2 is the integral along vertical segment $[\beta - iT, \beta + iT]$. The first term in the above identity is the residue of the second order pole of $Z(s)$ at $s = 0$, and is given by $\mathcal{M}_{m,d,l}(x)$. We use the standard bounds for the Riemann zeta function $\zeta(s)$ (see (2.2)), modulo multiplication by constants depending on d and ℓ

$$I_1, I_3 \ll_{m,d,\ell} \frac{\log T}{T}\left(\int_\beta^0 x^\sigma T^{\frac{-\sigma}{2}}d\sigma + \int_0^\alpha x^\sigma \log T d\sigma\right) \ll_{m,d,\ell} \frac{(\log T)^2}{T\log x}.$$

We next estimate the integral I_2 using Proposition 2.4.2.

$$I_2 \ll_{m,d,\ell} x^\beta \log T \int_0^T \frac{|\zeta(\beta+1+it)|}{|\beta+it|}dt \ll_{m,d,\ell} x^\beta T^{\frac{-1}{2}-\beta}(\log T)^2.$$

We next consider the case for the non-principal character $\chi \neq \chi_0$. We continue with the contour defined above. Using the bounds (2.5) for the Dirichlet L-function modulo m and Proposition 2.4.3, we obtain

$$I_1, I_3 \ll_{m,d,\ell} \frac{(\log T)^2}{T\log x} \text{ and } I_2 \ll_{m,d,\ell} x^\beta T^{\frac{-1}{2}-\beta}(\log T)^2.$$

We collect all the above estimate and take optimally $T = x^{\frac{2(k-1)}{3k-2}}\exp\left(c\sqrt{\log x}\right)$. This completes the proof of Lemma 4.1.2. □

Lemma 4.1.3. *For $x \geq 1$, we have*

$$\sum_{n\leq x}\mu_k(n)^2\prod_{p|n}\left(1-\frac{1}{\sqrt{p}}\right)^{-1} =\frac{x}{\zeta(k)}\prod_{p}\left(1+\frac{p^{k-1}-1}{(\sqrt{p}-1)(p^k-1)}\right) + \mathrm{O}\left(x^{\frac{k}{3k-2}+\epsilon}\right).$$

Proof. The arithmetic function $\mu_k(n)^2\prod_{p|n}\left(1-\frac{1}{\sqrt{p}}\right)^{-1}$ is multiplicative, so its Dirichlet series is given by

$$\begin{aligned}
F(s) &= \sum_{n=1}^{\infty}\frac{\mu_k(n)^2}{n^s}\prod_{p|n}\left(1-\frac{1}{p^{\frac{1}{2}}}\right)^{-1}\\
&=\prod_p\left(1+\left(1-\frac{1}{\sqrt{p}}\right)^{-1}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots+\frac{1}{p^{(k-1)s}}\right)\right)\\
&=\prod_p\left(1+\left(1-\frac{1}{\sqrt{p}}\right)^{-1}\left(\frac{1-\frac{1}{p^{(k-1)s}}}{p^s(1-p^{-s})}\right)\right)\\
&=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}\prod_p\left(1-\frac{1}{p^s}+\frac{\sqrt{p}}{\sqrt{p}-1}\frac{1-\frac{1}{p^{(k-1)s}}}{p^s}\right)\\
&=\frac{\zeta(s)}{\zeta(ks)}\prod_p\left(1-\frac{1}{p^{ks}}\right)^{-1}\left(1-\frac{1}{p^{ks}}+\frac{1}{(\sqrt{p}-1)p^s}-\frac{1}{(\sqrt{p}-1)p^{ks}}\right)\\
&=\frac{\zeta(s)}{\zeta(ks)}\prod_p\left(1+\frac{p^{(k-1)s}-1}{(\sqrt{p}-1)(p^{ks}-1)}\right).
\end{aligned}$$

Note that $F(s)$ is absolutely convergent for $\Re(s)>1$ and can be analytically continued for $\Re(s)>\frac{1}{k}$ except for a simple pole at $s=1$. Let $T\geq 2$ and $\alpha=1+\frac{1}{\log x}$. We apply Proposition 2.1.1 for the Dirichlet series $F(s)$. Therefore

$$\sum_{n\leq x}\mu_k(q)^2\prod_{p|n}\left(1-\frac{1}{\sqrt{p}}\right)^{-1}=\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}F(s)\frac{x^s}{s}ds+\mathrm{O}\left(R(T)\right), \tag{4.10}$$

where

$$R(T)\ll\frac{x^\alpha}{T}\sum_{n=1}^{\infty}\frac{2^{\omega(n)}}{n^\alpha|\log x/n|}\ll\frac{x^{1+\epsilon}\log x}{T}.$$

In the last step, we estimate the error term as in (4.7). To estimate the integral in

(4.10), we consider the rectangular contour with line segments connecting the points $\beta - iT, \beta - iT, \alpha - iT$, and $\alpha + iT$, where $\beta = \frac{1}{k}$. Since the integrand has a pole at $s = 1$ thus by Cauchy's residue theorem, we have

$$\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} F(s)\frac{x^s}{s}ds = \frac{x}{\zeta(k)}\prod_p\left(1+\frac{p^{k-1}-1}{(p^{\frac{1}{2}}-1)(p^k-1)}\right)+\sum_{j=1}^{3} I_j, \tag{4.11}$$

where I_1 and I_3 are integrals along the horizontal segments $[\beta + iT, \alpha + iT]$ and $[\beta - iT, \alpha - iT]$, respectively, and I_2 denotes the integral along the vertical segment $[\beta + iT, \beta + iT]$. We estimate the integrals in the above equation using the bounds in (2.2) and (2.4). Therefore,

$$\begin{aligned} I_1, I_3 &\ll_k \int_\beta^\alpha \frac{x^\sigma|\zeta(\sigma+iT)|}{|\sigma+iT||\zeta(k\sigma+ikT)|}d\sigma \\ &\ll_k \frac{\log T}{T}\left(\int_\beta^1 |\zeta(\sigma+iT)|x^\sigma d\sigma + \int_1^\alpha |\zeta(\sigma+iT)|x^\sigma d\sigma\right) \\ &\ll_k \frac{\log^2 T}{T}\left(\int_\beta^1 T^{\frac{1-\sigma}{2}}x^\sigma d\sigma + \int_1^\alpha x^\sigma d\sigma\right) \ll_k \frac{x\log^2 T}{T\log x}. \end{aligned} \tag{4.12}$$

Next, we estimate the integral I_2 using Proposition 2.4.1.

$$I_2 \ll_k x^\beta \log T \int_0^T \frac{|\zeta(\beta+it)|}{|\beta+it|}dt \ll_k x^\beta T^{\frac{1}{2}-\beta}\log^2 T. \tag{4.13}$$

Combining the estimates from (4.11), (4.12), (4.13) with the choice $T = x^{\frac{2(k-1)}{3k-2}}$ and inserting in (4.10) gives the required result. □

4.1.3 Weighted k-free Farey sums

We expand the Farey sums for the Farey fractions in $\mathscr{F}_{Q,k}^{(m)}$ using the Möbius and the k-free Möbius function. This is helpful for the reduction and estimation of the exponential sum for $\mathscr{F}_{Q,k}^{(m)}$.

Lemma 4.1.4. *Assume that f is any complex-valued function defined on the interval*

$[0,1]$, *and let* $\gamma_i \in \mathscr{F}_{Q,k}^{(m)}$ *for* $1 \le i \le \mathcal{N}(Q,k,m)$*. Then, we have*

$$\sum_{j=1}^{\mathcal{N}(Q,k,m)} f(\gamma_j) = \sum_{q \le Q} M_q\left(\frac{Q}{q}\right) \sum_{a \le q} f\left(\frac{a}{q}\right),$$

where

$$M_q(x) = \sum_{\substack{n \le x \\ qn \equiv b \pmod m}} \mu(n)\mu_k(qn)^2.$$

Proof. We write

$$\sum_{j=1}^{\mathcal{N}(Q,k,m)} f(\gamma_j) = \sum_{\substack{q \le Q \\ q \equiv b \pmod m}} \mu_k(q)^2 \sum_{\substack{a \le q \\ (a,q)=1}} f\left(\frac{a}{q}\right).$$

Employing Theorem 2.0.1, we obtain

$$\begin{aligned}
\sum_{j=1}^{\mathcal{N}(Q,k,m)} f(\gamma_j) &= \sum_{\substack{q \le Q \\ q \equiv b \pmod m}} \mu_k(q)^2 \sum_{a \le q} f\left(\frac{a}{q}\right) \sum_{\substack{d|a \\ d|q}} \mu(d) \\
&= \sum_{d \le Q} \mu(d) \sum_{\substack{q \le Q \\ q \equiv b \pmod m \\ d|q}} \mu_k(q)^2 \sum_{\substack{a \le q \\ d|a}} f\left(\frac{a}{q}\right) \\
&= \sum_{d \le Q} \mu(d) \sum_{\substack{q \le \frac{Q}{d} \\ qd \equiv b \pmod m}} \mu_k(qd)^2 \sum_{a \le q} f\left(\frac{a}{q}\right) = \sum_{q \le Q} M_q\left(\frac{Q}{q}\right) \sum_{a \le q} f\left(\frac{a}{q}\right).
\end{aligned}$$

□

Lemma 4.1.5. *Let* $f(x) = x - \lfloor x \rfloor - \frac{1}{2}$ *and* M_n *as in Lemma 4.1.4. For any real number* $u \in [0,1]$ *lying between two successive Farey fractions* γ_v *and* γ_{v+1} *in* $\mathscr{F}_{Q,k}^{(m)}$*, we have*

$$\sum_{j=1}^{\mathcal{N}(Q,k,m)} f(u+\gamma_j) = \mathcal{N}(Q,k,m)u - v - \frac{1}{2}.$$

Proof. Employing Lemma 4.1.4 and using the fact that

$$\sum_{a\leq n} f\left(u+\frac{a}{n}\right) = f(nu),$$

we can write

$$\begin{aligned}\sum_{j=1}^{\mathcal{N}(Q,k,m)} f(u+\gamma_j) &= \sum_{n\leq Q} f(nu)M_n\left(\frac{Q}{n}\right) = \sum_{n\leq Q}\left(nu-\lfloor nu\rfloor-\frac{1}{2}\right)M_n\left(\frac{Q}{n}\right)\\ &= \mathcal{N}(Q,k,m)u-\sum_{n\leq Q}\lfloor nu\rfloor M_n\left(\frac{Q}{n}\right)-\frac{1}{2}.\end{aligned}$$

Note that the sum on the right-hand side of the above equation counts the number of fractions in $\mathscr{F}_{Q,k}^{(m)}$ less than or equal to u. Therefore between γ_v and γ_{v+1} the above sum is equal to $\mathcal{N}(Q,k,m)u-v-\frac{1}{2}$. □

4.2 GRH and Farey fractions

In this section, we provide proof of Theorems 1.1.11 and 1.1.12.

Recall that the Farey sequence is the set of fractions a/q in $[0,1]$ such that $(a,q)=1$ with denominators at most Q. We detect this condition using the following identity

$$\sum_{d|n}\mu(d)=\begin{cases}1 & \text{if } n=1,\\ 0 & \text{if } n>1.\end{cases}$$

Thus, the counting of Farey fractions can be expressed in terms of sums involving $\mu(n)$. These Möbius sums are closely related to the zeros of the Riemann zeta function. In particular, the Riemann Hypothesis holds if and only if

$$\sum_{n\leq x}\mu(n)=o\left(x^{\frac{1}{2}+\epsilon}\right) \text{ for all } \epsilon>0.$$

This establishes a connection between the distribution of the Farey sequence and the Riemann Hypothesis. In particular, we have the results of Franel [40] and Landau [72] given in (1.9) and (1.10), respectively. For Farey sequences with denominators in a fixed arithmetic progression, Dirichlet characters arise naturally, leading to twisted sum

$$\sum_{n\leq x}\chi(n)\mu(n)=o\left(x^{\frac{1}{2}+\epsilon}\right)\text{ for all }\epsilon>0,$$

which is equivalent to the Generalized Riemann Hypothesis. Indeed, Davenport proposed in his problem list that analogous results to Franel and Landau should hold for the zeros of a fixed Dirichlet L-function. Motivated with this, Huxley [56] established analogous result to those of Farnel and Landau. Here, we extend this line of investigation and study the analogues of the results of Franel and Landau for the Farey fractions with k-free denominators lying in a fixed arithmetic progression.

The proof of Theorems 1.1.11 and 1.1.12 involves decomposing the weighted sum of Merten's function with congruence constraints in two different forms. To establish the bounds, we employ the Dirichlet hyperbola method alongside the non-trivial bounds for a twisted Möbius sum. We now proceed to prove Theorem 1.1.11.

4.2.1 Proof of Theorem 1.1.11

We first assume that

$$\sum_{j=1}^{\mathcal{N}(Q,k,m)} R_{\mathcal{N}(Q,k,m)}(\gamma_j)=\mathrm{O}_m\left(Q^{\frac{1}{2}+\epsilon}\right).$$

We apply Lemma 4.1.4 with $f(x)=e(x)$ and obtain

$$\sum_{v=1}^{\mathcal{N}(Q,k,m)} e(\gamma_v)=\sum_{q\leq Q}M_q\left(\frac{Q}{q}\right)\sum_{a\leq q}e\left(\frac{a}{q}\right)=M_1(Q),$$

where $M_1(Q)=M(Q)=\sum_{\substack{n\leq Q\\ n\equiv b \pmod m}}\mu(n)$. In the last step, we used the following identity

$$\sum_{a\leq q}e\left(\frac{a}{q}\right)=\begin{cases}1, & \text{if } q=1,\\ 0, & \text{otherwise.}\end{cases}$$

We have

$$\begin{aligned}
M(Q) &= \sum_{v=1}^{\mathcal{N}(Q,k,m)} e(\gamma_v) = \sum_{v=1}^{\mathcal{N}(Q,k,m)} e\left(\gamma_v - \frac{v}{\mathcal{N}(Q,k,m)} + \frac{v}{\mathcal{N}(Q,k,m)}\right) \\
&= \sum_{v=1}^{\mathcal{N}(Q,k,m)} e(R_{\mathcal{N}(Q,k,m)}(\gamma_v)) e\left(\frac{v}{\mathcal{N}(Q,k,m)}\right) \\
&= \sum_{v=1}^{\mathcal{N}(Q,k,m)} e\left(\frac{v}{\mathcal{N}(Q,k,m)}\right)\left(e(R_{\mathcal{N}(Q,k,m)}(\gamma_v)) - 1\right) \\
&\quad + \sum_{v=1}^{\mathcal{N}(Q,k,m)} e\left(\frac{v}{\mathcal{N}(Q,k,m)}\right).
\end{aligned}$$

This yields

$$\begin{aligned}
|M(Q)| &\le \sum_{v=1}^{\mathcal{N}(Q,k,m)} |e(R_{\mathcal{N}(Q,k,m)}(\gamma_v)) - 1| \le 2 \sum_{v=1}^{\mathcal{N}(Q,k,m)} |\sin \pi R_{\mathcal{N}(Q,k,m)}(\gamma_v)| \\
&\le 2\pi \sum_{v=1}^{\mathcal{N}(Q,k,m)} R_{\mathcal{N}(Q,k,m)}(\gamma_v) \ll_m Q^{\frac{1}{2}+\epsilon}.
\end{aligned}$$

Thus, GRH holds. For the converse, assume that GRH is true. We apply Lemma 4.1.4 with $f(x) = x - \lfloor x \rfloor - \frac{1}{2}$. We have

$$G(u) = \sum_{v=1}^{\mathcal{N}(Q,k,m)} f(u+\gamma_v) = \sum_{q \le Q} M_q\left(\frac{Q}{q}\right) \sum_{a \le q} f\left(u + \frac{a}{q}\right) = \sum_{q \le Q} M_q\left(\frac{Q}{q}\right) f(qu).$$

In here, for $f(x) = x - \lfloor x \rfloor - \frac{1}{2}$, we have used the fact that

$$\sum_{a \le q} f\left(u + \frac{a}{q}\right) = f(qu).$$

We denote

$$I := \int_0^1 (G(u))^2 du. \tag{4.14}$$

Case-I: If

$$G(u) = \sum_{q \le Q} M_q\left(\frac{Q}{q}\right) f(qu).$$

Substituting in (4.14), we have

$$\begin{aligned} I &= \int_0^1 \sum_{q_1 \le Q} f(q_1 u) M_{q_1}\left(\frac{Q}{q_1}\right) \sum_{q_2 \le Q} f(q_2 u) M_{q_2}\left(\frac{Q}{q_2}\right) du \\ &= \sum_{q_1, q_2 \le Q} M_{q_1}\left(\frac{Q}{q_1}\right) M_{q_2}\left(\frac{Q}{q_2}\right) \int_0^1 f(q_1 u) f(q_2 u) du. \end{aligned} \tag{4.15}$$

We now estimate the integral in the above identity. If $q_2 = 1$, then

$$\begin{aligned} \int_0^1 f(q_1 u) f(u) du &= \int_0^{q_1} f(u) f(q_1^{-1} u) du = \frac{1}{q_1} \sum_{m=0}^{q_1 - 1} \int_0^1 f(m+t) f\left(\frac{m}{q_1} + \frac{t}{q_1}\right) dt \\ &= \frac{1}{q_1} \int_0^1 f(t) f\left(q_1 \cdot \frac{t}{q_1}\right) dt = \frac{1}{q_1} \int_0^1 \left(t - \frac{1}{2}\right)^2 dt = \frac{1}{12 q_1}. \end{aligned} \tag{4.16}$$

If $(q_1, q_2) = 1$, then clearly $(q_2 m / q_1) \pmod 1$ for $m = 0, 1, \ldots, q_1 - 1$ yields each fraction j/q_1 exactly once. We have

$$\int_0^1 f(q_1 u) f(q_2 u) du = \frac{1}{q_1} \int_0^1 f(t) f\left(q_1 \cdot \frac{q_2 t}{q_1}\right) dt = \frac{1}{q_1} \int_0^1 f(t) f\left(q_2 t\right) dt = \frac{1}{12 q_1 q_2}. \tag{4.17}$$

In the last step, we have used (4.16). Now, if $\gcd(q_1, q_2) = q$ then $q_1 = q q_1'$ and $q_2 = q q_2'$, where $(q_1', q_2') = 1$. Thus, we have

$$\int_0^1 f(q_1 u) f(q_2 u) du = \int_0^1 f(q q_1' u) f(q q_2' u) du = \frac{1}{q} \int_0^q f(q_1' t) f(q_2' t) dt.$$

Using (4.17), we have

$$\int_0^1 f(q_1 u) f(q_2 u) du = \frac{1}{12 q_1' q_2'} = \frac{(\gcd(q_1, q_2))^2}{12 q_1 q_2}.$$

Hence, the above estimate with (4.15) gives

$$I = \frac{1}{12} \sum_{q_1, q_2 \le Q} M_{q_1}\left(\frac{Q}{q_1}\right) M_{q_2}\left(\frac{Q}{q_2}\right) \frac{(\gcd(q_1, q_2))^2}{q_1 q_2}. \tag{4.18}$$

If GRH holds, then by employing Lemma 4.1.1, we obtain

$$M_q\left(\frac{Q}{q}\right) = \sum_{\substack{d\leq \frac{Q}{q}\\ qd\equiv b \pmod m}} \mu(d)\mu_k(qd)^2 \ll_m \left(\frac{Q}{q}\right)^{\frac{1}{2}+\epsilon} \prod_{p|q}\left(1+\frac{1}{\sqrt{p}-1}\right).$$

The above estimate with (4.18) yields

$$I \leq CQ^{1+2\epsilon}\sum_{q_1,q_2\leq Q}\frac{(\gcd(q_1,q_2))^2}{(q_1q_2)^{\frac{3}{2}+\epsilon}} \leq CQ^{1+2\epsilon}\sum_{\delta\leq Q}\frac{1}{\delta^{1+\epsilon}}\sum_{\substack{q_1,q_2\leq\frac{Q}{\delta}\\(q_1,q_2)=1}}\frac{1}{(q_1q_2)^{\frac{3}{2}+\epsilon}} \leq CQ^{1+2\epsilon}, \tag{4.19}$$

where $C>0$ is constant depending on m.

