Rectilinear Crossing Number of Uniform Hypergraphs

by

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under the supervision of

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Declaration

This thesis contains the results from my original research. Contributions from other sources are clearly mentioned with the citations to the literature. This research work has been carried out under the supervision of **Dr. Saswata Shannigrahi** and **Dr. Anuradha Sharma**.

This thesis has not been submitted in part or in full, to any other university or institution for the award of any other degree.

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Certificate

This is to certify that the thesis titled **Rectilinear Crossing Number of Uniform Hypergraphs** being submitted by **Mr. Rahul Gangopadhyay** to the Indraprastha Institute of Information Technology-Delhi, for the award of the degree **Doctor of Philosophy**, is an original research work carried out by him under our supervision.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree/diploma.

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To My Parents and Dida

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Abstract

In this thesis, we study the d-dimensional rectilinear drawings of d-uniform hypergraphs in which each hyperedge contains exactly d vertices. A d-dimensional rectilinear drawing of a d-uniform hypergraph is a drawing of the hypergraph in $\mathbb{R}^d$ when its vertices are placed as points in general position and its hyperedges are drawn as the convex hulls of the corresponding d points. In such a drawing, a pair of hyperedges forms a crossing if they are vertex disjoint and contain a common point in their relative interiors. A special kind of d-dimensional rectilinear drawing of a d-uniform hypergraph is known as a d-dimensional convex drawing of it when its vertices are placed as points in general as well as in convex position in $\mathbb{R}^d$. The d-dimensional rectilinear crossing number of a d-uniform hypergraph is the minimum number of crossing pairs of hyperedges among all d-dimensional rectilinear drawings of it. Similarly, the d-dimensional convex crossing number of a d-uniform hypergraph is the minimum number of crossing pairs of hyperedges among all d-dimensional convex drawings of it.

We study two types of uniform hypergraphs in this thesis, namely, the complete d-uniform hypergraphs and the complete balanced d-partite d-uniform hypergraphs. We summarise the main results of this thesis as follows.

- We prove that the d-dimensional rectilinear crossing number of a complete d-uniform hypergraph having n vertices is $\Omega(2^d\sqrt{d})\dbinom{n}{2d}$.
- We prove that any 3-dimensional convex drawing of a complete 3-uniform hypergraph with n vertices contains $3\dbinom{n}{6}$ crossing pairs of hyperedges.
- We prove that there exist $\Theta\left(4^d/\sqrt{d}\right)\dbinom{n}{2d}$ crossing pairs of hyperedges in the d-dimensional rectilinear drawing of a complete d-uniform hypergraph having n vertices when all its vertices are placed over the d-dimensional moment curve.
- We prove that the d-dimensional rectilinear crossing number of a complete balanced d-partite d-uniform hypergraph having nd vertices is $\Omega\left(2^d\right)(n/2)^d\left((n-1)/2\right)^d$.

We also study the properties of different types of d-dimensional rectilinear drawings and d-dimensional convex drawings of the complete d-uniform hypergraph having $2d$ vertices by exploiting its relations with convex polytopes and k-sets.

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Chapter 1

Introduction

A *hypergraph* $H = (V, E)$ is a combinatorial object where V denotes the set of vertices and $E \subset 2^V$ denotes the set of hyperedges. Hypergraphs are extensively studied in the literature [13]. A hypergraph is called *uniform* if each hyperedge contains an equal number of vertices. A hypergraph is said to be d-uniform if each of its hyperedges contains d vertices. In particular, a graph is a 2-uniform hypergraph, i.e., each hyperedge contains two vertices. As a result, uniform hypergraphs are natural generalizations of graphs. Many combinatorial problems on graphs are generalized for uniform hypergraphs. For example, the 2-colorability of a uniform hypergraph is a generalization of the graph colorability problem and has been widely studied in the literature [12, 49]. Füredi [25], Alon et al. [6] studied the problem of matching in uniform hypergraphs. Lehel [39] studied the edge covering problem in uniform hypergraphs. Hypergraphs are also used to model various practical problems in different domains, e.g., RDBMS [24] and social networks [40].

Graph drawing is also an active area of research for many decades, with applications in various fields of computer science, e.g., CAD, database design and circuit schematics [33, 53]. Dey et al. [18] generalised the concept of graph drawing to the drawing of uniform hypergraphs. In this thesis, we address the problem of drawing a uniform hypergraph. Let us introduce some basic notations that are used throughout the thesis. A hypergraph $H = (V, E)$ is d-uniform if each of its hyperedges contains d vertices. A *complete* d-uniform hypergraph with n vertices, denoted by K_n^d, contains $\binom{n}{d}$ hyperedges. A d-uniform hypergraph $H = (V, E)$ is said to be *d-partite* if there exists a sequence $< X_1, X_2, \ldots, X_d >$ of disjoint sets such that $V = \bigcup_{i=1}^{d} X_i$ and $E \subseteq X_1 \times X_2 \times \ldots \times X_d$. The set X_i is called the i^{th} part of V. Such a d-partite d-uniform hypergraph is called *balanced* if each of the parts contains an equal number of vertices. A d-uniform d-partite hypergraph H is said to be *complete* if $|E| = |X_1 \times X_2 \times \ldots \times X_d|$. In particular, let $K^d_{d\times n'}$ denote the complete balanced d-partite d-uniform hypergraph with n' vertices in each part.

A *good drawing* of a graph is defined as a drawing of the graph in $\mathbb{R}^2$ such that its vertices are placed as points in general position and its edges are drawn as simple continuous curves (i.e., homeomorphic to a line segment) joining the corresponding two vertices. In such a drawing of a graph, two edges are said to be crossing if they do not share a common vertex and intersect each other at a point different from their vertices. The *crossing number* of a graph G, denoted by $cr(G)$, is defined as the minimum number of crossing pairs of edges among all such good drawings of it. For a graph G with n vertices and $m \geq 4n$ edges, Erdős et al. [21] conjectured that $cr(G) \geq cm^3/n^2$ for some constant $c > 0$. Ajtai et al. [4] and Leighton [38] independently proved the conjecture affirmatively and established a value of c to be 1/64.

Let $K_{m,n}$ denote a complete bipartite graph having m vertices in one part and n vertices in the other part. Let $K_{1,m,n}$ denote a complete tripartite graph having 1 vertex in the first part, m vertices in the second part and n vertices in the last part. Finding the crossing number of $K_{m,n}$ is also an active area for research. In 1944, Turan in his famous "brick factory problem" asked for the crossing number of a complete bipartite graph [54]. Kleitman [36] proved that $cr(K_{6,n}) = 6\lfloor n/2\rfloor \lfloor (n-1)/2\rfloor$. Zarankiewicz [57] conjectured that $cr(K_{m,n}) = \lfloor m/2\rfloor \lfloor (m-1)/2\rfloor \lfloor n/2\rfloor \lfloor (n-1)/2\rfloor$ and this number has been proven to be an upper bound on $cr(K_{m,n})$ [57]. The crossing numbers of some other special graphs have been widely studied in the literature [16, 37]. For example, Gethnerand et al. [28] studied the crossing number of balanced complete multipartite graphs. Ho [34] gave a lower bound on the crossing number of $K_{1,m,n}$. Glebsky et al. [29] studied the crossing number of cartesian product graphs. We summarise the crossing numbers of a few special graphs in Table 1.1.

A *rectilinear drawing* of a graph is a good drawing of it where each edge is drawn as a straight line segment connecting the two corresponding vertices. The *rectilinear crossing number* of a graph G, denoted by $\overline{cr}(G)$, is defined as the minimum number of crossing

Graph	Crossing Number	Reference
$K_{1,3,n}$	$2\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor + \lfloor n/2\rfloor$	[9]
$K_{1,4,n}$	$n(n-1)$	[35]
$K_{2,3,n}$	$4\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor + n$	[9]
$K_{6,n}$	$6\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor$	[36]
K_{11}	100	[48]

Table 1.1

pairs of edges among all rectilinear drawings of G. The currently best-known lower and upper bounds on the rectilinear crossing number of a complete graph with n vertices are $0.37997\binom{n}{4} + \Theta(n^3)$ and $0.380449186\binom{n}{4} + \Theta(n^3)$, respectively [10, 1]. The exact values of rectilinear crossing numbers of complete graphs with n vertices are known for $n \leq 27$ [11]. The rectilinear crossing numbers for a few special graphs have also been studied in the literature [14].

A *convex drawing* of a graph is a rectilinear drawing of it where all of its vertices are placed as the vertices of a convex polygon. The *convex crossing number* of a graph G, denoted by $cr^*(G)$, is defined as the minimum number of crossing pairs of edges among all such convex drawings of it. Shahrokhi et al. [50] studied the convex crossing number problem for graphs. They established an upper bound on $cr^*(G)$ with respect to $cr(G)$. In particular, they proved that $cr^*(G) = O((cr(G) + \sum_{v \in V} d_v^2)\log^2 n)$, where V is the set of n vertices of G and d_v denotes the degree of the vertex v.

Dey et al. [18] defined a *d-dimensional geometric d-hypergraph* $H = (V, E)$ as a collection of $(d-1)$-simplices, induced by some d-tuples of a vertex set in general position in $\mathbb{R}^d$. Similarly, Anshu et al. [8] defined a *d-dimensional rectilinear drawing* of a d-uniform hypergraph H as a drawing of it in $\mathbb{R}^d$ where its vertices are placed as points in general position in $\mathbb{R}^d$ (no d+1 points on a $(d-1)$-dimensional hyperplane) and each hyperedge is represented as a convex hull of the d corresponding vertices. In a d-dimensional rectilinear drawing of a d-uniform hypergraph, a pair of hyperedges is said to have an intersection if they contain a common point in their relative interiors [18].

The convex hull of a finite point set P is denoted by $Conv(P)$ and the affine hull of P is denoted by $Aff(P)$. The convex hulls $Conv(P)$ and $Conv(P')$ of two finite point sets P and P' *intersect* if they contain a common point in their relative interiors. The convex hulls

$Conv(P)$ and $Conv(P')$ *cross* if they are vertex disjoint and they intersect.

In a d-dimensional rectilinear drawing of a d-uniform hypergraph, a pair of hyperedges is said to be *crossing* if the hyperedges are vertex disjoint and contain a common point in their relative interiors [8, 18]. For u and w in the range $2 \leq u, w \leq d$, a $(u-1)$-simplex $Conv(U)$ spanned by a point set U containing u points and a $(w-1)$-simplex $Conv(W)$ spanned by a point set W containing w points (when these $u+w$ points are in general position in $\mathbb{R}^d$) *cross* if $Conv(U)$ and $Conv(W)$ intersect, and $U \cap W = \emptyset$ [18].

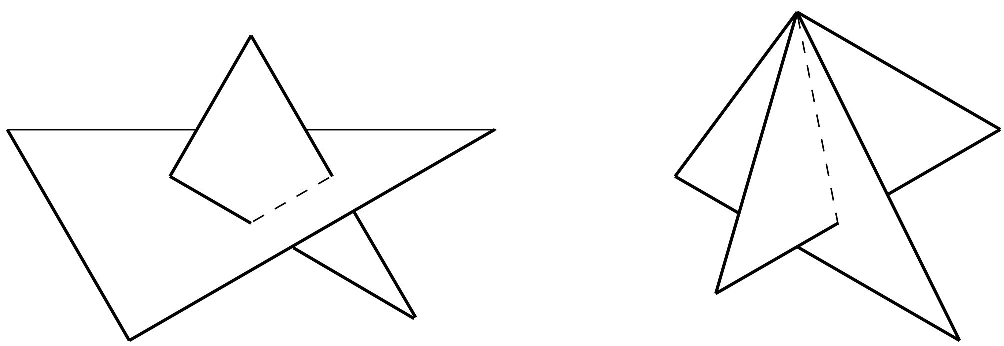

Figure 1.1: (left) Crossing simplices in $\mathbb{R}^3$, (right) Intersecting simplices in $\mathbb{R}^3$

The *d-dimensional rectilinear crossing number* of a d-uniform hypergraph H is defined as the minimum number of crossing pairs of hyperedges among all d-dimensional rectilinear drawings of H and it is denoted by $\overline{cr}_d(H)$ [8]. Dey et al. [17] proved the following results about the 3-dimensional geometric 3-hypergraph.

Lemma 1. *[17] (i) A* 3*-dimensional geometric* 3*-hypergraph can have at most* n^2 2*-simplices if there does not exist an intersecting pair of* 2*-simplices in the collection.*
(ii) A 3*-dimensional geometric* 3*-hypergraph can have at most* $\dfrac{3n^2}{2}$ 2*-simplices if there does not exist a crossing pair of* 2*-simplices in the collection.*

Later, Dey et al. [18] proved that a d-dimensional geometric d-hypergraph can have at most $O(n^{d-1})$ $(d-1)$-simplices if it does not have a crossing pair of $(d-1)$-simplices induced by n vertices. Anshu et al. [8] established the first non-trivial lower bound on $\overline{cr}_d(K_n^d)$ for $n \geq 2d$. In particular, they established that $\overline{cr}_d(K_{2d}^d) = \Omega\left(\dfrac{2^d \log d}{\sqrt{d}}\right)$. Since the set of crossing pairs of hyperedges due to a particular set of $2d$ vertices is disjoint from the set of crossing pairs of hyperedges due to another set of $2d$ vertices, it follows from this result that

$\overline{cr}_d(K_n^d) = \Omega\left(\frac{2^d \log d}{\sqrt{d}}\right)\binom{n}{2d}$ for $n \geq 2d$. They also proved that $\overline{cr}_4(K_8^4) = 4$. Note that we use $\log d$ to denote $\log_2 d$ in this thesis.

This thesis contains results about the d-dimensional rectilinear drawings of d-uniform hypergraphs. Unless specified otherwise, the dimension d used throughout the thesis is a sufficiently large integer even though several statements are true for smaller values of d as well. In Chapter 4, we improve the lower bound on $\overline{cr}_d(K_{2d}^d)$. In Chapter 5, we investigate the d-dimensional convex drawings of K_n^d. We also establish the first non-trivial lower bound on $\overline{cr}_d\left(K_{d\times n}^d\right)$ in Chapter 6. Let us introduce some basic definitions and theorems before discussing the main results of this thesis.

Let $V = \{v_1, v_2, \ldots, v_{2d}\}$ be the set of points corresponding to the set of vertices in a d-dimensional rectilinear drawing of the hypergraph K_{2d}^d. The points in V are said to be in *convex position* if there does not exist any point $v_i \in V$ (for some $1 \leq i \leq 2d$) such that v_i can be expressed as a convex combination of the points in $V \setminus \{v_i\}$. We define a *d-dimensional convex drawing* of a d-uniform hypergraph H as a d-dimensional rectilinear drawing of H such that the vertices of H are placed as points in general, as well as in convex position. The *d-dimensional convex crossing number* of H, denoted by $cr_d^*(H)$, is the minimum number of crossing pairs of hyperedges among all d-dimensional convex drawings of H. Note that the convex hull of the vertices of H in a d-dimensional convex drawing of it forms a d-dimensional *convex polytope*. All d-dimensional polytopes considered in this thesis are convex polytopes with vertices placed in general position in $\mathbb{R}^d$. The *d-dimensional moment curve* is defined as $\{(a, a^2, \ldots, a^d) : a \in \mathbb{R}\}$. Let us define the ordering between two points $p_i = (a_i, (a_i)^2, \ldots, (a_i)^d)$ and $p_j = (a_j, (a_j)^2, \ldots, (a_j)^d)$ on the d-dimensional moment curve by $p_i \prec p_j$ (p_i *precedes* p_j) if $a_i < a_j$. A d-dimensional convex polytope is said to be *t-neighborly polytope* if each subset of its vertex set having at most t vertices forms a face of the polytope. A $\lfloor d/2 \rfloor$-neighborly polytope is called *neighborly polytope* since any d-dimensional convex polytope can be at most $\lfloor d/2 \rfloor$-neighborly unless it is a d-simplex. The d-dimensional cyclic polytope is a special kind of d-dimensional neighborly polytope where all of its vertices are placed on the d-dimensional moment curve.

We summarize the main contributions of this thesis in Section 1.1. In order to prove them, we use a few theorems and techniques from combinatorial geometry. We introduce those theorems and techniques briefly.

Ham-Sandwich Theorem for measures. *[44] Let $\mu_1, \mu_2, \ldots, \mu_d$ be d finite Borel measures in $\mathbb{R}^d$ such that any hyperplane has measure 0 for each μ_i. There exists a hyperplane h in $\mathbb{R}^d$ that bisects each of these d measures, i.e., $\mu_i(h^+) = \mu_i(h^-) = \dfrac{\mu_i(\mathbb{R}^d)}{2}$ for each i in the range $1 \le i \le d$, where h^+ and h^- denote the open half-spaces created by h.*

The Ham-Sandwich theorem for measures can be proved using the Borsuk-Ulam theorem.

We now state the Ham-Sandwich theorem for finite point sets and refer to it as the Ham-Sandwich theorem in this thesis.

Ham-Sandwich Theorem. *[5, 43] There exists a $(d-1)$-dimensional hyperplane h which simultaneously bisects d finite point sets $P_1, P_2, \ldots, P_d$ in $\mathbb{R}^d$, such that each of the open half-spaces created by h contains at most $\left\lfloor \dfrac{|P_i|}{2} \right\rfloor$ points for each of the sets $P_i, 1 \le i \le d$.*

The Ham-Sandwich theorem is a direct consequence of the Ham-Sandwich theorem for measures. We replace each point in P_i by a ball of very small radius and apply the Ham-Sandwich theorem for measures to get the desired result.

Colored Tverberg Theorem with restricted dimensions. *[44, 55] Let $\{C_1, C_2, \ldots, C_{k+1}\}$ be a collection of $k+1$ disjoint finite point sets in $\mathbb{R}^d$. Each of these sets is assumed to be of cardinality at least $2r-1$, where r is a prime integer satisfying the inequality $r(d-k) \le d$. Then, there exist r disjoint sets $S_1, S_2, \ldots, S_r$ such that $S_i \subseteq \bigcup_{j=1}^{k+1} C_j$, $\bigcap_{i=1}^{r} Conv(S_i) \neq \emptyset$ and $|S_i \cap C_j| = 1$ for all i and j satisfying $1 \le i \le r$ and $1 \le j \le k+1$.*

Throughout this thesis, we use the Gale transformation extensively. The Gale transformation is a technique to convert a point sequence to a vector sequence in a lower dimension. In Chapter 2, we discuss it in detail. We also use k-set and related concepts extensively. We discuss these in Chapter 3.

1.1 Our Contributions

In Chapter 4, we improve the lower bound on $\overline{cr}_d(K_{2d}^d)$ to $\Omega(2^d)$ from $\Omega\left(\frac{2^d \log d}{\sqrt{d}}\right)$ that was proved in [8]. We use the Gale transformation and the Ham-Sandwich theorem to improve this lower bound. We further improve the lower bound on $\overline{cr}_d(K_{2d}^d)$ to $\Omega(2^d\sqrt{\log d})$ and then to $\Omega(2^d\sqrt{d})$ by using the properties of k-sets and balanced lines, respectively.

