

INDRAPRASTHA INSTITUTE OF INFORMATION TECHNOLOGY, DELHI

DOCTORAL THESIS

Higher Order Space and Time Discretizations of Maxwell's Equations

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(PhD20303)

*A thesis submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
to Indraprastha Institute of Information Technology, Delhi*

Advisor: Kaushik Kalyanaraman



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Certificate

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The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree/diploma.



June 2026

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Declaration

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*With profound gratitude,
this thesis is dedicated to my parents,
Mrs. PUSHPANJALI and Mr. ANIL KUMAR
in recognition of their constant support, encouragement,
and unwavering belief in me.*

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Abstract

COMPUTATIONAL electromagnetics involves the numerical solution of Maxwell's equations and has been one of the fundamental pillars of modern electrical engineering. Given the ubiquity of electrical and electronic devices at all power levels, from micro- to giga-watts in our everyday world, the importance of a provably correct and accurate finite element discretization of Maxwell's equations is both theoretically useful and practically relevant. In our work, we demonstrate such a structure-preserving higher-order numerical discretization for Maxwell's equations. Our work shows how modern finite elements, incorporating ideas from differential geometry and algebraic topology, can be used in conjunction with higher-order time discretization schemes and provide for highly accurate energy-conserving schemes.

In our work, we study a system of Maxwell's equations that describes the time evolution of electromagnetic fields with an additional electric scalar variable to make the system amenable to a compatible, mixed finite element spatial discretization. We demonstrate stability and energy conservation for the variational formulation of this Maxwell's system. We discuss two implicit, energy conserving schemes for its temporal discretization: the classical Crank-Nicolson scheme and an implicit leapfrog scheme. We show discrete stability and discrete energy conservation for the semi-discretization using these two time integration methods. We complete our discussion by showing that the error for the full discretization of the Maxwell's system with each of the two implicit time discretization schemes and with spatial discretization through a conforming sequence of de Rham finite element spaces converges quadratically in the step size of the time discretization and as an appropriate polynomial power of the mesh parameter in accordance with the choice of approximating polynomial spaces. Our results for the Crank-Nicolson method are generally well known but have not been demonstrated for this Maxwell's system. Our implicit leapfrog scheme is a new method to the best of our knowledge, and we provide a complete error analysis for it. Moreover, we show computational results to validate our theoretical claims using linear and quadratic Whitney forms for the finite element discretization for a model problem each in two and three spatial dimensions.

We then propose two energy conserving fourth-order time discretizations of the same three-field formulation of Maxwell's equations in conjunction with a spatial discretization using higher-order and compatible de Rham finite element spaces. Toward this end, we delineate two broad classes of strategies for general higher-order time discretizations which we term spatial and temporal strategies. We provide a description of these two strategies and develop fourth-order time accurate schemes in the context of our Maxwell's system. Moreover, our description can be used to prescribe similar fourth- or even higher-order time-integration methods for any linear (or quasi-linear) system of time-dependent partial differential equations. Our organizing principle in our proposed two strategies is to Taylor expand the unknown solution in time by assuming sufficient regularity. Then, in the spatial strategy, we use Maxwell's equations themselves to replace the fourth-order time derivatives in an appropriately truncated Taylor expansion with corresponding higher-order spatial derivatives. On the other hand, in the temporal strategy, we simply use higher-order finite difference schemes for the various higher-order time derivative terms in the truncated Taylor approximation. In both cases, we then defer to a standard finite element exterior calculus manner of compatible discretization for the spatial component of the Maxwell's solution.

Finally, we generalize our proposed time discretization scheme using spatial strategy to an arbitrarily higher (even) order implicit leapfrog scheme for time discretization of our Maxwell's system. We

use this in conjunction with an arbitrarily higher-order and compatible discretization using finite element spaces that form a de Rham complex. We prove stability, demonstrate energy conservation, and characterize the asymptotic convergence of the error for the time semidiscretization as well as for the full spatial and temporal discretization of this Maxwell's system.

Code Availability: Python code for all examples in this thesis to generate the various pieces output including tables and figures are available at the following GitHub archive:

https://github.com/archana180897/PhD_thesis.