Case-II: Next, we apply Lemma 4.1.5, which implies that between γ_v and γ_{v+1}, the value of G is given by the closed form formula $G(u) = -1/2+\mathcal{N}(Q,k,m)u-v$. Therefore

$$\begin{aligned}
I = &\sum_{v=1}^{\mathcal{N}(Q,k,m)}\int_{\gamma_{v-1}}^{\gamma_v}\left(\frac{1}{2}+u\mathcal{N}(Q,k,m)-v\right)^2 du\\
=&\frac{1}{3\mathcal{N}(Q,k,m)}\sum_{v=1}^{\mathcal{N}(Q,k,m)}\left(\left(\gamma_v\mathcal{N}(Q,k,m)-v+\frac{1}{2}\right)^3\right.\\
&\left.-\left(\gamma_{v-1}\mathcal{N}(Q,k,m)-v+\frac{1}{2}\right)^3\right)\\
=&\frac{1}{3\mathcal{N}(Q,k,m)}\sum_{v=1}^{\mathcal{N}(Q,k,m)}\left(\left(R_{\mathcal{N}(Q,k,m)}(\gamma_v)N(Q,k,m)+\frac{1}{2}\right)^3\right.\\
&\left.-\left(R_{\mathcal{N}(Q,k,m)}(\gamma_{v-1})\mathcal{N}(Q,k,m)-\frac{1}{2}\right)^3\right).
\end{aligned}$$

Using $R_{\mathcal{N}(Q,k,m)}(\gamma_0)\mathcal{N}(Q,k,m)-\frac{1}{2} = -\frac{1}{2} = R_{\mathcal{N}(Q,k,m)}(\gamma_{\mathcal{N}(Q,k,m)})\mathcal{N}(Q,k,m)-\frac{1}{2}$, we see that

$$I = \frac{1}{3\mathcal{N}(Q,k,m)}\sum_{v=1}^{\mathcal{N}(Q,k,m)}\left(\left(R_{\mathcal{N}(Q,k,m)}(\gamma_v)\mathcal{N}(Q,k,m)+\frac{1}{2}\right)^3\right.$$

$$- \left(R_{\mathcal{N}(Q,k,m)}(\gamma_v)\mathcal{N}(Q,k,m) - \frac{1}{2}\right)^3\Bigg)$$

$$= \mathcal{N}(Q,k,m) \sum_{v=1}^{\mathcal{N}(Q,k,m)} (R_{\mathcal{N}(Q,k,m)}(\gamma_v))^2 + \frac{1}{12}. \tag{4.20}$$

The above estimate with (4.19) gives

$$\mathcal{N}(Q,k,m) \sum_{v=1}^{\mathcal{N}(Q,k,m)} (R_{\mathcal{N}(Q,k,m)}(\gamma_v))^2 < CQ^{1+2\epsilon},$$

where C is a constant depending on ϵ. By the Schwarz inequality, we have

$$\begin{aligned}\sum_{v=1}^{\mathcal{N}(Q,k,m)} R_{\mathcal{N}(Q,k,m)}(\gamma_v) &\leq \left(\sum_{v=1}^{\mathcal{N}(Q,k,m)} 1\right)^{1/2} \left(\sum_{v=1}^{\mathcal{N}(Q,k,m)} (R_{\mathcal{N}(Q,k,m)}(\gamma_v))^2\right)^{1/2} \\ &\leq \left(\mathcal{N}(Q,k,m) \sum_{v=1}^{\mathcal{N}(Q,k,m)} (R_{\mathcal{N}(Q,k,m)}(\gamma_v))^2\right)^{1/2} \leq C^{1/2}Q^{1/2+\epsilon}.\end{aligned}$$

This completes the proof of Theorem 1.1.11.

4.2.2 Proof of Theorem 1.1.12

Employing (4.18) and (4.20), we have

$$\begin{aligned}\mathcal{N}(Q,k,m) &\sum_{v=1}^{\mathcal{N}(Q,k,m)} (R_{\mathcal{N}(Q,k,m)}(\gamma_v))^2 + \frac{1}{12} \\ &= \frac{1}{12} \sum_{q_1,q_2 \leq Q} M_{q_1}\left(\frac{Q}{q_1}\right) M_{q_2}\left(\frac{Q}{q_2}\right) \frac{(\gcd(q_1,q_2))^2}{q_1 q_2}.\end{aligned}$$

Therefore, we obtain

$$\sum_{v=1}^{\mathcal{N}(Q,k,m)} (R_{\mathcal{N}(Q,k,m)}(\gamma_v))^2$$

$$= \frac{1}{12\mathcal{N}(Q,k,m)} \left(\sum_{q_1,q_2\le Q} M_{q_1}\left(\frac{Q}{q_1}\right) M_{q_2}\left(\frac{Q}{q_2}\right) \frac{(\gcd(q_1,q_2))^2}{q_1q_2} - 1 \right).$$

This completes the proof of the first part of Theorem 1.1.12. To prove the second part, we use the above identity and the expression

$$M_q(x) = \sum_{\substack{n\le x \\ nq\equiv b \pmod m}} \mu(n)\mu_k(nq)^2$$

to obtain

$$\sum_{v=1}^{\mathcal{N}(Q,k,m)} (R_{\mathcal{N}(Q,k,m)}(\gamma_v))^2 = \frac{1}{12\mathcal{N}(Q,k,m)} \left(\sum_{q_1\le Q} \sum_{\substack{d_1\le \frac{Q}{q_1} \\ q_1d_1\equiv b \pmod m}} \mu(d_1)\mu_k(q_1d_1)^2 \sum_{q_2\le Q} \right.$$
$$\left. \times \sum_{\substack{d_2\le \frac{Q}{q_2} \\ q_2d_2\equiv b \pmod m}} \mu(d_2)\mu_k(q_2d_2)^2 \frac{(\gcd(q_1,q_2))^2}{q_1q_2} - 1 \right). \tag{4.21}$$

Let $\gcd(q_1,q_2) = \delta$ so that $q_1 = q_1'\delta$ and $q_2 = q_2'\delta$ with $(q_1',q_2') = 1$. The above identity can be expressed as

$$\sum_{v=1}^{\mathcal{N}(Q,k,m)} (R_{\mathcal{N}(Q,k,m)}(\gamma_v))^2 = \frac{1}{12\mathcal{N}(Q,k,m)} \left(\sum_{\delta\le Q} \sum_{q_1'\le \frac{Q}{\delta}} \frac{1}{q_1'} \sum_{\substack{d_1\le \frac{Q}{q_1'\delta} \\ q_1'd_1\delta\equiv b \pmod m}} \mu_k(q_1'd_1\delta)^2 \right.$$
$$\left. \times \mu(d_1) \sum_{\substack{q_2'\le \frac{Q}{\delta} \\ (q_1',q_2')=1}} \frac{1}{q_2'} \sum_{\substack{d_2\le \frac{Q}{q_2'\delta} \\ q_2'd_2\delta\equiv b \pmod m}} \mu(d_2)\mu_k(q_2'd_2\delta)^2 - 1 \right). \tag{4.22}$$

We apply Dirichlet hyperbola method to estimate the inner sum on the above identity

$$\begin{aligned}
S :=& \sum_{\substack{q\leq \frac{Q}{\delta}\\ (q,l)=1}} \frac{1}{q} \sum_{\substack{d\leq \frac{Q}{q\delta}\\ qd\delta\equiv b \pmod m}} \mu(d)\mu_k(qd\delta)^2\\
=& \sum_{\substack{q\leq \sqrt{\frac{Q}{\delta}}\\ (q,l)=1}} \frac{1}{q} \sum_{\substack{d\leq \frac{Q}{q\delta}\\ qd\delta\equiv b \pmod m}} \mu(d)\mu_k(qd\delta)^2 + \sum_{d\leq \sqrt{\frac{Q}{\delta}}} \mu(d) \sum_{\substack{q\leq \frac{Q}{d\delta}\\ qd\delta\equiv b \pmod m\\ (q,l)=1}} \frac{\mu_k(qd\delta)^2}{q}\\
&- \sum_{\substack{q\leq \sqrt{\frac{Q}{\delta}}\\ (q,l)=1}} \frac{1}{q} \sum_{\substack{d\leq \sqrt{\frac{Q}{\delta}}\\ qd\delta\equiv b \pmod m}} \mu(d)\mu_k(qd\delta)^2.
\end{aligned}$$

Employing Lemma 4.1.1 to the inner sum in the first and last terms, and Lemma 4.1.2 to the inner sum in the second term of the above identity, we obtain

$$\begin{aligned}
S &\ll_m \frac{Q}{\delta} \sum_{\substack{q\leq \sqrt{\frac{Q}{\delta}}\\ (q,l)=1}} \frac{1}{q^2} \exp\left(-c\frac{(\log Q/q\delta)^{3/5}}{(\log\log Q/q\delta)^{1/5}}\right) \prod_{p|q\delta}\left(\frac{\sqrt{p}}{\sqrt{p}-1}\right) \prod_{p|\ell}\left(1-\frac{1}{\sqrt{p}}\right)^{-1}\\
&\ll_m \frac{Q}{\delta} \exp\left(-c\frac{(\log Q/\delta)^{3/5}}{(\log\log Q/\delta)^{1/5}}\right) \prod_{p|\delta}\left(\frac{\sqrt{p}}{\sqrt{p}-1}\right) \prod_{p|\ell}\left(1-\frac{1}{\sqrt{p}}\right)^{-1}.
\end{aligned}$$

Inserting the above estimate into (4.22) gives

$$\begin{aligned}
\sum_{v=1}^{\mathcal{N}(Q,k,m)} (R_{\mathcal{N}(Q,k,m)}(\gamma_v))^2 &\ll_m \frac{Q^2}{\mathcal{N}(Q,k,m)} \sum_{\delta\leq Q} \frac{1}{\delta^{2-\epsilon}} \exp\left(-c\frac{(\log Q/\delta)^{3/5}}{(\log\log Q/\delta)^{1/5}}\right)\\
&\ll_m \sum_{\delta\leq\sqrt{Q}} \frac{1}{\delta^{2-\epsilon}} \exp\left(-c\frac{(\log Q/\delta)^{\frac{3}{5}}}{(\log\log Q/\delta)^{\frac{1}{5}}}\right)\\
&\quad + \sum_{\sqrt{Q}<\delta\leq Q} \frac{1}{\delta^{2-\epsilon}} \exp\left(-c\frac{(\log Q/\delta)^{\frac{3}{5}}}{(\log\log Q/\delta)^{\frac{1}{5}}}\right)\\
&\ll_m \exp\left(-c\frac{(\log Q)^{\frac{3}{5}}}{(\log\log Q)^{\frac{1}{5}}}\right).
\end{aligned}$$

This completes the proof unconditionally. We now estimate the sum on the right-hand side of (4.22) under the assumption of the GRH. Assuming GRH, we apply Lemma 4.1.1. Therefore,

$$\sum_{v=1}^{\mathcal{N}(Q,k,m)} (R_{\mathcal{N}(Q,k,m)}(\gamma_v))^2 \ll_m \frac{Q^{1+\epsilon}}{\mathcal{N}(Q,k,m)} \sum_{\delta \leq Q} \frac{1}{\delta^{1+\epsilon}} \sum_{q_1' \leq \frac{Q}{\delta}} \frac{1}{(q_1')^{\frac{3}{2}+\epsilon}} \sum_{q_2' \leq \frac{Q}{\delta}} \frac{1}{(q_2')^{\frac{3}{2}+\epsilon}}$$
$$\ll_m Q^{-1+\epsilon}.$$

This completes the proof of Theorem 1.1.12.

4.3 Equidistribution

The aim of this section is to establish Theorems 1.1.9 and 1.1.10 by studying the Weyl sum over $\mathfrak{F}_{Q,k}^{(m)}$ and quantitative equidistribution, respectively.

4.3.1 Weyl sum

The Weyl sums are central to various number-theoretic problems, including the zero-free region of the Riemann zeta function, the prime number theorem, and the Diophantine equations. The Weyl sums have been extensively studied in different forms by various authors. Specifically, the Weyl sum over the roots of quadratic congruences was studied in [33, 34]. The metric theory of Weyl sums appeared in [25]. For more details and problems on the Weyl sums, one may refer to [17, 26, 27] and references therein. In our first result, we establish an upper bound for the Weyl sum over Farey fractions with k-free denominators in residue classes. The Weyl sum for Farey fractions was dealt in [29, 41]. We study the Weyl sum over Farey fractions in $\mathscr{F}_{Q,k}^{(m)}$.

Theorem 4.3.1. *For $r \in \mathbb{Z} \setminus \{0\}$, we have*

$$\sum_{\gamma \in \mathscr{F}_{Q,k}^{(m)}} e(r\gamma) = \mathrm{O}_m\left(\min(Q, r^{\epsilon})\, Q \exp\left(-c\frac{(\log Q)^{3/5}}{(\log\log Q)^{1/5}}\right)\right),$$

where $c>0$ is some constant, $\epsilon>0$ is arbitrary small real number, and $e(x)=e^{2\pi ix}$.

Proof of Theorem 4.3.1. We have

$$\sum_{\gamma\in\mathfrak{F}_{Q,k}^{(m)}} e(r\gamma)=\sum_{\substack{q\le Q\\ q\equiv b \pmod m}}\mu_k(q)^2\sum_{\substack{1\le a\le q\\ (a,q)=1}} e\left(\frac{ar}{q}\right)$$

By Theorem 2.0.1, we see that

$$\begin{aligned}\sum_{\gamma\in\mathfrak{F}_{Q,k}^{(m)}} e(r\gamma)&=\sum_{\substack{q\le Q\\ q\equiv b \pmod m}}\mu_k(q)^2\sum_{1\le a\le q} e\left(\frac{ar}{q}\right)\sum_{d\mid \gcd(a,q)}\mu(d)\\
&=\sum_{d\le Q}\mu(d)\sum_{\substack{q\le Q\\ q\equiv b \pmod m\\ d\mid q}}\mu_k(q)^2\sum_{\substack{1\le a\le q\\ d\mid a}} e\left(\frac{ar}{q}\right)\\
&=\sum_{d\le Q}\mu(d)\sum_{\substack{q\le \frac{Q}{d}\\ qd\equiv b \pmod m}}\mu_k(qd)^2\sum_{1\le a\le q} e\left(\frac{ar}{q}\right)\\
&=\sum_{d\le Q}\mu(d)\sum_{\substack{q\le \frac{Q}{d}\\ qd\equiv b \pmod m\\ q\mid r}}q\mu_k(qd)^2=\sum_{\substack{q\le Q\\ q\mid r}}q\sum_{\substack{d\le \frac{Q}{q}\\ qd\equiv b \pmod m}}\mu(d)\mu_k(qd)^2.\end{aligned}$$

We use Lemma 4.1.1 to estimate the inner sum above, and we find that

$$\sum_{\gamma\in\mathfrak{F}_{Q,k}^{(m)}} e(r\gamma)\ll_m Q\exp\left(-c\frac{(\log Q)^{3/5}}{(\log\log Q)^{1/5}}\right)\sum_{\substack{q\le Q\\ q\mid r}}\mu_k(q)^2\prod_{p\mid q}\left(1-\frac{1}{\sqrt{p}}\right)^{-1}. \tag{4.23}$$

If r is small with respect to Q then clearly $\prod_{p\mid q}\left(1-\frac{1}{\sqrt{p}}\right)^{-1}\le 2^{\omega(q)}$, where $\omega(q)$ counts the distinct prime divisors of q. Therefore, we have

$$\sum_{\substack{q\le Q\\ q\mid r}}\mu_k(q)^2\prod_{p\mid q}\left(1-\frac{1}{\sqrt{p}}\right)^{-1}\ll\sum_{q\mid r}2^{\omega(q)}\ll r^{\epsilon}.$$

If r is large with respect to Q then using Lemma 4.1.3, we obtain

$$\sum_{\substack{q \leq Q \\ q \mid r}} \mu_k(q)^2 \prod_{p \mid q} \left(1 - \frac{1}{\sqrt{p}}\right)^{-1} \ll Q.$$

Combining the above estimates with (4.23) yields

$$\sum_{\gamma \in \mathfrak{F}_{Q,k}^{(m)}} e(r\gamma) \ll_m \min\left(Q, r^{\epsilon}\right) Q \exp\left(-c \frac{(\log Q)^{3/5}}{(\log\log Q)^{1/5}}\right).$$

This completes the proof of Theorem 4.3.1. □

4.3.2 Proof of Theorem 1.1.9

Employing Theorem 4.3.1, we see that

$$\lim_{Q \to \infty} \frac{1}{\mathcal{N}(Q,k,m)} \sum_{\gamma \in \mathfrak{F}_{Q,k}^{(m)}} e(r\gamma) = 0 \text{ for all integers } r \neq 0.$$

Using Theorem 1.1.2 completes the proof of Theorem 1.1.9.

4.3.3 Discrepancy

In this section, we establish the quantitative aspect of equidistribution. In order to achieve this, we study discrepancy of the sequence $\left(\mathscr{F}_{Q,k}^{(m)}\right)_{Q \geq 1}$.

Proof of Theorem 1.1.10. Let $\epsilon > 0$ be arbitrarily small real number, and set $\alpha = 1/Q - \epsilon$ to obtain a lower bound for $D^*_{\mathcal{N}(Q,k,m)}\left(\mathscr{F}_{Q,k}^{(m)}\right)$. Let $A(\alpha; \mathcal{N}(Q,k,m))$ be the number of terms of the sequence $\left(\mathscr{F}_{Q,k}^{(m)}\right)_{Q \geq 1}$ that do not exceed α. Since $\gamma \geq \frac{1}{Q}$ for all $\gamma \in \mathscr{F}_{Q,k}^{(m)}$, it follows that $A(1/Q - \epsilon; \mathcal{N}(Q,k,m)) = 0$. Since

$$D^*_{\mathcal{N}(Q,k,m)}\left(\mathscr{F}_{Q,k}^{(m)}\right) = \sup_{0 \leq \alpha \leq 1} R_{\mathcal{N}(Q,k,m)}(\alpha),$$

where

$$R_{\mathcal{N}(Q,k,m)}(\alpha) = \left| \frac{A(\alpha; \mathcal{N}(Q,k,m))}{\mathcal{N}(Q,k,m)} - \alpha \right|,$$

we therefore have

$$D^*_{\mathcal{N}(Q,k,m)}\left(\mathscr{F}^{(m)}_{Q,k}\right) \geq R_{\mathcal{N}(Q,k,m)}(\alpha) = R_{\mathcal{N}(Q,k,m)}\left(\frac{1}{Q} - \epsilon\right) = \frac{1}{Q} - \epsilon$$

for all $\epsilon > 0$. Since $\epsilon > 0$ is arbitrary, one can thus deduce that

$$D^*_{\mathcal{N}(Q,k,m)}\left(\mathscr{F}^{(m)}_{Q,k}\right) \geq \frac{1}{Q}. \tag{4.24}$$

We next estimate the upper bound for the discrepancy. For any $\alpha \in [0,1]$, we write

$$\begin{aligned}
\mathcal{A}(\alpha; \mathcal{N}(Q,k,m)) - \alpha\mathcal{N}(Q,k,m) &= \sum_{\substack{\gamma \in \mathscr{F}^{(m)}_{Q,k} \\ \gamma \leq \alpha}} 1 - \alpha\mathcal{N}(Q,k,m) \\
&= \sum_{\substack{q \leq Q \\ q \equiv b \pmod m}} \mu_k(q)^2 \sum_{\substack{a \leq q\alpha \\ (a,q)=1}} 1 - \alpha \sum_{\substack{q \leq Q \\ q \equiv b \pmod m}} \mu_k(q)^2 \sum_{\substack{a \leq q \\ (a,q)=1}} 1.
\end{aligned}$$

Employing Theorem 2.0.1, we see that

$$\begin{aligned}
&\mathcal{A}(\alpha; \mathcal{N}(Q,k,m)) - \alpha\mathcal{N}(Q,k,m) \\
&= \sum_{\substack{q \leq Q \\ q \equiv b \pmod m}} \mu_k(q)^2 \sum_{a \leq q\alpha} \sum_{\substack{d|a \\ d|q}} \mu(d) - \alpha \sum_{\substack{q \leq Q \\ q \equiv b \pmod m}} \mu_k(q)^2 \sum_{a \leq q} \sum_{\substack{d|a \\ d|q}} \mu(d) \\
&= \sum_{d \leq Q} \mu(d) \sum_{\substack{q \leq Q \\ q|d}} \mu_k(q)^2 \left(\sum_{\substack{a \leq q\alpha \\ d|a}} 1 - \alpha \sum_{\substack{a \leq q \\ d|a}} 1 \right) \\
&= \sum_{d \leq Q} \mu(d) \sum_{\substack{q \leq \frac{Q}{d} \\ qd \equiv b \pmod m}} \mu_k(qd)^2 \left(\sum_{a \leq q\alpha} 1 - \alpha \sum_{a \leq q} 1 \right) \\
&= \sum_{d \leq Q} \mu(d) \sum_{\substack{q \leq \frac{Q}{d} \\ qd \equiv b \pmod m}} \mu_k(qd)^2 (\lfloor q\alpha \rfloor - \alpha q)
\end{aligned}$$

$$= -\sum_{d\leq Q}\mu(d)\sum_{\substack{q\leq \frac{Q}{d}\\ qd\equiv b \pmod m}}\mu_k(qd)^2\{q\alpha\}.$$

Next, we take the modulus of both sides. Therefore,

$$\begin{aligned}|\mathcal{A}(\alpha;\mathcal{N}(Q,k,m))-\alpha\mathcal{N}(Q,k,m)| &= \left|\sum_{d\leq Q}\mu(d)\sum_{\substack{q\leq \frac{Q}{d}\\ qd\equiv b \pmod m}}\mu_k(qd)^2\{q\alpha\}\right|\\ &\ll \sum_{q\leq Q}\mu_k(q)^2\left|\sum_{\substack{d\leq \frac{Q}{q}\\ qd\equiv b \pmod m}}\mu(d)\mu_k(qd)^2\right|. \quad (4.25)\end{aligned}$$

By employing Lemma 4.1.1, the above sum can be expressed as

$$\begin{aligned}&|\mathcal{A}(\alpha;\mathcal{N}(Q,k,m))-\alpha\mathcal{N}(Q,k,m)|\\ &\ll_m \sum_{q\leq Q}\mu_k(q)^2\frac{Q}{q}\prod_{p|q}\left(1+\frac{1}{\sqrt{p}-1}\right)\exp\left(-c\sqrt{\log(Q/q)}\right)\\ &\ll_m \sum_{q\leq Q}\mu_k(q)^2\prod_{p|q}\left(1+\frac{1}{\sqrt{p}-1}\right)\sum_{d\leq\frac{Q}{q}}\exp\left(-c\sqrt{\log d}\right)\\ &\ll_m \sum_{d\leq Q}\exp\left(-c\sqrt{\log d}\right)\sum_{q\leq\frac{Q}{d}}\mu_k(q)^2\prod_{p|q}\left(1+\frac{1}{\sqrt{p}-1}\right).\end{aligned}$$

To estimate the inner-sum, we apply Lemma 4.1.3 and obtain

$$\begin{aligned}&|\mathcal{A}(\alpha;\mathcal{N}(Q,k,m))-\alpha\mathcal{N}(Q,k,m)|\\ &\ll_m \sum_{d\leq Q}\exp\left(-c\sqrt{\log d}\right)\frac{Q}{d\zeta(k)}\prod_{p}\left(1+\frac{p^{k-1}-1}{(p^{\frac{1}{2}}-1)(p^k-1)}\right)\\ &\ll_m \frac{Q}{\zeta(k)}\sum_{d\leq Q}\frac{1}{d\exp\left(c\sqrt{\log d}\right)} \ll_m Q. \quad (4.26)\end{aligned}$$

Therefore,

$$R_{\mathcal{N}(Q,k,m)}(\alpha) = \frac{1}{\mathcal{N}(Q,k,m)} \left|A(\alpha;\mathcal{N}(Q,k,m)) - \alpha\mathcal{N}(Q,k,m)\right| \ll_m \frac{1}{Q},$$

uniformly in $\alpha \in [0,1]$. Hence

$$D^*_{\mathcal{N}(Q,k,m)}\left(\mathscr{F}^{(m)}_{Q,k}\right) \ll_m \frac{1}{Q}. \tag{4.27}$$

Therefore, using (4.24), (4.27), and Theorem 1.1.3, we obtain

$$\frac{1}{Q} \ll_m D_{\mathcal{N}_{Q,P}}(\mathscr{F}^{(m)}_{Q,k}) \ll_m \frac{1}{Q}.$$

This completes the proof of Theorem 1.1.10. □

4.4 Correlation measure

In this section, we investigate if the ν-level correlations of the sequence $\left(\mathscr{F}^{(m)}_{Q,k}\right)_{Q\geq 1}$ are Poissonian or not. Our primary aim is to compute the ν-level correlation measure for all $\nu \geq 2$. In particular, we derive Theorems 4.4.1 and 1.1.13. We begin by deriving a closed-form formula for the exponential sum over the Farey fractions whose denominators are k-free and lie in an arithmetic progression, which is crucial for establishing correlation measure.