Theorem 1. *The number of crossing pairs of hyperedges in a d-dimensional rectilinear drawing of K_{2d}^d is $\Omega(2^d d^{3/2})$ if the vertices of K_{2d}^d are not in convex position.*

Theorem 2. *The d-dimensional rectilinear crossing number of a complete d-uniform hypergraph having $2d$ vertices is $\Omega(2^d\sqrt{d})$, i.e., $\overline{cr}_d(K_{2d}^d) = \Omega(2^d\sqrt{d})$.*

In Chapter 5, we investigate a special d-dimensional convex drawing of K_{2d}^d where all of its vertices are placed on the d-dimensional moment curve. For such a d-dimensional convex drawing of K_{2d}^d, we count the exact number of crossing pairs of hyperedges in it. This number is denoted by c_d^m. We also prove that the 3-dimensional convex crossing number of K_6^3 is 3. We then investigate some special types of d-dimensional rectilinear drawings of K_{2d}^d. We summarize the results obtained in Chapter 5 here.

Theorem 3. *Let c_d^m be the number of crossing pairs of hyperedges in a d-dimensional convex drawing of K_{2d}^d where all of its vertices are placed on the d-dimensional moment curve. The value of c_d^m is*

$$c_d^m = \begin{cases} \binom{2d-1}{d-1} - \sum\limits_{i=1}^{\frac{d}{2}} \binom{d}{i}\binom{d-1}{i-1} & if\ d\ is\ even \\ \binom{2d-1}{d-1} - 1 - \sum\limits_{i=1}^{\left\lfloor\frac{d}{2}\right\rfloor} \binom{d-1}{i}\binom{d}{i} & if\ d\ is\ odd \end{cases}$$

$$= \Theta\left(\frac{4^d}{\sqrt{d}}\right)$$

Theorem 4. *The number of crossing pairs of hyperedges in a 3-dimensional rectilinear drawing of K_6^3 is 3 when all the vertices of K_6^3 are in convex as well as general position in $\mathbb{R}^3$.*

Theorem 5. *For any constant $t \geq 1$ independent of d, the number of crossing pairs of hyperedges in a d-dimensional rectilinear drawing of K_{2d}^d is $\Omega(2^d d^{3/2})$ if the vertices of K_{2d}^d are placed as the vertices of a d-dimensional t-neighborly polytope that is not $(t+1)$-neighborly.*

Theorem 6. *For any constant $t' \geq 0$ independent of d, the number of crossing pairs of hyperedges in a d-dimensional rectilinear drawing of K_{2d}^d is $\Omega(2^d d^{5/2})$ if the vertices of K_{2d}^d are placed as the vertices of a d-dimensional $(\lfloor d/2 \rfloor - t')$-neighborly polytope.*

In Chapter 6, we deal with a generalized version of the rectilinear crossing number problem for bipartite graphs. We investigate the d-dimensional rectilinear drawing of the complete balanced d-partite d-uniform hypergraph with nd vertices. Using Colored Tverberg Theorem with restricted dimensions, we first prove that $\overline{cr}_d\left(K_{d\times n}^d\right) = \Omega\left((8/3)^{d/2}\right)(n/2)^d\left((n-1)/2\right)^d$ for $n \geq 3$. By using the Gale transformation and the Ham-Sandwich theorem, we then improve this bound to $\Omega\left(2^d\right)(n/2)^d\left((n-1)/2\right)^d$ for $n \geq 3$. In summary, we prove the following.

Theorem 7. *For $n \geq 3$, $\overline{cr}_d\left(K_{d\times n}^d\right) = \Omega\left(2^d\right)(n/2)^d\left((n-1)/2\right)^d$.*

In Chapter 7, we summarize the results in this thesis and discuss some open problems.

1.2 Organization of the Thesis

The thesis is organised as follows.

- **Introduction**

  We survey the literature on the rectilinear drawings of graphs in a plane. Thereafter, we mention that the concept of rectilinear drawings of graphs in a plane can be generalized to the d-dimensional rectilinear drawings of d-uniform hypergraphs. We then survey the literature on the d-dimensional rectilinear drawings of d-uniform hypergraphs. Finally, we define the problems that are studied in this thesis and briefly discuss the results obtained by us.

- **Gale Transformation**

  We use Gale transformation and its properties for the proofs in this thesis. In Chapter 2, we study Gale transformation in detail. In particular, we mention the properties of a Gale transform of a point set and discuss their proofs. These properties are well-known but their proofs are not written in detail elsewhere as per our knowledge.

- **Balanced Lines, j-Facets and k-Sets**

  We use the concepts of balanced lines, j-facets and k-sets to obtain the results of this thesis. First, we study the concepts of j-edges and k-sets of planar point sets and discuss the relation between them in Observation 1. The proof of Observation 1 is well-known but we produce the proof in the thesis for the sake of completeness. We then discuss the concept of a balanced line and prove Observation 2 which is used later. Finally, we discuss the concepts of j-facets and k-sets of a point set in $\mathbb{R}^3$ and also discuss their properties in Observation 3, Observation 4 and Observation 5 that are used later in the thesis.

- **Rectilinear Crossing Number of Complete d-Uniform Hypergraphs**

  In this chapter, we first reproduce the results obtained by Anshu et al. [8]. We then improve the bound obtained by Anshu et al. [8] by proving that the d-dimensional rectilinear crossing number of a complete d-uniform hypergraph with $2d$ vertices is $\Omega(2^d)$. In Section 4.3, we prove Theorem 1. Subsequently, we improve the lower bound on the d-dimensional rectilinear crossing number of a complete d-uniform hypergraph with $2d$ vertices to $\Omega(2^d\sqrt{\log d})$. Finally, we prove Theorem 2 in this chapter.

- **Convex Crossing Number of Complete d-Uniform Hypergraphs**

  In this chapter, we first reproduce the proof of Gale's evenness criterion. We then compute a Gale transform of $d+3$ points on the d-dimensional moment curve. Using these results, we prove a non-trivial lower bound on c_d^m. We then prove Theorem 3 and Theorem 4. Finally, we prove Theorem 5 and Theorem 6 in Section 5.3.

- **Rectilinear Crossings in Complete Balanced d-Partite d-Uniform Hypergraphs**

In this chapter, we first prove a non-trivial lower bound on the d-dimensional rectilinear crossing number of the complete balanced d-partite d-uniform hypergraph having nd vertices by using the Colored Tverberg theorem with restricted dimensions. In the subsequent section, we improve this lower bound by proving Theorem 7.

- **Conclusions**

  In this chapter, we summarise the results of the thesis. We then state some open problems.

1.3 List of Symbols

K_n^d	A complete d-uniform hypergraph with n vertices
$K_{d\times n'}^d$	A complete balanced d-partite d-uniform hypergraph with n' vertices in each part
$\overline{cr}_d(H)$	The d-dimensional rectilinear crossing number of a d-uniform hypergraph H
$Conv(P)$	The convex hull of the point set P
$Aff(P)$	The affine hull of the point set P
c_d^m	The number of crossing pairs of hyperedges in a d-dimensional convex drawing of K_{2d}^d where all of its vertices are placed on the d-dimensional moment curve
$D(P)$	A Gale transform of the point set P
$\overline{D(P)}$	An affine Gale diagram of the point set P
$e'_k(S)$	The number of k-sets of a planar point set S
$E'_j(S)$	The number of j-edges of a planar point set S
$e_k(S)$	The number of k-sets of a point set S in $\mathbb{R}^3$
$E_j(S)$	The number of j-facets of a point set S in $\mathbb{R}^3$

Chapter 2

Gale Transformation

The *Gale transformation* is a useful technique to deal with the properties of high dimensional point sets [43]. Consider a sequence of $m > d+1$ points $P = < p_1, p_2, \ldots, p_m >$ in $\mathbb{R}^d$ such that the affine hull of the points is $\mathbb{R}^d$. Let the i^{th} point p_i be represented as $(x_1^i, x_2^i, \ldots, x_d^i)$. To compute a Gale transform of P, let us consider the $(d+1) \times m$ matrix $M(P)$ whose i^{th} column is $\begin{bmatrix} x_1^i & x_2^i & \ldots & x_d^i & 1 \end{bmatrix}^T$.

$$M(P) = \begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^m \\ x_2^1 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & \vdots \\ x_d^1 & x_d^2 & \cdots & x_d^m \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

Since there exists a set of $d+1$ points in P that is affinely independent, the rank of $M(P)$ is $d+1$. Therefore, the dimension of the null space of $M(P)$ is $m-d-1$. Let $\{(b_1^1, b_2^1, \ldots, b_m^1)$, $(b_1^2, b_2^2, \ldots, b_m^2), \ldots, (b_1^{m-d-1}, b_2^{m-d-1}, \ldots, b_m^{m-d-1})\}$ be a set of $m-d-1$ vectors that spans the null space of $M(P)$. A Gale transform $D(P)$ is the sequence of vectors $< g_1, g_2, \ldots, g_m >$ where $g_i = (b_i^1, b_i^2, \ldots, b_i^{m-d-1})$ for each i satisfying $1 \le i \le m$. Note that $D(P)$ can also be treated as a point sequence in $\mathbb{R}^{m-d-1}$. We denote vectors and points as row vectors in this thesis.

We define a *linear separation* of $D(P)$ to be a partition of $D(P)$ into two disjoint sets of vectors $D^+(P)$ and $D^-(P)$ contained in the opposite open half-spaces created by a linear hyperplane (i.e., a hyperplane passing through the origin). A linear separation of $D(P)$ is called *proper* if one of the sets among $D^+(P)$ and $D^-(P)$ contains $\left\lceil \frac{m}{2} \right\rceil$ vectors and the other contains $\left\lfloor \frac{m}{2} \right\rfloor$ vectors. We list the following properties of $D(P)$. For the sake of completeness, we mention the proofs of these properties.

Property 1. *[43] Every set of $m-d-1$ vectors in $D(P)$ spans $\mathbb{R}^{m-d-1}$ if and only if the points in P are in general position.*

Proof. ($\Rightarrow$) Without loss of generality, let us assume that the first $d+1$ points in P are not in general position. This implies that there exist real numbers $\mu_1, \mu_2, \ldots, \mu_{d+1}$, not all of them zero, satisfying the following equation.

$$\begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^m \\ x_2^1 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & \vdots \\ x_d^1 & x_d^2 & \cdots & x_d^m \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_{d+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.1}$$

It is evident from Equation 2.1 that the vector $(\mu_1,\ \mu_2,\ \ldots,\ \mu_{d+1},\ 0,\ 0,\ \ldots,\ 0)$ lies in the null space of the row space of $M(P)$. This implies that $(\mu_1,\ \mu_2,\ \ldots,\ \mu_{d+1},\ 0,\ 0,\ \ldots, 0)$ $= \alpha_1(b_1^1, b_2^1,\ \ldots, b_m^1) + \alpha_2(b_1^2,\ b_2^2,\ \ldots, b_m^2) + \ldots + \alpha_{m-d-1}\ (b_1^{m-d-1},\ b_2^{m-d-1},\ \ldots,\ b_m^{m-d-1})$ for some real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{m-d-1}$, not all of them zero. In other words, there exist α_1, $\alpha_2,\ \ldots, \alpha_{m-d-1}$, not all of them zero, such that $\alpha_1 b_j^1 + \alpha_2 b_j^2 + \ldots + \alpha_{m-d-1} b_j^{m-d-1} = 0$ for $j = d+2, d+3, \ldots, m$. This shows that the last $m-d-1$ vectors in $D(P)$ lie on the hyperplane $\sum_{i=1}^{m-d-1} \alpha_i x_i = 0$. This implies that there exists a set of $m-d-1$ vectors in $D(P)$ that does not span $\mathbb{R}^{m-d-1}$, leading to a contradiction.

($\Leftarrow$) Without loss of generality, let us assume that the first $m-d-1$ vectors in $D(P)$, i.e., $(b_1^1,\ b_1^2,\ \ldots,\ b_1^{m-d-1})$, $(b_2^1,\ b_2^2,\ \ldots,\ b_2^{m-d-1})$, $\ldots$, $(b_{m-d-1}^1,\ b_{m-d-1}^2,\ \ldots,\ b_{m-d-1}^{m-d-1})$ do not span $\mathbb{R}^{m-d-1}$.

This implies that there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_{m-d-1}$, not all of them zero, such that $\lambda_1(b_1^1, b_1^2, \ldots, b_1^{m-d-1}) + \lambda_2(b_2^1, b_2^2, \ldots, b_2^{m-d-1}) + \ldots + \lambda_{m-d-1}(b_{m-d-1}^1, b_{m-d-1}^2, \ldots, b_{m-d-1}^{m-d-1}) = \vec{0}$. Let us consider the vector $(\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_{m-d-1}, \lambda_{m-d} = 0,\ \ldots,\ \lambda_m = 0)$. It is easy to see that $\lambda_1(b_1^1,\ b_1^2,\ \ldots,\ b_1^{m-d-1})\ +\lambda_2(b_2^1,\ b_2^2,\ \ldots,\ b_2^{m-d-1}) +\ \ldots + \lambda_m(b_m^1,\ b_m^2,\ \ldots,\ b_m^{m-d-1}) = \vec{0}$. This implies that $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ lies in the row space of $M(P)$. This further implies that there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{d+1}$, not all of them zero, such that the following linear equations hold for each i satisfying $1 \leq i \leq m$.

$$\alpha_1 x_1^i + \alpha_2 x_2^i + \ldots + \alpha_d x_d^i + \alpha_{d+1} = \lambda_i$$

This implies that the last $d+1$ points in P, i.e., $p_{m-d}, p_{m-d+1}, \ldots, p_m$ lie on the hyperplane $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_d x_d + \alpha_{d+1} = 0$. This further implies that the points in P are not in general position, leading to a contradiction. $\square$

Property 2. *[43] Consider two integers u and v satisfying $1 \leq u, v \leq d-1$ and $u+v+2 = m$. If the points in P are in general position in $\mathbb{R}^d$, there exists a bijection between the crossing pairs of u- and v-simplices formed by some points in P and the linear separations of $D(P)$ into $D^+(P)$ and $D^-(P)$ such that $|D^+(P)| = u+1$ and $|D^-(P)| = v+1$.*

Proof. Let σ be a u-simplex that crosses a v-simplex ν, such that $1 \leq u \leq d-1$, $1 \leq v \leq d-1$ and $u+v+2 = m$. Without loss of generality, we assume that σ is spanned by the first $u+1$ points $\{p_1, p_2, \ldots p_{u+1}\}$ of P and ν is spanned by the remaining $v+1$ points $\{p_{u+2}, p_{u+3}, \ldots, p_m\}$ of P. As there exists a crossing between σ and ν, we know that there exists a point p belonging to the relative interiors of both σ and ν. This implies that there exist real numbers $\lambda_k > 0$, $1 \leq k \leq m$, satisfying the following equations:

$$p = \sum_{i=1}^{u+1} \lambda_i p_i = \sum_{j=u+2}^{m} \lambda_j p_j$$

$$\sum_{i=1}^{u+1} \lambda_i = \sum_{j=u+2}^{m} \lambda_j = 1$$

Therefore, we obtain the following equation.

$$\begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^m \\ x_2^1 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & \vdots \\ x_d^1 & x_d^2 & \cdots & x_d^m \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{u+1} \\ -\lambda_{u+2} \\ \vdots \\ -\lambda_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.2}$$

It is evident from Equation 2.2 that the vector $(\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_{u+1},\ -\lambda_{u+2},\ \ldots,\ -\lambda_m)$ lies in the null space of the row space of $M(P)$. This implies that $(\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_{u+1},\ -\lambda_{u+2},\ \ldots, -\lambda_m) = \alpha_1(b_1^1, b_2^1,\ \ldots, b_m^1) + \alpha_2(b_1^2,\ b_2^2,\ \ldots, b_m^2) + \ldots + \alpha_{m-d-1}\ (b_1^{m-d-1},\ b_2^{m-d-1},\ \ldots,\ b_m^{m-d-1})$ for some real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{m-d-1}$, not all of them zero. In other words, there exist $\alpha_1, \alpha_2, \ldots, \alpha_{m-d-1}$, not all of them zero, such that $\alpha_1 b_i^1 + \alpha_2 b_i^2 + \ldots + \alpha_{m-d-1}\ b_i^{m-d-1} > 0$ for $i = 1, 2, \ldots, u+1$, and $\alpha_1 b_j^1 + \alpha_2 b_j^2 + \ldots + \alpha_{m-d-1} b_j^{m-d-1} < 0$ for $j = u+2, u+3, \ldots, m$. This shows that the hyperplane $\sum_{i=1}^{m-d-1} \alpha_i x_i = 0$ separates the first $u+1$ vectors in $D(P)$ from the remaining $v+1$ vectors in it.

In the other direction, let us assume without loss of generality that the hyperplane

$$\sum_{i=1}^{m-d-1} \alpha_i' x_i = 0$$

separates the first $u+1$ vectors in $D(P)$ from the remaining $v+1$ vectors. This implies that there exists a vector $(\mu_1', \mu_2', \ldots, \mu_m') = \alpha_1'(b_1^1, b_2^1, \ldots, b_m^1) + \alpha_2'(b_1^2,\ b_2^2,\ \ldots,\ b_m^2) + \ldots + \alpha_{m-d-1}'(b_1^{m-d-1},\ b_2^{m-d-1},\ \ldots,\ b_m^{m-d-1})$ such that the signs of μ_i' for $1 \le i \le u+1$ are opposite to the signs of μ_j' for $u+2 \le j \le m$. Without loss of generality, let us assume that $\mu_i' > 0$ for $1 \le i \le u+1$ and $\mu_j' < 0$ for $u+2 \le j \le m$. As this vector $(\mu_1', \mu_2', \ldots, \mu_m')$ lies in the null space of the row space of $M(P)$, it satisfies the following equation.

$$\begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^m \\ x_2^1 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & \vdots \\ x_d^1 & x_d^2 & \cdots & x_d^m \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \mu'_1 \\ \mu'_2 \\ \vdots \\ \mu'_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.3}$$

From Equation 2.3, we obtain the following.

$$\sum_{i=1}^{u+1} \mu'_i p_i = \sum_{j=u+2}^{m} -\mu'_j p_j$$

$$\sum_{i=1}^{u+1} \mu'_i = \sum_{j=u+2}^{m} -\mu'_j$$

Rearranging the above equations, we obtain the following.

$$\sum_{i=1}^{u+1} \frac{\mu'_i}{\sum_{i=1}^{u+1} \mu'_k} p_i = \sum_{j=u+2}^{m} \frac{\mu'_j}{\sum_{j=u+2}^{m} \mu'_k} p_j$$

$$\sum_{i=1}^{u+1} \frac{\mu'_i}{\sum_{i=1}^{u+1} \mu'_i} = \sum_{j=u+2}^{m} \frac{\mu'_j}{\sum_{j=u+2}^{m} \mu'_j} = 1$$

It shows that there exists a crossing between the u-simplex spanned by the first $u+1$ points of P and the v-simplex spanned by the remaining $v+1$ points of P. □

We now prove the following property that is a slight variation of Property 2. We use this property in the proof of Theorem 7.