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

Maxwell's equations are a set of coupled partial differential equations (PDEs) that describe the relationship between electric fields, magnetic fields, electric charges, and currents. They are widely used in electrical, power, and communication engineering for designing electrical machines, understanding field behavior, and analyzing electromagnetic wave propagation. Additionally, these equations are applied in electronics and electromagnetics to study components such as capacitors and inductors, electromagnetic interference and field distribution, and this enables accurate design and efficient operation of modern electrical systems. Mathematically, Maxwell's equations form a well-structured system of linear hyperbolic partial differential equations. Among the different classes of linear PDEs, hyperbolic equations help model wave propagation with finite speeds and admit a unique solution for a well-posed initial value problem. Moreover, being linear and hyperbolic, their solutions can help provide physical insights for time-domain electromagnetic simulations and communication engineering applications.

In the field of computational electromagnetics, researchers have long been satisfied with using lower-order time discretization methods and more generally even, numerical PDE solutions have often relied predominantly on lower-order time-integration schemes. The era of lower-order time schemes has been essentially due to computational considerations coupled with an engineering principle of a refinement in computations without resorting to higher-order methods. However, in the current era of scientific machine learning, there is in fact a demand for higher-order accurate schemes which may enable also long-time simulations, and this necessitates requiring to study and characterise higher-order time-integration methods necessary. Higher-order computations have often been prohibitively expensive on CPU like computational architectures due to high computational cost and their limited parallelism. In contrast, GPU computing, with its massive parallel processing capability, does make solutions using higher-order methods more feasible and subsequently allow for investigations into making them more efficient as well. As a concrete example, an “absurdly” higher-order variant of the classical Runge–Kutta scheme, of the order 1000, has indeed been used in the solution of Burger's equation for the purpose of training a neural network in one of the earliest works in Physics-informed neural networks [RPK19].

1.2 Notes on Literature and State of the Art

Computational solution schemes for Maxwell's equations has an exceedingly rich history and we will not be able to do justice to the vast compendium of available literature and its various contributors. Nevertheless, we wish to highlight some of the important ones and certainly this is biased as well as limited by our depth of knowledge of the history. The field of computational electromagnetics probably had its first big break with the development of the finite difference time domain (FDTD) scheme in [Yee66] which led to further developments later on in mimetic finite difference schemes for Maxwell's equations [HS99; Bos01; LMS14; Adl+21]. Posing Maxwell's equations in the language of differential forms [Des81], and a combinatorial and geometric discretization of exterior calculus as discrete exterior calculus (DEC) [Hir03] has led to a plethora of methods for Maxwell's equations such as direct geometric discretizations popularized by [Ton01a; Ton01b; Ton02], and solutions of Maxwell's equations using DEC by [Räb14; RMR15; CC17; SP18; Zha+23] to illustrate a few. Then there are time domain finite element methods for the solution of Maxwell's equations in various transient and time harmonic formulations. One can begin tracing their history from works by [Néd78; Néd80; Néd86] which introduced in the context of computational electromagnetics what we now refer to as Whitney forms. Then there are several contribution by Monk such as [Mon92a; Mon92b; MM95], his book [Mon03] and the tutorial-style article [CM12] which have provided analyses for finite element methods for time domain Maxwell's equations. Bossavit's popularization of the "rationale for using Whitney elements" in computational electromagnetism through his various works over the decades [Bos88b; Bos88a; Bos90; Bos92], via his book [Bos98a] as well as his insights on the discretization of electromagnetism posed via calculus on manifolds [Bos91; Bos98b; Bos10] have inspired several newer generations of researchers in the field including our own foray into this enterprise. We wish to also point to the various works by Hiptmair [Hip99; Hip02; Hip15], works by Christiansen works such as [Chr09], and many other broad finite element exterior calculus works such as [RB09a; RB09b; CMO11; BR14; CR16]. All of these provide foundations on which our work in this thesis stands on. We finally wish to also point out that error analysis for a full discretization of Maxwell's equations using finite elements in space and finite difference methods in time are still incomplete and prior to our work in this thesis, were only clarified for the very simple backward Euler implicit scheme for time differencing as in [AA19].

1.3 Finite Element Method for Linear PDEs

The finite element method is a powerful numerical technique used to approximate solutions to a wide range of engineering and scientific problems involving partial differential equations and complex geometries. The finite element method has its roots in the 1950s when engineers began using it to solve complex problems in elasticity and structural mechanics. Since then, the spectrum of applications has continuously increased, and today it is fairly diverse. The finite element method has indeed evolved over the years to become one of the most widely used numerical techniques for solving complex engineering problems described by partial differential equations. Its success is attributed to the rigorous mathematical framework developed over the past decades, enabling error analysis and understanding of different finite element methods. The concept of approximating solutions of differential equations through a discrete algorithm, often referred to as a "black box", is a fundamental to the usage of the finite element method especially by the engineering