4.4.1 Exponential sum over $\mathfrak{F}^{(m)}_{Q,k}$

Lemma 4.4.1. *Let $r \in \mathbb{Z}$, we have*

$$\sum_{\gamma\in\mathfrak{F}^{(m)}_{Q,k}} e(r\gamma) = \sum_{\substack{q\leq Q\\ q|r}} qM_q\left(\frac{Q}{q}\right), \tag{4.28}$$

where $M_q(x) = \sum_{\substack{d\leq x\\ qd\equiv b \pmod m}} \mu(d)\mu_k(qd)^2$.

Proof. We begin by considering the left-hand side of (4.28) and apply Theorem 2.0.1

to detect the coprimality of a and q. Therefore, we have

$$\begin{aligned}
\sum_{\gamma\in\mathfrak{F}_{Q,k}^{(m)}} e(r\gamma) &= \sum_{\substack{q\le Q\\ q\equiv b \pmod m}} \mu_k(q)^2 \sum_{\substack{1\le a\le q\\ (a,q)=1}} e\left(\frac{ar}{q}\right)\\
&= \sum_{\substack{q\le Q\\ q\equiv b \pmod m}} \mu_k(q)^2 \sum_{1\le a\le q} e\left(\frac{ar}{q}\right) \sum_{d|\gcd(a,q)} \mu(d)\\
&= \sum_{d\le Q}\mu(d) \sum_{\substack{q\le Q\\ q\equiv b \pmod m\\ d|q}} \mu_k(q)^2 \sum_{\substack{1\le a\le q\\ d|a}} e\left(\frac{ar}{q}\right)\\
&= \sum_{d\le Q}\mu(d) \sum_{\substack{q\le \frac{Q}{d}\\ qd\equiv b \pmod m}} \mu_k(qd)^2 \sum_{1\le a\le q} e\left(\frac{ar}{q}\right)\\
&= \sum_{d\le Q}\mu(d) \sum_{\substack{q\le \frac{Q}{d}\\ qd\equiv b \pmod m\\ q|r}} q\mu_k(qd)^2 = \sum_{\substack{q\le Q\\ q|r}} qM_q\left(\frac{Q}{q}\right).
\end{aligned}$$

In the second last step, we have used the identity

$$\sum_{n=1}^{m} e(nl/m) = \begin{cases} m, & \text{if } m|l,\\ 0, & \text{otherwise.}\end{cases}$$

This completes the proof of result. □

4.4.2 Correlation measure

To state our results on the correlation measure, we first fix some notations and define certain transformations. Let

$$\tilde{\Omega}_{n,e,k,\Delta} = \left\{(x,y) : 0 < x \le y \le 1,\ y \ge \frac{1}{\mathcal{C}(\Lambda,k,1)},\ 0 < ye_j - x\Delta_j \le \frac{1}{n_j}\right\}.$$

$\mathcal{C}(\Lambda,k,1) = \frac{\Lambda}{\mathscr{C}(k,1)}$. We define another map T on $\mathbb{R}^{\nu-1}$ and its inverse T^{-1} as follows:

$$T(x_1,\dots,x_{\nu-1}) = (x_1-x_2, x_2-x_3,\dots,x_{\nu-2}-x_{\nu-1},x_{\nu-1}), \tag{4.29}$$

$$T^{-1}(x_1,\ldots,x_{\nu-1})=(x_1+\cdots+x_{\nu-1},x_2+\cdots+x_{\nu-1},\ldots,x_{\nu-2}+x_{\nu-1},x_{\nu-1}).$$

We are now ready to state our result on the ν-level correlations. Recall that

$$\mathcal{S}^{(\nu)}(\mathfrak{B})=\lim_{Q\to\infty}\mathcal{S}^{\nu}_{\mathfrak{F}^{(m)}_{Q,k}}(\mathfrak{B}),$$

where

$$\begin{aligned}\mathcal{S}^{\nu}_{\mathfrak{F}^{(m)}_{Q,k}}(\mathfrak{B})=&\frac{1}{\mathcal{N}(Q,k,m)}\#\Big\{(\gamma_1,\ldots,\gamma_\nu)\in\left(\mathfrak{F}^{(m)}_{Q,k}\right)^{\nu}:\gamma_i\text{ distinct},\\&(\gamma_1-\gamma_2,\ldots,\gamma_{\nu-1}-\gamma_\nu)\in\frac{1}{\mathcal{N}(Q,k,m)}\mathfrak{B}+\mathbb{Z}^{\nu-1}\Big\}.\end{aligned}$$

We assume that the constant $\Xi(k,n_j,e_j,\Delta_j)>0$ in Remark 2.6.1 is positive.

Theorem 4.4.1. *Let $k,\nu\geq 2$ be integers. All ν-level correlation measure of the sequence $\left(\mathscr{F}^{(1)}_{Q,k}\right)_{Q\geq1}$ exist. For any box $\mathfrak{B}\subset(0,\Lambda)^{\nu-1}$, the ν-level correlation measure is given by*

$$\begin{aligned}\mathcal{S}^{(\nu)}(\mathfrak{B})=\frac{1}{\mathscr{C}(k,1)}&\sum_{1\leq n_j\leq\mathcal{C}(\Lambda,k,1)}\mu(n_1)\cdots\mu(n_{\nu-1})\sum_{\substack{1\leq\Delta_j\leq(\nu-1)\mathcal{C}(\Lambda,k,1)\\1\leq e_j\leq\nu\mathcal{C}(\Lambda,k,1)}}\\&\times\Xi(k,n_j,e_j,\Delta_j)\iint_{\tilde{\Omega}_{n,e,k,\Delta}}g^{\chi}_{k,e,\Delta}(x,y)dxdy,\end{aligned}$$

where $\Xi(k,n_j,e_j,\Delta_j)>0$ is a positive constant depending on k,n_j,e_j,Δ_j,

$$\mathscr{C}(k,1)=\frac{1}{2L(k,\chi_0)}\prod_p\left(1-\frac{p^{k-1}-1}{p(p^k-1)}\right)$$

and

$$g^{\chi}_{k,e,\Delta}(x,y)=\chi_{\mathfrak{B}}\circ T\left(g^{(1)}_{k,e,\Delta}(x,y),\ldots,g^{(\nu-1)}_{k,e,\Delta}(x,y)\right).$$

Remark 4.4.1. *Note that for the ν-level correlation to be Poissonian, we must have $\mathcal{S}^{(\nu)}(\mathfrak{B})=vol(\mathfrak{B})$ for all boxes $\mathfrak{B}$. Using the above expression, we observe that the sequence $\left(\mathscr{F}^{(1)}_{Q,k}\right)_{Q\geq1}$ does not have Poissonian ν-level correlations for all $\nu\geq2$. Let $\Lambda>0$ be a real number such that $\Lambda^{-2\nu+1}>2^{\nu-1}(\nu(\nu-1))^{\nu-1}(\mathscr{C}(k,1))^{-3\nu+2}$, and let $\mathfrak{B}=(0,\Lambda/2]^{\nu-1}$. Then, clearly, $\mathcal{S}^{(\nu)}(\mathfrak{B})<vol(\mathfrak{B})$.*

Recall that the pair correlation function of the sequence $\left(\mathscr{F}_{Q,k}^{(m)}\right)_{Q\geq 1}$ exists and is given by

$$\mathfrak{g}_{m,k}(\lambda) = \frac{6}{\lambda^2\pi^2\phi^2(m)} \sum_{1\leq n<\frac{\lambda}{\mathscr{C}(k,m)}} F_k(n) \log\left(\frac{\lambda}{n\mathscr{C}(k,m)}\right),$$

where $F_k(n)$ is as in (1.22).

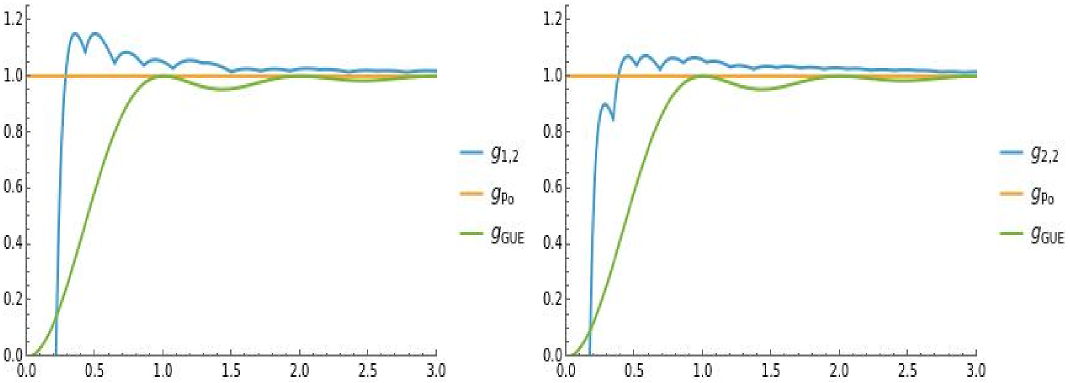

FIGURE 4.1: The graphs of pair correlation functions $\mathfrak{g}_{1,2}(\lambda), \mathfrak{g}_{2,2}(\lambda), g_{Po}(\lambda) \equiv 1$ and $g_{GUE}(\lambda) = 1 - \left(\frac{\sin\pi\lambda}{\pi\lambda}\right)^2$.

Figure 1 shows the graphs of $\mathfrak{g}_{1,2}(\lambda)$ and $\mathfrak{g}_{2,2}(\lambda)$. For comparison, we also plot the graphs of the pair correlation functions of GUE model and Poisson case, given respectively by

$$g_{GUE}(\lambda) = 1 - \left(\frac{\sin\pi\lambda}{\pi\lambda}\right)^2, \; g_{Po}(\lambda) = 1.$$

The graphs of $\mathfrak{g}_{1,2}(\lambda)$ and $\mathfrak{g}_{2,2}(\lambda)$ show a strong repulsion between the elements of the sequences $\mathfrak{F}_{Q,2}^{(1)}$ and $\mathfrak{F}_{Q,2}^{(2)}$, respectively–even more robust than the repulsion among the zeros of the Riemann zeta function. As $\lambda \to \infty$, repulsion decreases, and the distribution tends to become constant.

Outline of Proof: To establish the correlation measure of the sequence $\left(\mathscr{F}_{Q,k}^{(m)}\right)_{Q\geq 1}$, we reduce the problem of counting the tuples described in (1.11) to estimating an exponential sum over $\mathscr{F}_{Q,k}^{(m)}$. This is achieved by expressing the Fourier series for the smooth real-valued function H with support contained in $\mathfrak{B}$. Furthermore, we rewrite the exponential sum in terms of a Möbius sum and utilize the Poisson summation formula for the coefficients of the Fourier series. Given that the support of H is contained within $\mathfrak{B}$, several changes of variables lead to establishing estimates for weighted lattice point counting with several coprimality and k-free restrictions. As a result, the contribution from principal Dirichlet character yields the correlation measure, while for the non-principal character, we provide an estimate for character sum twisted by a continuously differentiable function and the characteristic function for the k-free numbers. By applying this result, the sum over non-principal characters approaches zero as $Q \to \infty$.

4.4.3 Proof of Theorem 4.4.1

In order to establish the ν-level correlation measure for the sequence of Farey fractions with k-free denominators q that run through a given arithmetic progression, we need to estimate, for any positive real number Λ, the quantity

$$\begin{aligned}\mathcal{S}^{\nu}_{\mathfrak{F}^{(1)}_{Q,k}}(\Lambda) =& \frac{1}{\mathcal{N}(Q,k,1)} \# \Big\{ (\gamma_1, \dots, \gamma_\nu) \in \left(\mathfrak{F}^{(1)}_{Q,k}\right)^{\nu} : \gamma_i \text{ distinct}, \\ & (\gamma_1 - \gamma_2, \dots, \gamma_{\nu-1} - \gamma_\nu) \in \frac{1}{\mathcal{N}(Q,k,1)} \mathfrak{B} + \mathbb{Z}^{\nu-1} \Big\}.\end{aligned}$$

To estimate this, we build upon the ideas introduced in [16] making several necessary and technical modifications on the way. We begin with smooth correlation measure as defined in (2.15). For a smooth real valued function H on $\mathbb{R}^{\nu-1}$ such that $\operatorname{Supp}(H) \subset (0, \Lambda)^{\nu-1}$, define

$$f(y) = \sum_{r \in \mathbb{Z}^{\nu-1}} H(\mathcal{N}(Q,k,1)(y + r)), \ y \in \mathbb{R}^{\nu-1},$$

and

$$S^{(\nu)}_{Q,k} = \sum_{\gamma_i \in \mathfrak{F}^{(1)}_{Q,k}, \text{distinct}} f(\gamma_1 - \gamma_2, \dots, \gamma_{\nu-1} - \gamma_\nu). \tag{4.30}$$

Since Supp$H \subset (0, \Lambda)$, the condition $\gamma_i \neq \gamma_j$ for $i \neq j$ can be removed for Q large enough that $\mathcal{N}(Q, k, 1) > \Lambda$. Let

$$f(y) = \sum_{r \in \mathbb{Z}^{\nu-1}} c_r e(r \cdot y)$$

be the Fourier series expansion of f, with the Fourier coefficients

$$\begin{aligned} c_r &= \int_{[0,1)^{\nu-1}} f(x) e(-r \cdot x) dx = \sum_{n \in \mathbb{Z}^{\nu-1}} \int_{[0,1)^{\nu-1}} H(\mathcal{N}(Q,k,1)(x+n)) e(-r \cdot x) dx \\ &= \int_{\mathbb{R}^{\nu-1}} H(\mathcal{N}(Q,k,1) v) e(-r \cdot v) dv = \frac{1}{(\mathcal{N}(Q,k,1))^{\nu-1}} \widehat{H}\left(\frac{r}{\mathcal{N}(Q,k,1)}\right), \end{aligned} \tag{4.31}$$

where $\widehat{H}$ is the Fourier transform of H. Then by (4.30), we have

$$\begin{aligned} S_{Q,k}^{(\nu)} &= \sum_{\gamma_1, \ldots, \gamma_\nu \in \mathfrak{F}_{Q,k}^{(1)}} f(\gamma_1 - \gamma_2, \ldots, \gamma_{\nu-1} - \gamma_\nu) \\ &= \sum_{\substack{\gamma_1, \ldots, \gamma_\nu \in \mathfrak{F}_{Q,k}^{(1)} \\ r_1, \ldots, r_{\nu-1} \in \mathbb{Z}}} c_r e(r \cdot (\gamma_1 - \gamma_2, \ldots, \gamma_{\nu-1} - \gamma_\nu)) \\ &= \sum_{\substack{\gamma_1, \ldots, \gamma_\nu \in \mathfrak{F}_{Q,k}^{(1)} \\ r_1, \ldots, r_{\nu-1} \in \mathbb{Z}}} c_r e(r_1 \gamma_1) e((r_2 - r_1)\gamma_2) \ldots e((r_{\nu-1} - r_{\nu-2})\gamma_{\nu-1}) e(r_{\nu-1}\gamma_\nu). \end{aligned} \tag{4.32}$$

For $m = 1$, by applying Lemma 4.4.1 to the above identity yields

$$\begin{aligned} S_{Q,k}^{(\nu)} &= \sum_{r = (r_1, \ldots, r_{\nu-1}) \in \mathbb{Z}^{\nu-1}} c_r \sum_{\substack{d_1 \mid r_1 \\ d_2 \mid r_2 - r_1 \\ \cdots \\ d_{\nu-1} \mid r_{\nu-1} - r_{\nu-2} \\ d_\nu \mid r_{\nu-1}}} d_1 \cdots d_\nu \mathcal{M}_{d_1}\left(\frac{Q}{d_1}\right) \cdots \mathcal{M}_{d_\nu}\left(\frac{Q}{d_\nu}\right) \\ &= \sum_{1 \le d_1, \ldots, d_\nu \le Q} d_1 \cdots d_\nu \mathcal{M}_{d_1}\left(\frac{Q}{d_1}\right) \cdots \mathcal{M}_{d_\nu}\left(\frac{Q}{d_\nu}\right) \sum_{\substack{d_1 \mid r_1 \\ d_2 \mid r_2 - r_1 \\ \cdots \\ d_{\nu-1} \mid r_{\nu-1} - r_{\nu-2} \\ d_\nu \mid r_{\nu-1}}} c_r, \end{aligned}$$

where $\mathcal{M}_q(x)=\sum_{n\le x}\mu(n)\mu_k(qn)^2$. The divisibility conditions in the inner-sum of the above identity can be expressed as

$$\begin{aligned}
r_1 &= l_1d_1\\
r_2 &= l_1d_1+l_2d_2\\
&\cdots\\
r_{\nu-1} &= l_1d_1+\cdots+l_{\nu-1}d_{\nu-1}=l_\nu d_\nu
\end{aligned}$$

for some $l_1,\ldots,l_\nu\in\mathbb{Z}$. We denote $d=(d_1,\ldots,d_{\nu-1})\in\square_Q^{\nu-1}:=[1,Q]^{\nu-1}\cap\mathbb{Z}^{\nu-1}$, $l=(l_1,\ldots,l_{\nu-1})$. We obtain

$$\begin{aligned}
S_{Q,k}^{(\nu)} &= \sum_{d\in\square_Q^{\nu-1}} d_1\cdots d_{\nu-1}\mathcal{M}_{d_1}\left(\frac{Q}{d_1}\right)\cdots\mathcal{M}_{d_{\nu-1}}\left(\frac{Q}{d_{\nu-1}}\right)\\
&\quad\times\sum_{l\in\mathbb{Z}^{\nu-1}} c_{d_1l_1,d_1l_1+d_2l_2,\ldots,d_1l_1+\cdots+d_{\nu-1}l_{\nu-1}}\sum_{d_\nu|d_1l_1+\cdots+d_{\nu-1}l_{\nu-1}} d_\nu\mathcal{M}_{d_\nu}\left(\frac{Q}{d_\nu}\right). \qquad (4.33)
\end{aligned}$$

By using (4.31) and Lemma 4.4.1, the two inner sums in (4.33) take the form

$$\begin{aligned}
&\sum_{\substack{l\in\mathbb{Z}^{\nu-1}\\ \gamma\in\mathfrak{F}_{Q,k}^{(1)}}} c_{d_1l_1,d_1l_1+d_2l_2,\ldots,d_1l_1+\cdots+d_{\nu-1}l_{\nu-1}}e(-\gamma d\cdot l)\\
&=\sum_{\substack{l\in\mathbb{Z}^{\nu-1}\\ \gamma\in\mathfrak{F}_{Q,k}^{(1)}}}\int_{\mathbb{R}^{\nu-1}} e\left(-\gamma\sum_{i=1}^{\nu-1}d_il_i-\sum_{j=1}^{\nu-1}x_j(d_1l_1+\cdots+d_jl_j)\right)H(\mathcal{N}(Q,k,1)x)dx\\
&=\sum_{\substack{l\in\mathbb{Z}^{\nu-1}\\ \gamma\in\mathfrak{F}_{Q,k}^{(1)}}}\int_{\mathbb{R}^{\nu-1}} e\left(-\sum_{i=1}^{\nu-1}d_il_i(x_i+\cdots+x_{\nu-1}+\gamma)\right)H(\mathcal{N}(Q,k,1)x)dx\\
&=\sum_{\substack{l\in\mathbb{Z}^{\nu-1}\\ \gamma\in\mathfrak{F}_{Q,k}^{(1)}}}\int_{\mathbb{R}^{\nu-1}}\\
&\times e\left(-\sum_{i=1}^{\nu-1}d_il_i(x_i+\cdots+x_{\nu-1})\right)H(\mathcal{N}(Q,k,1)(x_1,\ldots,x_{\nu-2},x_{\nu-1}-\gamma))dx. \qquad (4.34)
\end{aligned}$$