Property 3. *[43] Let h be a linear hyperplane, i.e., a hyperplane passing through the origin, in $\mathbb{R}^{m-d-1}$ such that it partitions the vectors in $D(P)$. Let $D^+(P) \subset D(P)$ and $D^-(P) \subset D(P)$ denote two sets of vectors such that $|D^+(P)|, |D^-(P)| \geq 2$ and the vectors in $D^+(P)$ and $D^-(P)$ lie in the opposite open half-spaces h^+ and h^- created by h, respectively. Then, the convex hull of the point set $P_a = \{p_i | p_i \in P, g_i \in D^+(P)\}$ and the convex hull of the point set*

$P_b = \{p_j | p_j \in P, g_j \in D^-(P)\}$ *cross.*

Proof. Let us assume that the hyperplane h is given by the equation $\sum_{i=1}^{m-d-1} \alpha_i x_i = 0$ such that $\alpha_i \neq 0$ for at least one i, and h^+ (h^-) is the positive (negative) open half-space created by h with an orientation assigned to it. Let $D^0(P) = \{g_k | g_k \in D(P), g_k \textit{ lies on } h\}$. This implies that there exists a vector $(\mu_1, \mu_2, \ldots, \mu_m) = \alpha_1(b_1^1, b_2^1, \ldots, b_m^1) + \alpha_2(b_1^2, b_2^2, \ldots, b_m^2) + \ldots + \alpha_{m-d-1}(b_1^{m-d-1}, b_2^{m-d-1}, \ldots, b_m^{m-d-1})$ such that $\mu_i > 0$ for each $g_i \in D^+(P)$, $\mu_j < 0$ for each $g_j \in D^-(P)$ and $\mu_k = 0$ for each $g_k \in D^0(P)$. Since this vector $(\mu_1, \mu_2, \ldots, \mu_m)$ lies in the null space of $M(P)$, it satisfies the following equation.

$$\begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^m \\ x_2^1 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & \vdots \\ x_d^1 & x_d^2 & \cdots & x_d^m \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

From the equation above, we obtain the following.

$$\sum_{i:g_i \in D^+(P)} \mu_i p_i = \sum_{j:g_j \in D^-(P)} -\mu_j p_j, \quad \sum_{i:g_i \in D^+(P)} \mu_i = \sum_{j:g_j \in D^-(P)} -\mu_j$$

Rearranging the equations above, we obtain the following.

$$\sum_{i:g_i \in D^+(P)} \frac{\mu_i}{\sum_{i:g_i \in D^+(P)} \mu_i} p_i = \sum_{j:g_j \in D^-(P)} \frac{\mu_j}{\sum_{j:g_j \in D^-(P)} \mu_j} p_j$$

$$\sum_{i:g_i \in D^+(P)} \frac{\mu_i}{\sum_{i:g_i \in D^+(P)} \mu_i} = \sum_{j:g_j \in D^-(P)} \frac{\mu_j}{\sum_{j:g_j \in D^-(P)} \mu_j} = 1$$

It shows that $Conv(P_a)$ and $Conv(P_b)$ cross. $\square$

Property 4. *[43] The points in P are in convex position in $\mathbb{R}^d$ if and only if there is no linear hyperplane h with exactly one vector from $D(P)$ in one of the open half-spaces created by h.*

Proof. ($\Rightarrow$) Without loss of generality, let us assume that the hyperplane

$$\sum_{i=1}^{m-d-1} \alpha'_i x_i = 0$$

ensures that exactly one vector from $D(P)$ lies in an open half-space created by it. This implies that there exists a vector $(\mu'_1, \mu'_2, \ldots, \mu'_m) = \alpha'_1(b_1^1, b_2^1, \ldots, b_m^1) + \alpha'_2(b_1^2,\ b_2^2,\ \ldots,\ b_m^2)$ $+ \ldots + \alpha'_{m-d-1}(b_1^{m-d-1},\ b_2^{m-d-1},\ \ldots,\ b_m^{m-d-1})$ such that $\mu'_i \geq 0$ for $1 \leq i \leq m-1$ and $\mu'_m < 0$. As this vector $(\mu'_1, \mu'_2, \ldots, \mu'_m)$ lies in the null space of the row space of $M(P)$, it satisfies the following equation.

$$\begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^m \\ x_2^1 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & \vdots \\ x_d^1 & x_d^2 & \cdots & x_d^m \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \mu'_1 \\ \mu'_2 \\ \vdots \\ \mu'_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.4}$$

From Equation 2.4, we obtain the following.

$$\sum_{i=1}^{m-1} \mu'_i p_i = -\mu'_m p_m$$

$$\sum_{i=1}^{m-1} \mu'_i = -\mu'_m$$

Rearranging the above equations, we obtain the following.

$$\sum_{i=1}^{m-1} \frac{\mu'_i}{\sum_{i=1}^{m-1} \mu'_i} p_i = p_m$$

$$\sum_{i=1}^{m-1} \frac{\mu'_i}{\sum_{i=1}^{m-1} \mu'_i} = 1$$

It shows that p_m can be expressed as the convex combination of $P \setminus \{p_m\}$, leading to a contradiction.

($\Leftarrow$) Let us assume that the points in P are not in convex position. Without loss of generality, we assume that p_m can be expressed as the convex combination of the points in $P \setminus \{p_m\} = \{p_1, p_2, \ldots p_{m-1}\}$. This implies that there exist real numbers $\lambda_k \geq 0$, $1 \leq k \leq m-1$, satisfying the following equations:

$$\sum_{i=1}^{m-1} \lambda_i p_i = p_m$$

$$\sum_{i=1}^{m-1} \lambda_i = 1$$

Therefore, we obtain the following equation.

$$\begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^m \\ x_2^1 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & \vdots \\ x_d^1 & x_d^2 & \cdots & x_d^m \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{m-1} \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.5}$$

It is evident from Equation 2.5 that the vector $(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, -1)$ lies in the null space of $M(P)$. This implies that $(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, -1) = \alpha_1(b_1^1, b_2^1, \ldots, b_m^1) + \alpha_2(b_1^2, b_2^2, \ldots, b_m^2)$ $+ \ldots + \alpha_{m-d-1}$ $(b_1^{m-d-1}, b_2^{m-d-1}, \ldots, b_m^{m-d-1})$ for some real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{m-d-1}$, not all of them zero. In other words, there exist α_1, α_2, $\ldots$, α_{m-d-1}, not all of them zero, such that $\alpha_1 b_i^1 + \alpha_2 b_i^2 + \ldots + \alpha_{m-d-1}$ $b_i^{m-d-1} \geq 0$ for $i = 1, 2, \ldots, m-1$, and $\alpha_1 b_m^1 + \alpha_2 b_m^2 + \ldots + \alpha_{m-d-1} b_m^{m-d-1} < 0$. This shows that the hyperplane $\sum\limits_{i=1}^{m-d-1} \alpha_i x_i = 0$ ensures that exactly one vector from $D(P)$ lies in an open half-space created by it, leading to a contradiction. $\square$

Let us state the definition of an *acyclic vector configuration.*

Acyclic Vector Configuration. *[58] A vector configuration* $V' = \{v_1', v_2', \ldots, v_n'\} \subset \mathbb{R}^d$ *is said to be acyclic if there does not exist any non-zero vector* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ *in* $\mathbb{R}^n$ *such that* $\alpha_i \geq 0$ *for each* i *satisfying* $1 \leq i \leq n$ *and* $\sum_{i=1}^{n} \alpha_i v_i' = \vec{0}$.

In Lemma 2, we mention a property of an acyclic vector configuration that is used to prove Property 5 and Property 6 of the Gale transformation. In order to prove Lemma 2, we mention the following version of the Farkas' Lemma. Recall that we denote vectors as row vectors in this thesis.

Farkas' Lemma. *[58] Let A be a $(d+1) \times n$ matrix with each element from $\mathbb{R}$, and let z be a vector in $\mathbb{R}^{d+1}$. Either there exists a vector $x \in \mathbb{R}^n$ with $Ax^T = z^T$, $x \geq \vec{0}$, or there exists a vector $c \in \mathbb{R}^{d+1}$ with $cA \geq \vec{0}$ and $cz^T < 0$, but not both.*

Lemma 2. *[58] $V' = \{v'_1, v'_2, \ldots, v'_n\} \subset \mathbb{R}^d$ is an acyclic vector configuration if and only if there exists a hyperplane h passing through the origin such that all the vectors in V' are contained in one of the open half-spaces created by h.*

Proof. ($\Leftarrow$) Let the i^{th} vector $v'_i \in V'$ be represented as $(v'_{i1}, v'_{i2}, \ldots, v'_{id})$. Let us consider the $(d+1) \times n$ matrix $M(V')$ whose i^{th} column is $\begin{bmatrix} v'_{i1} v'_{i2} \ldots v'_{id} 1 \end{bmatrix}^T$.

$$M(V') = \begin{bmatrix} v'_{11} & v'_{21} & \cdots & v'_{n1} \\ v'_{12} & v'_{22} & \cdots & v'_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ v'_{1d} & v'_{2d} & \cdots & v'_{nd} \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

Suppose that the vector configuration $V' = \{v'_1, v'_2, \ldots, v'_n\} \subset \mathbb{R}^d$ is not acyclic. Since the vector configuration $V' = \{v'_1, v'_2, \ldots, v'_n\} \subset \mathbb{R}^d$ is not acyclic, there exists a non-zero vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\mathbb{R}^n$ such that $\alpha_i \geq 0$ for each i satisfying $1 \leq i \leq n$ and $\sum_{i=1}^{n} \alpha_i v'_i = \vec{0}$. For $1 \leq i \leq n$, let us define $\mu_i = \dfrac{\alpha_i}{\sum_{i=1}^{n} \alpha_i}$. Note that each $\mu_i \geq 0$ and $(\mu_1, \mu_2, \ldots, \mu_n)$ satisfies the following equation.

$$\begin{bmatrix} v'_{11} & v'_{21} & \cdots & v'_{n1} \\ v'_{12} & v'_{22} & \cdots & v'_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ v'_{1d} & v'_{2d} & \cdots & v'_{nd} \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2.6}$$

Let us denote the vector $(0, 0, \ldots, 0, 1) \in \mathbb{R}^{d+1}$ by z. The Farkas' lemma implies that there does not exist a vector $c = (c_1, c_2, \ldots, c_d, c_0) \in \mathbb{R}^{d+1}$ with $cM(V') \geq \vec{0}$ and $cz^T = c_0 < 0$. This implies that there does not exist a hyperplane h passing through the origin such that all the vectors in V' are contained in one of the open half-spaces created by h.

($\Rightarrow$) Let us assume that the vector configuration $V' = \{v'_1, v'_2, \ldots, v'_n\} \subset \mathbb{R}^d$ is acyclic. This implies that there does not exist a non-zero vector $(\mu_1, \mu_2, \ldots, \mu_n)$ which satisfies Equation 2.6. The Farkas' lemma implies that there exists a vector $c = (c_1, c_2, \ldots, c_d, c_0) \in \mathbb{R}^{d+1}$ with $cM(V') \geq \vec{0}$ and $cz^T = c_0 < 0$. This further implies that there exists a hyperplane h passing through the origin such that all the vectors in V' are contained in one of the open half-spaces created by h. $\square$

Two nonempty convex sets C and D in $\mathbb{R}^d$ are said to be *properly separated* if there exists a $(d-1)$-dimensional hyperplane h such that C and D lie in the opposite closed half-spaces determined by h, and C and D are not both contained in the hyperplane h [32].

Proper Separation Theorem. *[30, 32] Two nonempty convex sets C and D in $\mathbb{R}^d$ can be properly separated if and only if their relative interiors are disjoint.*

Let Q be a d-dimensional convex polytope.

Face of a Convex Polytope. *[30] A face of the convex polytope Q is defined as follows:*

- *Q itself is a face.*
- *Any subset of Q of the form $Q \cap h$ is a face of Q, where h is a $(d-1)$-dimensional hyperplane such that Q is contained in one of the closed half-spaces created by h.*

We consider the following property of $D(P)$.

Property 5. *[30] Let the points in P be in general, as well as in convex position in $\mathbb{R}^d$. Note that $Conv(P)$ is a d-dimensional polytope. A t-element ($t \leq d$) subset $P' = \{p_1, p_2, \ldots, p_t\} \subset P$ forms a $(t-1)$-dimensional face of $Conv(P)$ if and only if the relative interior of the convex hull of the points in $D(P) \setminus \{g_1, g_2, \ldots, g_t\}$ contains the origin.*

Proof. ($\Rightarrow$) Let us assume that the relative interior of $Conv(\{g_{t+1}, g_{t+2}, \ldots, g_m\})$ does not contain the origin. The Proper Separation theorem implies that there exists a hyperplane $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_{m-d-1} x_{m-d-1} = 0$ such that the points $g_{t+1}, g_{t+2}, \ldots, g_m$ lie in the same closed half-space created by the hyperplane and not all of these points lie on the hyperplane. This implies that there exists a vector $(\mu_1,\ \mu_2,\ \ldots,\ \mu_t,\ \mu_{t+1} \geq 0,\ \mu_{t+2} \geq 0,\ \ldots, \mu_m \geq 0)$ $= \alpha_1(b_1^1, b_2^1,\ \ldots, b_m^1) + \alpha_2(b_1^2,\ b_2^2,\ \ldots, b_m^2) + \ldots + \alpha_{m-d-1}\ (b_1^{m-d-1},\ b_2^{m-d-1},\ \ldots,\ b_m^{m-d-1})$ for some real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{m-d-1}$, not all of them zero. Since the vector $(\mu_1,\ \mu_2,\ \ldots,\ \mu_t,$ $\mu_{t+1} \geq 0,\ \mu_{t+2} \geq 0,\ \ldots, \mu_m \geq 0)$ lies in the null space of the row space of $M(P)$, it satisfies the following equation.

$$\begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^m \\ x_2^1 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & \vdots \\ x_d^1 & x_d^2 & \cdots & x_d^m \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_t \\ \mu_{t+1} \\ \vdots \\ \mu_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.7}$$

From Equation 2.7, we obtain the following.

$$-\sum_{i=1}^{t} \mu_i p_i = \sum_{j=t+1}^{m} \mu_j p_j$$

$$-\sum_{i=1}^{t} \mu_i = \sum_{j=t+1}^{m} \mu_j$$

Also, note that $\mu_j \geq 0$ for each j satisfying $t+1 \leq j \leq m$.

Rearranging the above equations, we obtain the following.

$$\sum_{i=1}^{t} \frac{\mu_i}{\sum\limits_{i=1}^{t} \mu_i} p_i = \sum_{j=t+1}^{m} \frac{\mu_j}{\sum\limits_{j=t+1}^{m} \mu_j} p_j$$

This implies that $Conv(\{p_{t+1}, p_{t+2}, \ldots, p_m\}) \cap Aff(\{p_1, p_2, \ldots, p_t\}) \neq \emptyset$. This further implies that $\{p_1, p_2, \ldots, p_t\}$ does not form a $(t-1)$-dimensional face of P, leading to a contradiction.

($\Leftarrow$) Let us assume that $\{p_1, p_2, \ldots, p_t\}$ does not form a $(t-1)$-dimensional face of P. This implies that $Conv(\{p_{t+1}, p_{t+2}, \ldots, p_m\}) \cap Aff(\{p_1, p_2, \ldots, p_t\}) \neq \emptyset$. This implies that there exist real numbers λ_i, $1 \leq i \leq m$, satisfying the following equations:

$$\sum_{i=1}^{t} \lambda_i p_i = \sum_{j=t+1}^{m} \lambda_j p_j$$

$$\sum_{i=1}^{t} \lambda_i = \sum_{j=t+1}^{m} \lambda_j = 1$$

$$\lambda_j \geq 0 \qquad for\ each\ t+1 \leq j \leq m$$

Therefore, we obtain the following equation.

$$\begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^m \\ x_2^1 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & \vdots \\ x_d^1 & x_d^2 & \cdots & x_d^m \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} -\lambda_1 \\ -\lambda_2 \\ \vdots \\ -\lambda_t \\ \lambda_{t+1} \\ \vdots \\ \lambda_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.8}$$

It is evident from Equation 2.8 that the vector $(-\lambda_1, -\lambda_2, \ldots, -\lambda_t, \lambda_{t+1}, \ldots, \lambda_m)$ lies in the null space of the row space of $M(P)$. This implies that $(-\lambda_1,\ -\lambda_2,\ \ldots,\ -\lambda_t,\ \lambda_{t+1},\ \ldots, \lambda_m) = \alpha_1(b_1^1, b_2^1, \ldots, b_m^1) + \alpha_2(b_1^2, b_2^2, \ldots, b_m^2) + \ldots + \alpha_{m-d-1}\,(b_1^{m-d-1}, b_2^{m-d-1}, \ldots, b_m^{m-d-1})$, for some real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{m-d-1}$, not all of them zero. In other words, there exist $\alpha_1,\ \alpha_2,\ \ldots, \alpha_{m-d-1}$, not all of them zero, such that $\alpha_1 b_j^1 + \alpha_2 b_j^2 + \ldots + \alpha_{m-d-1} b_j^{m-d-1} \geq 0$ for $j = t+1, t+2, \ldots, m$. This implies that the points $g_{t+1}, g_{t+2}, \ldots, g_m$ lie in the same closed half-space created by the hyperplane $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_{m-d-1} x_{m-d-1} = 0$. Since the points in P are in general position in $\mathbb{R}^d$, all the points in $\{g_{t+1}, g_{t+2}, \ldots, g_m\}$ can not lie on the hyperplane. The Proper Separation theorem implies that the relative interior of the convex hull of the point set $\{g_{t+1}, g_{t+2}, \ldots, g_m\}$ does not contain the origin, leading to a contradiction. $\square$

Property 6. *[30] Let the points in P be in general, as well as in convex position in $\mathbb{R}^d$. A d-dimensional polytope formed by the convex hull of P is t-neighborly ($2 \leq t \leq \lfloor d/2 \rfloor$) if and only if each of the linear separations of $D(P)$ contains at least $t+1$ points in each of the open half-spaces created by the corresponding linear hyperplanes.*

Proof. ($\Rightarrow$) Note that P is the vertex set of a t-neighborly d-dimensional polytope having m vertices. Consider a Gale transform $D(P)$ of P. Without loss of generality, assume for the sake of contradiction that there is a linear separation of $D(P)$ into two sets of size $m-t$ and t. Property 3 of the Gale transformation implies that the convex hull of some t points of P crosses with the convex hull of the remaining $m-t$ points. This is a contradiction to the fact that every set of t vertices of P forms a face of $Conv(P)$.