community. In such a typical understanding, a discretization of a problem domain is performed using elemental subdomains which are connected to form a mesh, and computations are performed using such a mesh. The elemental domains can be simplicial or polyhedral but generally restricted to be convex polytopes, and any curvatures in the domain are then suitably approximated using piecewise linear segments. The interpolation capability of finite elements and proper mathematical modelling are crucial for achieving accurate results and maintaining properties like continuity, stability, and consistency. The finite element method relies on functional analysis tools, such as Hilbert and Banach spaces, Sobolev spaces, and various mathematical theorems, to establish the mathematical foundations and ensure the validity of the approximations. The approach of using weak formulations in the continuous domain, which involves test functions and variational formulations, ensures that the strong and weak forms are equivalent at the continuous level. The use of weak formulations allows for greater flexibility in handling various boundary conditions and simplifying the treatment of complicated geometries. The finite element method has become a mature and essential tool in computational science, offering a systematic way to solve a wide array of differential equations and analyze complex systems.

Finite element theory especially its presentation under the unified finite element exterior calculus is already now standard and well described in many works such as [Arn18; BS08; EG21a; EG21b; EG21c]. Here, in this section and the next chapter, we aim to present a simplified version of the theory and clarified with ample pictorial illustrations for an enhanced understanding. We intend our presentation to help complement the more rigorous textbook descriptions. We discuss finite element method on Poisson's Equation in two dimensions (2d) and three dimensions (3d) with various boundary conditions in a detailed way in Section 2.1, and this will serve as an introduction to the finite element method. In that section, we explain the finite element techniques for the numerical solution of two-dimensional partial differential equations. The fundamental idea behind doing so is to first transform the boundary value issue into a variational equation, and then look for an approximate solution to this equation in the space of continuous piecewise linear functions. The differential equation under discussion is first rewritten as a variational equation for the process of deriving a finite element approach. In our example, the variational formulation is set up by taking inner products with so-called test functions. Then, we recast the equation in the variational form by suitably employing the generalized Stokes' theorem, that is, integration by parts, to move second-order derivatives to first-order ones. This allows for the spaces of solutions to therefore be expanded and we can seek an approximation, for example, in the space of piecewise continuous linear functions. We concentrate on linear second-order elliptic equations using the Poisson equation as our primary model and our outlined numerical approaches can then be readily adopted for other problems. In our approach, we have tried to clearly explain the fundamental ideas such as approximation using piecewise polynomial spaces and the variational formulation. We encapsulate these steps to devise a finite element method in a flowchart in Figure 1.1, and we elaborate on this with the help of the following example. Consider the 1D Poisson's equation with homogeneous Dirichlet boundary conditions:

$$-\frac{d^2u}{dx^2} = f \text{ on } \Omega = [0, 1], \quad u(\partial\Omega) = 0. \quad (1.1)$$

First we write the variational formulation corresponding to strong PDE Equation (1.1) and so we have, for all choices of a test function v from a suitable choice of a function space that makes sense,

that:

$$\begin{aligned}
& \left\langle -\frac{d^2 u}{dx^2}, v \right\rangle = \langle f, v \rangle \quad , \quad v \in L^2([0, 1]), \\
& \Rightarrow \int_0^1 -\frac{d^2 u}{dx^2} v \, dx = \int_0^1 f v \, dx, \\
& \Rightarrow \int_0^1 -\frac{d^2 u}{dx^2} v \, dx = \left[-\frac{du}{dx} v \right]_0^1 + \int_0^1 \frac{du}{dx} \frac{dv}{dx} \, dx, \quad (\text{using integration by-parts}) \\
& \Leftrightarrow \int_0^1 \frac{du}{dx} \frac{dv}{dx} \, dx + \left[-\frac{du}{dx} v \right]_0^1 = \int_0^1 f v \, dx \quad , \quad v \in \dot{H}^1([0, 1]).
\end{aligned}$$

Now, we need to choose a appropriate function space that makes the problem well posed, and for the Dirichlet boundary conditions we seek $u \in V = \dot{H}^1([0, 1])$. We will then pick a finite-dimensional subspace $V_h \subseteq V$. Finally, we will use a basis, say, $\{\phi_i\}_{i=0}^{\dim V_h}$ of V_h that results into a linear system:

$$\left[\int \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \right] [u_i] = \left[\int f \phi_j \right].$$