We take $y_i = d_i(x_i + \cdots + x_{\nu-1}),\ i = 1, \ldots, \nu-1$ with $y = (y_1, \ldots, y_{\nu-1}) \in \mathbb{R}^{\nu-1}$ and set

$$\begin{aligned} H_{\mathcal{N}(Q,k,1);d,\gamma}(y) =& H\left(\mathcal{N}(Q,k,1)\left(\frac{y_1}{d_1} - \frac{y_2}{d_2}\right), \ldots, \mathcal{N}(Q,k,1)\left(\frac{y_{\nu-2}}{d_{\nu-2}} - \frac{y_{\nu-1}}{d_{\nu-1}}\right),\right. \\ & \left.\mathcal{N}(Q,k,1)\left(\frac{y_{\nu-1}}{d_{\nu-1}} - \gamma\right)\right). \end{aligned}$$

Therefore, the identity in (4.34) can be expressed as follows

$$\begin{aligned} \frac{1}{d_1 \cdots d_{\nu-1}} & \sum_{\gamma \in \mathfrak{F}^{(1)}_{Q,k}} \sum_{l \in \mathbb{Z}^{\nu-1}} \int_{\mathbb{R}^{\nu-1}} e(-l \cdot y) H_{\mathcal{N}(Q,k,1);d,\gamma}(y) dy \\ &= \frac{1}{d_1 \cdots d_{\nu-1}} \sum_{\gamma \in \mathfrak{F}^{(1)}_{Q,k}} \sum_{l \in \mathbb{Z}^{\nu-1}} \widehat{H}_{\mathcal{N}(Q,k,1);d,\gamma}(l). \end{aligned}$$

Employing the Poisson summation formula to the inner sum of the above identity and inserting it back into (4.33), we obtain

$$S^{(\nu)}_{Q,k} = \sum_{d \leq \square_Q^{\nu-1}} \mathcal{M}_{d_1}\left(\frac{Q}{d_1}\right) \cdots \mathcal{M}_{d_{\nu-1}}\left(\frac{Q}{d_{\nu-1}}\right) \sum_{\gamma \in \mathfrak{F}^{(1)}_{Q,k}} \sum_{l \in \mathbb{Z}^{\nu-1}} H_{\mathcal{N}(Q,k,1);d,\gamma}(l).$$

As $\mathrm{Supp} H \subset (0, \Lambda)^{\nu-1}$, we have

$$0 < \mathcal{N}(Q,k,1)\left(\frac{l_j}{d_j} - \frac{l_{j+1}}{d_{j+1}}\right) < \Lambda',\ j = 1, \ldots, \nu-2.$$

The above inequalities implies $l_j d_{j+1} - l_{j+1} d_j \geq 1$ and

$$\Lambda > \frac{\mathcal{N}(Q,k,1)(l_j d_{j+1} - l_{j+1} d_j)}{d_j d_{j+1}} \geq \frac{\mathcal{N}(Q,k,1)}{d_j d_{j+1}}.$$

Therefore, for all $Q \geq Q_0(\Lambda)$ using above inequality we get

$$\frac{Q^2}{d_j d_{j+1}} = \frac{Q^2}{\mathcal{N}(Q,k,1)} \cdot \frac{\mathcal{N}(Q,k,1)}{d_j d_{j+1}} < \frac{Q^2 \Lambda}{\mathcal{N}(Q,k,1)} < \frac{\Lambda}{\mathscr{C}(k,1)} =: \mathcal{C}(\Lambda, k, 1).$$

Note that both $Q/d_j \geq 1$ and $Q/d_{j+1} \geq 1$. Therefore, for all $Q \geq Q_0(\Lambda)$, we have $1 \leq \frac{Q}{d_j} \leq \mathcal{C}(\Lambda, k, 1),\ j = 1 \ldots, \nu - 1$. Similarly, we obtain

$$\frac{Q}{q} \leq \mathcal{C}(\Lambda, k, 1). \tag{4.35}$$

Hence,

$$\begin{aligned} S^{(\nu)}_{Q,k} &= \sum_{d \leq \square_Q^{\nu-1}} \sum_{l \in \mathbb{Z}^{\nu-1}} \sum_{1 \leq n_j \leq Q/d_j} (\mu(n_1) \cdots \mu(n_{\nu-1}))(\mu_k(n_1 d_1)^2 \cdots \mu_k(n_{\nu-1} d_{\nu-1})^2) \\ &\quad \times \sum_{\substack{a/q \in \mathfrak{F}^{(1)}_{Q,k} \\ q \geq Q/\mathcal{C}(\Lambda,k,1)}} H_{\mathcal{N}(Q,k,1);d,\gamma}(l) \\ &= \sum_{1 \leq n_j \leq \mathcal{C}(\Lambda,k,1)} \mu(n_1) \cdots \mu(n_{\nu-1}) \sum_{1 \leq d_j \leq Q/n_j} \mu_k(n_1 d_1)^2 \cdots \mu_k(n_{\nu-1} d_{\nu-1})^2 \\ &\quad \times \sum_{l \in \mathbb{Z}^{\nu-1}} \sum_{\substack{a/q \in \mathfrak{F}^{(1)}_{Q,k} \\ q \geq Q/\mathcal{C}(\Lambda,k,1)}} H_{\mathcal{N}(Q,k,1);d,\gamma}(l). \end{aligned} \tag{4.36}$$

We set $\Delta_j = q l_j - a d_j$ for $j = 1, \ldots, \nu - 1$. Consequently, l_j is uniquely determined as $l_j = \frac{\Delta_j + a d_j}{q}$. This in turn implies that

$$\frac{l_j}{d_j} - \frac{l_{j+1}}{d_{j+1}} = \frac{\Delta_j + a d_j}{q d_j} - \frac{\Delta_{j+1} + a d_{j+1}}{q d_{j+1}} = \frac{1}{q}\left(\frac{\Delta_j}{d_j} - \frac{\Delta_{j+1}}{d_{j+1}}\right),\ j = 1, \ldots, \nu - 2.$$

Moreover,

$$\frac{l_{\nu-1}}{d_{\nu-1}} - \frac{a}{q} = \frac{\Delta_{\nu-1}}{q d_{\nu-1}}.$$

Also, d_j satisfy the congruence $d_j \equiv -\bar{a}\Delta_j \pmod{q},\ j = 1, \ldots, \nu - 1$, where $1 \leq \bar{a} \leq q$ such that $a\bar{a} = 1 \pmod{q}$. Since $\operatorname{Supp} H \subset (0, \Lambda)^{\nu-1}$, we get

$$\begin{aligned} 0 < \frac{\mathcal{N}(Q,k,1)\Delta_j}{q d_j} &= \mathcal{N}(Q,k,1)\left(\frac{l_j}{d_j} - \frac{a}{q}\right) \\ &= \mathcal{N}(Q,k,1)\left(\frac{l_j}{d_j} - \frac{l_{j+1}}{d_{j+1}}\right) + \cdots + \mathcal{N}(Q,k,1)\left(\frac{l_{\nu-1}}{d_{\nu-1}} - \frac{a}{q}\right) \\ &< (\nu - j)\Lambda'. \end{aligned}$$

For $Q \geq Q_0(\Lambda)$, the above inequalities give

$$1 \leq \Delta_j \leq \frac{qd_j(\nu - j)\Lambda}{\mathcal{N}(Q,k,1)} \leq \frac{Q^2(\nu - j)\Lambda}{\mathcal{N}(Q,k,1)} \leq (\nu - j)\mathcal{C}(\Lambda,k,1);$$

thus, we have $1 \leq \Delta_1, \ldots, \Delta_{\nu-1} \leq (\nu-1)\mathcal{C}(\Lambda,k,1)$. Therefore, (4.36) becomes

$$\begin{aligned} S_{Q,k}^{(\nu)} = & \sum_{1 \leq n_j \leq \mathcal{C}(\Lambda,k,1)} \mu(n_1)\cdots\mu(n_{\nu-1}) \sum_{1 \leq \Delta_j \leq (\nu-1)\mathcal{C}(\Lambda,k,1)} \sum_{1 \leq d_j \leq Q/n_j} \mu_k(n_1 d_1)^2 \cdots \\ & \times \mu_k(n_{\nu-1}d_{\nu-1})^2 \sum_{\substack{a/q \in \mathfrak{F}_{Q,k}^{(1)}, q \geq Q/\mathcal{C}(\Lambda,k,1) \\ d_j \equiv -\bar{a}\Delta_j \pmod{q}}} \\ & \times H\left(\frac{\mathcal{N}(Q,k,1)}{q}\left(\frac{\Delta_1}{d_1} - \frac{\Delta_2}{d_2}, \ldots, \frac{\Delta_{\nu-2}}{d_{\nu-2}} - \frac{\Delta_{\nu-1}}{d_{\nu-1}}, \frac{\Delta_{\nu-1}}{d_{\nu-1}}\right)\right). \end{aligned}$$

We simplify the above expression by employing the linear transformation T defined in (4.29). We set $\tilde{H} = H \circ T$, which is smooth and $\operatorname{Supp}\tilde{H} \subset (0, (\nu-1)\Lambda] \times \cdots \times (0, \Lambda]$. The above identity then becomes

$$\begin{aligned} S_{Q,k}^{(\nu)} = & \sum_{1 \leq n_j \leq \mathcal{C}(\Lambda,k,1)} \mu(n_1)\cdots\mu(n_{\nu-1}) \sum_{1 \leq \Delta_j \leq (\nu-1)\mathcal{C}(\Lambda,k,1)} \sum_{1 \leq d_j \leq Q/n_j} \mu_k(n_1 d_1)^2 \cdots \\ & \times \mu_k(n_{\nu-1}d_{\nu-1})^2 \sum_{\substack{a/q \in \mathfrak{F}_{Q,k}^{(1)} \\ q \geq Q/\mathcal{C}(\Lambda,k,1) \\ d_j \equiv -\bar{a}\Delta_j \pmod{q}}} \tilde{H}\left(\frac{\mathcal{N}(Q,k,1)}{q}\left(\frac{\Delta_1}{d_1}, \frac{\Delta_2}{d_2}, \ldots, \frac{\Delta_{\nu-1}}{d_{\nu-1}}\right)\right). \end{aligned}$$

We define $e_j = \frac{d_j + \bar{a}\Delta_j}{q}$, $j = 1, \ldots, \nu-1$. Note that e_j is an integer since $d_j \equiv -\bar{a}\Delta_j \pmod{q}$. As $d_j, \bar{a}$, and Δ_j are all integers, it follows that $e_j \geq 1$. Moreover, using (4.35), we obtain $1 \leq e_j \leq \nu\mathcal{C}(\Lambda,k,1)$, $j = 1, \ldots, \nu-1$. For each value of e_j, with a, q, and Δ_j fixed, we obtain a unique value of d_j; in particular, $d_j = qe_j - \bar{a}\Delta_j$. Also, with fixed e_j and Δ_j and variable $a/q \in \mathfrak{F}_{Q,k}^{(1)}$, in order for d_j to belong to the set $\{1, \ldots, \lfloor Q/n_j \rfloor\}$, a and q must satisfy $\frac{Q}{n_j\mathcal{C}(\Lambda,k,1)} \leq qe_j - \bar{a}\Delta_j \leq \frac{Q}{n_j}$. We consider the region

$$\Omega_{n,e,k,\Delta} = \left\{(x,y) : 0 < x \leq y \leq 1,\ y \geq \frac{1}{\mathcal{C}(\Lambda,k,1)},\ \frac{n_j^{-1}}{\mathcal{C}(\Lambda,k,1)} \leq ye_j - x\Delta_j \leq \frac{1}{n_j}\right\}.$$

We next set the functions $f_{k,e,\Delta}, f^{(j)}_{k,e,\Delta}$ defined on $\Omega_{n,e,k,\Delta}$ as follows

$$f_{k,e,\Delta}(x,y) = \tilde{H}\left(f^{(1)}_{k,e,\Delta}(x,y), \ldots, f^{(\nu-1)}_{k,e,\Delta}(x,y)\right)$$

and

$$f^{(j)}_{k,e,\Delta}(x,y) = \frac{\mathcal{N}(Q,k,1)\Delta_j}{y(ye_j - x\Delta_j)},\ j = 1, \ldots, \nu-1.$$

We also set $a' = \bar{a}$ and note that $a'/q \in \mathfrak{F}^{(1)}_{Q,k}$ with $q \geq Q/\mathcal{C}(\Lambda,k,1)$ as $a/q \in \mathfrak{F}^{(1)}_{Q,k}$. Therefore

$$S^{(\nu)}_{Q,k} = \sum_{1\leq n_j \leq \mathcal{C}(\Lambda,k,1)} \mu(n_1)\cdots\mu(n_{\nu-1}) \sum_{\substack{1\leq \Delta_j \leq (\nu-1)\mathcal{C}(\Lambda,k,1)\\ 1\leq e_j\leq \nu\mathcal{C}(\Lambda,k,1)}} \sum_{\substack{(a',q)\in Q\Omega_{n,e,k,\Delta}\\ (a',q)=1,\mu_k(q)^2=1\\ \mu_k(n_j(qe_j-a'\Delta_j))^2=1}} f_{k,e,\Delta}(a',q). \tag{4.37}$$

To estimate the inner sum in the above identity, we apply Lemma 2.6.4

$$\sum_{\substack{(a',q)\in Q\Omega_{n,e,k,\Delta}\\ (a',q)=1\mu_k(q)^2=1\\ \mu_k(n_j(qe_j-a'\Delta_j))^2=1}} f_{k,e,\Delta}(a',q) = \Xi(k,n_j,e_j,\Delta_j)\iint_{Q\Omega_{n,e,k,\Delta}} f_{k,e,\Delta}(x,y) + \mathrm{O}\left(Q^{1+\frac{1}{k}}\log^2 Q\right). \tag{4.38}$$

By invoking the above estimate in (4.37), and making the change of variables $(u,v) = (Qx,Qy)$ in the main term of (4.38), we obtain

$$\begin{aligned} S^{(\nu)}_{Q,k} =& Q^2 \sum_{1\leq n_j\leq \mathcal{C}(\Lambda,k,1)} \mu(n_1)\cdots\mu(n_{\nu-1}) \sum_{\substack{1\leq \Delta_j\leq(\nu-1)\mathcal{C}(\Lambda,k,1)\\ 1\leq e_j\leq \nu\mathcal{C}(\Lambda,k,1)}} \Xi(k,n_j,e_j,\Delta_j)\\ &\times \mathcal{I}_k(r,e,\Delta) + \mathrm{O}\left(Q^{1+\frac{1}{k}+\epsilon}\log^2 Q\right), \end{aligned} \tag{4.39}$$

where

$$\mathcal{I}_k(r,e,\Delta) = \iint_{\Omega_{n,e,k,\Delta}} g_{k,e,\Delta}(x,y)dxdy, \tag{4.40}$$

$$g_{k,e,\Delta}(x,y) = \tilde{H}\left(g^{(1)}_{k,e,\Delta}(x,y), \ldots, g^{(\nu-1)}_{k,e,\Delta}(x,y)\right)$$

and

$$g^{(j)}_{k,e,\Delta}(x,y) = \frac{\mathcal{N}(Q,k,1)\Delta_j}{Q^2 y(ye_j - x\Delta_j)},\ j = 1, \ldots, \nu - 1.$$

Employing Proposition 4.1.1 and the inequality

$$|\tilde{H}(v) - \tilde{H}(w)| \leq \|\tilde{H}'\||v - w| \leq 2\|H'\||v - w|,$$

we observe that (4.39) holds true when $g^{(j)}_{k,e,\Delta}$ is replaced by

$$g^{(j)}_{k,e,\Delta}(x,y) = \frac{\mathscr{C}(k,1)\Delta_j}{y(ye_j - x\Delta_j)},\ j = 1, \ldots, \nu - 1$$

in the formula for $g_{k,e,\Delta}$. Therefore

$$\begin{aligned} S^{(\nu)}_{Q,k} =& Q^2 \sum_{1 \leq n_j \leq \mathcal{C}(\Lambda,k,1)} \mu(n_1) \cdots \mu(n_{\nu-1}) \sum_{\substack{1 \leq \Delta_j \leq (\nu-1)\mathcal{C}(\Lambda,k,1) \\ 1 \leq e_j \leq \nu \mathcal{C}(\Lambda,k,1)}} \Xi(k, n_j, e_j, \Delta_j) \\ & \times \mathcal{I}_k(r,e,\Delta) + \mathrm{O}\left(Q^{1+\frac{1}{k}+\epsilon} \log^2 Q\right), \end{aligned}$$

where $\mathcal{I}_k(r,e,\Delta)$ is as in (4.40). We also note that the region can be extended to

$$\tilde{\Omega}_{n,e,k,\Delta} = \left\{(x,y) : 0 < x \leq y \leq 1,\ y \geq \frac{1}{\mathcal{C}(\Lambda,k,1)},\ 0 < ye_j - x\Delta_j \leq \frac{1}{n_j}\right\}.$$

If $(x,y) \in \tilde{\Omega}_{n,e,k,\Delta} \setminus \Omega_{n,e,k,\Delta}$, there is some j such that $|ye_j - x\Delta_j| < 1/n_j\mathcal{C}(\Lambda,k,1)$, which implies that

$$|g^{(j)}_{k,e,\Delta}(x,y)| \geq n_j \Delta_j \mathscr{C}(k,1)\mathcal{C}(\Lambda,k,1) \geq \mathscr{C}(k,1)\mathcal{C}(\Lambda,k,1) = \Lambda.$$

This in turn implies that $g_{k,e,\Delta} = 0$ on $\tilde{\Omega}_{n,e,k,\Delta} \setminus \Omega_{n,e,k,\Delta}$. Therefore

$$\begin{aligned} S^{(\nu)}_{Q,k} =& Q^2 \sum_{1 \leq n_j \leq \mathcal{C}(\Lambda,k,1)} \mu(n_1) \cdots \mu(n_{\nu-1}) \sum_{\substack{1 \leq \Delta_j \leq (\nu-1)\mathcal{C}(\Lambda,k,1) \\ 1 \leq e_j \leq \nu \mathcal{C}(\Lambda,k,1)}} \Xi(k, n_j, e_j, \Delta_j) \\ & \times \iint_{\tilde{\Omega}_{n,e,k,\Delta}} g_{k,e,\Delta}(x,y)\,dx\,dy + \mathrm{O}\left(Q^{1+\frac{1}{k}+\epsilon} \log^2 Q\right). \end{aligned} \tag{4.41}$$

Using Proposition 4.1.1 with the above formula yields

$$\frac{S_{Q,k}^{(\nu)}}{\mathcal{N}(Q,k,1)} = \frac{1}{\mathscr{C}(k,1)} \sum_{1\le n_j\le \mathcal{C}(\Lambda,k,1)} \mu(n_1)\cdots\mu(n_{\nu-1}) \sum_{\substack{1\le \Delta_j\le (\nu-1)\mathcal{C}(\Lambda,k,1)\\ 1\le e_j\le \nu\mathcal{C}(\Lambda,k,1)}} \times \Xi(k,n_j,e_j,\Delta_j) \iint_{\tilde{\Omega}_{n,e,k,\Delta}} g_{k,e,\Delta}(x,y)dxdy + \mathrm{O}\left(Q^{-1+\frac{1}{k}+\epsilon}\log^2 Q\right).$$

By the standard approximation argument, we next approximate the characteristic function $\chi_{\mathfrak{B}}$ of a box $\mathfrak{B}\subset(0,\Lambda)^{\nu-1}$ from above and from below by H. Thus, we have

$$\mathcal{S}^{\nu}(\mathfrak{B}) = \lim_{Q\to\infty}\frac{S_{Q,k}^{(\nu)}}{\mathcal{N}(Q,k,1)} = \frac{1}{\mathscr{C}(k,1)} \sum_{1\le n_j\le \mathcal{C}(\Lambda,k,1)} \mu(n_1)\cdots\mu(n_{\nu-1}) \sum_{\substack{1\le \Delta_j\le (\nu-1)\mathcal{C}(\Lambda,k,1)\\ 1\le e_j\le \nu\mathcal{C}(\Lambda,k,1)}} \times \Xi(k,n_j,e_j,\Delta_j) \iint_{\tilde{\Omega}_{n,e,k,\Delta}} g^{\chi}_{k,e,\Delta}(x,y)dxdy,$$

where

$$g^{\chi}_{k,e,\Delta}(x,y) = \chi_{\mathfrak{B}}\circ T\left(g^{(1)}_{k,e,\Delta}(x,y),\ldots,g^{(\nu-1)}_{k,e,\Delta}(x,y)\right).$$

This completes the proof of Theorem 4.4.1. Using similar arguments as in the above proof, we can establish the correlation measure for the sequence $\left(\mathfrak{F}_{Q,k}^{(m)}\right)$ for $m\ge 2$.