($\Leftarrow$) Let us assume that $Conv(P)$ is not a t-neighborly d-dimensional polytope. Without loss of generality, we assume that the first t points $p_1, p_2, \ldots, p_t$ do not span a face of $Conv(P)$. Property 5 implies that the relative interior of $Conv(\{g_{t+1}, g_{t+2}, \ldots, g_m\})$ does not contain the origin. Since the vectors in $D(P)$ are in general position in $\mathbb{R}^{m-d-1}$, this implies that there does not exist a non-zero vector $(\alpha_1, \alpha_2, \ldots, \alpha_{m-t})$ such that $\alpha_i \geq 0$ for each i satisfying $1 \leq i \leq m-t$ and $\sum_{i=1}^{m-t} \alpha_i g_{t+i} = \vec{0}$. This implies that $\{g_{t+1}, g_{t+2}, \ldots, g_m\}$ forms an acyclic vector configuration in $\mathbb{R}^{m-d-1}$. Lemma 2 implies that there exists a hyperplane h passing through the origin such that all the vectors in $\{g_{t+1}, g_{t+2}, \ldots, g_m\}$ are contained in one of the open half-spaces created by h. This implies that the other open half-space created by h contains at most t points of $D(P)$, leading to a contradiction. $\square$

We obtain an *affine Gale diagram* [43] of P by considering a hyperplane $\bar{h}$ that is not parallel to any vector in $D(P)$ and is not passing through the origin. For each $1 \leq i \leq m$, we extend the vector $g_i \in D(P)$ either in the direction of g_i or in its opposite direction until it cuts $\bar{h}$ at the point $\overline{g_i}$. We color $\overline{g_i}$ as *white* if the projection is in the direction of g_i, and *black* otherwise. The sequence of m points $\overline{D(P)} = <\overline{g_1}, \overline{g_2}, \ldots, \overline{g_m}>$ in $\mathbb{R}^{m-d-2}$ along with the color of each point is defined as an affine Gale diagram of P.

We define a *separation* of $\overline{D(P)}$ to be a partition of $\overline{D(P)}$ into two disjoint sets of points $\overline{D^+(P)}$ and $\overline{D^-(P)}$ contained in the opposite open half-spaces created by a hyperplane. We restate Property 2 using these definitions and notations.

Property 7. *[43] Consider two integers $1 \leq u, v \leq d-1$ satisfying $u + v + 2 = m$. If the points in P are in general position in $\mathbb{R}^d$, there exists a bijection between the crossing pairs of u- and v-simplices formed by some points in P and the partitions of the points in $\overline{D(P)}$ into $\overline{D^+(P)}$ and $\overline{D^-(P)}$ such that the number of white points in $\overline{D^+(P)}$ plus the number of black points in $\overline{D^-(P)}$ is $u+1$ and the number of white points in $\overline{D^-(P)}$ plus the number of black points in $\overline{D^+(P)}$ is $v+1$.*

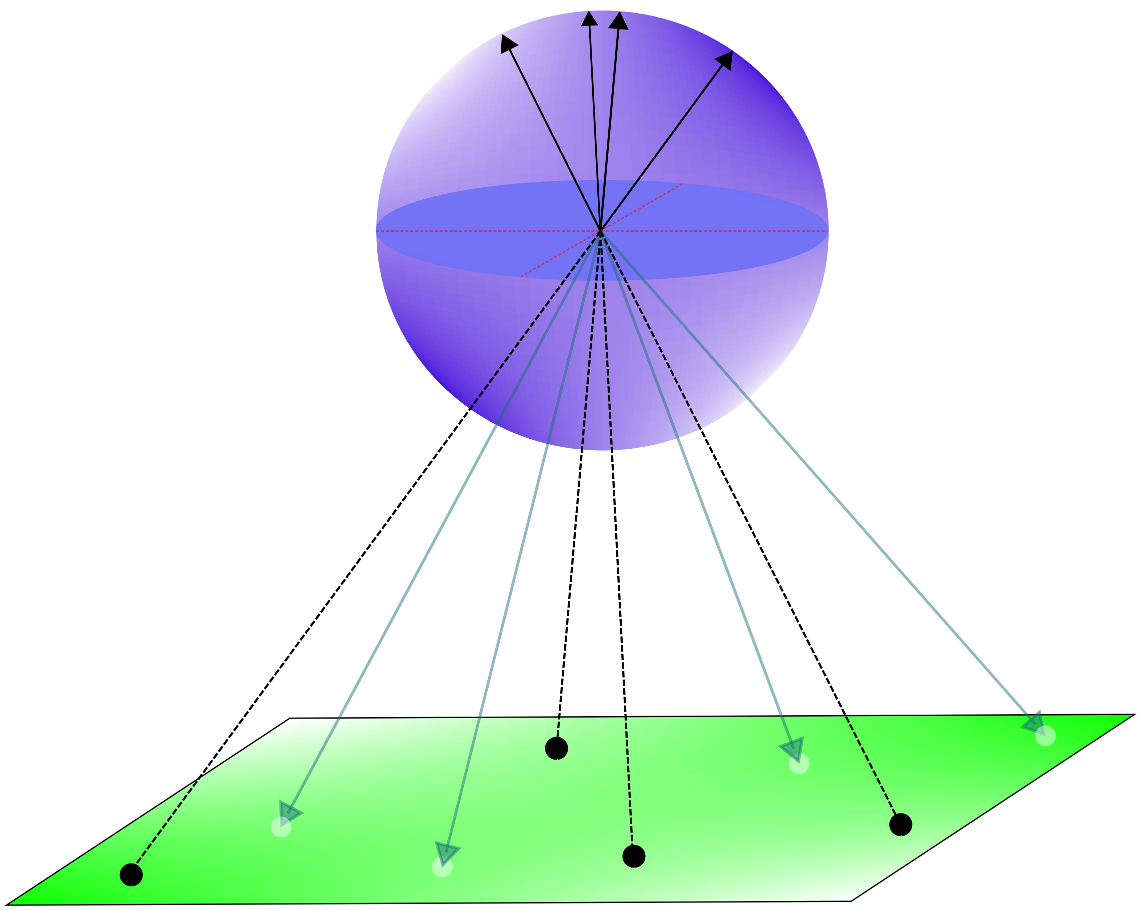

Figure 2.1: An affine Gale diagram of 8 points in $\mathbb{R}^4$

Chapter 3

Balanced Lines, j-Facets and k-Sets

In this chapter, we state the definitions and discuss the properties of j-facets and k-sets of a finite set of points in $\mathbb{R}^2$ and $\mathbb{R}^3$. The concepts of j-edges and k-sets of planar point sets were first studied by Lovász [42] and Erdős et al. [22]. They have been extensively studied in discrete geometry since then. We also state the definition of a balanced line and discuss some of its properties in this chapter.

3.1 j-Edges and k-Sets in $\mathbb{R}^2$

Consider a set S containing s points in general position in $\mathbb{R}^2$.

j-Edge. *[43] A j-edge of S is an directed line spanned by* 2 *points of S such that exactly j points of S lie in the left open half-space created by it.*

k-Set. *[43] A k-set of S is a subset of S of size k that can be separated from the rest of the points by a line that does not pass through any of the points in S.*

We observe the following on the relation between the number of $(k-1)$-edges and k-sets of S. Let us denote the number of k-sets of S by $e'_k(S)$. We also denote the number of $(k-1)$-edges of S by $E'_{k-1}(S)$. We use this observation in the proof of Lemma 11.

Observation 1. *[56] For each k satisfying $1 \leq k \leq s-1$, $e'_k(S) = E'_{k-1}(S)$.*

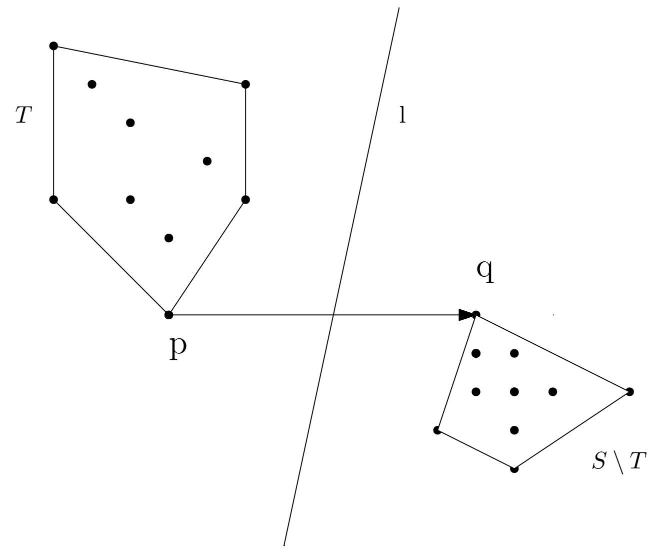

Figure 3.1: k-Set and $(k-1)$-Edge

Proof. Let T be a k-set of S which can be separated from $S \setminus T$ by the line l as shown in Figure 3.1. It is easy to observe that there exists a unique ordered pair of points (p, q) where $p \in T$ and $q \in S \setminus T$ such that $T \setminus \{p\}$ lies completely in the left open half-space created by the directed line $\overrightarrow{pq}$ and $(S \setminus T) \setminus \{q\}$ is completely contained in the other open half-space created by the directed line $\overrightarrow{pq}$. This implies that the directed line $\overrightarrow{pq}$ is a $(k-1)$-edge of S, since there exist $k-1$ points in its left open half-space. The directed line $\overrightarrow{pq}$ is a $(k-1)$-edge of S corresponding to this particular k-set.

On the other hand, let the directed line $\overrightarrow{pq}$ be a $(k-1)$-edge of S as shown in Figure 3.1. If we rotate the directed line $\overrightarrow{pq}$ counter-clockwise around the midpoint of the line segment $[p, q]$, we obtain a line l which does not pass through any of the points in S and k points (i.e., all the points in $\{p\} \cup (T \setminus \{p\})$) are contained in one of its open half-spaces. This implies that T is a k-set of S.

The above argument establishes a bijection between $(k-1)$-edges and k-sets of S. This implies that $e'_k(S) = E'_{k-1}(S)$ for each k satisfying $1 \leq k \leq s-1$. $\square$

3.2 Balanced Lines in $\mathbb{R}^2$

We now introduce the concept of a balanced line. Consider a set R containing r points in general position in $\mathbb{R}^2$, such that $\lceil r/2 \rceil$ points are colored white and $\lfloor r/2 \rfloor$ points are colored black. Let us state the definitions of a *balanced line* and an *almost balanced directed line* of R, and discuss their properties that are used in the proof of Theorem 2.

Balanced Line. *[47] A balanced line l of R is a straight line that passes through a white and a black point in R and the number of black points is equal to the number of white points in each of the open half-spaces created by l.*

Note that a balanced line exists only when r is even. The following lemma gives a non-trivial lower bound on the number of balanced lines of R.

Lemma 3. *[47] When r is even, the number of balanced lines of R is at least $r/2$.*

We extend the definition of a balanced line to define an *almost balanced directed line* of R.

Almost Balanced Directed Line. *When r is even, an almost balanced directed line l of R is a balanced line with direction assigned from the black point to the white point it passes through. When r is odd, an almost balanced directed line l of R is a directed straight line that passes through a white and a black point in R such that the number of black points is equal to the number of white points in the left open half-space created by l.*

The following observation follows from Lemma 3.

Observation 2. *The number of almost balanced directed lines of R is at least $\lfloor r/2 \rfloor$.*

Proof. As already mentioned, an almost balanced directed line is a balanced line if r is even. When r is even, Lemma 3 therefore implies that there exist $\lfloor r/2 \rfloor$ balanced lines.

Let us assume that r is odd. As mentioned earlier, we assume that R contains r points in general position in $\mathbb{R}^2$ such that $\lceil r/2 \rceil$ points are colored white and $\lfloor r/2 \rfloor$ points are colored black. We remove one white point from R. Let us denote this new set by R'. Lemma 3 implies that the number of balanced lines of R' is at least $\lfloor r/2 \rfloor$. Note that the removed point can not lie on any of these balanced lines, since the points in R are in general position in $\mathbb{R}^2$. Consider a balanced line of R'. The removed point must lie in one of the open half-spaces created by it. Note that we can assign a direction to each balanced line of R' such that the removed point lies in the right open half-space created by each of the balanced lines of R'. This therefore implies that the number of almost balanced directed lines of R is at least $\lfloor r/2 \rfloor$. $\square$

3.3 j-Facets and k-Sets in $\mathbb{R}^3$

Let us introduce the concepts of j-facets and k-sets of a set of points in general position in $\mathbb{R}^3$. Consider a set S containing s points in general position in $\mathbb{R}^3$. Let us first state the definitions of a *j-facet* and an *$(\leq j)$-facet* of S for some integer $j \geq 0$. We then state the definitions of a *k-set* and an *$(\leq k)$-set* of S for some integer $k \geq 1$, and discuss their properties that are used in the proofs of Theorems 1, 5 and 6.

j-Facet. *[7] A j-facet of S is an oriented 2-dimensional hyperplane spanned by 3 points of S such that exactly j points of S lie in the positive open half-space created by it.*

Let us denote the number of j-facets of S by $E_j(S)$.

$(\leq j)$-Facet. *[7] An $(\leq j)$-facet of S is an oriented 2-dimensional hyperplane h spanned by 3 points of S such that at most j points of S lie in the positive open half-space created by it.*

Almost Halving Triangle. *An almost halving triangle of S is a j-facet of S such that $|j - (s - j - 3)|$ is at most one.*

When s is odd, note that an almost halving triangle is a *halving triangle* containing an equal number of points in each of the open half-spaces created by it. The following lemma gives a non-trivial lower bound on the number of halving triangles of S. In fact, it is shown in [51] that this lemma is equivalent to Lemma 3.

Lemma 4. *[51] When s is odd, the number of halving triangles of S is at least $\lfloor s/2 \rfloor^2$.*

The following observation follows from Lemma 4.

Observation 3. *The number of almost halving triangles of S is at least $\lfloor (s-1)/2 \rfloor^2$.*

Proof. Consider a set S containing s points in general position in $\mathbb{R}^3$. As already mentioned, an almost halving triangle is a halving triangle if S is odd. When s is odd, Lemma 4 implies that there exist $\lfloor s/2 \rfloor^2$ halving triangles.

Let us assume that S is even. We remove one point from S. Let us denote this new set by S'. Lemma 4 implies that the number of halving triangles of S' is at least $\lfloor (s-1)/2 \rfloor^2$. Note that the removed vertex can not lie on any of the hyperplanes which created those $\lfloor (s-1)/2 \rfloor^2$ halving triangles of S' since the points in S are in general position in $\mathbb{R}^3$. Consider a halving triangle of S'. The removed vertex must lie in one of the open half-spaces created by the hyperplane corresponding to the halving triangle. This implies that each halving triangle of S' corresponds to a unique almost halving triangle of S. This further implies that the number of almost halving triangles of S is at least $\lfloor (s-1)/2 \rfloor^2$. □

We consider the following lemma which gives a non-trivial lower bound on the number of $(\leq j)$-facets of S.

Lemma 5. *[3] For $j < s/4$, the number of $(\leq j)$-facets of S is at least $4\binom{j+3}{3}$.*

k-Set. *[43] A k-set of S is a non-empty subset of S having size k that can be separated from the rest of the points by a 2-dimensional hyperplane that does not pass through any of the points of S.*

Let us denote the number of k-sets of S by $e_k(S)$.

$(\leq k)$-Set. *[7] A subset $T \subseteq S$ is called an $(\leq k)$-set if $1 \leq |T| \leq k$ and T can be separated from $S \setminus T$ by a 2-dimensional hyperplane that does not pass through any of the points of S.*

Andrzejak et al. [7] gave the following lemma regarding the number of the j-facets and the k-sets of S.

Lemma 6. *[7] $e_1(S) = (E_0(S)/2) + 2$, $e_{s-1}(S) = (E_{s-3}(S)/2) + 2$, and $e_k(S) = (E_{k-1}(S) + E_{k-2}(S))/2 + 2$ for each k in the range $2 \leq k \leq s-2$.*

The following observation follows from Observation 3 and Lemma 6.

Observation 4. *There exist a total of $\Omega(s^2)$ k-sets of S such that $min\{k, s-k\}$ is at least $\lceil (s-1)/2 \rceil$.*

Proof. Consider a set S containing s points in general position in $\mathbb{R}^3$. Let us assume that s is even. Lemma 6 implies that $e_{\lceil (s-1)/2 \rceil}(S) = (E_{\lfloor (s-1)/2 \rfloor}(S) + E_{\lfloor (s-1)/2 \rfloor - 1}(S))/2 + 2$. For even s, a $(\lfloor (s-1)/2 \rfloor - 1)$-facet is an almost halving triangle. Observation 3 implies that the number of almost halving triangles of S is $\Omega(s^2)$. This implies that $e_{\lceil (s-1)/2 \rceil}(S) > E_{\lfloor (s-1)/2 \rfloor}(S)/2 = \Omega(s^2)$.

Let us now assume that s is odd. Lemma 6 implies that $e_{(s-1)/2}(S) = (E_{(s-3)/2}(S) + E_{(s-5)/2}(S))/2 + 2$. For odd s, a $((s-3)/2)$-facet is a halving triangle. Observation 3 implies that the number of halving triangles of S is $\Omega(s^2)$. This implies that $e_{(s-1)/2}(S) > E_{(s-3)/2}(S)/2 = \Omega(s^2)$. $\square$

The following observation follows from Lemma 5 and Lemma 6.

Observation 5. *The number of* $(\leq \lceil s/4 \rceil)$*-sets of* S *is* $\Omega(s^3)$.

Proof. Lemma 6 implies that the number of $(\leq \lceil s/4 \rceil)$-sets of S is at least $\sum_{i=0}^{\lceil s/4 \rceil - 1} E_i(S)/2$. This further implies that $(\leq \lceil s/4 \rceil)$-sets of S is at least $\sum_{i=0}^{\lfloor s/4 \rfloor - 1} E_i(S)/2$. Lemma 5 implies that $\sum_{i=0}^{\lfloor s/4 \rfloor - 1} E_i(S)/2$ is at least $4\binom{\lfloor s/4 \rfloor + 2}{3} = \Omega(s^3)$. $\square$

Chapter 4

Rectilinear Crossing Number of Complete d-Uniform Hypergraphs

As mentioned in the introduction, we present the proofs of Theorem 1 and Theorem 2 in this chapter. Let us recall that a d-dimensional rectilinear drawing of K_{2d}^{d} is a drawing of it in $\mathbb{R}^d$ such that all its $2d$ vertices are placed as points in general position and each of the $\binom{2d}{d}$ hyperedges is drawn as the convex hull of d corresponding vertices. In such a drawing of K_{2d}^{d}, two hyperedges are said to be crossing if they are vertex disjoint and contain a common point in their relative interiors. Recall that the d-dimensional rectilinear crossing number of K_{2d}^{d}, denoted by $cr_d\left(K_{2d}^{d}\right)$, is defined as the minimum number of crossing pairs of hyperedges among all d-dimensional rectilinear drawings of it.

4.1 Motivation and Previous Works

As mentioned earlier, Anshu et al. [8] proved the first non-trivial lower bound of $\Omega\left(\frac{2^d \log d}{\sqrt{d}}\right)$ on $\overline{cr}_d(K_{2d}^{d})$. They used the Gale transformation to reduce the crossing number problem to a linear separation problem. For a given set of $d+4$ points in $\mathbb{R}^d$, Property 2 of the Gale transformation ensures that there exists a bijection between the crossing pairs of $\left\lfloor \frac{d+4}{2} \right\rfloor$-and $\left\lceil \frac{d+4}{2} \right\rceil$-simplices in $\mathbb{R}^d$ and the proper linear separations of $d+4$ vectors in $\mathbb{R}^3$ [43].