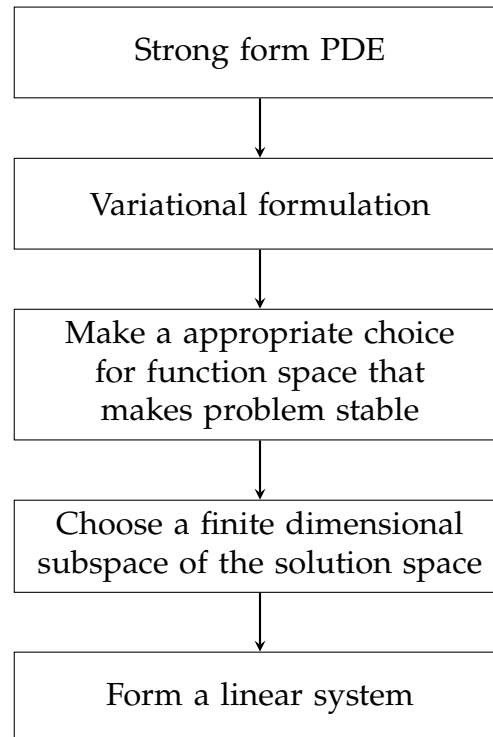


FIGURE 1.1: Finite Element Solution Workflow

In the case of 2d and 3d problems, we can solve the scalar Laplacian with Dirichlet or Neumann boundary conditions via both *standard formulation* and *mixed formulation* whereas vector Laplacians

can be reliably and stably solved using finite elements only via a *mixed formulation*. By mixed finite element formulation we mean a method which converts a system of second-order PDEs into a system involving only first-order PDEs, and potentially with some additional equality-type constraints. These methods involve using different finite element spaces for the various variables and a Lagrange multipliers approach to enforce constraints, leading to a system of constrained linear system of equations. One of the challenges in mixed finite element methods is selecting appropriate finite element spaces to ensure convergence and accuracy. Not all combinations of spaces will lead to reliable approximations, and the choice of spaces may be influenced by the specific problem being solved, the PDE being approximated, and the constraints involved. This involves analyzing stability, coercivity, and consistency conditions to guarantee convergence of the numerical solution to the true solution of the underlying PDE.

1.3.1 Weak Derivatives

Classical differentiation requires a function to be sufficiently smooth, which is often too restrictive for many problems arising in analysis and partial differential equations. Weak derivatives provide a generalized notion of differentiation that allows less regular functions to be studied while preserving many of the essential properties of classical derivatives.

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ denote the space of infinitely differentiable functions with compact support in Ω . Observe that $\mathcal{D}(\Omega)$ is a vector space and that if $\phi \in \mathcal{D}(\Omega)$, then $\frac{\partial \phi}{\partial x_i} \in \mathcal{D}(\Omega)$ for each $i = 1, \dots, n$. A multi-index is an n -tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of non-negative integers. The order of α is defined by

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n.$$

For a sufficiently smooth function u , we write

$$D^\alpha u = \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \left(\frac{\partial}{\partial x_2} \right)^{\alpha_2} \dots \left(\frac{\partial}{\partial x_n} \right)^{\alpha_n} = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}.$$

A measurable function $f : \Omega \rightarrow \mathbb{R}$ is said to be locally integrable if $\int_K |f| dx < \infty$ for every compact set

$K \subset \Omega$. The space of all locally integrable functions on Ω is denoted by $L^1_{\text{loc}}(\Omega)$, where integrability is understood in the Lebesgue sense.

Let $u \in L^1_{\text{loc}}(\Omega)$. We say that u admits a weak derivative corresponding to the multi-index α if there exists a function $u_\alpha \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_{\Omega} u D^\alpha \phi dx = (-1)^{|\alpha|} \int_{\Omega} u_\alpha \phi dx \quad \forall \phi \in \mathcal{D}(\Omega).$$

The function u_α is called the weak derivative of u corresponding to α and is denoted by $D^\alpha u$. The above integrals are well-defined since u and u_α are locally integrable and every test function $\phi \in \mathcal{D}(\Omega)$ has compact support. Moreover, weak derivatives are unique upto equality almost everywhere.

A function u is said to be weakly differentiable if all its first-order weak derivatives exist.