4.4.4 Proof of Theorem 1.1.13

To prove Theorem 1.1.13, we need to estimate, for any positive real number Λ, the quantity

$$S^{(2)}_{\mathfrak{F}_{Q,k}^{(m)}}(\Lambda) = \frac{1}{\mathcal{N}(Q,k,m)}\#\left\{(\gamma_1,\gamma_2)\in\left(\mathfrak{F}_{Q,k}^{(m)}\right)^2 : \gamma_1\ne\gamma_2, \gamma_1-\gamma_2\in \frac{1}{\mathcal{N}(Q,k,m)}(0,\Lambda)+\mathbb{Z}\right\}, \tag{4.42}$$

as $Q \to \infty$. Let H be any continuously differentiable function with Supp $H \subset (0, \Lambda)$. To estimate (4.42), we consider (4.32) with $\nu = 2$. We obtain

$$S_{Q,k}^{(2)} = \sum_{r \in \mathbb{Z}} c_r \sum_{\gamma_1 \in \mathfrak{F}_{Q,k}^{(m)}} e(r\gamma_1) \sum_{\gamma_2 \in \mathfrak{F}_{Q,k}^{(m)}} e(r\gamma_2). \tag{4.43}$$

We employ Lemma 4.4.1 into the above identity and express it as

$$\begin{aligned} S_{Q,k}^{(2)} &= \sum_{r \in \mathbb{Z}} c_r \sum_{d_1, d_2 \leq Q} \mu(d_1)\mu(d_2) \sum_{\substack{q_1 \leq \frac{Q}{d_1}, q_2 \leq \frac{Q}{d_2} \\ [q_1, q_2] | r \\ q_1 d_1 \equiv b \pmod m \\ q_2 d_2 \equiv b \pmod m}} q_1 q_2 \mu_k(q_1 d_1)^2 \mu_k(q_2 d_2)^2 \\ &= \sum_{d_1, d_2 \leq Q} \mu(d_1)\mu(d_2) \sum_{\substack{q_1 \leq \frac{Q}{d_1}, q_2 \leq \frac{Q}{d_2} \\ q_1 d_1 \equiv b \pmod m \\ q_2 d_2 \equiv b \pmod m}} q_1 q_2 \mu_k(q_1 d_1)^2 \mu_k(q_2 d_2)^2 \sum_{r \in \mathbb{Z}} c_{r[q_1, q_2]}, \end{aligned} \tag{4.44}$$

where $[q_1, q_2]$ is least common multiple of q_1 and q_2. In order to estimate the innermost sum, for each $y > 0$ we consider the function

$$H_y(x) = \frac{1}{y} H\left(\frac{x\mathcal{N}(Q, k, m)}{y}\right), \ x \in \mathbb{R}.$$

Then

$$\widehat{H}_y(z) = \frac{1}{\mathcal{N}(Q, k, m)} \widehat{H}\left(\frac{yz}{\mathcal{N}(Q, k, m)}\right). \tag{4.45}$$

Since $c_r = \frac{1}{(\mathcal{N}(Q,k,m))} \widehat{H}\left(\frac{r}{\mathcal{N}(Q,k,m)}\right)$, by (4.45) and Proposition 2.1.3, we obtain

$$\sum_{r \in \mathbb{Z}} c_{[q_1, q_2]r} = \sum_{r \in \mathbb{Z}} \widehat{H}_{[q_1, q_2]}(r) = \sum_{r \in \mathbb{Z}} H_{[q_1, q_2]}(r) = \sum_{r \in \mathbb{Z}} \frac{1}{[q_1, q_2]} H\left(\frac{r\mathcal{N}(Q, k, m)}{[q_1, q_2]}\right). \tag{4.46}$$

Using the above identity into (4.44), we obtain

$$S_{Q,k}^{(2)} = \sum_{d_1, d_2 \leq Q} \mu(d_1)\mu(d_2) \sum_{\substack{q_1 \leq \frac{Q}{d_1}, q_2 \leq \frac{Q}{d_2} \\ q_1 d_1 \equiv b \pmod m \\ q_2 d_2 \equiv b \pmod m}} \gcd(q_1, q_2) \mu_k(q_1 d_1)^2 \mu_k(q_2 d_2)^2$$

$$\times \sum_{r\in\mathbb{Z}} H\left(\frac{r\mathcal{N}(Q,k,m)}{[q_1,q_2]}\right). \tag{4.47}$$

Let $\gcd(q_1,q_2)=\delta$, so that $q_1=\delta q_1'$, $q_1=\delta q_1'$ with $(q_1',q_2')=1$. Then (4.47) becomes

$$\begin{aligned} S^{(2)}_{Q,k} = &\sum_{\delta\le Q}\delta \sum_{d_1,d_2\le\frac{Q}{\delta}} \mu(d_1)\mu(d_2) \sum_{\substack{q_1'\le\frac{Q}{\delta d_1},q_2'\le\frac{Q}{\delta d_2}\\ q_1'\delta d_1\equiv b \pmod m\\ q_2'\delta d_2\equiv b \pmod m\\ (q_1',q_2')=1}} \mu_k(q_1'd_1\delta)^2\mu_k(q_2'd_2\delta)^2 \\ &\times \sum_{r\in\mathbb{Z}} H\left(\frac{r\mathcal{N}(Q,k,m)}{q_1'q_2'\delta}\right) \\ = &\sum_{\delta\le Q}\delta \sum_{d_1,d_2\le\frac{Q}{\delta}} \mu(d_1)\mu(d_2) \sum_{\substack{q_1\le\frac{Q}{\delta d_1},q_2\le\frac{Q}{\delta d_2}\\ q_1\delta d_1\equiv b \pmod m\\ q_2\delta d_2\equiv b \pmod m\\ (q_1,q_2)=1}} \mu_k(q_1d_1\delta)^2\mu_k(q_2d_2\delta)^2 \\ &\times \sum_{r\in\mathbb{Z}} H\left(\frac{r\mathcal{N}(Q,k,m)}{q_1q_2\delta}\right). \end{aligned} \tag{4.48}$$

For the non-zero contribution from H, using the fact that $\mathrm{Supp}H\subset(0,\Lambda)$ and Proposition 4.1.1, one must have

$$0<\frac{\mathcal{N}(Q,k,m)r}{q_1q_2\delta}<\Lambda, \tag{4.49}$$

which implies

$$\delta d_1d_2r<\frac{\Lambda}{\mathscr{C}(k,m)}=:\mathcal{C}(\Lambda,k,m).$$

By applying the above estimate and observing that

$$H\left(\frac{\mathcal{N}(Q,k,m)r}{q_1q_2\delta}\right)=H\left(\frac{Q^2\mathscr{C}(k,m)r}{q_1q_2\delta}\right)+\mathrm{O}_m\left(\frac{r}{q_1q_2\delta}Q^{\frac{2(2k-1)}{3k-2}}\right),$$

the sum in (4.48) can be expressed as

$$\begin{aligned}
S_{Q,k}^{(2)} = & \sum_{\substack{d_1,d_2,\delta,r\geq 1\\ \delta d_1 d_2 r<\mathcal{C}(\Lambda,k,m)}} \delta\mu(d_1)\mu(d_2) \sum_{\substack{q_1\leq \frac{Q}{\delta d_1}, q_2\leq \frac{Q}{\delta d_2}\\ q_1\delta d_1\equiv b \pmod m\\ q_2\delta d_2\equiv b \pmod m\\ (q_1,q_2)=1}} \mu_k(q_1 d_1\delta)^2\mu_k(q_2 d_2\delta)^2\\
& \times H\left(\frac{Q^2\mathscr{C}(k,m)r}{q_1q_2\delta}\right) + \mathrm{O}_m\left(Q^{\frac{2(2k-1)}{3k-2}}(\log Q)^2\right)\\
= & \frac{1}{\phi^2(m)} \sum_{\substack{\chi (\bmod\ m)\\ \chi' (\bmod\ m)}} \sum_{\substack{d_1,d_2,\delta,r\geq 1\\ \delta d_1 d_2 r<\mathcal{C}(\Lambda,k,m)}} \delta\chi(\delta d_1\bar{b})\chi'(\delta d_2\bar{b})\mu_k(\delta)^2\mu(d_1)\mu(d_2)\\
& \times \sum_{\substack{q_1\leq \frac{Q}{\delta d_1}, q_2\leq \frac{Q}{\delta d_2}\\ (q_1,q_2)=1}} \chi(q_1)\chi'(q_2)\mu_k(q_1 d_1\delta)^2\mu_k(q_2 d_2\delta)^2 H\left(\frac{Q^2\mathscr{C}(k,m)r}{q_1q_2\delta}\right)\\
& + \mathrm{O}_m\left(Q^{\frac{2(2k-1)}{3k-2}}(\log Q)^2\right). \qquad (4.50)
\end{aligned}$$

Next, we deal with the cases of principal and non-principal characters separately.

Case-I: If $\chi = \chi_0$ and $\chi' = \chi_0'$ then we have

$$\begin{aligned}
S_{Q,k}^{(2)}(\chi_0,\chi_0') = & \sum_{\substack{d_1,d_2,\delta,r\geq 1\\ \delta d_1 d_2 r<\mathcal{C}(\Lambda,k,m)\\ (d_1 d_2\delta,m)=1}} \delta\mu_k(\delta)^2\mu(d_1)\mu(d_2) \sum_{\substack{q_1\leq \frac{Q}{\delta d_1}, q_2\leq \frac{Q}{\delta d_2}\\ (q_1q_2,m)=1=(q_1,q_2)}} \mu_k(q_1 d_1\delta)^2\mu_k(q_2 d_2\delta)^2\\
& \times H\left(\frac{Q^2\mathscr{C}(k,m)r}{q_1q_2\delta}\right). \qquad (4.51)
\end{aligned}$$

To estimate the inner sum in the above identity, we employ Lemma 2.6.2 which counts the k-free lattice points with some weight and congruence constraints. Note that, since Supp $H \subset (0,\Lambda)$, for the non-zero contribution from H, one has $0 < \frac{Q^2\mathscr{C}(k,m)r}{x_1x_2\delta} < \Lambda$. For $0 < x_1 \leq \frac{Q}{\delta d_1}$ and $0 < x_2 \leq \frac{Q}{\delta d_2}$, we obtain

$$\frac{1}{x_1} \leq \frac{\mathcal{C}(\Lambda,k,m)}{rd_2Q} \text{ and } \frac{1}{x_2} \leq \frac{\mathcal{C}(\Lambda,k,m)}{rd_1Q}. \qquad (4.52)$$

Using (4.52) and the necessary condition for the non-zero contribution of H, we get

$$\left|\frac{\partial H}{\partial x_1}(x_1,x_2)\right| \ll \frac{1}{Q} \text{ and } \left|\frac{\partial H}{\partial x_2}(x_1,x_2)\right| \ll \frac{1}{Q}.$$

Hence

$$\|DH\|_\infty \ll \frac{1}{Q}.$$

Employing Lemma 2.6.2 with $r_1 = r_2 = m$, $\delta_1 = d_1\delta$, $\delta_2 = d_2\delta$, and $f(a,b) = H\left(\frac{Q^2\mathscr{C}(k,m)r}{ab\delta}\right)$, the inner-sum in (4.51) is expressed as

$$\begin{aligned}
&\sum_{\substack{q_1\leq\frac{Q}{\delta d_1},q_2\leq\frac{Q}{\delta d_2}\\(q_1q_2,m)=1,(q_1,q_2)=1}} \mu_k(q_1d_1\delta)^2\mu_k(q_2d_2\delta)^2 H\left(\frac{Q^2\mathscr{C}(k,m)r}{q_1q_2\delta}\right)\\
&= \frac{6P^k_{m,m}(d_1\delta,d_2\delta)}{\pi^2}\int_0^{\frac{Q}{\delta d_1}}\int_0^{\frac{Q}{\delta d_2}} H\left(\frac{Q^2\mathscr{C}(k,m)r}{xy\delta}\right)dxdy + \mathrm{O}_m\left(\tau(m)Q^{1+\frac{1}{k}}\log^2 Q\right).
\end{aligned} \tag{4.53}$$

Using the fact that Supp$H \subset (0,\lambda)$ and applying the change of variable $\lambda = \frac{\mathscr{C}(k,m)r}{xy\delta}$ the integral in (4.53) can be expressed as

$$\begin{aligned}
\int_0^{\frac{Q}{\delta d_1}}\int_0^{\frac{Q}{\delta d_2}} H\left(\frac{Q^2\mathscr{C}(k,m)r}{xy\delta}\right)dxdy =& Q^2\int_0^{\frac{1}{\delta d_1}}\int_0^{\frac{1}{\delta d_2}} H\left(\frac{\mathscr{C}(k,m)r}{xy\delta}\right)dxdy\\
=& \frac{Q^2r\mathscr{C}(k,m)}{\delta}\int_0^{\frac{1}{\delta d_1}}\int_{\frac{rd_2\mathscr{C}(k,m)}{x}}^{\Lambda}\frac{H(\lambda)}{\lambda^2x}d\lambda dx\\
=& \frac{Q^2r\mathscr{C}(k,m)}{\delta}\int_{r\delta d_1d_2\mathscr{C}(k,m)}^{\Lambda}\int_{\frac{rd_2\mathscr{C}(k,m)}{\lambda}}^{\frac{1}{\beta d_1}}\frac{H(\lambda)}{\lambda^2x}dxd\lambda\\
=& \frac{Q^2\mathscr{C}(k,m)r}{\delta}\int_{r\delta d_1d_2\mathscr{C}(k,m)}^{\Lambda}\frac{H(\lambda)}{\lambda^2}\\
&\times\log\left(\frac{\lambda}{r\delta d_1d_2\mathscr{C}(k,m)}\right)d\lambda.
\end{aligned}$$

The above identity with (4.51) and (4.53) gives

$$\begin{aligned}
S^{(2)}_{Q,k}(\chi_0,\chi_0') =& \frac{6Q^2\mathscr{C}(k,m)}{\pi^2}\sum_{\substack{d_1,d_2,\delta,r\geq1\\ \delta d_1d_2r<\mathcal{C}(\Lambda,k,m)\\ (d_1d_2\delta,m)=1}} r\mu_k(\delta)^2\mu(d_1)\mu(d_2)P^k_{m,m}(d_1\delta,d_2\delta)\\
&\times\int_{r\delta d_1d_2\mathscr{C}(k,m)}^{\Lambda}\frac{H(\lambda)}{\lambda^2}\log\left(\frac{\lambda}{r\delta d_1d_2\mathscr{C}(k,m)}\right)d\lambda + \mathrm{O}_m\left(Q^{1+\frac{1}{k}}\log^2 Q\right)
\end{aligned}$$

$$
\begin{aligned}
=&\frac{6Q^2\mathscr{C}(k,m)}{\pi^2}\sum_{1\leq n<\mathcal{C}(\Lambda,k,m)}\int_{n\mathscr{C}(k,m)}^{\Lambda}\frac{H(\lambda)}{\lambda^2}\log\left(\frac{\lambda}{n\mathscr{C}(k,m)}\right)d\lambda\\
&\times\sum_{\substack{\delta d_1 d_2 r=n\\(d_1 d_2\delta,m)=1}} r\mu_k(\delta)^2\mu(d_1)\mu(d_2)P^k_{m,m}(d_1\delta,d_2\delta)+\mathrm{O}_m\left(Q^{1+\frac{1}{k}}\log^2 Q\right)\\
=&\frac{6Q^2\mathscr{C}(k,m)}{\pi^2}\int_0^{\Lambda}\frac{H(\lambda)}{\lambda^2}\sum_{1\leq n<\mathcal{C}(\Lambda,k,m)}F_k(n)\log\left(\frac{\lambda}{n\mathscr{C}(k,m)}\right)d\lambda\\
&+\mathrm{O}_m\left(Q^{1+\frac{1}{k}}\log^2 Q\right), \qquad (4.54)
\end{aligned}
$$

where

$$
F_k(n)=\sum_{\substack{\delta d_1 d_2 r=n\\(d_1 d_2\delta,m)=1}} r\mu_k(\delta)^2\mu(d_1)\mu(d_2)P^k_{m,m}(d_1\delta,d_2\delta).
$$

Case-II: Suppose at least one of χ or χ' is non-principal Dirichlet character. Without loss of generality, suppose χ' is non-principal character. The case when χ is non-principal follows similarly.

$$
\begin{aligned}
S^{(2)}_{Q,k}(\chi,\chi')=&\sum_{\substack{d_1,d_2,\delta,r\geq 1\\ \delta d_1 d_2 r<\mathcal{C}(\Lambda,k,m)}}\delta\chi(\delta d_1\bar{b})\chi'(\delta d_2\bar{b})\mu_k(\delta)^2\mu(d_1)\mu(d_2)\sum_{\substack{q_1\leq\frac{Q}{\delta d_1},q_2\leq\frac{Q}{\delta d_2}\\(q_1,q_2)=1}}\chi(q_1)\chi'(q_2)\\
&\times\mu_k(q_1 d_1\delta)^2\mu_k(q_2 d_2\delta)^2 H\left(\frac{Q^2\mathscr{C}(k,m)r}{q_1q_2\delta}\right)\\
=&\sum_{\substack{d_1,d_2,\delta,r\geq 1\\ \delta d_1 d_2 r<\mathcal{C}(\Lambda,k,m)}}\delta\chi(\delta d_1\bar{b})\chi'(\delta d_2\bar{b})\mu_k(\delta)^2\mu(d_1)\mu(d_2)\sum_{q_1\leq\frac{Q}{\delta d_1}}\chi(q_1)\mu_k(q_1 d_1\delta)^2\\
&\times\sum_{\substack{q_2\leq\frac{Q}{\delta d_2}\\(q_2,q_1)=1}}\chi'(q_2)\mu_k(q_2 d_2\delta)^2 H\left(\frac{Q^2\mathscr{C}(k,m)r}{q_1q_2\delta}\right). \qquad (4.55)
\end{aligned}
$$

In order to estimate the inner sum in the above identity, we use Lemma 2.3.1 with

$f\left(\frac{M}{q_2}\right) = H\left(\frac{Q^2\mathscr{C}(k,m)r}{q_1q_2\delta}\right)$ and obtain

$$\sum_{\substack{q_2\le \frac{Q}{\delta d_2}\\ (q_2,q_1)=1}} \chi'(q_2)\mu_k(q_2d_2\delta)^2 H\left(\frac{Q^2\mathscr{C}(k,m)r}{q_1q_2\delta}\right) \ll_m \tau(q_1)\left(\frac{Q}{\delta d_2}\right)^{\frac{1}{k}} \log\frac{Q}{\delta d_2},$$

and this in conjunction with (4.55) yields

$$S^{(2)}_{Q,k}(\chi,\chi') \ll_m Q^{1+\frac{1}{k}+\epsilon}\log Q. \tag{4.56}$$

We collect the estimates from (4.54) and (4.56) and insert them into (4.50). We obtain

$$\begin{aligned} S^{(2)}_{Q,k} =& \frac{6Q^2\mathscr{C}(k,m)}{\pi^2\phi^2(m)}\int_0^\Lambda \frac{H(\lambda)}{\lambda^2}\sum_{1\le n<\mathcal{C}(\Lambda,k,m)} F_k(n)\log\left(\frac{\lambda}{n\mathscr{C}(k,m)}\right)d\lambda \\ &+ \mathrm{O}_m\left(Q^{1+\frac{1}{k}+\epsilon}\log^2 Q\right) \\ =& Q^2\mathscr{C}(k,m)\int_0^\Lambda H(\lambda)\mathfrak{g}_{m,k}(\lambda)d\lambda + \mathrm{O}_m\left(Q^{1+\frac{1}{k}+\epsilon}\log^2 Q\right). \end{aligned}$$

Therefore

$$\frac{S^{(2)}_{Q,k}}{\mathcal{N}(Q,k)} = \int_0^\Lambda H(\lambda)\mathfrak{g}_{m,k}(\lambda)d\lambda + \mathrm{O}_m\left(Q^{-1+\frac{1}{k}+\epsilon}\log^2 Q\right).$$

We next approximate the characteristic function of the interval $(0,\Lambda)$ from below and above by the smooth functions with compact support in $(0,\Lambda)$, using the standard approximation argument, to obtain

$$\mathcal{S}^2((0,\Lambda)) = \lim_{Q\to\infty} S^{(2)}_{\mathfrak{F}^{(m)}_{Q,k}}(\Lambda) = \int_0^\Lambda \mathfrak{g}_{m,k}(\lambda)d\lambda.$$

This completes the proof of Theorem 1.1.13.

5

Distribution of index of Farey sequences

In this chapter, we study the distribution of the index function by evaluating the moments of Farey indices with $\mathcal{B}$-free Farey denominators in an arithmetic progression. Similar divisibility constraint with $\mathcal{B}$-free Farey denominators in an arithmetic progression was imposed by the authors in [1] where they obtain absolute bounds on the discrepancy of Farey fractions with such denominators. The notion of $\mathcal{B}$-free numbers was introduced by Erdős in [35] as a generalization of square-free numbers.

Definition 5.0.1 ($\mathcal{B}$-free numbers)**.** *Given an infinite set* $\mathcal{B} = \{b_1, b_2, \cdots\}$ *of integers greater than* 1*, we say that a positive integer is* $\mathcal{B}$*-free if it is not divisible by any element of* $\mathcal{B}$*.*

Denote the set of all primes by $\mathcal{P}$. For positive integer $k \geq 2$, $\mathcal{B} = \{p^k : p \in \mathcal{P}\}$ gives rise to k-free integers. It is known [35] that for all large enough N and some

$0 < c < 1$, the interval $[N, N + N^c]$ contains at least one $\mathcal{B}$-free number.

First, we recall the definition of the index of Farey fractions. Let $\gamma' = \frac{a'}{q'} < \gamma = \frac{a}{q} < \gamma'' = \frac{a''}{q''}$ be three consecutive Farey fractions in $\mathcal{F}_Q$. Then, the index of the Farey fraction γ is defined as

$$\nu_Q(\gamma) = \frac{q' + q''}{q} = \frac{a' + a''}{a}. \tag{5.1}$$

The l-th moment of Farey indices with $\mathcal{B}$-free Farey denominator in an arithmetic progression is given by

$$\mathcal{M}_{l,\mathcal{B}}(u, m, Q) := \sum_{\substack{\gamma = \frac{a}{q} \in \mathcal{F}_Q \\ q \equiv u \pmod{m} \\ q \text{ is } \mathcal{B}\text{-free}}} \nu_Q(\gamma)^l. \tag{5.2}$$

Here, we establish asymptotic formulas for the above sum for first, second, and higher moments of the set

$$\mathcal{B} = \left\{ p \in \mathcal{P} : \sum_p \frac{1}{p^\sigma} < \infty \text{ for some } \sigma < \theta, \text{ where } 1/2 < \theta < 1 \right\}. \tag{5.3}$$

In particular, we provide proofs for Theorems 1.2.1, 1.2.2, and 1.2.3.