In order to calculate the lower bound on c_d, Anshu et al. [8] chose a set of $d+4$ vertices from the set of $2d$ vertices of K^d_{2d} in $\mathbb{R}^d$. A Gale transform of these $d+4$ vertices is a set of $d+4$ vectors in general position in $\mathbb{R}^3$. Using the Ham-Sandwich theorem, Anshu et al. [8] proved the existence of $\Theta(\log d)$ distinct proper linear separations of the set of $d+4$ vectors mentioned above. Each proper linear separation of $d+4$ vectors in $\mathbb{R}^3$ corresponds to a crossing between $\left\lfloor \frac{d+4}{2} \right\rfloor$ and $\left\lceil \frac{d+4}{2} \right\rceil$ simplices in $\mathbb{R}^d$. They extended the crossings between the lower dimensional simplices to the crossings between $(d-1)$-simplices to get the bound $\overline{cr}_d(K^d_{2d}) = \Omega\left(\frac{2^d \log d}{\sqrt{d}}\right)$. In particular, they showed that $\overline{cr}_4(K^4_8) \geq 4$. They also constructed an arrangement of 8 vertices of K^4_8 in $\mathbb{R}^4$ having 4 crossing pairs of hyperedges. This arrangement established that $\overline{cr}_4(K^4_8) = 4$. In this section, we reproduce the proofs in detail.

Lemma 7. *[8] The 4-dimensional rectilinear crossing number of a complete 4-uniform hypergraph with 8 vertices is 4, i.e.,* $\overline{cr}_4\,(K^4_8) = 4$.

Proof. In a 4-dimensional rectilinear drawing of K^4_8, its vertices are represented as points in general position in $\mathbb{R}^4$. Let $P = \{p_1, p_2, \ldots, p_8\}$ be a set of 8 points in general position in $\mathbb{R}^4$. Let $D(P) = \{g_1, g_2, \ldots, g_8\}$ a Gale transform of P, be a collection of 8 vectors in $\mathbb{R}^3$. As already mentioned, the vectors in $D(P)$ can be treated as points in $\mathbb{R}^3$. We use the Ham-Sandwich theorem to obtain the proper linear separations of $D(P)$. As the points in $D(P)$ are in $\mathbb{R}^3$, we use three colors, namely, c_0, c_1 and c_2. The coloring argument proceeds as follows.

We color the origin with c_0 and all the 8 points of $D(P)$ with color c_1. The Ham-Sandwich theorem guarantees that any separating hyperplane passes through the origin. Property 1 of the Gale transformation guarantees that at most 2 vectors in $D(P)$ can lie on the separating hyperplane. It is easy to observe that we can rotate the separating hyperplane to obtain a proper linear separation of the vectors in $D(P)$. Let us assume without loss of generality that the proper linear separation obtained in this way is $\{\{g_1, g_2, g_3, g_4\}, \{g_5, g_6, g_7, g_8\}\}$.

After obtaining the first proper linear separation, we color the points in $\{g_1,\ g_2, g_3, g_4\}$ with c_1 and the points in $\{g_5, g_6, g_7, g_8\}$ with c_2. The origin is colored with c_0. We apply the Ham-Sandwich theorem to obtain a separating hyperplane passing through the origin. Since at most 2 vectors in $D(P)$ can lie on the separating hyperplane, we obtain a new proper linear separation of $D(P)$ by rotating the separating hyperplane, if necessary. Without loss of generality, let us assume that the new partition of $D(P)$ is $\{\{g_1, g_2, g_5, g_6\}, \{g_3, g_4, g_7, g_8\}\}$. Note that the pairs of points $\{g_1, g_2\}$,$\{g_3, g_4\}$,$\{g_5, g_6\}$ and $\{g_7, g_8\}$ remained together in both the partitions obtained previously.

We color $\{g_1, g_2\}$ with color c_1 and rest of the six points with color c_2. The origin is colored with c_0. We again obtain a new proper linear separation as g_1 and g_2 get separated. The proper linear separation obtained in this way can be of two types: (i) all the four pairs of points, i.e., $\{g_1, g_2\}$, $\{g_3, g_4\}$, $\{g_5, g_6\}$ and $\{g_7, g_8\}$ get separated, (ii) two of the three pairs of points, i.e., $\{g_3, g_4\}$,$\{g_5, g_6\}$ and $\{g_7, g_8\}$ remain together.

In Case (i), let us assume without loss of generality that the proper linear separation obtained is $\{\{g_1, g_3, g_5, g_7\}, \{g_2, g_4, g_6, g_8\}\}$. Note that three out of four points in $\{g_1, g_2, g_3, g_5\}$ have remained together in all the three partitions obtained till now. In this case, we color $\{g_1, g_2, g_3, g_5\}$ with c_1, the rest of the four points with color c_2 and the origin with c_0 to obtain a new proper linear separation of $D(P)$. In Case (ii), we color one of the two unseparated pairs with color c_1 and the rest with color c_2. We also color the origin with color c_0 to obtain a new proper linear separation of $D(P)$.

Since Property 2 implies that each of these four proper linear separations of vectors in $D(P)$ corresponds to a unique crossing pair of 3-simplices, the above argument shows that $\overline{cr}_4\left(K_8^4\right) \geq 4$. Anshu et al. [8] created a particular 4-dimensional rectilinear drawing of K_8^4 having 4 crossing pairs of hyperedges. In the following, we mention the placement of 8 points in general position in $\mathbb{R}^4$ to obtain the above mentioned 4-dimensional rectilinear drawing of K_8^4 with 4 crossing pairs of hyperedges. The coordinates of the points are listed in the following table [8].

Point	Coordinate
p_1	$(1,0,0,0)$
p_2	$(-1/4,1,0,0)$
p_3	$(-1/4,-1/3,1,0)$
p_4	$(-1/4,-1/3,-1/2,4/5)$
p_5	$(-1/4,-1/3,-1/2,-4/5)$
p_6	$(-1/28,1/16,0,3/40)$
p_7	$(-1/5,1/100,3/200,1/10)$
p_8	$(-6/25,0,-1/20,1/20)$

$\square$

Lemma 8. *[8] The d-dimensional rectilinear crossing number of a complete d-uniform hypergraph with 2d vertices is* $\Omega\left(\frac{2^d \log d}{\sqrt{d}}\right)$, *i.e.,* $\overline{cr}_d\left(K_{2d}^d\right) = \Omega\left(\frac{2^d \log d}{\sqrt{d}}\right)$.

Proof. In a d-dimensional rectilinear drawing of K_{2d}^d its vertices are represented as points in general position in $\mathbb{R}^d$. Let $P' = \{p_1, p_2, \ldots, p_{2d}\}$ be a set of $2d$ points in general position in $\mathbb{R}^d$. Let us consider a subset containing $d+4$ points of P'. Without loss of generality, let the subset be $P = \{p_1, p_2, \ldots, p_{d+4}\} \subset P'$. Let $D(P) = \{g_1, g_2, \ldots, g_{d+4}\}$, a collection of $d+4$ vectors in $\mathbb{R}^3$, be a Gale transform of P. As already mentioned, the vectors in $D(P)$ can be treated as points in $\mathbb{R}^3$. We use the Ham-Sandwich theorem to obtain the proper linear separations of $D(P)$. As the points in $D(P)$ are in $\mathbb{R}^3$, we use three colors, namely, c_0, c_1 and c_2. The coloring argument proceeds as follows.

We color the origin with c_0 and all the points in $D(P)$ with c_1. The color of the origin remains unchanged throughout the process. By the Ham-Sandwich theorem and rotating the separating hyperplane if needed (as mentioned before), we obtain a proper linear separation of $D(P)$ into $D_{11}(P)$ and $D_{12}(P)$ having $\lfloor (d+4)/2 \rfloor$ and $\lceil (d+4)/2 \rceil$ vectors, respectively.

We then color all the vectors in $D_{11}(P)$ with c_1 and all the vectors in D_{12} with c_2. The Ham-Sandwich theorem guarantees that we obtain a partition $D_{21}(P)$ and $D_{22}(P)$ of $D(P)$. Note that at least $\lfloor (d+4)/4 \rfloor$ points of $D(P)$ have stayed together in both the partitions.

Next, we color these $\lfloor (d+4)/4 \rfloor$ points of $D(P)$ with c_1 and rest of the points with c_2 to obtain a new proper linear separation of $D(P)$ into $D_{31}(P)$ and $D_{32}(P)$. Note that

$\lfloor (d+4)/8 \rfloor$ points of $D(P)$ have stayed together in all three partitions obtained till now.

In particular, in the k^{th} step, we obtain a proper linear separation of $D(P)$ into $D_{k1}(P)$ and $D_{k2}(P)$. Note that the k^{th} proper linear separation of $D(P)$ is distinct from all the $k-1$ proper linear separations obtained before. It is easy to observe that $\lfloor (d+4)/2^k \rfloor$ points of $D(P)$ have stayed together in all the k proper linear separations obtained so far.

We then color these $\lfloor (d+4)/2^k \rfloor$ points of $D(P)$ with c_1 and rest of the points with c_2 to obtain the $(k+1)^{th}$ proper linear separation of $D(P)$. We keep on coloring in this way until only a pair of points stays together. This implies we can keep on coloring for $\Theta(\log d)$ times without repeating any of the previous proper linear separations of $D(P)$. Property 2 implies that each of these $\Theta(\log d)$ proper linear separations of vectors in $D(P)$ corresponds to a unique pair of crossing $(\lfloor (d+4)/2 \rfloor - 1)$-simplex and $(\lceil (d+4)/2 \rceil - 1)$-simplex. Any such crossing pair of simplices can be extended to a crossing pair of $(d-1)$-simplices in $\binom{d-4}{\lfloor (d-4)/2 \rfloor} = \Theta\left(\frac{2^d}{\sqrt{d}}\right)$ distinct ways. This implies that $\overline{cr}_d\left(K_{2d}^d\right) = \Omega\left(\frac{2^d \log d}{\sqrt{d}}\right)$. $\square$

4.2 Lower Bound by Gale Transformation and Ham-Sandwich Theorem

In this section, we improve the lower bound on c_d by using the Gale transformation and Corollary 1 mentioned below. Let P denote the set of $2d$ vertices of K_{2d}^d that are in general position in $\mathbb{R}^d$. Let us recall that two nonempty convex sets C and D in $\mathbb{R}^d$ are said to be properly separated if there exists a $(d-1)$-dimensional hyperplane h such that C and D lie in the opposite closed half-spaces determined by h, and C and D are not both contained in the hyperplane h [32]. Let us recall the proper separation theorem that is mentioned earlier in Chapter 2.

Proper Separation Theorem. *[32] Two nonempty convex sets C and D in $\mathbb{R}^d$ can be properly separated if and only if their relative interiors are disjoint.*

Lemma 9. *Consider a set A that contains at least $d+1$ points in general position in $\mathbb{R}^d$. Let B and C be its disjoint subsets such that $|B| = b$, $|C| = c$, $2 \leq b, c \leq d$ and $b + c \geq d + 1$. If the $(b-1)$-simplex formed by B and the $(c-1)$-simplex formed by C form a crossing pair,* *then the u-simplex ($u \geq b-1$) formed by a point set $B' \supseteq B$ and the v-simplex ($v \geq c-1$) formed by a point set $C' \supseteq C$ satisfying $B' \cap C' = \emptyset$, $|B'|, |C'| \leq d$ and $B', C' \subset A$ also form a crossing pair.*

Proof. For the sake of contradiction, we assume that there exist a u-simplex and a v-simplex, formed respectively by the disjoint point sets $B' \supseteq B$ and $C' \supseteq C$, that do not cross. We consider two cases.

Case 1. Let us assume that $Conv(B') \cap Conv(C') = \emptyset$. It clearly leads to a contradiction since $Conv(B) \cap Conv(C) \neq \emptyset$.

Case 2. Let us assume that $Conv(B') \cap Conv(C') \neq \emptyset$. Since the relative interiors of $Conv(B')$ and $Conv(C')$ are disjoint, the Proper Separation theorem implies that there exists a $(d-1)$-dimensional hyperplane h such that $Conv(B')$ and $Conv(C')$ lie in the opposite closed half-spaces determined by h. It implies that $Conv(B)$ and $Conv(C)$ also lie in the opposite closed half-spaces created by h. Since the relative interiors of $Conv(B)$ and $Conv(C)$ are not disjoint and they lie in the opposite closed halfspaces of h, it implies that all $b+c \geq d+1$ points in $B \cup C$ lie on h. This leads to a contradiction since the points in $B \cup C$ are in general position in $\mathbb{R}^d$. $\square$

Corollary 1. *Consider two disjoint point sets $U, V \subset P$ such that $|U| = p$, $|V| = q$, $2 \leq p, q \leq d$ and $p+q \geq d+1$. If the $(p-1)$-simplex formed by U crosses the $(q-1)$-simplex formed by V, then the $(d-1)$-simplices formed by any two disjoint point sets $U' \supseteq U$ and $V' \supseteq V$ satisfying $|U'| = |V'| = d$ also form a crossing pair.*

Since Corollary 1 is a special case of Lemma 9, its proof is immediate from Lemma 9.

Lemma 10. *The d-dimensional rectilinear crossing number of a complete d-uniform hypergraph with $2d$ vertices is $\Omega(2^d)$.*

Proof. Consider the hypergraph K_{2d}^d whose vertices are in general position in $\mathbb{R}^d$, and let A be any subset of $d+3$ vertices selected from these vertices. The Gale transform $D(A)$ of the point set A contains $d+3$ vectors in $\mathbb{R}^2$, which can also be considered as a sequence of $d+3$ points (as mentioned earlier). In order to apply the Ham-Sandwich Theorem (mentioned in the Introduction) in $\mathbb{R}^2$, we assign the points in $D(A)$ to P_1 and the origin to P_2 to obtain a line l passing through the origin that bisects the points in $D(A)$ such that each partition (open half-space) contains at most $\lfloor \frac{1}{2}|D(A)| \rfloor$ points from $D(A)$. Since the points in A are in general position, every pair of vectors in $D(A)$ spans $\mathbb{R}^2$. Hence, at most one point from $D(A)$ can lie on l. As a consequence, l can be rotated using the origin as the center of rotation to obtain a proper linear separation of $D(A)$ into 2 subsets l_1^+ and l_1^- of size $\lfloor \frac{d+3}{2} \rfloor$ and $\lceil \frac{d+3}{2} \rceil$, respectively, such that l_1^+ denotes the left (counter-clockwise) side and l_1^- denotes the right (clockwise) side of l. Property 2 implies that this proper linear separation corresponds to a crossing pair of a $(\lfloor \frac{d+3}{2} \rfloor - 1)$-simplex and a $(\lceil \frac{d+3}{2} \rceil - 1)$-simplex in $\mathbb{R}^d$. We observe from Corollary 1 that this crossing pair of simplices can be used to obtain $\binom{d-3}{\lfloor \frac{d-3}{2} \rfloor}$ distinct crossing pairs of $(d-1)$-simplices formed by the vertices of the hypergraph K_{2d}^d.

We rotate l clockwise using the origin as the center of rotation, until one of the $d+3$ points in $D(A)$ moves from one side of the line l to the other side. Since every pair of vectors in $D(A)$ spans $\mathbb{R}^2$, it can be observed that exactly one point of $D(A)$ can change its side at any particular time during the rotation of l. We further rotate l clockwise to obtain another new partition $\{l_2^+, l_2^-\}$, each having at least $\lfloor \frac{d+1}{2} \rfloor$ points, at the instance a point in either l_1^+ or l_1^- changes its side. This new linear separation corresponds to a crossing pair of simplices in $\mathbb{R}^d$, which can be used to obtain at least $\binom{d-3}{\lfloor \frac{d-5}{2} \rfloor}$ distinct crossing pairs of $(d-1)$-simplices formed by the vertices of the hypergraph K_{2d}^d. Note that all the crossing pairs of simplices obtained by extending the partitions $\{l_1^+, l_1^-\}$ and $\{l_2^+, l_2^-\}$ are distinct. Continuing in this manner for any $1 \leq k \leq \lfloor \frac{d-3}{2} \rfloor - 1$, we rotate l clockwise to obtain a new partition $\{l_{k+1}^+, l_{k+1}^-\}$, each having at least $\lfloor \frac{d-2k+3}{2} \rfloor$ points, at any time a point in either l_k^+ or l_k^- changes its side. Therefore, the corresponding crossing pair of simplices in $\mathbb{R}^d$ can be extended to crossing pairs of $(d-1)$-simplices in at least $\binom{d-3}{d-\lfloor \frac{d-2k+3}{2} \rfloor} = \binom{d-3}{\lfloor \frac{d-2k-3}{2} \rfloor}$ distinct ways. Hence, the number of crossing pairs of $(d-1)$-simplices obtained using this method is at least $\binom{d-3}{\lfloor \frac{d-3}{2} \rfloor} + \binom{d-3}{\lfloor \frac{d-5}{2} \rfloor} + \binom{d-3}{\lfloor \frac{d-7}{2} \rfloor} + \ldots + \binom{d-3}{1} = \Theta(2^d)$. $\square$

4.3 Improved Lower Bound

In the following, we state Carathéodory's Theorem which is used in the proof of Theorem 1.

Carathéodory's Theorem. *[43] Let $X \subseteq \mathbb{R}^d$. Then, each point in the convex hull of X can be expressed as a convex combination of at most $d+1$ points in X.*

Theorem 1. *The number of crossing pairs of hyperedges in a d-dimensional rectilinear drawing of K_{2d}^d is $\Omega(2^d d^{3/2})$ if the vertices of K_{2d}^d are not in convex position.*

Proof. Since the points in V are not in convex position in $\mathbb{R}^d$, we assume without loss of generality that v_{d+2} can be expressed as a convex combination of the points in $V \setminus \{v_{d+2}\}$. The Carathéodory's theorem implies that v_{d+2} can be expressed as a convex combination of $d+1$ points in $V \setminus \{v_{d+2}\}$. Without loss of generality, we assume these $d+1$ points to be $\{v_1, v_2, \ldots, v_{d+1}\}$.

Consider the set of points $V' = \{v_1, v_2, \ldots, v_{d+5}\} \subset V$. Note that a Gale transform $D(V')$ of it is a collection of $d+5$ vectors in $\mathbb{R}^4$. Property 4 of the Gale transformation implies that there exists a linear hyperplane h that partitions $D(V')$ in such a way that one of the open half-spaces created by h contains exactly one vector of $D(V')$. Since the points in V' are in general position in $\mathbb{R}^4$, Property 1 implies that at most three vectors of $D(V')$ lie on h. Since the vectors in $D(V')$ are in general position, it can be easily seen that we can slightly rotate h to obtain a linear hyperplane h' that partitions $D(V')$ such that one of the open half-spaces created by h' contains $d+4$ vectors and the other one contains exactly one vector.