5.1 General setup and outline of the proof

In this section, we will provide general setup and outline our strategy to prove Theorems 1.2.1, 1.2.2, and 1.2.3. We first express the arithmetic progression in terms of Dirichlet characters. Further, we distinguish between the cases of the first, second, and higher moments. For fixed positive integers m and u with $(m, u) = 1$ and $u\bar{u} \equiv 1 \pmod{m}$, using Proposition 2.3.1, the l-th moment in (5.2) can be written as

$$\mathcal{M}_{l,\mathcal{B}}(u, m, Q) = \frac{1}{\phi(m)} \sum_{\substack{\gamma = \frac{a}{q} \in \mathcal{F}_Q \\ \mu_{\mathcal{B}}(q) = 1}} \nu_Q(\gamma)^l \sum_{\chi \pmod{m}} \chi(\bar{u}q)$$

$$
\begin{aligned}
&= \frac{1}{\phi(m)} \sum_{\chi \pmod m} \chi(\bar{u}) \sum_{\substack{\gamma=\frac{a}{q}\in\mathcal{F}_Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\nu_Q(\gamma)^l \\
&= \frac{1}{\phi(m)} \sum_{\chi \pmod m} \chi(\bar{u}) \mathcal{M}_{l,\mathcal{B}}(\chi, Q),
\end{aligned}
$$

where

$$
\mathcal{M}_{l,\mathcal{B}}(\chi, Q) := \sum_{\substack{\gamma=\frac{a}{q}\in\mathcal{F}_Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(s)\nu_Q(\gamma)^l = \sum_{\substack{q\le Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q) \sum_{\gamma=\frac{a}{q}\in\mathcal{F}_Q} \nu_Q(\gamma)^l. \tag{5.4}
$$

We express the partial sum involving index separately for $l = 1, 2$ and $l \geq 3$.

First moment

(i) For $l = 1$, we use the definition of index (5.1) and the fact that $q' > Q - q$, since a'/q' and a/q are consecutive Farey fractions. We have

$$
\sum_{\gamma=\frac{a}{q}\in\mathcal{F}_Q} \nu_Q(\gamma) = \frac{2}{q} \sum_{\substack{q'=Q-q+1 \\ \gcd(q',q)=1}}^{Q} q' = 2\sum_{d|q} \mu(d) \left\lfloor \frac{Q}{d} \right\rfloor - \phi(q) + \epsilon(q), \tag{5.5}
$$

where $\epsilon(q)$ is unit function. This leads to

$$
\mathcal{M}_{1,\mathcal{B}}(\chi, Q) = 2 \sum_{\substack{q\le Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q) \sum_{d|q} \mu(d) \left\lfloor \frac{Q}{d} \right\rfloor - \sum_{\substack{q\le Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q) + 1. \tag{5.6}
$$

We now express the inner sum in the first term of the above identity in terms of Euler totient function. Note that the summands in the first and second terms of (5.6) are multiplicative. We represent their Dirichlet series via an Euler product and employ Lemma 2.1.1. Distinguishing between principal and non-principal Dirichlet characters, we then invoke Cauchy's residue theorem together with the bounds of the Riemann zeta function and Dirichlet L-function to establish the asymptotic formula for the first moment.

Second moment

(ii) For $l = 2$, we write the mean value of square of the index using deficiency. Recall, that the index $\nu_Q(\gamma)$ can take at most two values $\lfloor 2Q/q \rfloor$ or $\lfloor 2Q/q \rfloor - 1$, and the deficiency $\delta(q)$ is the number of Farey fractions $\gamma \in \mathcal{F}_Q$ with denominator q such that $\nu_Q(\gamma) = \left\lfloor \frac{2Q}{q} \right\rfloor - 1$. Moreover, it is well known that there are $\phi(q)$ fractions in $\mathcal{F}_Q$ with denominator q. This yields

$$\begin{aligned} T(q) := \sum_{\gamma = \frac{a}{q} \in \mathcal{F}_Q} \nu_Q(\gamma) &= (\phi(q) - \delta(q)) \left\lfloor \frac{2Q}{q} \right\rfloor + \delta(q) \left(\left\lfloor \frac{2Q}{q} \right\rfloor - 1 \right) \\ &= \phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor - \delta(q), \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \sum_{\gamma = \frac{a}{q} \in \mathcal{F}_Q} \nu_Q(\gamma)^2 &= (\phi(q) - \delta(q)) \left\lfloor \frac{2Q}{q} \right\rfloor^2 + \delta(q) \left(\left\lfloor \frac{2Q}{q} \right\rfloor - 1 \right)^2 \\ &= \phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor^2 - \delta(q) \left(2 \left\lfloor \frac{2Q}{q} \right\rfloor - 1 \right). \end{aligned} \tag{5.8}$$

Note that from (5.5) and (5.7), we get an alternate expression for $\delta(q)$ given by

$$\delta(q) = \phi(q) \left(\left\lfloor \frac{2Q}{q} \right\rfloor + 1 \right) - 2 \sum_{d \mid q} \mu(d) \left\lfloor \frac{Q}{d} \right\rfloor - \epsilon(q). \tag{5.9}$$

Consequently from (5.4) and (5.8), we have

$$\begin{aligned} \mathcal{M}_{2,\mathcal{B}}(\chi, Q) &= \sum_{\substack{q \le Q \\ \mu_{\mathcal{B}}(q) = 1}} \chi(q) \left(\phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor^2 - \delta(q) \left(2 \left\lfloor \frac{2Q}{q} \right\rfloor - 1 \right) \right) \\ &= X_\chi(Q) - 2Y_\chi(Q) + \sum_{\substack{q \le Q \\ \mu_{\mathcal{B}}(q) = 1}} \chi(q) \delta(q), \end{aligned} \tag{5.10}$$

where

$$X_\chi(Q) := \sum_{\substack{q \le Q \\ \mu_{\mathcal{B}}(q) = 1}} \chi(q) \phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor^2, \ Y_\chi(Q) := \sum_{\substack{q \le Q \\ \mu_{\mathcal{B}}(q) = 1}} \chi(q) \delta(q) \left\lfloor \frac{2Q}{q} \right\rfloor.$$

To estimate $X_\chi(Q)$ and $Y_\chi(Q)$, we extend the range of summation from $1 \leq q \leq Q$ to $1 \leq q \leq 2Q$ and write

$$X_\chi(Q) = \sum_{\substack{q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor^2 - \sum_{\substack{Q<q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q),$$

and

$$Y_\chi(Q) = \sum_{\substack{q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\delta(q) \left\lfloor \frac{2Q}{q} \right\rfloor - \sum_{\substack{Q<q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\delta(q).$$

We express $\delta(q)$ in terms of the Möbius function and the Euler totient function using (5.9). We further write the Dirichlet series as an Euler product, since the summands are multiplicative. Using a similar approach to the first moment, we derive the asymptotic formula for the second moment.

Higher moments

(iii) For $l \geq 3$, we use the fact that for any three consecutive Farey fractions $\gamma' = \frac{a'}{q'} < \gamma = \frac{a}{q} < \gamma'' = \frac{a''}{q''}$ their denominators are related (see [47]) as

$$q'' = \left\lfloor \frac{Q+q'}{q} \right\rfloor q - q'. \tag{5.11}$$

This together with definition (5.1) enable us to write

$$\begin{aligned}
\sum_{\gamma=\frac{a}{q}\in\mathcal{F}_Q} \nu_Q(\gamma)^l &= \sum_{\substack{Q-q<q'<Q \\ (q',q)=1}} \left\lfloor \frac{Q+q'}{q} \right\rfloor^l = \sum_{\substack{Q-q<q'<Q \\ (q',q)=1}} \left(\left(\frac{Q+q'}{q}\right)^l + \mathrm{O}\left(\left(\frac{Q+q'}{q}\right)^{l-1}\right)\right) \\
&= \sum_{\substack{Q-q<q'<Q \\ (q',q)=1}} \left(\frac{Q+q'}{q}\right)^l + \mathrm{O}\left(\sum_{\substack{q \leq Q \\ \mu_{\mathcal{B}}(q)=1}} \sum_{\substack{Q-q<q'<Q \\ (q',q)=1}} \left(\frac{Q+q'}{q}\right)^{l-1} \right) \\
&= \sum_{\substack{Q-q<q'<Q \\ (q',q)=1}} \left(\left(\frac{2Q}{q}\right)^l - \left(\frac{2Q}{q}\right)^l + \left(\frac{Q+q'}{q}\right)^l \right) + \mathrm{O}\left(Q^{l-1}\right).
\end{aligned}$$

As a result, substituting in (5.4), we obtain

$$\mathcal{M}_{l,\mathcal{B}}(\chi, Q) = \sum_{\substack{q\leq Q\\ \mu_{\mathcal{B}}(q)=1}} \chi(q) \sum_{\substack{Q-q<q'<Q\\ (q',q)=1}} \left(\left(\frac{2Q}{q}\right)^{l} - \left(\frac{2Q}{q}\right)^{l} + \left(\frac{Q+q'}{q}\right)^{l}\right) + \mathrm{O}\left(Q^{l-1}\right). \tag{5.12}$$

The estimates for (5.6), (5.10), and (5.12) will be handled separately in later sections.

5.2 First Moment

Proof of Theorem 1.2.1. Using (5.6), the first moment can be expressed as

$$\mathcal{M}_{1,\mathcal{B}}(\chi, Q) = 2 \sum_{\substack{q\leq Q\\ \mu_{\mathcal{B}}(q)=1}} \chi(q) \sum_{d|q} \mu(d) \left\lfloor \frac{Q}{d} \right\rfloor - \sum_{\substack{q\leq Q\\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q) + 1 = 2S_1 - S_2 + 1. \tag{5.13}$$

We evaluate the first sum on the right side of (5.13) as

$$\begin{aligned} S_1 &= Q \sum_{d\leq Q} \frac{\mu(d)}{d} \sum_{\substack{q\leq Q\\ d|q\\ \mu_{\mathcal{B}}(q)=1}} \chi(q) + \mathrm{O}\left(\sum_{q\leq Q} \tau(q)\right) \\ &= Q \sum_{d\leq Q} \frac{\mu(d)}{d} \sum_{\substack{q\leq Q\\ d|q}} \chi(q)\mu_{\mathcal{B}}(q) + \mathrm{O}\left(Q\log Q\right) \\ &= Q \sum_{d\leq Q} \frac{\mu(d)\chi(d)\mu_{\mathcal{B}}(d)}{d} \sum_{n\leq \frac{Q}{d}} \chi(n)\mu_{\mathcal{B}}(n) + \mathrm{O}\left(Q\log Q\right). \end{aligned} \tag{5.14}$$

We first estimate the inner sum in (5.14). Since $\chi(n)\mu_{\mathcal{B}}(n)$ is multiplicative, its Dirichlet series is given by

$$F(s) = \sum_{n=1}^{\infty} \frac{\chi(n)\mu_{\mathcal{B}}(n)}{n^s} = \prod_p \left(1 + \frac{\chi(p)\mu_{\mathcal{B}}(p)}{p^s} + \frac{\chi(p^2)\mu_{\mathcal{B}}(p^2)}{p^{2s}} + \cdots\right)$$

$$
\begin{aligned}
&= \prod_{p \notin \mathcal{B}} \left(1 + \frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} + \cdots \right) = \prod_{p \notin \mathcal{B}} \left(\frac{1}{1 - \frac{\chi(p)}{p^s}} \right) \\
&= L(s,\chi) \prod_{p \in \mathcal{B}} \left(1 - \frac{\chi(p)}{p^s}\right), \qquad (5.15)
\end{aligned}
$$

having analytic continuation to the half plane $\Re(s) > \theta$. Moreover, the term containing the product on the far right side of (5.15) is bounded in any half plane $\Re(s) > \sigma$ with $\sigma > \theta$. Also, there is a constant $P_{k,\mathcal{B}} > 0$ such that

$$
\left| \prod_{\substack{p \in \mathcal{B} \\ p \nmid m}} \left(1 - \frac{1}{p^s}\right) \right| \leq P_{k,\mathcal{B}}.
$$

Let $T \geq 2$, $\alpha = 1 + \frac{1}{\log Q}$. We use Lemma 2.1.1, where we put $x = \frac{Q}{d} + \frac{1}{2}$,

$$
S_{11} := \sum_{n \leq \frac{Q}{d}} \chi(n) \mu_{\mathcal{B}}(n) = \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{\left(\frac{Q}{d} + \frac{1}{2}\right)^s F(s)}{s} ds + R(T), \qquad (5.16)
$$

and

$$
|R(T)| \ll \frac{Q^\alpha}{d^\alpha T} \sum_{n=1}^{\infty} \frac{n^{-\alpha}}{\left|\log \frac{\frac{Q}{d} + \frac{1}{2}}{n}\right|}.
$$

We divide the sum into three subsums over the following ranges of n: $n \leq \frac{Q}{2d}$, $\frac{Q}{2d} < n < \frac{3Q}{2d}$, and $n \geq \frac{3Q}{2d}$. For $n \leq \frac{Q}{2d}$ and $n \geq \frac{3Q}{2d}$, it is clear that $\left|\log \frac{\frac{Q}{d} + \frac{1}{2}}{n}\right| \geq \log \frac{3}{2}$. Hence the first and last subsums are $\mathrm{O}\left(1/(\alpha - 1)\right)$. For values of n satisfying $\frac{Q}{2d} < n < \frac{3Q}{2d}$ the middle subsum is

$$
\ll \left(\frac{Q}{d}\right)^{-\alpha} \sum_{\frac{-Q}{2d} < n \leq \frac{Q}{2d}} \frac{1}{\left|\log \frac{\frac{Q}{d} + \frac{1}{2}}{\frac{Q}{d} + n}\right|} \ll \left(\frac{Q}{d}\right)^{1-\alpha} \sum_{\frac{-Q}{2d} < n \leq \frac{Q}{2d}} \frac{1}{|n - \frac{1}{2}|} \ll \left(\frac{Q}{d}\right)^{1-\alpha} \log Q.
$$

Therefore,

$$
|R(T)| \ll \frac{Q^\alpha \log Q}{d^\alpha T}. \qquad (5.17)
$$

We now consider the integral in (5.16), which we estimate by shifting the line of integration into a rectangular contour with line segments joining the point $\alpha - iT$,

$\alpha+iT$, $\theta+iT$, and $\theta-iT$. We first consider the principle character χ_0 modulo m. In this case, By Cauchy's residue theorem, we have

$$\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\frac{\left(\frac{Q}{d}+\frac{1}{2}\right)^s F(s)}{s}ds=\frac{Q}{d}\prod_{\substack{p\in\mathcal{B}\\ p\nmid m}}\left(1-\frac{1}{p}\right)\prod_{p\mid m}\left(1-\frac{1}{p}\right)+\sum_{j=1}^{3}I_j, \tag{5.18}$$

where I_1 and I_3 are the integrals along the horizontal segments $[\alpha-iT,\theta-iT]$ and $[\theta+iT,\alpha+iT]$, respectively and I_2 is the integral over the vertical segment $[\theta-iT,\theta+iT]$. The first term in the right hand side is obtained from the residue of the simple pole at $s=1$, giving rise to the main term in the asymptotic formula in the statement of Theorem 1.2.1. To estimate the I_j's, one finds that the bounds provided in (2.2), for similar integrals, apply to our case as well, modulo multiplication by constants depending on m and $\mathcal{B}$. Therefore,

$$|I_1|,|I_3|\ll_{m,\mathcal{B}}\frac{\log Q\log^2 T}{d^\alpha T^{1/2}}\left(\frac{Q}{dT^{1/2}}-\frac{Q^\theta}{d^\theta T^{\theta/2}}+\frac{Q^\alpha}{d^\alpha}-\frac{Q}{d}\right), \tag{5.19}$$

and

$$\begin{aligned}|I_2| &\ll_{m,\mathcal{B}}\int_{-T}^{T}\frac{|\left(\frac{Q}{d}+\frac{1}{2}\right)^{\theta+it}||\zeta(\theta+it)|}{|\theta+it|}dt\ll_{m,\mathcal{B}}\left(\frac{Q}{d}\right)^\theta\int_0^T\frac{|\zeta(\theta+it)|}{|\theta+it|}dt\\ &\ll_{m,\mathcal{B}}(Q/d)^\theta\log T,\end{aligned} \tag{5.20}$$

where we used Lemma 2.4.1 to estimate the zeta integral. Next, we consider the case for a non-principle character $\chi\neq\chi_0$. We continue with the rectangular contour defined above. Using bounds for $L(s,\chi)$ (see (2.5) and (2.6)), we obtain

$$|I_1|,|I_3|\ll_{m,\mathcal{B}}\frac{Q^\alpha\log Q\log T}{d^\alpha T}\ ;\ |I_2|\ll_{m,\mathcal{B}}\frac{Q^\theta\log T}{d^\theta}.$$

Therefore, inserting (5.19) and (5.20) into (5.18) and choosing $T=Q^2$, for $\chi=\chi_0$, we have

$$S_{11}=\frac{Q}{d}\prod_{\substack{p\in\mathcal{B}\\ p\nmid m}}\left(1-\frac{1}{p}\right)\prod_{p\mid m}\left(1-\frac{1}{p}\right)+\mathrm{O}_{m,\mathcal{B}}\left(\frac{Q^\theta\log Q}{d^\theta}\right), \tag{5.21}$$

and for $\chi \neq \chi_0$, we obtain

$$S_{11} = \mathrm{O}_{m,\mathcal{B}}\left(\frac{Q^\theta \log Q)}{d^\theta}\right). \tag{5.22}$$

For $\chi = \chi_0$, invoking (5.21) in (5.14) gives

$$\begin{aligned} S_1 = \sum_{\substack{q\leq Q\\ \mu_{\mathcal{B}}(q)=1}} \chi_0(q) \sum_{d|q} \mu(d) \left\lfloor \frac{Q}{d} \right\rfloor &= Q^2 \prod_{\substack{p\in\mathcal{B}\\ p\nmid m}} \left(1-\frac{1}{p}\right) \prod_{p|m}\left(1-\frac{1}{p}\right) \sum_{d\leq Q} \frac{\chi_0(d)\mu(d)\mu_{\mathcal{B}}(d)}{d^2} \\ &+ \mathrm{O}_{m,\mathcal{B}}\left(Q^{1+\theta}\log Q \sum_{d\leq Q}\frac{1}{d^{1+\theta}}\right). \end{aligned} \tag{5.23}$$

One can write

$$\sum_{d\leq Q} \frac{\chi_0(d)\mu(d)\mu_{\mathcal{B}}(d)}{d^2} = \sum_{d=1}^{\infty} \frac{\chi_0(d)\mu(d)\mu_{\mathcal{B}}(d)}{d^2} + \mathrm{O}_{m,\mathcal{B}}\left(\frac{1}{Q}\right), \tag{5.24}$$

and

$$\begin{aligned} \sum_{d=1}^{\infty} \frac{\chi_0(d)\mu(d)\mu_{\mathcal{B}}(d)}{d^2} &= \prod_p \left(1 + \frac{\chi_0(p)\mu(p)\mu_{\mathcal{B}}(p)}{p^2} + \frac{\chi_0(p^2)\mu(p^2)\mu_{\mathcal{B}}(p^2)}{p^4} + \cdots\right) \\ &= \prod_p \left(1 - \frac{\chi_0(p)\mu_{\mathcal{B}}(p)}{p^2}\right) = \prod_{\substack{p\notin\mathcal{B}\\ p\nmid m}} \left(1-\frac{1}{p^2}\right) \\ &= \frac{1}{\zeta(2)} \prod_{\substack{p\in\mathcal{B}\\ p\nmid m}} \left(1-\frac{1}{p^2}\right)^{-1} \prod_{p|m}\left(1-\frac{1}{p^2}\right)^{-1}. \end{aligned} \tag{5.25}$$

Now, using (5.23) and (5.25), for the principle character $\chi = \chi_0$, we have

$$S_1 = \frac{Q^2}{\zeta(2)} \prod_{\substack{p\in\mathcal{B}\\ p\nmid m}} \left(1+\frac{1}{p}\right)^{-1} \prod_{p|m}\left(1+\frac{1}{p}\right)^{-1} + \mathrm{O}_{m,\mathcal{B}}\left(Q^{1+\theta}\log Q\right). \tag{5.26}$$

Likewise for $\chi \neq \chi_0$, using (5.22) in (5.14) gives

$$S_1 \ll \mathrm{O}_{m,\mathcal{B}}\left(Q^{1+\theta}\log Q\right).$$

Next, we first examine the second sum S_2 in (5.13) for $\chi = \chi_0$. Employing the formula $\phi(n) = \sum_{d|n} \mu(d)n/d$, we have

$$\begin{aligned} S_2 &= \sum_{\substack{q\leq Q \\ \mu_{\mathcal{B}}(q)=1}} \chi_0(q) \sum_{d|q} \frac{\mu(d)q}{d} = \sum_{d\leq Q} \frac{\mu(d)}{d} \sum_{\substack{q\leq Q \\ d|q}} \chi_0(q)\mu_{\mathcal{B}}(q)q \\ &= \sum_{d\leq Q} \chi_0(d)\mu(d)\mu_{\mathcal{B}}(d) \sum_{n\leq \frac{Q}{d}} \chi_0(n)\mu_{\mathcal{B}}(n)n. \end{aligned} \tag{5.27}$$

Using partial summations on the inner sum on the far RHS (5.27) and then applying (5.21), we have

$$\begin{aligned} \sum_{n\leq \frac{Q}{d}} \chi_0(n)\mu_{\mathcal{B}}(n)n = \left(\frac{Q}{d}\right)^2 \prod_{\substack{p\in\mathcal{B} \\ p\nmid m}} \left(1-\frac{1}{p}\right) \prod_{p|m} \left(1-\frac{1}{p}\right) + \mathrm{O}_{m,\mathcal{B}}\left(\frac{Q^{1+\theta}(\log Q)^{3/2}}{d^{1+\theta}}\right) \\ - \int_1^{\frac{Q}{d}} \sum_{n\leq t} \chi_0(n)\mu_{\mathcal{B}}(n)dt. \end{aligned}$$