Consider a hyperplane parallel to h'. We project the vectors in $D(V')$ on this hyperplane to obtain an affine Gale diagram $\overline{D(V')}$. Note that $\overline{D(V')}$ contains $d+4$ points of the same color and one point of the other color in $\mathbb{R}^3$. Without loss of generality, let us assume that the majority color is white. Also, note that the points in $\overline{D(V')}$ are in general position in $\mathbb{R}^3$ since the corresponding vectors in the Gale transform $D(V')$ are in general position in $\mathbb{R}^4$.

Consider the set W containing $d+4$ white points of $\overline{D(V')}$ in $\mathbb{R}^3$. Observation 4 implies that there exist $\Omega(d^2)$ distinct k-sets of W such that $min\{k, d+4-k\}$ is at least $\lceil (d+3)/2 \rceil$. Each of these k-sets corresponds to a unique linear separation of $D(V')$ having at least $\lceil (d+3)/2 \rceil$ vectors in each of the open half-spaces created by the corresponding linear hyperplane. Property 2 of the Gale transformation implies that there exists a unique crossing pair of u-simplex and v-simplex corresponding to each of these linear separations of $D(V')$, such that $u+v+2 = d+5$ and $min\{u+1, v+1\} \geq \lceil (d+3)/2 \rceil$. It follows from

Corollary 1 that each such crossing pair of u-simplex and v-simplex can be extended to obtain at least $\binom{d-5}{d-\lceil (d+3)/2\rceil}$ crossing pairs of $(d-1)$-simplices formed by the hyperedges in E. Therefore, the total number of crossing pairs of hyperedges in such a d-dimensional rectilinear drawing of K_{2d}^d is at least $\Omega(d^2)\binom{d-5}{d-\lceil (d+3)/2\rceil} = \Omega\left(2^d d^{3/2}\right)$. □

In the following, we improve the lower bound on $\overline{cr}_d\left(K_{2d}^d\right)$ to $\Omega(2^d\sqrt{\log d})$ using the properties of k-sets of $\mathbb{R}^2$.

Lemma 11. *The d-dimensional rectilinear crossing number of a complete d-uniform hypergraph with $2d$ vertices is $\Omega(2^d\sqrt{\log d})$, i.e., $\overline{cr}_d\left(K_{2d}^d\right) = \Omega(2^d\sqrt{\log d})$.*

Proof. Consider a subset $V' = \{v_1, v_2, \ldots, v_{d+4}\} \subset V$ having $d+4$ points. The Gale transform $D(V')$ is a set of $d+4$ vectors in $\mathbb{R}^3$. Using the similar procedure employed by Anshu et al. [8], we apply the Ham-Sandwich theorem to obtain $\Omega(\log d)$ proper linear separations of $D(V')$. Consider an affine Gale diagram $\overline{D(V')}$. Let us ignore the colors of the points in $\overline{D(V')}$. Note that each of the $\Omega(\log d)$ proper linear separations of $D(V')$ corresponds to a k-set of $\overline{D(V')}$ for some k satisfying $1 \le k \le d+3$. For each of these $\Omega(\log d)$ k-sets, we can obtain a distinct $(k-1)$-edge in $\overline{D(V')}$ as mentioned in the proof of Observation 1. We need at least $\Omega(\sqrt{\log d})$ distinct points of $\overline{D(V')}$ to span these $\Omega(\log d)$ lines. This implies that there exists a set of $\Omega(\sqrt{\log d})$ $(k-1)$-edges such that each of them contains a unique point of $\overline{D(V')}$ for some k satisfying $1 \le k \le d+3$. Let us denote the collection of these $\Omega(\sqrt{\log d})$ $(k-1)$-edges by L.

Let l_1 be a member of L. Without loss of generality, let us assume that $\overline{g_1}$ is the unique point contained in l_1. Without loss of generality, let us also assume that l_1 contains $\overline{g_2}$. We rotate l_1 counter-clockwise around the mid-point of $[\overline{g_1}, \overline{g_2}]$ to obtain the line $l_1(0)$. Note that $l_1(0)$ does not contain any point of $\overline{D(V')}$. We obtain a partition of the points in $\overline{D(V')}$ by $l_1(0)$. This partition corresponds to a linear separation of vectors in $D(V')$ such that each of the open half-spaces contains at least $\lfloor (d+1)/2 \rfloor$ vectors. We now rotate l_1 counter-clockwise with respect to $\overline{g_1}$ until it meets another point $\overline{g_j}$ of $\overline{D(V')}$ and let us denote this line by l_1^1. We then rotate the line l_1^1 counter-clockwise around the mid-point of $[\overline{g_1}, \overline{g_j}]$ as mentioned in the proof of Observation 1 to obtain the line $l_1^1(0)$. Note that $l_1^1(0)$ does not contain any point of

$\overline{D(V')}$. We obtain a new partition of the points in $\overline{D(V')}$ by $l_1^1(0)$. This partition corresponds to a linear separation of vectors in $D(V')$. We now rotate l_1^1 counter-clockwise around $\overline{g_1}$ until it meets the next point and let us denote this line by l_1^2. We then rotate l_1^2 counter-clockwise around the mid points of its two endpoints in $\overline{D(V')}$ to obtain $l_1^2(0)$. We obtain a new partition of the points in $\overline{D(V')}$ by $l_1^2(0)$. In general, we rotate l_1^j around the mid-points of its two endpoints in $\overline{D(V')}$ to obtain the line $l_1^j(0)$. $l_1^j(0)$ creates a new partition of the points in $\overline{D(V')}$. We then rotate l_1^j counter-clockwise around $\overline{g_1}$ to obtain a new $(k-1)$-edge l_1^{j+1} for some k satisfying $1 \leq k \leq d+3$. Note that we can keep on rotating like this until all the points in $\overline{D(V')} \setminus \{g_1, g_2\}$ are covered. Note that we obtain $d+3$ distinct proper linear separations while rotating the line with respect to g_1. Let us denote the linear separation of $D(V')$ that corresponds to the line $l_1^j(0)$ by $\{D_j^+(V'), D_j^-(V')\}$. Note that $\left|\left|D_j^+(V')\right| - \left|D_{j+1}^+(V')\right|\right| \leq 4$ for j satisfying $0 \leq j \leq d+2$. Each of these proper linear separations corresponds to distinct crossing pairs of u-simplex and v-simplex where $u+v = d+2$ and $1 \leq u, v \leq d-1$. It follows from Corollary 1 that each such crossing pair of u-simplex and v-simplex can be extended to obtain at least $\binom{d-4}{d-u-1}$ crossing pairs of $(d-1)$-simplices formed by the hyperedges in E. The total number of crossing pairs of hyperedges obtained in this way is at least $\binom{d-4}{d-\lfloor (d+1)/2 \rfloor} + \binom{d-4}{d-\lfloor (d+1)/2 \rfloor - 4} + \ldots + \binom{d-4}{d-4} = \Omega(2^d)$.

For each of the $\Omega(\sqrt{\log d})$ $(k-1)$-edges in L, we obtain $\Omega(2^d)$ crossing pairs of hyperedges in a similar way. Note that Observation 1 implies that partitions of points in $\overline{D(V')}$ obtained during the rotation of a line in L are distinct from the partitions of points in $\overline{D(V')}$ obtained during the rotation of another line in L. This implies that for each of the lines in L, we obtain $\Omega(2^d)$ distinct crossing pairs of hyperedges. This proves that $\overline{cr}_d(K_{2d}^d) = \Omega(2^d\sqrt{\log d})$. □

We further improve the lower bound on c_d from $\Omega(2^d\sqrt{\log d})$ to $\Omega\left(2^d\sqrt{d}\right)$ using the properties of balanced lines in the following.

Theorem 2. *The d-dimensional rectilinear crossing number of a complete d-uniform hypergraph having* $2d$ *vertices is* $\Omega(2^d\sqrt{d})$, *i.e.,* $\overline{cr}_d\left(K_{2d}^d\right) = \Omega(2^d\sqrt{d})$.

Proof. Consider a set $V' = \{v_1, v_2, \ldots, v_{d+4}\} \subset V$, whose Gale transform $D(V')$ is a set of $d+4$ vectors in $\mathbb{R}^3$. As mentioned before, the vectors in $D(V')$ can be treated as points in

$\mathbb{R}^3$. In order to apply the Ham-Sandwich theorem to obtain a proper linear separation of $D(V')$, we keep the origin in a set and all the points in $D(V')$ in another set. The Ham-Sandwich theorem implies that there exists a linear hyperplane h such that each of the open half-spaces created by it contains at most $\lfloor (d+4)/2 \rfloor$ vectors of $D(V')$. Since the vectors in $D(V')$ are in general position in $\mathbb{R}^3$, note that at most two vectors in $D(V')$ can lie on h and no two vectors in $D(V')$ lie on a line passing through the origin. As a result, it can be easily seen that we can slightly rotate h to obtain a linear hyperplane h' which creates a proper linear separation of $D(V')$. Consider a hyperplane parallel to h' and project the vectors in $D(V')$ on this hyperplane to obtain an affine Gale diagram $\overline{D(V')}$. Note that $\overline{D(V')}$ contains $\lfloor (d+4)/2 \rfloor$ points of the same color and $\lceil (d+4)/2 \rceil$ points of the other color in $\mathbb{R}^2$. Without loss of generality, let us assume that the majority color is white. Also, note that the points in $\overline{D(V')}$ are in general position in $\mathbb{R}^2$.

Observation 2 implies that there exist at least $\lfloor (d+4)/2 \rfloor$ almost balanced directed lines of $\overline{D(V')}$. Consider an almost balanced directed line that passes through a white and a black point in $\overline{D(V')}$. Consider the middle point p of the straight line segment connecting these two points. We rotate the almost balanced directed line slightly counter-clockwise around p to obtain a partition of $\overline{D(V')}$ by a directed line that does not pass through any point of $\overline{D(V')}$. Note that this partition of $\overline{D(V')}$ corresponds to a unique linear separation of $D(V')$ having at least $\lfloor (d+2)/2 \rfloor$ vectors in each of the open half-spaces created by the corresponding linear hyperplane. This implies that there exist at least $\lfloor (d+4)/2 \rfloor$ distinct linear separations of $D(V')$ such that each such linear separation contains at least $\lfloor (d+2)/2 \rfloor$ vectors in each of the open half-spaces created by the corresponding linear hyperplane. Property 2 of the Gale transformation implies that there exists a unique crossing pair of u-simplex and v-simplex corresponding to each linear separation of $D(V')$, such that $u+v+2 = d+4$ and $min\{u+1, v+1\} \geq \lfloor (d+2)/2 \rfloor$. It follows from Corollary 1 that each such crossing pair of u-simplex and v-simplex can be extended to obtain at least $\binom{d-4}{d-\lfloor (d+2)/2 \rfloor} = \Omega\left(2^d/\sqrt{d}\right)$ crossing pairs of $(d-1)$-simplices formed by the hyperedges in E. Therefore, the total number of crossing pairs of hyperedges in a d-dimensional rectilinear drawing of K_{2d}^d is at least $\lfloor (d+4)/2 \rfloor \, \Omega\left(2^d/\sqrt{d}\right) = \Omega\left(2^d\sqrt{d}\right)$. $\square$

Chapter 5

Convex Crossing Number of Complete d-Uniform Hypergraphs

5.1 Motivation and Previous Works

In this chapter, we investigate some d-dimensional convex drawings of K_{2d}^d. Our main focus in this chapter is to derive a closed form expression on the number of crossing pairs of hyperedges when all $2d$ vertices of K_{2d}^d are placed on the d-dimensional moment curve. Note that the points placed on the d-dimensional moment curve are in general, as well as in convex position in $\mathbb{R}^d$. Also, recall that a d-dimensional cyclic polytope is a polytope whose vertices lie on the d-dimensional moment curve. The d-dimensional moment curve plays an important role in discrete geometry.

Our motivation to work on this particular d-dimensional convex drawing of K_{2d}^d is the Upper Bound theorem [45], which states that the d-dimensional cyclic polytope has the maximum number of faces (of any given dimension i in the range $1 \leq i \leq d-1$) among all d-dimensional convex polytopes having an equal number of vertices. Gale's evenness criterion [27, 43] provides a necessary and sufficient condition to determine the number of facets ($(d-1)$-dimensional faces) of the d-dimensional cyclic polytope. Let us recall that there exists a natural ordering among the points on the d-dimensional moment curve. Given two points $p_i = (a_i, (a_i)^2, \ldots, (a_i)^d)$ and $p_j = (a_j, (a_j)^2, \ldots, (a_j)^d)$ on the d-dimensional moment curve, we say $p_i \prec p_j$ (p_i precedes p_j) if $a_i < a_j$.

Gale's Evenness Criterion. *[43] Let V' be the set of vertices of a d-dimensional cyclic polytope with the usual ordering on the d-dimensional moment curve. Let $F = \{v_1', v_2', \ldots, v_d'\} \subset V'$ be a set of d vertices of the d-dimensional cyclic polytope such that $v_1' \prec v_2' \prec \ldots \prec v_d'$. F spans a facet ($(d-1)$-dimensional face) of the cyclic polytope if and only if the number of vertices $v_i' \in F$ with the ordering $u' \prec v_i' \prec v'$ is even for each pair of vertices $u', v' \in V' \setminus F$.*

Lemma 12. *[43] The number of facets of a d-dimensional cyclic polytope with $n \geq d+1$ vertices is*

$$\begin{cases} \binom{n-\lfloor d/2 \rfloor}{\lfloor d/2 \rfloor} + \binom{n-\lfloor d/2 \rfloor - 1}{\lfloor d/2 \rfloor - 1} & \textit{if } d \textit{ is even,} \\ 2\binom{n-\lfloor d/2 \rfloor - 1}{\lfloor d/2 \rfloor} & \textit{if } d \textit{ is odd.} \end{cases}$$

Proof. The Gale's evenness criterion implies that counting the number of facets of a d-dimensional cyclic polytope with $n \geq d+1$ vertices is equivalent to counting the number of ways of placing d black points and $n-d$ white points in a row such that there exists an even number of black points between every two white points. Let us call an arrangement of d black points and $n-d$ white points in a row a valid arrangement if there exist an even number of black points between every two consecutive white points.

Let us first consider the case when d is odd. Let d be $2k+1$. Note that there can be odd number of black points at the beginning or at the end but not both in a valid arrangement. Let us consider the case when there are odd number of black points at the beginning. We ignore the first black point. We are then left with $2k$ black points and $n-2k-1$ white points. Also, note that each contiguous segment of remaining black points contains an even number of black points. The total number of ways we can obtain such an arrangement is $\binom{n-k-1}{k}$, since by deleting every second black point from the remaining $2k$ black points we obtain a one-to-one correspondence with selecting k positions for the black points out of $n-2k-1+k = n-k-1$ positions. We repeat the same argument when there are odd number of black points at the end by ignoring the last black point. This implies that the number of facets of a d-dimensional cyclic polytope with $n \geq d+1$ vertices is $2\binom{n-\lfloor d/2 \rfloor - 1}{\lfloor d/2 \rfloor}$ when d is odd.

Let us now consider the case when d is even. Let d be $2k$. Note that there can be odd number of black points or even number of black points at the beginning in a valid arrangement. Let us assume that there are odd number of black points at the beginning. This implies that there are odd number of black points at the end. We ignore the first and the last black points. We are then left with $2k-2$ black points and $n-2k-2$ white points. Note that each contiguous segment of the remaining black points contains an even number of black points. The total number of ways we can obtain such an arrangement is $\binom{n-k-1}{k-1}$, since by deleting every second black point from the remaining $2k-2$ black points we obtain a one-to-one correspondence with selecting $k-1$ positions for the black points out of $n-2k+k-1=n-k-1$ positions. Let us now assume that there are an even number of black points at the beginning and the end. In this case, each contiguous segment of black points contains an even number of black points. The total number of ways we can obtain such an arrangement is $\binom{n-k}{k}$, since by deleting every second black point from the $2k$ black points we obtain a one-to-one correspondence with selecting k positions for black points out of $n-2k+k=n-k$ positions. This implies that the number of facets of a d-dimensional cyclic polytope with $n \geq d+1$ vertices is $\binom{n-\lfloor d/2 \rfloor}{\lfloor d/2 \rfloor} + \binom{n-\lfloor d/2 \rfloor - 1}{\lfloor d/2 \rfloor - 1}$ when d is even. $\square$

It can be noted that the cyclic polytope is a neighborly polytope [27, 43]. Also, it is easy to verify that any $(d-1)$-dimensional hyperplane cuts the d-dimensional moment curve in at most d points [43]. The Upper Bound theorem also guarantees that any d-dimensional neighborly polytope whose vertices are in general position in $\mathbb{R}^d$ has the maximum number of faces (of any given dimension i for any i satisfying $1 \leq i \leq d-1$) among all d-dimensional convex polytopes having the same number of vertices [45]. The Upper Bound theorem motivated us to investigate the d-dimensional convex drawings of K_{2d}^d when its vertices are placed in general position as the vertices of a neighborly polytope.

In this chapter, we determine a Gale transform of $d+3$ points placed on the d-dimensional moment curve. This result helps us to obtain a lower bound on the number of crossing pairs of hyperedges when all $2d$ vertices of K_{2d}^d are placed on the d-dimensional moment curve. Let us recall that c_d^m denotes the number of crossing pairs of hyperedges in a d-dimensional convex drawing of K_{2d}^d when its $2d$ vertices are placed on the d-dimensional moment curve. We obtain the exact value of c_d^m. We also prove that the number of crossing pairs of hyperedges among all 3-dimensional rectilinear drawings of K_n^3 is maximized when its vertices are placed on the 3-dimensional moment curve.

5.2 Crossings in Cyclic Polytope

In this section, we obtain the value of c_d^m. Let us recall that c_d^m is the number of crossing pairs of hyperedges of K_{2d}^d, when all the $2d$ vertices of K_{2d}^d are placed on the d-dimensional moment curve. As mentioned in Section 5.1, we first prove a lower bound on c_d^m using the Gale transform and show later that this bound can be improved by using other techniques to obtain the exact value of c_d^m. Let $A = \big\langle (a_1, (a_1)^2, \ldots, (a_1)^d),\ (a_2, (a_2)^2, \ldots, (a_2)^d),\ \ldots,\ (a_{d+3}, (a_{d+3})^2, \ldots, (a_{d+3})^d) \big\rangle$, where $a_1 < a_2 < \ldots < a_{d+3}$, be a subset of $d+3$ vertices selected from the set of $2d$ vertices of K_{2d}^d. We obtain the following.

Lemma 13. *The following sequence of* 2*-dimensional vectors* $D(A) = \big\langle g_1, g_2, \ldots, g_{d+3} \big\rangle$ *can be obtained by the Gale transform of* $A = \big\langle (a_1, (a_1)^2, \ldots, (a_1)^d),\ (a_2, (a_2)^2, \ldots, (a_2)^d),\ \ldots,\ (a_{d+3}, (a_{d+3})^2, \ldots, (a_{d+3})^d) \big\rangle$.

$$g_i = \begin{cases} \left((-1)^{d+1} \dfrac{\prod\limits_{j \in \{1,2,\cdots,d+1\} \setminus \{i\}} (a_{d+2} - a_j)}{\prod\limits_{k \in \{1,2,\cdots,d+1\} \setminus \{i\}} (a_k - a_i)}, (-1)^{d+1} \dfrac{\prod\limits_{j \in \{1,2,\cdots,d+1\} \setminus \{i\}} (a_{d+3} - a_j)}{\prod\limits_{k \in \{1,2,\cdots,d+1\} \setminus \{i\}} (a_k - a_i)} \right) & if\ i \notin \{d+2, d+3\} \\ (1,0) & if\ \ i = d+2 \\ (0,1) & if\ \ i = d+3 \end{cases}$$

Proof. Let us consider the following matrix M(A).

$$M(A) = \begin{bmatrix} a_1 & a_2 & \cdots & a_{d+3} \\ (a_1)^2 & (a_2)^2 & \cdots & (a_{d+3})^2 \\ \vdots & \vdots & \vdots & \vdots \\ (a_1)^d & (a_2)^d & \cdots & (a_{d+3})^d \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

To obtain the basis of the null space, we need to find solutions of the following $d+1$ equations involving $d+3$ variables $\gamma_1, \gamma_2, \ldots, \gamma_{d+3}$.

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_{d+3} \\ (a_1)^2 & (a_2)^2 & \cdots & (a_{d+3})^2 \\ \vdots & \vdots & \vdots & \vdots \\ (a_1)^d & (a_2)^d & \cdots & (a_{d+3})^d \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{d+3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{5.1}$$

Rearranging Equation 5.1, we get the following:

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{d+1} \end{bmatrix} = - \begin{bmatrix} a_1 & a_2 & \cdots & a_{d+1} \\ (a_1)^2 & (a_2)^2 & \cdots & (a_{d+1})^2 \\ \vdots & \vdots & \vdots & \vdots \\ (a_1)^d & (a_2)^d & \cdots & (a_{d+1})^d \\ 1 & 1 & \cdots & 1 \end{bmatrix}^{-1} \begin{bmatrix} a_{d+2} & a_{d+3} \\ (a_{d+2})^2 & (a_{d+3})^2 \\ \vdots & \vdots \\ (a_{d+2})^d & (a_{d+3})^d \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{d+2} \\ \gamma_{d+3} \end{bmatrix}$$

Setting $\gamma_{d+2} = 1$ and $\gamma_{d+3} = 0$, we obtain the vector $v_1 = (\gamma_1, \gamma_2, \ldots, \gamma_{d+2}, \gamma_{d+3})$ for every i satisfying $1 \leq i \leq d+1$.

$$\gamma_i = (-1)^{d+1} \frac{\prod\limits_{j \in \{1,2,\cdots,d+1\} \setminus \{i\}} (a_{d+2} - a_j)}{\prod\limits_{k \in \{1,2,\cdots,d+1\} \setminus \{i\}} (a_k - a_i)}.$$

Setting $\gamma_{d+2} = 0$ and $\gamma_{d+3} = 1$, we obtain the vector $v_2 = (\gamma_1, \gamma_2, \ldots, \gamma_{d+2}, \gamma_{d+3})$ for every i satisfying $1 \leq i \leq d+1$.

$$\gamma_i = (-1)^{d+1} \frac{\prod\limits_{j \in \{1,2,\cdots,d+1\} \setminus \{i\}} (a_{d+3} - a_j)}{\prod\limits_{k \in \{1,2,\cdots,d+1\} \setminus \{i\}} (a_k - a_i)}.$$

Note that the vectors v_1 and v_2 are linearly independent and form a basis of the null space of the row space of M(A). Hence, the result follows. $\square$

Note that for every $1 \leq i \leq d+3$, each vector g_i in $D(A)$ is represented as an ordered pair (b_i, c_i) where $b_i, c_i \in \mathbb{R}$. We denote the slope of the vector g_i as $s_i = \dfrac{c_i}{b_i}$. In order to count the number of linear separations, we observe the following properties of these vectors.

Observation 6. *The sequence of 2-dimensional vectors $D(A) = \langle g_1, g_2, \ldots, g_{d+3} \rangle$ having slopes $\langle s_1, s_2, \ldots, s_{d+3} \rangle$ satisfies the following properties.*

(i) For any $1 \leq i \leq d+1$, g_i lies in the first (third) quadrant if $d+1+i$ is odd (even).

(ii) $\infty = s_{d+3} > s_1 > s_2 > \ldots > s_{d+1} > s_{d+2} = 0$.

Lemma 14. *The number of crossing pairs of hyperedges in a d-dimensional convex drawing of K^d_{2d} where all of its vertices are placed on the d-dimensional moment curve is $\Omega(2^d\sqrt{d})$, i.e., $c^m_d = \Omega(2^d\sqrt{d})$.*

Proof. Consider the vectors in $D(A)$, that can also be considered as a sequence of $d+3$ points in $\mathbb{R}^2$. We apply the Ham-Sandwich Theorem by assigning the points in $D(A)$ to P_1 and the origin to P_2 to obtain a line l passing through the origin that bisects the points in $D(A)$ into two partitions, each containing at most $\lfloor \frac{1}{2}|D(A)| \rfloor$ points. Since at most one point from $D(A)$ can lie on l, it can be rotated using the origin as the center of rotation to obtain a proper linear separation of $D(A)$ into 2 subsets l_1^+ (left or counter-clockwise side) and l_1^- (right or clockwise side) of size $\lfloor \frac{d+3}{2} \rfloor$ and $\lceil \frac{d+3}{2} \rceil$, respectively. This proper linear separation corresponds to a crossing pair of a $(\lfloor \frac{d+3}{2} \rfloor - 1)$-simplex and a $(\lceil \frac{d+3}{2} \rceil - 1)$-simplex in $\mathbb{R}^d$, as mentioned in Property 2. It follows from Corollary 1 that this crossing pair of simplices can be used to obtain $\binom{d-3}{\lfloor \frac{d-3}{2} \rfloor}$ distinct crossing pairs of $(d-1)$-simplices formed by the vertices of the hypergraph K^d_{2d}. We rotate l clockwise using the origin as the center of rotation until one of the $d+3$ points in $D(A)$ moves from one side of the line to the other side to obtain new subsets $\{l_2^+, l_2^-\}$, each having at least $\lfloor \frac{d+1}{2} \rfloor$ points. This new linear separation $\{l_2^+, l_2^-\}$ corresponds to a crossing pair of simplices in $\mathbb{R}^d$, which can be used to obtain at least $\binom{d-3}{\lfloor \frac{d-5}{2} \rfloor}$ distinct crossing pairs of $(d-1)$-simplices formed by the vertices of the hypergraph K^d_{2d}. Note that all the crossing pairs of simplices obtained by extending the partitions $\{l_1^+, l_1^-\}$ and $\{l_2^+, l_2^-\}$ are distinct. Since all the $d+3$ points of A lie on the d-dimensional moment curve, Observation 6 implies that the sequence of vectors in $D(A)$, excluding g_{d+2} and g_{d+3}, lie alternatively in the first and third quadrants with increasing slopes. As a consequence, another clockwise rotation of l results in a point in $D(A)$ changing its side at some point of time from a side having more than or equal to $\lceil \frac{d+3}{2} \rceil$ points to the other side. This creates a new partition $\{l_3^+,\ l_3^-\}$, each containing at least $\lfloor \frac{d+3}{2} \rfloor$ points. We continue rotating l clockwise until we obtain the partition $\{l_1^+, l_1^-\}$ again. In this way, we obtain at least $2\lfloor \frac{d+3}{2} \rfloor$ distinct partitions of $D(A)$ such that each subset in a partition contains at least $\lfloor \frac{d+1}{2} \rfloor$ points.

Hence, the number of crossing pairs of hyperedges spanned by the vertices of K_{2d}^d placed on the d-dimensional moment curve is at least $2\left\lfloor\frac{d+3}{2}\right\rfloor\binom{d-3}{\left\lceil\frac{d-5}{2}\right\rceil} = \Theta(2^d\sqrt{d})$. $\square$

We now use Lemma 15 and Lemma 16 to prove Theorem 3 that implies $c_d^m = \Theta\left(\frac{4^d}{\sqrt{d}}\right)$.

Lemma 15. *[15] Let $p_1 \prec p_2 \prec \ldots \prec p_{\left\lfloor\frac{d}{2}\right\rfloor+1}$ and $q_1 \prec q_2 \prec \ldots \prec q_{\left\lceil\frac{d}{2}\right\rceil+1}$ be two distinct point sequences on the d-dimensional moment curve such that $p_i \neq q_j$ for any $1 \leq i \leq \left\lfloor\frac{d}{2}\right\rfloor+1$ and $1 \leq j \leq \left\lceil\frac{d}{2}\right\rceil+1$. The $\left\lfloor\frac{d}{2}\right\rfloor$-simplex and the $\left\lceil\frac{d}{2}\right\rceil$-simplex, formed respectively by these point sequences, cross if and only if every interval (q_j, q_{j+1}) contains exactly one p_i and every interval (p_i, p_{i+1}) contains exactly one q_j.*

Lemma 16. *[18] Let P and Q be two vertex-disjoint $(d-1)$-simplices such that each of the $2d$ vertices belonging to these simplices lies on the d-dimensional moment curve. If P and Q cross, then there exist a $\left\lfloor\frac{d}{2}\right\rfloor$-simplex $U \subsetneq P$ and another $\left\lceil\frac{d}{2}\right\rceil$-simplex $V \subsetneq Q$ such that U and V cross.*

Proof. Let us define an interval $I_{ik} = \{p_j|\ p_j = (a_j, (a_j)^2, \ldots, (a_j)^d)\ \textit{and}\ a_i \leq a_j \leq a_k\}$ on the d-dimensional moment curve. For the sake of contradiction, let us assume that P and Q cross but there do not exist a $\left\lfloor\frac{d}{2}\right\rfloor$-simplex $U \subsetneq P$ and another $\left\lceil\frac{d}{2}\right\rceil$-simplex $V \subsetneq Q$ such that U and V cross. We color the vertices of P by red and the vertices of Q by blue. Lemma 15 implies that there is no chain of length $d+2$ with alternating colors. This further implies that the set of all vertices of P and Q can be partitioned into at most $d+1$ monochromatic alternating intervals. Thus, the set of monochromatic red intervals can be separated from the set of monochromatic blue intervals by a hyperplane passing through d points on the d-dimensional moment curve. This implies that the set of red points and the set of blue points can be separated by a hyperplane. This further implies that P and Q lie in the different open half-spaces created by this hyperplane. This contradicts our assumption that P and Q cross. $\square$

Theorem 3. *Let c_d^m be the number of crossing pairs of hyperedges in a d-dimensional convex drawing of K_{2d}^d where all of its vertices are placed on the d-dimensional moment curve. The* *value of* c_d^m *is*

$$c_d^m = \begin{cases} \binom{2d-1}{d-1} - \sum_{i=1}^{\frac{d}{2}} \binom{d}{i}\binom{d-1}{i-1} & if\ d\ is\ even \\ \binom{2d-1}{d-1} - 1 - \sum_{i=1}^{\left\lfloor \frac{d}{2} \right\rfloor} \binom{d-1}{i}\binom{d}{i} & if\ d\ is\ odd \end{cases}$$

$$= \Theta\left(\frac{4^d}{\sqrt{d}}\right)$$

Proof. Let $\{C, D\}$ be a pair of disjoint vertex sets, each having d vertices of K_{2d}^d placed on the d-dimensional moment curve. Without loss of generality, let us assume that C contains the first vertex (i.e., the vertex corresponding to the minimum value of t) of K_{2d}^d. Note that the number of such unordered pairs $\{C, D\}$ is $\frac{1}{2}\binom{2d}{d} = \binom{2d-1}{d-1}$. Let us color the vertices in C and D by red and blue, respectively, to obtain d partitions created by the red vertices. In particular, the first $d-1$ of these partitions are between two adjacent red vertices, and the last one is after the last red vertex. It implies from Corollary 1 and Lemma 15 that the pair of $(d-1)$-simplices formed by the vertices in C and D cross if there exists a sequence of $d+2$ vertices with alternating colors. Similarly, we obtain from Lemma 15 and Lemma 16 that the pair of $(d-1)$-simplices formed by the vertices in C and D do not cross if there does not exist any sequence of $d+2$ vertices with alternating colors.

When d is even, the number of disjoint vertex sets $\{C, D\}$ that do not contain any subsequence of length $d+2$ having alternating colors is equal to the number of ways d blue vertices can be distributed among d partitions such that at most $\frac{d}{2}$ of the partitions are non-empty. This number is equal to $\sum_{i=1}^{\frac{d}{2}} \binom{d}{i}\binom{d-1}{i-1}$. When d is odd, the number of disjoint vertex sets $\{C, D\}$ that do not contain any subsequence of length $d+2$ having alternating colors is equal to the number of ways d blue vertices can be distributed among d partitions such that at most $\left\lfloor \frac{d}{2} \right\rfloor$ of the first $d-1$ partitions are non-empty. This number is equal to $\sum_{i=1}^{\left\lfloor \frac{d}{2} \right\rfloor} \binom{d-1}{i}\left(\binom{d-1}{i-1} + \binom{d-1}{i}\right) + 1$.

Hence, the total number of crossing pairs of $(d-1)$-simplices spanned by the $2d$ vertices placed on the d-dimensional moment curve is

$$c_d^m = \begin{cases} \binom{2d-1}{d-1} - \sum_{i=1}^{\frac{d}{2}} \binom{d}{i}\binom{d-1}{i-1} & \textit{if } d \textit{ is even.} \\ \binom{2d-1}{d-1} - 1 - \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} \binom{d-1}{i}\binom{d}{i} & \textit{if } d \textit{ is odd.} \end{cases}$$

□

In the following, we show that 3-dimensional convex crossing number of K_6^3 is 3, i.e., $c_3^* = 3$. It is easy to see that $c_2^* = 1$. However, we are not aware of the exact values of c_d^* for $d > 3$.

Theorem 4. *The number of crossing pairs of hyperedges in a 3-dimensional rectilinear drawing of K_6^3 is 3 when all the vertices of K_6^3 are in convex as well as general position in $\mathbb{R}^3$.*

Proof. Let A be the set of vertices of K_6^3 that are in convex as well as general position in $\mathbb{R}^3$. Let $D(A)$ denote the Gale transform of A. Since the points in A are in general position, Property 1 of the Gale transformation shows that the 6 vectors in $D(A)$ are in general position in $\mathbb{R}^2$. Since the points in A are also in convex position, Property 4 of the Gale transformation implies that these vectors can be partitioned by a line l passing through the origin in two possible ways, i.e., the number of vectors in the opposite open half-spaces created by l can be either 4 and 2, or 3 and 3. Note that the second case is also known as a proper linear separation that corresponds to a crossing pair of 2-simplices spanned by the points in A. Without loss of generality, let us assume that l partitions the vectors in $D(A)$ in such a way that one of the open half-spaces created by l contains 4 vectors and the other contains 2 vectors. We rotate l clockwise using the origin as the center of rotation until one vector changes its side. Since Property 4 of the Gale transformation shows that l cannot partition the vectors such that there exists 1 vector on one of its side, this new partition obtained by rotating l is a proper linear separation. We again rotate l clockwise using the origin as the axis of rotation until one vector changes its side to obtain a new partition having 4 vectors on one side and 2 on the other side. We continue rotating l clockwise till we reach the first partition to obtain three proper linear separations of the vectors in $D(A)$. $\square$

5.3 Crossings in Other Convex Polytopes

In this section, we consider K^d_{2d} having the vertex set $V = \{v_1, v_2, \ldots, v_{2d}\}$ and the hyperedge set E having $\binom{2d}{d}$ hyperedges formed by these $2d$ vertices.

Theorem 5. *For any constant $t \geq 1$ independent of d, the number of crossing pairs of hyperedges in a d-dimensional rectilinear drawing of K^d_{2d} is $\Omega(2^d d^{3/2})$ if the vertices of K^d_{2d} are placed as the vertices of a d-dimensional t-neighborly polytope that is not $(t+1)$-neighborly.*

Proof. Consider the points in V that form the vertex set of a d-dimensional t-neighborly polytope which is not $(t+1)$-neighborly. Property 6 of the Gale transformation implies that there exists a linear hyperplane $\widetilde{h}$ such that one of the open half-spaces created by it contains $t+1$ vectors of $D(V)$. Without loss of generality, we denote the set of these $t+1$ vectors by $D^+(V)$. It implies that one of the closed half-spaces created by $\widetilde{h}$ contains $2d-t-1$ vectors of $D(V)$. If $d-2$ vectors of $D(V)$ do not lie on $\widetilde{h}$, we rotate $\widetilde{h}$ around the lower dimensional hyperplane spanned by the vectors on $\widetilde{h}$ till some new vector $g_i \in D(V)$ lies on it. We keep rotating $\widetilde{h}$ in this way till $d-2$ vectors of D(V) lie on it. Property 6 of the Gale transformation also implies that none of these $d-2$ vectors belongs to the set $D^+(V)$. After rotating $\widetilde{h}$ in the above mentioned way, we obtain a partition of $D(V)$ by a linear hyperplane $\widetilde{h}'$ such that one of the open half-spaces created by it contains $t+1$ vectors and the other one contains $d+1-t$ vectors. This implies that there exist a t-simplex and a $(d-t)$-simplex created by the vertices in V such that they form a crossing. We choose any three vertices from the rest of the $d-2$ vertices in V and add these three vertices to the d+2 vertices corresponding to this crossing pair of simplices. This implies that the t-neighborly sub-polytope formed by the convex hull of the $d+5$ vertices is not $(t+1)$-neighborly.

Without loss of generality, let the vertex set of this sub-polytope be $V' = \{v_1, v_2, \ldots, v_{d+5}\}$. Note that a Gale transform $D(V')$ of it is a collection of $d+5$ vectors in $\mathbb{R}^4$. Property 6 of the Gale transformation implies that there exists a linear hyperplane h such that one of the open half-spaces created by it contains exactly $t+1$ vectors of $D(V')$. As described in the proof of Theorem 1, it follows from Property 1 that at most three vectors can lie on h. Since the vectors in $D(V')$ are in general position, we can slightly rotate h to obtain a linear hyperplane h' such that one of the open half-spaces created by h' contains $t+1$ vectors and the other one contains $d+4-t$ vectors.

Consider a hyperplane parallel to h' and project the vectors in $D(V')$ on this hyperplane to obtain an affine Gale diagram $\overline{D(V')}$. Note that $\overline{D(V')}$ contains $d+4-t$ points of the same color and $t+1$ points of the other color in $\mathbb{R}^3$. Without loss of generality, let us assume that these $d+4-t$ points of the same color are white. Also, note that the points in $\overline{D(V')}$ are in general position in $\mathbb{R}^3$.