Applying (5.21) again to the integrand above, we have

$$\begin{aligned} \int_1^{\frac{Q}{d}} \sum_{n\leq t} \chi_0(n)\mu_{\mathcal{B}}(n)dt &= \int_1^{\frac{Q}{d}} \left(t \prod_{\substack{p\in\mathcal{B} \\ p\nmid m}} \left(1-\frac{1}{p}\right) \prod_{p|m} \left(1-\frac{1}{p}\right) + \mathrm{O}_{m,\mathcal{B}}\left(t^{\theta}(\log t)^{3/2}\right) \right) dt \\ &= \frac{Q^2}{2d^2} \prod_{\substack{p\in\mathcal{B} \\ p\nmid m}} \left(1-\frac{1}{p}\right) \prod_{p|m} \left(1-\frac{1}{p}\right) + \mathrm{O}_{m,\mathcal{B}}\left(\frac{Q^{1+\theta}(\log Q)^{3/2}}{d^{1+\theta}}\right). \end{aligned}$$

Hence

$$\sum_{n\leq \frac{Q}{d}} \chi_0(n)\mu_{\mathcal{B}}(n)n = \frac{Q^2}{2d^2} \prod_{\substack{p\in\mathcal{B} \\ p\nmid k}} \left(1-\frac{1}{p}\right) \prod_{p|k} \left(1-\frac{1}{p}\right) + \mathrm{O}_{m,\mathcal{B}}\left(\frac{Q^{1+\theta}(\log Q)^{3/2}}{d^{1+\theta}}\right). \tag{5.28}$$

This along with (5.24), (5.25) and (5.27) yields an estimate for S_2 for the principal character χ_0

$$\begin{aligned} S_2 =& \frac{Q^2}{2}\prod_{\substack{p\in\mathcal{B}\\ p\nmid m}}\left(1-\frac{1}{p}\right)\prod_{p\mid m}\left(1-\frac{1}{p}\right)\sum_{d\le Q}\frac{\chi_0(d)\mu(d)\mu_{\mathcal{B}}(d)}{d^2}\\ &+\mathrm{O}_{m,\mathcal{B}}\left(Q^{1+\theta}(\log Q)^{3/2}\sum_{d\le Q}\frac{1}{d^{1+\theta}}\right)\\ =&\frac{Q^2}{2\zeta(2)}\prod_{\substack{p\in\mathcal{B}\\ p\nmid m}}\left(1+\frac{1}{p}\right)^{-1}\prod_{p\mid m}\left(1+\frac{1}{p}\right)^{-1}+\mathrm{O}_{m,\mathcal{B}}\left(Q^{1+\theta}(\log Q)^{3/2}\right). \end{aligned} \tag{5.29}$$

For all other characters, an estimate is as below

$$\begin{aligned} S_2 &= \sum_{d\le Q}\chi(d)\mu(d)\mu_{\mathcal{B}}(d)\sum_{n\le\frac{Q}{d}}\chi(n)\mu_{\mathcal{B}}(n)n\\ &=\sum_{d\le Q}\chi(d)\mu(d)\mu_{\mathcal{B}}(d)\sum_{j\le\frac{Q}{d}}\sum_{j\le n\le\frac{Q}{d}}\chi(n)\mu_{\mathcal{B}}(n)\\ &\ll_{m,\mathcal{B}} Q^{1+\theta}\log Q\sum_{d\le Q}\frac{1}{d^{1+\theta}}\ll_{m,\mathcal{B}} Q^{1+\theta}\log Q. \end{aligned} \tag{5.30}$$

Inserting the estimates from (5.26), (5.29), (5.22), and (5.30) into (5.13) proves Theorem 1.2.1. □

5.3 Second Moment

To estimate the second moment, we need an asymptotic formula for the deficiency sum in (5.10).

5.3.1 Deficiency

In this section, we discuss the average order of deficiency when the denominator of Farey fractions is $\mathcal{B}$-free.

Theorem 5.3.1. *For a fixed positive integer m, let χ be a Dirichlet character modulo*

m and a set $\mathcal{B}$ of prime numbers such that $\sum_{p\in\mathcal{B}} \dfrac{1}{p^{\sigma}} < \infty$ for some $\sigma < \theta$, where $1/2 < \theta < 1$. Then, for all large positive integers Q, we have

$$\sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\delta(q) = \begin{cases} \mathrm{O}_{m,\mathcal{B}}\left(Q^{1+\theta}(\log Q)^2\right), & \chi\neq\chi_0 \\ \\ 2Q^2\prod_{\substack{p\in\mathcal{B}\\ p\nmid m}}\left(1+\frac{1}{p}\right)^{-1}\prod_{p\mid m}\left(1+\frac{1}{p}\right)^{-1} - \frac{3Q^2}{\zeta(2)}\prod_{\substack{p\in\mathcal{B}\\ p\nmid m}} & \\ \times\left(1+\frac{1}{p}\right)^{-1}\prod_{p\mid m}\left(1+\frac{1}{p}\right)^{-1} + \mathrm{O}_{m,\mathcal{B}}\left(Q^{1+\theta}(\log Q)^2\right), & \chi=\chi_0. \end{cases}$$

Proof. Using (5.7), we have

$$\sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\delta(q) = \sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q)\left\lfloor\frac{2Q}{q}\right\rfloor - \sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}} \chi(q)T(q). \tag{5.31}$$

We begin with estimation of the first sum on the right side of the above equation.

Non-principal Dirichlet character: For $\chi\neq\chi_0$, we write

$$\begin{aligned} \sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q)\left\lfloor\frac{2Q}{q}\right\rfloor &= \sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\left\lfloor\frac{2Q}{q}\right\rfloor\sum_{d\mid q}\mu(d)\frac{q}{d} \\ &= \sum_{d\le Q}\frac{\mu(d)}{d}\sum_{\substack{q\le Q\\ d\mid q}}\chi(q)\mu_{\mathcal{B}}(q)q\left\lfloor\frac{2Q}{q}\right\rfloor \\ &= \sum_{d\le Q}\chi(d)\mu(d)\mu_{\mathcal{B}}(d)\sum_{n\le\frac{Q}{d}}\chi(n)\mu_{\mathcal{B}}(n)n\left\lfloor\frac{2Q}{nd}\right\rfloor \\ &= \sum_{d\le Q}\chi(d)\mu(d)\mu_{\mathcal{B}}(d)U_{\chi}\left(\frac{2Q}{d}\right), \end{aligned} \tag{5.32}$$

where

$$U_{\chi}(y) = \sum_{r\le\frac{y}{2}}\chi(r)\mu_{\mathcal{B}}(r)r\left\lfloor\frac{y}{r}\right\rfloor.$$

Define $A(n) = \sum_{r \leq n} \chi(r)\mu_{\mathcal{B}}(r)$ and

$$b(n) = \begin{cases} n \left\lfloor \frac{y}{n} \right\rfloor, & \text{if } n \leq \left\lfloor \frac{y}{2} \right\rfloor, \\ \\ 0, & \text{if } n > \left\lfloor \frac{y}{2} \right\rfloor. \end{cases}$$

We have

$$\begin{aligned} U_\chi(y) = \sum_{n \leq \lfloor \frac{y}{2} \rfloor} \chi(n)\mu_{\mathcal{B}}(n)b(n) &= \sum_{n \leq \lfloor \frac{y}{2} \rfloor} (A(n) - A(n-1))b(n) = \sum_{n \leq \lfloor \frac{y}{2} \rfloor} A(n)b(n) \\ &- \sum_{n \leq \lfloor \frac{y}{2} \rfloor - 1} A(n)b(n+1) = \sum_{n \leq \lfloor \frac{y}{2} \rfloor} A(n)(b(n) - b(n+1)). \end{aligned}$$

The sum $A(n)$ is similar to S_{11} where the former sum is upto n and the latter upto Q/d. Hence, from (5.22), $A(n) \ll_{k,\mathcal{B}} n^\theta \log n$ and

$$\begin{aligned} |b(n) - b(n+1)| &\leq \left\lfloor \frac{y}{n+1} \right\rfloor + n \left(\left\lfloor \frac{y}{n} \right\rfloor - \left\lfloor \frac{y}{n+1} \right\rfloor \right) \\ &\leq \frac{y}{n+1} + n \left(\left\lfloor \frac{y}{n} \right\rfloor - \left\lfloor \frac{y}{n+1} \right\rfloor \right). \end{aligned}$$

This yields

$$U_\chi\left(\frac{2Q}{d}\right) \ll_{k,\mathcal{B}} \frac{Q^{1+\theta} \log Q}{d^{1+\theta}} \sum_{n \leq \lfloor \frac{Q}{d} \rfloor} \frac{1}{n} \ll_{k,\mathcal{B}} \frac{Q^{1+\theta}(\log Q)^2}{d^{1+\theta}}.$$

Combining this with (5.32), we have

$$\sum_{\substack{q \leq Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor \ll_{m,\mathcal{B}} Q^{1+\theta}(\log Q)^2 \sum_{d \leq Q} \frac{1}{d^{1+\theta}} \ll_{m,\mathcal{B}} Q^{1+\theta}(\log Q)^2. \tag{5.33}$$

Inserting the above bound into (5.31) and applying Theorem 1.2.1 to the second term in (5.31), gives the required result for the case $\chi \neq \chi_0$.

Principal Dirichlet character: For $\chi = \chi_0$, we extend the range from $1 \leq q \leq Q$

to $1 \leq q \leq 2Q$, and write

$$\sum_{\substack{q \leq Q \\ \mu_{\mathcal{B}}(q)=1}} \chi_0(q)\phi(q) \left[\frac{2Q}{q}\right] = \sum_{\substack{q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi_0(q)\phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor - \sum_{\substack{Q < q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi_0(q)\phi(q). \tag{5.34}$$

From (5.29), we have

$$\sum_{\substack{Q<q\leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi_0(q)\phi(q) = \frac{3Q^2}{2\zeta(2)} \prod_{\substack{p \in \mathcal{B} \\ p \nmid m}} \left(1 + \frac{1}{p}\right)^{-1} \prod_{p \mid m} \left(1 + \frac{1}{p}\right)^{-1} + \mathrm{O}_{m,\mathcal{B}}\left(Q^{1+\theta} (\log Q)^{\frac{3}{2}}\right). \tag{5.35}$$

The first sum on the right side is solved as

$$\begin{aligned} \sum_{\substack{q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi_0(q)\phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor &= \sum_{\substack{q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi_0(q)\phi(q) \sum_{\substack{n \leq 2Q \\ q \mid n}} 1 \\ &= \sum_{n \leq 2Q} \sum_{q \mid n} \chi_0(q)\phi(q)\mu_{\mathcal{B}}(q) = \sum_{n \leq 2Q} f_{\chi_0}(n), \end{aligned}$$

where

$$f_{\chi_0}(n) := \sum_{r \mid n} \chi_0(r)\phi(r)\mu_{\mathcal{B}}(r).$$

Since $f_{\chi_0} = \chi_0\phi\mu_{\mathcal{B}} * 1$, the Dirichlet series for f_{χ_0} is given by

$$\begin{aligned} F(s) = \sum_{n=1}^{\infty} \frac{f_{\chi_0}(n)}{n^s} &= \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) \left(\sum_{n=1}^{\infty} \frac{\chi_0(n)\phi(n)\mu_{\mathcal{B}}(n)}{n^s}\right) \\ &= \zeta(s-1) \prod_{\substack{p \in \mathcal{B} \\ p \nmid m}} \left(1 - \frac{1}{p^{s-1}}\right)\left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p \mid m} \left(1 - \frac{1}{p^{s-1}}\right)\left(1 - \frac{1}{p^s}\right)^{-1}. \end{aligned}$$

which is absolutely convergent for $\Re(s) > 2$. Moreover, the product term over primes in $\mathcal{B}$ is bounded for $\Re(s) > 1 + \theta$. Now, use Lemma 2.1.1 with $x = 2Q + \frac{1}{3}$, and sum over $n \leq 2Q$, we get

$$\sum_{n \leq 2Q} f_{\chi_0}(n) = \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{\left(2Q + \frac{1}{3}\right)^s F(s)}{s} ds + R(T), \tag{5.36}$$

where

$$|R(T)| \ll \frac{Q^\alpha}{T}\sum_{n=1}^{\infty}\frac{n}{n^\alpha\left|\log\frac{2Q+\frac{1}{3}}{n}\right|}.$$

Let $\alpha = 2+\frac{1}{\log Q}$ and using the same argument as in (5.17), we have

$$|R(T)| \ll \frac{Q^2\log Q}{T}.$$

The integrand in (5.36) has a simple pole at $s = 2$. We deform the line integral into a rectangular contour with vertices $\alpha \pm iT$ and $1 \pm iT$. By Cauchy's Residue theorem, we have

$$\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\frac{\left(2Q+\frac{1}{3}\right)^s F(s)}{s}ds = \frac{\left(2Q+\frac{1}{3}\right)^2}{2}\prod_{\substack{p\in\mathcal{B}\\ p\nmid m}}\left(1+\frac{1}{p}\right)^{-1}\prod_{p\mid m}\left(1+\frac{1}{p}\right)^{-1}+\sum_{j=1}^{3}I_j,$$

where the first term of right side is due to the residue of simple pole at the point $s = 2$ and I_1, I_2, and I_3 are the integrals along the lines $[\alpha-iT, 1-iT]$, $[1-iT, 1+iT]$, and $[1+iT, \alpha+iT]$, respectively.

Estimation of I_1 and I_3: Using (2.2), we have

$$\begin{aligned}|I_1|, |I_3| &\ll_{m,\mathcal{B}} \int_1^\alpha \frac{\left|\left(2Q+\frac{1}{3}\right)^{\sigma+iT}\right||\zeta(\sigma-1+iT)|}{|\sigma+iT|}d\sigma \ll_{m,\mathcal{B}} \log T\int_1^\alpha (QT^{-1/2})^\sigma d\sigma\\ &\ll_{m,\mathcal{B}} \frac{(Q^2+QT^{1/2})\log T\log(QT^{-1/2})}{T}.\end{aligned}$$

Estimation of I_2: Since $\zeta(s-1) \ll T^{(1-\sigma)/2}\log T$ if $1 \le \sigma \le 2$, we have

$$|I_2| \ll_{m,\mathcal{B}} \int_{-T}^{T}\frac{|(2Q+\frac{1}{3})^{1+it}||\zeta(it)|}{|1+it|}dt \ll_{m,\mathcal{B}} QT^{1/2}(\log T)^2.$$

Collecting all estimates and choosing $T = Q^{2/3}$, we get

$$\sum_{n\le 2Q} f_{\chi_0}(n) = 2Q^2\prod_{\substack{p\in\mathcal{B}\\ p\nmid m}}\left(1+\frac{1}{p}\right)^{-1}\prod_{p\mid m}\left(1+\frac{1}{p}\right)^{-1}+\mathrm{O}_{m,\mathcal{B}}\left(Q^{4/3}(\log Q)^2\right). \quad (5.37)$$

We get the required result for χ_0 by inserting (5.37) and (5.35) in (5.34). □

5.3.2 Proof of Theorem 1.2.2

To find the second moment, we first expand the terms $X_\chi(Q)$ and $Y_\chi(Q)$ in (5.10). Since $\lfloor 2Q/s \rfloor = 1$ when $Q < s \leq 2Q$, we extend the range from $1 \leq s \leq Q$ to $1 \leq s \leq 2Q$ and we have

$$\begin{aligned}
X_\chi(Q) = \sum_{\substack{q \leq Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor^2 &= \sum_{\substack{q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor^2 - \sum_{\substack{Q<q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q) \\
&= \sum_{\substack{q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor \left(\left\lfloor \frac{2Q}{q} \right\rfloor + 1 \right) - \sum_{\substack{q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor \\
&\quad - \sum_{\substack{Q<q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q) \\
&= 2 \sum_{\substack{q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \frac{\chi(q)\phi(q)}{q} \sum_{\substack{n \leq 2Q \\ q|n}} n - \sum_{\substack{q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor - \sum_{\substack{Q<q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q) \\
&= 2 \sum_{n \leq 2Q} n h_\chi(n) - \sum_{\substack{q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor - \sum_{\substack{Q<q \leq 2Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q), \qquad (5.38)
\end{aligned}$$

where

$$h_\chi(n) = \sum_{q|n} \frac{\chi(q)\phi(q)\mu_{\mathcal{B}}(q)}{q}. \qquad (5.39)$$

From (5.5) and (5.7), we have

$$\begin{aligned}
\delta(q) &= \phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor - T(q) \\
&= \phi(q) \left\lfloor \frac{2Q}{q} \right\rfloor - 2 \sum_{d|q} \mu(d) \left\lfloor \frac{Q}{d} \right\rfloor + \phi(q) - \epsilon(q) \\
&= \phi(q) \left(\left\lfloor \frac{2Q}{q} \right\rfloor + 1 \right) - 2 \sum_{d|q} \mu(d) \left\lfloor \frac{Q}{d} \right\rfloor - \epsilon(q). \qquad (5.40)
\end{aligned}$$

Next, we consider the expression $Y_\chi(Q)$. The equation (5.40) and the well known estimate $\sum_{q\le Q}\tau(q)/q \ll (\log Q)^2$ leads to

$$\begin{aligned}
Y_\chi(Q) = & \sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q)\left\lfloor\frac{2Q}{q}\right\rfloor\left(\left\lfloor\frac{2Q}{q}\right\rfloor+1\right) - 2Q\sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}}\frac{\chi(q)\phi(q)}{q}\left\lfloor\frac{2Q}{q}\right\rfloor \\
& + \mathrm{O}\left(Q(\log Q)^2\right) \\
= & \sum_{\substack{q\le 2Q\\ \mu_{\mathcal{B}}(q)=1}} \chi(q)\phi(q)\left\lfloor\frac{2Q}{q}\right\rfloor\left(\left\lfloor\frac{2Q}{q}\right\rfloor+1\right) - 2\sum_{\substack{Q<q\le 2Q\\ \mu_{\mathcal{B}}(q)=1}}\chi(q)\phi(q) \\
& - 2Q\sum_{\substack{q\le 2Q\\ \mu_{\mathcal{B}}(q)=1}}\frac{\chi(q)\phi(q)}{q}\left\lfloor\frac{2Q}{q}\right\rfloor + 2Q\sum_{\substack{Q<q\le 2Q\\ \mu_{\mathcal{B}}(q)=1}}\frac{\chi(q)\phi(q)}{q} + \mathrm{O}\left(Q(\log Q)^2\right) \\
=& 2\sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}}\frac{\chi(q)\phi(q)}{q}\sum_{\substack{n\le 2Q\\ q|n}} n - 2\sum_{\substack{Q<q\le 2Q\\ \mu_{\mathcal{B}}(q)=1}}\chi(q)\phi(q) - 2Q\sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}}\frac{\chi(q)\phi(q)}{q}\sum_{\substack{n\le 2Q\\ q|n}} 1 \\
& + 2Q\sum_{\substack{Q<q\le 2Q\\ \mu_{\mathcal{B}}(q)=1}}\frac{\chi(q)\phi(q)}{q} + \mathrm{O}\left(Q(\log Q)^2\right) \\
=& 2\sum_{n\le 2Q}(n-Q)h_\chi(n) - 2\sum_{\substack{Q<q\le 2Q\\ \mu_{\mathcal{B}}(q)=1}}\chi(q)\phi(q) + 2Q\sum_{\substack{Q<q\le 2Q\\ \mu_{\mathcal{B}}(q)=1}}\frac{\chi(q)\phi(q)}{q} \\
& + \mathrm{O}\left(Q(\log Q)^2\right). \qquad (5.41)
\end{aligned}$$

From (5.10), (5.38), and (5.41), we have

$$\begin{aligned}
\mathcal{M}_{2,\mathcal{B}}(\chi,Q) =& 2\sum_{n\le 2Q}(2Q-n)h_\chi(n) - \sum_{\substack{q\le 2Q\\ \mu_{\mathcal{B}}(q)=1}}\chi(q)\phi(q)\left\lfloor\frac{2Q}{q}\right\rfloor + 3\sum_{\substack{Q<q\le 2Q\\ \mu_{\mathcal{B}}(q)=1}}\chi(q)\phi(q) \\
& - 4Q\sum_{\substack{Q<q\le 2Q\\ \mu_{\mathcal{B}}(q)=1}}\frac{\chi(q)\phi(q)}{q} + \sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}}\chi(q)\delta(q) + \mathrm{O}\left(Q(\log Q)^2\right) \\
=& 2M_1 - M_2 + 3M_3 - 4QM_4 + M_5 + \mathrm{O}\left(Q(\log Q)^2\right). \qquad (5.42)
\end{aligned}$$

One can obtain M_5 from Theorem 5.3.1. Estimation of M_2 is in (5.33) and (5.37) for non-principal and principal Dirichlet characters, respectively.