Let us consider the set W consisting of $d+4-t$ white points of $\overline{D(V')}$. Observation 4 implies that there exist $\Omega(d^2)$ distinct k-sets of W such that $min\{k, d+4-t-k\}$ is at least $\lceil (d+3-t)/2 \rceil$. Each of these k-sets corresponds to a unique linear separation of $D(V')$ such that it contains at least $\lceil (d+3-t)/2 \rceil$ vectors in each of the open half-spaces created by the corresponding linear hyperplane. Property 2 of the Gale transformation implies that there exists a unique crossing pair of u-simplex and v-simplex corresponding to each of these linear separations of $D(V')$, such that $u+v+2=d+5$ and $min\{u+1, v+1\} \geq \lceil (d+3-t)/2 \rceil$. It follows from Corollary 1 that each such crossing pair of u-simplex and v-simplex can be extended to obtain at least $\binom{d-5}{d-\lceil (d+3-t)/2 \rceil}$ crossing pairs of $(d-1)$-simplices formed by the hyperedges in E. Therefore, the total number of crossing pairs of hyperedges in such a d-dimensional rectilinear drawing of K_{2d}^d is at least $\Omega(d^2)\binom{d-5}{d-\lceil (d+3-t)/2 \rceil} = \Omega\left(2^d d^{3/2}\right)$. □

Theorem 6. *For any constant $t' \geq 0$ independent of d, the number of crossing pairs of hyperedges in a d-dimensional rectilinear drawing of K_{2d}^d is $\Omega(2^d d^{5/2})$ if the vertices of K_{2d}^d are placed as the vertices of a d-dimensional $(\lfloor d/2 \rfloor - t')$-neighborly polytope.*

Proof. Since the points in V form the vertex set of a d-dimensional $(\lfloor d/2 \rfloor - t')$-neighborly polytope, consider a sub-polytope of it formed by the convex hull of the vertex set V' containing any d+5 points of V. Without loss of generality, let V' be $\{v_1, v_2, \ldots, v_{d+5}\}$. Note that a Gale transform $D(V')$ of it is a collection of $d+5$ vectors in $\mathbb{R}^4$ and an affine Gale diagram $\overline{D(V')}$ of it is a collection of $d+5$ points in $\mathbb{R}^3$. In this proof, we ignore the colors of these points. However, note that the points in $\overline{D(V')}$ are in general position in $\mathbb{R}^3$.

Consider the set $\overline{D(V')}$. It follows from Observation 5 that the number of $(\leq \lceil (d+5)/4 \rceil)$ -sets of $\overline{D(V')}$ is $\Omega(d^3)$. For each k in the range $1 \leq k \leq \lceil (d+5)/4 \rceil$, a k-set of $\overline{D(V')}$ corresponds to a unique linear separation of $D(V')$. Property 6 of the Gale transformation implies that each of these $\Omega(d^3)$ linear separations of $D(V')$ contains at least $\lfloor d/2 \rfloor - t' + 1$ vectors in each of the open half-spaces created by the corresponding linear hyperplane. Property 2 of the Gale transformation implies that there exists a unique crossing pair of u-simplex and v-simplex corresponding to each linear separation of $D(V')$, such that $u + v + 2 = d + 5$ and $min\{u + 1, v + 1\} \geq \lfloor d/2 \rfloor - t' + 1$. It follows from Corollary 1 that each such crossing pair of u-simplex and v-simplex can be extended to obtain at least $\binom{d-5}{d - \lfloor d/2 \rfloor + t' - 1} = \Omega\left(2^d/\sqrt{d}\right)$ crossing pairs of $(d-1)$-simplices formed by the hyperedges in E. Therefore, the total number of crossing pairs of hyperedges in such a d-dimensional rectilinear drawing of K_{2d}^d is at least $\Omega(d^3)\Omega\left(2^d/\sqrt{d}\right) = \Omega\left(2^d d^{5/2}\right)$. $\square$

Chapter 6

Rectilinear Crossings in Complete Balanced d-Partite d-Uniform Hypergraphs

6.1 Motivation and Previous Works

In this chapter, we establish a non-trivial lower bound on the d-dimensional rectilinear crossing number of the complete balanced d-uniform d-partite hypergraph having nd vertices. Finding the rectilinear crossing number of complete bipartite graphs (i.e., complete 2-uniform bipartite hypergraphs) is an active area of research [36]. Let $K_{n,n}$ denote the complete bipartite graph having n vertices in each partition. The best-known lower and upper bounds on $\overline{cr}(K_{n,n})$ are $\dfrac{n(n-1)}{5} \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$ and $\lfloor n/2 \rfloor^2 \lfloor (n-1)/2 \rfloor^2$, respectively [36, 57]. Nahas [46] improved the lower bound on $\overline{cr}(K_{n,n})$ to $\dfrac{n(n-1)}{5} \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor + 9.9 \times 10^{-6} n^4$ for sufficiently large n.

Let us recall that a hypergraph H is called d-uniform if each hyperedge contains d vertices. Let us also recall that a d-uniform hypergraph $H = (V, E)$ is said to be d-partite if there exists a sequence $< X_1, X_2, \ldots, X_d >$ of disjoint sets such that $V = \bigcup\limits_{i=1}^{d} X_i$ and $E \subseteq X_1 \times X_2 \times \ldots \times X_d$. We call X_i to be the i^{th} part of V. Moreover, such a d-partite d-uniform hypergraph is called balanced if $|X_1| = |X_2| = \ldots = |X_d|$ and *complete* if $|E| = |X_1 \times X_2 \times \ldots \times X_d|$.

Let us also recall that $K^d_{d\times n}$ denotes the complete balanced d-partite d-uniform hypergraph with n vertices in each part. For $t \geq 2$, let us denote by $K^d_{k_1\times n_1+k_2\times n_2+\ldots+k_t\times n_t}$ the complete d-partite d-uniform hypergraph if $\sum_{i=1}^{t} k_i = d$, $n_i \neq n_{i+1}$ for all i in the range $1 \leq i \leq t-1$, and each of the first $k_1 > 0$ parts contains n_1 vertices, each of the next $k_2 > 0$ parts contains n_2 vertices, ..., each of the final $k_t > 0$ parts contains n_t vertices.

We first use the Colored Tverberg theorem with restricted dimensions and Corollary 1 to observe the lower bound on $\overline{cr}_d\left(K^d_{d\times n}\right)$ mentioned in Observation 7. Let us introduce a few more definitions and notations used in its proof. Two d-uniform hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are *isomorphic* if there is a bijection $f : V_1 \to V_2$ such that any set of d vertices $\{u_1, u_2, \ldots, u_d\}$ is a hyperedge in E_1 if and only if $\{f(u_1), f(u_2), \ldots, f(u_d)\}$ is a hyperedge in E_2. A hypergraph $H' = (V', E')$ is called an *induced sub-hypergraph* of $H = (V, E)$ if $V' \subseteq V$ and E' contains all hyperedges of E spanned by the vertices in V'. A $(u-1)$-simplex which is a convex hull of a set U containing u points $(1 \leq u \leq d+1)$ in general position in $\mathbb{R}^d$ is denoted by $Conv(U)$. Recall that a $(u-1)$-simplex $Conv(U)$ spanned by a point set U containing u points and a $(w-1)$-simplex $Conv(W)$ spanned by the point set W containing w points *cross* if $U \cap W = \phi$ and they contain a common point in their relative interiors. For the sake of completeness, we again mention the Colored Tverberg theorem with restricted dimensions.

Colored Tverberg Theorem with restricted dimensions. *[44, 55] Let $\{C_1, C_2, \ldots, C_{k+1}\}$ be a collection of $k+1$ disjoint finite point sets in $\mathbb{R}^d$. Each of these sets is assumed to be of cardinality at least $2r-1$, where r is a prime integer satisfying the inequality $r(d-k) \leq d$. Then, there exist r disjoint sets $S_1, S_2, \ldots, S_r$ such that $S_i \subseteq \bigcup_{j=1}^{k+1} C_j$, $\bigcap_{i=1}^{r} Conv(S_i) \neq \emptyset$ and $|S_i \cap C_j| = 1$ for all i and j satisfying $1 \leq i \leq r$ and $1 \leq j \leq k+1$.*

We prove the following observation to establish a lower bound on $\overline{cr}_d\left(K^d_{d\times n}\right)$. We improve this lower bound later in this chapter.

Observation 7. $\overline{cr}_d\left(K^d_{d\times n}\right) = \Omega\left((8/3)^{d/2}\right)(n/2)^d\left((n-1)/2\right)^d$ *for* $n \geq 3$.

Proof. Let us consider the hypergraph $H = K^d_{d\times n}$ such that its vertices are in general position in $\mathbb{R}^d$. Let $H' = K^d_{(\lceil d/2\rceil+1)\times 3+(\lfloor d/2\rfloor-1)\times 2}$ be an induced sub-hypergraph of it containing

3 vertices from each of the first $\lceil d/2 \rceil + 1$ parts and 2 vertices from each of the remaining $\lfloor d/2 \rfloor - 1$ parts. Let C_i denote the i^{th} part of the vertex set of H' for each i in the range $1 \leq i \leq \lceil d/2 \rceil + 1$. Note that $C_1, C_2, \ldots, C_{\lceil d/2 \rceil+1}$ are disjoint sets in $\mathbb{R}^d$ and each of them contains 3 vertices. Clearly, these sets satisfy the condition of Colored Tverberg theorem with restricted dimensions for $k = \lceil d/2 \rceil$ and $r = 2$. Since the vertices of H' are in general position in $\mathbb{R}^d$, Colored Tverberg theorem with restricted dimensions implies that there exists a crossing pair of $\lceil d/2 \rceil$-simplices spanned by $U \subseteq \bigcup_{j=1}^{\lceil d/2 \rceil+1} C_j$ and $W \subseteq \bigcup_{j=1}^{\lceil d/2 \rceil+1} C_j$ such that $U \cap W = \emptyset$ and $|U \cap C_j| = 1$, $|W \cap C_j| = 1$ for each j in the range $1 \leq j \leq \lceil d/2 \rceil + 1$. Corollary 1 implies that U and W can be extended to form $2^{\lfloor d/2 \rfloor - 1}$ distinct crossing pairs of $(d-1)$-simplices, where each $(d-1)$-simplex contains exactly one vertex from each part of H'. This implies that $\overline{cr}_d(H') \geq 2^{\lfloor d/2 \rfloor - 1}$. Note that each crossing pair of hyperedges corresponding to these $(d-1)$-simplices is contained in $(n-2)^{\lceil d/2 \rceil+1}$ distinct induced sub-hypergraphs of H, each of which is isomorphic to H'. Moreover, there are $\binom{n}{3}^{\lceil d/2 \rceil+1} \binom{n}{2}^{\lfloor d/2 \rfloor-1}$ distinct induced sub-hypergraphs of H, each of which is isomorphic to H'. This implies $\overline{cr}_d\left(K^d_{d\times n}\right) \geq 2^{\lfloor d/2 \rfloor - 1}\binom{n}{3}^{\lceil d/2 \rceil+1}\binom{n}{2}^{\lfloor d/2 \rfloor-1} \Big/ (n-2)^{\lceil d/2 \rceil+1}$ $= n^d(n-1)^d/6^{\lceil d/2 \rceil+1} = \Omega\left((8/3)^{d/2}\right)(n/2)^d\left((n-1)/2\right)^d$. □

In Section 6.2, we improve the lower bound on $\overline{cr}_d\left(K^d_{d\times n}\right)$. To the best of our knowledge, this is the first non-trivial lower bound on this number.

6.2 Lower Bound on the d-Dimensional Rectilinear Crossing Number of $K^d_{d\times n}$

In this section, we use Property 3 and the Ham-Sandwich theorem to improve the previously observed lower bound on the d-dimensional rectilinear crossing number of $K^d_{d\times n}$ for $n \geq 3$.

Theorem 7. *$\overline{cr}_d\left(K^d_{d\times n}\right) = \Omega\left(2^d\right)(n/2)^d\left((n-1)/2\right)^d$ for $n \geq 3$.*

Proof. Let us consider the hypergraph $H = K^d_{d\times n}$ such that all of its vertices are in general position in $\mathbb{R}^d$. Let $H' = K^d_{2\times 3+(d-2)\times 2}$ be an induced sub-hypergraph of it containing 3 vertices from each of the first 2 parts and 2 vertices from each of the remaining $(d-2)$ parts of the vertex set of H. Let $P = < p_1, p_2, p_3, \ldots, p_{2d+1}, p_{2d+2} >$ be a sequence of the vertices of H' such that $\{p_1, p_2, p_3\}$ belongs to the first partition L_1, $\{p_4, p_5, p_6\}$ belongs to the second partition L_2 and $\{p_{2k+1}, p_{2k+2}\}$ belongs to the k^{th} partition L_k for each k in the range $3 \leq k \leq d$. We consider a Gale transform of P and obtain a sequence of $2d+2$ vectors $D(P) = < v_1, v_2, v_3, \ldots, v_{2d+1}, v_{2d+2} >$ in $\mathbb{R}^{d+1}$. It follows from Property 1 of the Gale transformation that any set containing $d+1$ of these vectors spans $\mathbb{R}^{d+1}$. As mentioned before, $D(P)$ can also be considered as a sequence of points in $\mathbb{R}^{d+1}$. In order to apply Ham-Sandwich theorem in $\mathbb{R}^{d+1}$, we color the origin with color c_0, $\{v_1, v_2, v_3\}$ with color c_1, $\{v_4, v_5, v_6\}$ with color c_2 and $\{v_{2k+1}, v_{2k+2}\}$ with color c_k for each k in the range $3 \leq k \leq d$. The Ham-Sandwich theorem guarantees that there exists a hyperplane h such that it passes through the origin and bisects the set colored with c_i for each i in the range $1 \leq i \leq d$. Note that at most d points of $D(P)$ lie on the linear hyperplane h, since any set of $d+1$ vectors in $D(P)$ spans $\mathbb{R}^{d+1}$. This implies that there exist at least $d+2$ points in $D(P)$ that lie either in the positive open half-space h^+ or in the negative open half-space h^- created by h with an orientation assigned to it. Let $D^+(P)$ and $D^-(P)$ be the two sets of points lying in h^+ and h^-, respectively. The Ham-Sandwich theorem ensures that at most d points of $D(P)$ can lie in one of h^+ and h^-. This implies that $|D^+(P)| \geq 2$ and $|D^-(P)| \geq 2$. Moreover, note that 2 points having the same color cannot lie in the same open half-space. Property 3 implies that there exist a $(u-1)$-simplex $Conv(P_a)$ spanned by the vertices of $P_a \subset P$ and a $(w-1)$-simplex $Conv(P_b)$ spanned by the vertices of $P_b \subset P$ such that the following properties are satisfied.

(I) $P_a \cap P_b = \emptyset$

(II) $Conv(P_a)$ and $Conv(P_b)$ cross.

(III) $2 \leq |P_a|, |P_b| \leq d$, $|P_a| + |P_b| \geq d+2$

(IV) $|P_a \cap L_i| \leq 1$ for each i in the range $1 \leq i \leq d$

(V) $|P_b \cap L_i| \leq 1$ for each i in the range $1 \leq i \leq d$

Corollary 1 implies that the crossing between two lower-dimensional simplices $Conv(P_a)$ and $Conv(P_b)$ can be extended to a crossing pair of $(d-1)$-simplices spanned by vertex sets $U', W' \subset P$ satisfying $U' \supseteq P_a$ and $W' \supseteq P_b$, respectively. In fact, it is always possible to add vertices to P_a and P_b in such a way that following conditions hold for U' and W'.

(I) $U' \cap W' = \emptyset$

(II) $Conv(U')$ and $Conv(W')$ cross.

(III) $|U'| = |W'| = d$

(IV) $|U' \cap L_i| = 1$ for each i in the range $1 \leq i \leq d$

(V) $|W' \cap L_i| = 1$ for each i in the range $1 \leq i \leq d$

The argument above establishes the fact that $\overline{cr}_d(H') \geq 1$. Note that H contains $\binom{n}{3}^2 \binom{n}{2}^{d-2}$ distinct induced sub-hypergraphs, each of which is isomorphic to H'. Since each crossing pair of hyperedges is contained in $(n-2)^2$ distinct induced sub-hypergraphs of H, each of which is isomorphic to H', we obtain $\overline{cr}_d\left(K^d_{d\times n}\right) \geq \binom{n}{3}^2 \cdot \binom{n}{2}^{d-2} \Big/ (n-2)^2 = \dfrac{n^d(n-1)^d}{9\cdot 2^d} = \Omega\left(2^d\right)(n/2)^d\left((n-1)/2\right)^d$. □

Chapter 7

Conclusions

In this chapter, we summarize the contributions in this thesis. In Chapter 4, we gave a lower bound on the d-dimensional rectilinear crossing number of a complete d-uniform hypergraph with n vertices by using the Gale transformation and the Ham-Sandwich theorem. In Chapter 5, we investigated the d-dimensional convex drawing of a complete d-uniform hypergraph when all of its vertices are placed on the d-dimensional moment curve. In particular, we proved that the 3-dimensional convex crossing number of a complete 3-uniform hypergraph with n vertices is $3\binom{n}{6}$. We also investigated different types of d-dimensional rectilinear drawings of a complete d-uniform hypergraph having $2d$ vertices in convex as well as general position in $\mathbb{R}^d$. In Chapter 6, we established a non-trivial lower bound on the d-dimensional rectilinear crossing number of a complete balanced d-partite d-uniform hypergraph having nd vertices. We list some open problems related to the d-dimensional rectilinear drawings of the d-uniform hypergraphs.

- We already showed that the number of crossing pairs of hyperedges in a d-dimensional rectilinear drawing of K_n^d is asymptotically maximum when all of its vertices are placed on the d-dimensional moment curve. It is an interesting problem to produce a d-dimensional rectilinear drawing of K_n^d which maximizes the number of crossing pairs of hyperedges. The Upper Bound theorem [45] states that the d-dimensional cyclic polytope (i.e., the polytope whose vertices are all placed on the d-dimensional moment curve) has the maximum number of faces of any given dimension among all d-dimensional convex polytopes having the same number of vertices. Inspired by this result, we conjecture the following.

Conjecture 1. *The placement of n vertices on the d-dimensional moment curve maximizes the number of crossing pairs of hyperedges in a d-dimensional convex drawing of K_n^d.*

- Garey and Johnson [26] showed that given a graph G and an integer M, determining whether the crossing number of G is less than or equal to M is NP-complete. The same proof can be modified to show that determining whether the rectilinear crossing number of a graph G is less than or equal to M is NP-hard. For $d \geq 3$ and an integer N, it is an interesting open problem to prove that determining whether the d-dimensional rectilinear crossing number of a d-uniform hypergraph is less than or equal to N is NP-hard.
- There is a significant gap between the lower bound and the upper bound on the d-dimensional rectilinear crossing number of K_{2d}^d. It is an interesting problem to reduce this gap.
- It is an exciting problem to establish a non-trivial upper bound on the d-dimensional rectilinear crossing number of a complete balanced d-partite d-uniform hypergraph having n vertices in each part.
- Guy [31] noted that in a rectilinear drawing of a complete graph, the number of crossing pairs of edges is minimum when the convex hull of its vertices forms a triangle. A rigorous proof of this claim can be found in [2]. No such result is known for the d-dimensional rectilinear drawings of K_{2d}^d. Proving a similar result for the d-dimensional rectilinear drawings of K_{2d}^d will improve the lower bound on the d-dimensional rectilinear crossing number of K_{2d}^d.

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