Case (i): For $\chi \neq \chi_0$, from (5.30), we have

$$M_3 \ll Q^{1+\theta} \log Q, \tag{5.43}$$

and

$$\begin{aligned} M_4 &= \sum_{\substack{Q<s\leq 2Q \\ \mu_{\mathcal{B}}(s)=1}} \frac{\chi(s)}{s} \sum_{d|s} \frac{\mu(d)s}{d} = \sum_{d\leq 2Q} \frac{\chi(d)\mu(d)\mu_{\mathcal{B}}(d)}{d} \sum_{\frac{Q}{d}<n\leq \frac{2Q}{d}} \chi(n)\mu_{\mathcal{B}}(n) \\ &\ll_{m,\mathcal{B}} Q^{\theta} \log Q \sum_{d\leq 2Q} \frac{1}{d^{1+\theta}} \ll_{m,\mathcal{B}} Q^{\theta} \log Q. \end{aligned} \tag{5.44}$$

Case (ii): For $\chi = \chi_0$, from (5.29)

$$M_3 = \frac{3Q^2}{2\zeta(2)} \prod_{\substack{p\in\mathcal{B} \\ p\nmid m}} \left(1+\frac{1}{p}\right)^{-1} \prod_{p|m} \left(1+\frac{1}{p}\right)^{-1} + \mathrm{O}_{m,\mathcal{B}}\left(Q^{1+\theta}(\log Q)^{3/2}\right), \tag{5.45}$$

and using (5.21), (5.24), and (5.25), we obtain

$$M_4 = \frac{Q}{\zeta(2)} \prod_{\substack{p\in\mathcal{B} \\ p\nmid m}} \left(1+\frac{1}{p}\right)^{-1} \prod_{p|m} \left(1+\frac{1}{p}\right)^{-1} + \mathrm{O}_{m,\mathcal{B}}\left(Q^{\theta}(\log Q)^{\frac{3}{2}}\right). \tag{5.46}$$

We are left to estimate M_1. For this we observe that

$$M_1 = \sum_{n=1}^{\infty} \max\left(\frac{2Q}{n}-1, 0\right) n h_{\chi}(n),$$

where $h_{\chi}(n)$ is in (5.42). Using the well known formula

$$\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^{s+1}}{s(s+1)} ds = \max(x-1, 0),\ x>0, \tag{5.47}$$

we write

$$M_1 = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{(2Q)^{s+1}}{s(s+1)} \left(\frac{h_{\chi}(n)}{n^s}\right) ds.$$

The Dirichlet series of $h_{\chi}(n)$ is absolutely convergent on the line $\Re(s) = 2$ and is given by

$$H(s) = \sum_{n=1}^{\infty} \frac{h_{\chi}(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{\chi(n)\phi(n)\mu_{\mathcal{B}}(n)}{n^{s+1}} = \frac{\zeta(s)L(s,\chi)}{L(s+1,\chi)}$$

$$\times \prod_{p\in\mathcal{B}} \left(\sum_{n=0}^{\infty} \frac{\chi(p^n)\phi(p^n)}{p^{ns+n}} \right)^{-1} = \frac{\zeta(s)L(s,\chi)}{L(s+1,\chi)} \prod_{p\in\mathcal{B}} \left(1 - \frac{\chi(p)}{p^s}\right) \left(1 - \frac{\chi(p)}{p^{s+1}}\right)^{-1}.$$

This yields

$$M_1 = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{(2Q)^{s+1}\zeta(s)L(s,\chi)}{s(s+1)L(s+1,\chi)} \prod_{p\in\mathcal{B}} \left(1 - \frac{\chi(p)}{p^s}\right) \left(1 - \frac{\chi(p)}{p^{s+1}}\right)^{-1} ds. \quad (5.48)$$

Non-principal Dirichlet character: For $\chi \neq \chi_0$, the above integrand has simple poles at points $s = 0$ and $s = 1$, so we shift the path of integration from $2 - iT$ to $2 + iT$ into a contour that contains the horizontal line segments from $2 - iT$ to $\theta - iT$, and from $\theta + iT$ to $2 + iT$ and the vertical line segments from $\theta - iT$ to $\theta + iT$, and from $2 - iT$ to $2 + iT$. By Cauchy's residue theorem, we have

$$\frac{1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} \frac{(2Q)^{s+1}\zeta(s)L(s,\chi)}{s(s+1)L(s+1,\chi)} \prod_{p\in\mathcal{B}} \left(1 - \frac{\chi(p)}{p^s}\right) \left(1 - \frac{\chi(p)}{p^{s+1}}\right)^{-1} ds$$

$$= \frac{2Q^2 L(1,\chi)}{L(2,\chi)} \prod_{p\in\mathcal{B}} \left(1 - \frac{\chi(p)}{p}\right) \left(1 - \frac{\chi(p)}{p^2}\right)^{-1} + \sum_{j=1}^{5} I_j, \quad (5.49)$$

where the first term is the residue of the integrand at s=1, I_1 and I_5 are the integrals along the vertical segments $(2 - i\infty, 2 - iT]$ and $[2 + iT, 2 + i\infty)$, respectively, I_2 and I_4 are the integrals along the horizontal axis $[2 - it, \theta - iT]$ and $[\theta + iT, 2 + iT]$, respectively, and I_3 is the integral over the vertical segment $[\theta - iT, \theta + iT]$.

Estimation of I_1 and I_5: Since $|\zeta(s)| \ll 1$ and $|L(s,\chi) \ll_m 1$ uniformly on $\Re(s) = 2$, we obtain

$$|I_1|, |I_5| \ll_{m,\mathcal{B}} \int_T^{\infty} \frac{Q^3}{t^2} dt \ll_{m,\mathcal{B}} \frac{Q^3}{T}.$$

Estimation of I_2 and I_4: Since $|L(\sigma + iT, \chi)| \gg_m \frac{1}{\log T}$ (see (2.6)) and using (2.2) and (2.5), we have

$$\begin{aligned}
|I_2|, |I_4| &\ll_{m,\mathcal{B}} \log T \int_{\theta}^{2} \frac{Q^{\sigma+1}|\zeta(\sigma+iT)||L(\sigma+iT,\chi)|}{|\sigma+iT||\sigma+1+iT|} d\sigma \\
&\ll_{m,\mathcal{B}} \frac{(\log T)^5}{T^{\frac{127+89\theta}{108}}} \int_{\theta}^{1} Q^{\sigma+1} d\sigma + \frac{(\log T)^3}{T^2} \int_{1}^{2} Q^{\sigma+1} d\sigma \\
&\ll_{m,\mathcal{B}} \frac{Q^2 \log Q (\log T)^5}{T^{\frac{127+89\theta}{108}}} + \frac{Q^3 \log Q (\log T)^3}{T^2}.
\end{aligned}$$

Estimation of I_3: Since $|L(\theta+iT,\chi)| \gg_m \frac{1}{\log T}$ (see (2.6)) and using (2.2), (2.5), we have

$$\begin{aligned}
|I_3| \ll_{m,\mathcal{B}} & \, Q^{1+\theta} \log T \int_{0}^{T} \frac{|\zeta(\theta+it)||L(\theta+it,\chi)|}{|\theta+it||1+\theta+it|} dt \\
\ll_{m,\mathcal{B}} & \, Q^{1+\theta} T^{\frac{89(1-\theta)}{108}} (\log T)^5 \int_{0}^{T} \frac{1}{|\theta+it||1+\theta+it|} dt \ll_{m,\mathcal{B}} \frac{Q^{1+\theta}(\log T)^5}{T^{\frac{19+89\theta}{108}}}.
\end{aligned}$$

Collecting all estimates in (5.49) and putting $T = Q^2$, we obtain

$$M_1 = \frac{2Q^2 L(1,\chi)}{L(2,\chi)} \prod_{p\in\mathcal{B}} \left(1 - \frac{\chi(p)}{p}\right)\left(1 - \frac{\chi(p)}{p^2}\right)^{-1} + \mathrm{O}_{m,\mathcal{B}}(Q). \tag{5.50}$$

Principal Dirichlet character: For $\chi = \chi_0$, (5.48) yields

$$\begin{aligned}
M_1 =& \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{(2Q)^{s+1}\zeta(s)^2}{s(s+1)\zeta(s+1)} \prod_{\substack{p\in\mathcal{B} \\ p\nmid m}} \left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^{s+1}}\right)^{-1} \\
& \times \prod_{p|m} \left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^{s+1}}\right)^{-1} ds.
\end{aligned}$$

Take $\alpha = 2$. We deform the path of integration as defined for $\chi \neq \chi_0$. Denote

$$Z(s) := \frac{(2Q)^{s+1}\zeta^2(s)}{s(s+1)\zeta(s+1)} \prod_{\substack{p\in\mathcal{B} \\ p\nmid m}} \left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^{s+1}}\right)^{-1} \prod_{p|m} \left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^{s+1}}\right)^{-1},$$

By Cauchy's residue theorem, we have

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Z(s)ds = \mathrm{Res}_{s=1}Z(s) + \sum_{j=1}^{5} I_j, \tag{5.51}$$

where I_j's are same as in non-principal Dirichlet character case above. The residue of the second order pole at $s=1$ of $Z(s)$ is given by

$$\begin{aligned}\mathrm{Res}_{s=1}Z(s) =& \frac{2Q^2}{\zeta(2)}\left(\log 2Q - \frac{\zeta'(2)}{\zeta(2)} - \frac{3}{2} + 2\gamma + \sum_{\substack{p\in\mathcal{B}\\ p\nmid m}}\frac{p\log p}{p^2-1} + \sum_{p\mid m}\frac{p\log p}{p^2-1}\right)\\ &\times\prod_{\substack{p\in\mathcal{B}\\ p\nmid m}}\frac{p}{p+1}\prod_{p\mid m}\frac{p}{p+1}.\end{aligned}$$

Estimation of I_1 and I_5: Since $|\zeta(s)| \ll 1$ uniformly on $\Re(s)=2$, we obtain

$$|I_1|, |I_5| \ll_{m,\mathcal{B}} \int_T^{\infty}\frac{Q^3}{t^2}dt \ll_{m,\mathcal{B}} \frac{Q^3}{T}.$$

Estimation of I_2 and I_4: Since $|\zeta(\sigma+iT)| \gg_m \frac{1}{\log T}$ (see (2.3)) and using (2.2), we have

$$\begin{aligned}|I_2|, |I_4| &\ll_{m,\mathcal{B}} \log T\int_{\theta}^{2}\frac{Q^{\sigma+1}|\zeta(\sigma+iT)|^2}{|\sigma+iT||\sigma+1+iT|}d\sigma\\ &\ll_{m,\mathcal{B}} \frac{(\log T)^3}{T^{1+\theta}}\int_{\theta}^{1}Q^{\sigma+1}d\sigma + \frac{(\log T)^3}{T^2}\int_1^2 Q^{\sigma+1}d\sigma\\ &\ll_{m,\mathcal{B}} \frac{Q^2\log Q(\log T)^3}{T^{1+\theta}} + \frac{Q^3\log Q(\log T)^3}{T^2}.\end{aligned}$$

Estimation of I_3: Using (2.2), we have

$$\begin{aligned}|I_3| &\ll_{m,\mathcal{B}} Q^{1+\theta}\log T\int_0^T\frac{|\zeta^2(\theta+it)|}{|\theta+it||1+\theta+it|}dt\\ &\ll_{m,\mathcal{B}} Q^{1+\theta}(\log T)^3\int_0^T\frac{1}{(\theta+t)^{1+\theta}}dt \ll_{m,\mathcal{B}} \frac{Q^{1+\theta}(\log T)^3}{T^{\theta}}.\end{aligned}$$

Collecting all estimates in (5.51) and putting $T = Q^2$, we obtain

$$\begin{aligned}M_1 =& \frac{2Q^2}{\zeta(2)}\left(\log 2Q - \frac{\zeta'(2)}{\zeta(2)} - \frac{3}{2} + 2\gamma + \sum_{\substack{p\in\mathcal{B}\\ p\nmid m}} \frac{p\log p}{p^2-1} + \sum_{p\mid m}\frac{p\log p}{p^2-1}\right)\\ &\times \prod_{\substack{p\in\mathcal{B}\\ p\nmid m}}\left(1+\frac{1}{p}\right)^{-1}\prod_{p\mid m}\left(1+\frac{1}{p}\right)^{-1} + \mathrm{O}_{m,\mathcal{B}}\left(Q\right). \qquad (5.52)\end{aligned}$$

Collecting all estimates from Theorem 5.3.1, (5.33), (5.43), (5.44), and (5.50), we obtain the statement of Theorem 1.2.2 for non-principal character. For $\chi = \chi_0$, the result follows from collecting estimates from Theorem 5.3.1, (5.37), (5.45), (5.46), and (5.52).

5.4 Higher Moments

Proof of Theorem 1.2.3. From (5.12), we have

$$\begin{aligned}\mathcal{M}_{l,\mathcal{B}}(\chi, Q) &= \sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}}\chi(q)\sum_{\substack{Q-q<q'<Q\\ (q',q)=1}}\left(\left(\frac{2Q}{q}\right)^l - \left(\frac{2Q}{q}\right)^l + \left(\frac{Q+q'}{q}\right)^l\right) + \mathrm{O}\left(Q^{l-1}\right)\\ &= \sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}}\chi(q)\sum_{\substack{Q-q<q'<Q\\ (q',q)=1}}\left(\frac{2Q}{q}\right)^l + \sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}}\chi(q)\sum_{\substack{Q-q<q'<Q\\ (q',q)=1}}\\ &\quad\times\left(\left(\frac{Q+q'}{q}\right)^l - \left(\frac{2Q}{q}\right)^l\right) + \mathrm{O}\left(Q^{l-1}\right) = S_1 + S_2 + \mathrm{O}\left(Q^{l-1}\right).\end{aligned} \qquad (5.53)$$

We begin with the first term on the right side of (5.53) as

$$S_1 = 2^lQ^l\sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}}\frac{\chi(q)}{q^l}\sum_{Q-q<q'<Q}\sum_{d\mid(q',q)}\mu(d) = 2^lQ^l\sum_{\substack{q\le Q\\ \mu_{\mathcal{B}}(q)=1}}\frac{\chi(q)}{q^l}\sum_{d\mid q}\frac{\mu(d)q}{d}$$

$$\begin{aligned}
&= 2^l Q^l \left(\sum_{q=1}^{\infty} \frac{\chi(q)\mu_{\mathcal{B}}(q)}{q^l} \sum_{d|q} \frac{\mu(d)q}{d} + \mathrm{O}_{m,\mathcal{B}}\left(\frac{1}{Q}\right)\right) \\
&= 2^l Q^l \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \sum_{\substack{q=1 \\ d|q}}^{\infty} \frac{\chi(q)\mu_{\mathcal{B}}(q)}{q^{l-1}} + \mathrm{O}_{m,\mathcal{B}}\left(Q^{l-1}\right) \\
&= 2^l Q^l \sum_{d=1}^{\infty} \frac{\chi(d)\mu(d)\mu_{\mathcal{B}}(d)}{d^l} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^{l-1}} + \mathrm{O}_{m,\mathcal{B}}\left(Q^{l-1}\right) \\
&= \frac{2^l Q^l L(l-1,\chi)}{L(l,\chi)} \prod_{p\in\mathcal{B}} \left(\sum_{n=0}^{\infty} \frac{\chi(p^n)}{(p^{l-1})^n}\right)^{-1} \left(\sum_{n=0}^{\infty} \frac{\chi(p^n)\mu(n)}{(p^l)^n}\right)^{-1} + \mathrm{O}_{m,\mathcal{B}}\left(Q^{l-1}\right).
\end{aligned} \tag{5.54}$$

For the second sum, observe that

$$\begin{aligned}
|S_2| &= \left| \sum_{\substack{q\le Q \\ \mu_{\mathcal{B}}(q)=1}} \chi(q) \sum_{\substack{Q-q<q'<Q \\ (q',q)=1}} \left(\left(\frac{Q+q'}{q}\right)^l - \left(\frac{2Q}{q}\right)^l\right)\right| \\
&\le \sum_{\substack{q\le Q \\ \mu_{\mathcal{B}}(q)=1}} \left| \sum_{\substack{Q-q<q'<Q \\ (q',q)=1}} \left(\left(\frac{Q+q'}{q}\right)^l - \left(\frac{2Q}{q}\right)^l\right)\right| \\
&\le \sum_{q\le Q} \sum_{Q-q<q'<Q} \frac{(2Q)^l - (Q+q')^l}{q^l} \ll \sum_{q\le Q} \frac{Q^{l-1}}{q^{l-2}} \\
&\ll \begin{cases} Q^2 \log Q, & \text{if } l=3, \\ Q^{l-1}, & \text{if } l\ge 4. \end{cases}
\end{aligned} \tag{5.55}$$

Inserting (5.54) and (5.55) in (5.53), we obtain the required result. □

6

Conclusion and Future directions

In this section, we conclude with a discussion of ongoing work and future research directions. The aim of this thesis is to study the distribution of Farey sequences with various restrictions on denominators and numerators. In future work, we are also interested in studying the distribution of Farey sequences whose denominators and numerators lie in a sparse set. For instance, the sequence of Farey fractions with prime denominators, as the set of primes is sparse, has Poissonian distribution.

6.1 Ongoing work

Let $g \geq 2$ and m be fixed positive integers with $(m, g-1) = 1$. If $n \in \mathbb{N}$ then representing n in the number system to base g:

$$n = \sum_{j=0}^{v} a_j(n) g^j, \ 0 \leq a_j \leq g-1, \ a_v \geq 1.$$

We denote

$$A(s) = \left\{ n = \sum_{j=0}^{v} a_j(n) g^j, \ 0 \leq a_j \leq g-1, \ a_v \geq 1 \mid \sigma(n) = s \right\}$$

and

$$A(r,m) = \left\{ n = \sum_{j=0}^{v} a_j(n) g^j, \ 0 \leq a_j \leq g-1, \ a_v \geq 1 \mid \sigma(n) \equiv r \pmod{m} \right\},$$

where $r \in \mathbb{Z}, \ 0 \leq s \leq (g-1)(v+1)$, and

$$\sigma(n) = \sum_{j=0}^{v} a_j(n).$$

Numerous authors [38, 42, 76, 77, 91] have investigated various arithmetical questions - such as properties of divisibility, distribution in arithmetic progressions, Weyl sums, character sums, etc concerning the sets $A(s)$ and $A(r,m)$. We define the sets of Farey fractions whose denominators are positive integers with a fixed digit sum and which lie in an arithmetic progression, denoted by $\mathcal{F}_{Q,A(s)}$ and $\mathcal{F}_{Q,A(r,m)}$, respectively, as follows:

$$\mathcal{F}_{Q,A(s)} = \left\{ \frac{a}{q} : 1 \leq a \leq q \leq Q, \ (a,q) = 1, \ q \in A(s) \right\}$$

and

$$\mathcal{F}_{Q,A(r,m)} = \left\{ \frac{a}{q} : 1 \leq a \leq q \leq Q, \ (a,q) = 1, \ q \in A(r,m) \right\}.$$

In this project with Igor Shparlinski, we are working on the distribution of Farey fractions with digital restrictions, namely $\mathcal{F}_{Q,A(s)}$ and $\mathcal{F}_{Q,A(r,m)}$. In particular, we want to study the quantitative equidistribution and pair correlation measure of the sequences $\left(\mathcal{F}_{Q,A(s)}\right)_Q$ and $\left(\mathcal{F}_{Q,A(r,m)}\right)_Q$. Furthermore, we want to explore whether these sequences behave like a uniformly random sequence or exhibit non-Poissonian behavior. To achieve this, we need to study the Weyl sums over $\mathcal{F}_{Q,A(s)}$ and $\mathcal{F}_{Q,A(r,m)}$ and establish an asymptotic formula for weighted visible lattice points with digital restrictions, which are of independent interest.

6.2 Future directions

In Chapter 3, we studied the distribution of the polynomial Farey sequence $(\mathcal{F}_{Q,P})_Q$, which is defined by

$$\mathcal{F}_{Q,P} = \left\{ \frac{a}{q} : 1 \leq a \leq q \leq Q,\ (P(a), q) = 1 \right\},$$

where $P(x) \in \mathbb{Z}[x]$. We proved that the lim sup of the pair correlation measure of this sequence is bounded, and for the specific polynomial $P(x) = x(x+1)$, we showed that the pair correlation measure exists and established an explicit formula for the pair correlation function, which is non-Poissonian.

Furthermore, we are interested in investigating the pair correlation measure of the polynomial Farey sequence $(\mathcal{F}_{Q,P})_Q$ for polynomials $P(x) \in \mathbb{Z}[x]$ of degree ≥ 3. It is then natural to have an explicit formula for the corresponding pair correlation function. For such polynomials, the problem presents several analytic difficulties, since the exponential sum over $\mathcal{F}_{Q,P}$ becomes more delicate.

Let $g \geq 2$ be a fixed integer. For any choice of $\mathcal{D} \subset \{0, 1, \ldots, g-1\}$, let

$$\boldsymbol{A} = \left\{ n = \sum_{0 \leq i \leq v} a_i(n) g^i \ :\ a_i(n) \in \mathcal{D},\ v \geq 0 \right\}$$

be the set of positive integers with restricted g-ary digits. Shparlinski, together with Banks [10] and with Banks and Conflitti [9], as well as numerous other authors, has considered various arithmetical questions concerning integers whose g-ary digits are restricted. We define the set of Farey fractions whose denominators and numerators are restricted g-ary digits as follows:

$$\mathcal{F}_{Q,\boldsymbol{A}} = \left\{ \frac{a}{q} : 1 \leq a \leq q \leq Q,\ (a,q) = 1,\ a, q \in \boldsymbol{A} \right\}.$$

It would be interesting to analyze the distribution of Farey fractions with denominators and numerators are restricted g-ary digits. In particular, our aim is to study the equidistribution and pair correlation measure of the sequence $(\mathcal{F}_{Q,\boldsymbol{A}})_Q$.

Let us recall the definition of the index of Farey fractions. Let $\gamma' = \frac{a'}{q'} < \gamma = \frac{a}{q} < \gamma'' = \frac{a''}{q''}$ be three consecutive Farey fractions in $\mathcal{F}_Q$. Then the index of the Farey fraction γ is defined as

$$\nu_Q(\gamma) = \frac{q' + q''}{q} = \frac{a' + a''}{a}.$$

In [3], the authors studied the moments of the index of Farey fractions, and in [3] this was done for Farey fractions with square-free denominators. In Chapter 5, we investigate the moments of the index of Farey fractions with $\mathcal{B}$-free denominators that lie in a fixed arithmetic progression. The case of k-free denominators will appear in a forthcoming paper.

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