As an illustration, consider the case $n = 1$ with $\Omega = (0, 2)$, and define

$$u(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 \leq x < 2 \end{cases} \quad \text{and} \quad v(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & 1 \leq x < 2. \end{cases}$$

We claim that v is the weak derivative of u . To verify this, let $\phi \in \mathcal{D}(0, 2)$. Then

$$\int_0^2 u(x) \phi'(x) dx = \int_0^1 x \phi'(x) dx + \int_1^2 \phi'(x) dx.$$

Using integration by parts on the first integral and the Fundamental Theorem of Calculus on the second, we obtain

$$\int_0^2 u(x) \phi'(x) dx = [x \phi(x)]_0^1 - \int_0^1 \phi(x) dx + [\phi(x)]_1^2.$$

Since ϕ has compact support in $(0, 2)$, we have $\phi(0) = \phi(2) = 0$. Therefore,

$$\int_0^2 u(x) \phi'(x) dx = - \int_0^1 \phi(x) dx = - \int_0^2 v(x) \phi(x) dx.$$

Hence,

$$\int_0^2 u(x) \phi'(x) dx = - \int_0^2 v(x) \phi(x) dx,$$

for all $\phi \in \mathcal{D}(0, 2)$, which shows that $v = D^1 u$ in the weak sense.

This example illustrates that weak derivatives may exist even when a function is not classically differentiable everywhere. In particular, weak differentiation allows functions with mild singularities or discontinuities in their classical derivatives to be treated within a unified analytical framework.

Remark 1.1. Throughout this thesis, we primarily work with classical (strong) derivatives in order to simplify the presentation. Nevertheless, the results and arguments developed in the subsequent chapters extend naturally to the corresponding weak formulations by replacing classical derivatives with the appropriate weak derivatives.

1.3.2 Stability Theorem for Mixed Formulation

In this section we discuss about the stability and well-posedness of the mixed formulation of PDEs, first we introduce some notations that is used throughout this section. The space $V^k \subseteq W^k$ denotes either of the vector spaces of scalar functions or vector fields, and the space $\mathfrak{H}^k$ denotes the space of harmonics. The operator d denotes either of the operators grad, curl, or $-\text{div}$ from vector calculus and then $L^k u = dd^*u + d^*du$. The $*$ in a superscript denotes the adjoint of the vector space or the

operator as appropriate.

$$\text{Domain of } L^k = \{u \in V^k \cap V^{k*} \mid du \in V^{k+1*}, d^*u \in V^{k-1}\}, \quad (1.2)$$

where V^k and V^{k*} are the domain spaces of d and d^* respectively. In order to formulate a problem which is solvable for any $f \in W^k$, we use the orthogonal projection $P_{\mathfrak{H}}$ of W^k onto $\mathfrak{H}^k$. We can formulate the following *strong form* of the Hodge Laplace equation, which is well-posed.

Strong formulation: If u is a solution to the problem: given $f \in W^k$, find $u \in D(L^k)$ such that

$$Lu = f - P_{\mathfrak{H}}f, \quad u \perp \mathfrak{H}^k. \quad (1.3)$$

We will consider two kinds of weak formulation of the above problem.

Primal weak formulation: This is also known as *standard weak formulation* and it directly corresponds to the strong form. For the above problem we can represent it as:

Given $f \in W^k$, find $u \in V^k \cap V^{k*}$ such that $u \perp \mathfrak{H}$ and

$$\langle du, dv \rangle + \langle d^*u, d^*v \rangle = \langle f - P_{\mathfrak{H}}f, v \rangle, \quad v \in V^k \cap V^{k*}. \quad (1.4)$$

Mixed weak formulation: For this weak formulation of above problem we introduce two new variables $\sigma = d^*u \in V^{k-1}$ and $p = P_{\mathfrak{H}}f \in \mathfrak{H}^k$, so the above problem is formulated as:

Given $f \in W^k$, find $\sigma \in V^{k-1}$, and $u \in V^k$ such that

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, d\tau \rangle &= 0, \quad \tau \in V^{k-1}, \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= 0, \quad v \in V^k, \\ \langle u, q \rangle &= 0, \quad q \in \mathfrak{H}^k. \end{aligned} \quad (1.5)$$

Unlike the other two formulations, the mixed weak formulation does not treat the operators d and d^* symmetrically. Only the operator d appears in this formulation, the terms involving d^* are done away with using the integration-by-parts formula. Despite their different appearances, for a so-called sufficiently regular solution, the three formulations are equivalent. We now state some theorems regarding the well-posedness of these problem formulations.

Theorem 1.2 (Equivalence of formulations, [Arn18]). *Let $f \in W^k$. An element $u \in W^k$ solves the strong formulation of the abstract Hodge Laplace problem if and only if it solves the primal weak formulation. Moreover, in this case, if we set $\sigma = d^*u$ and $p = P_{\mathfrak{H}}f$, then the triple (σ, u, p) solves the mixed weak formulation. Finally, if some (σ, u, p) solves the mixed weak formulation, then u solves the other two formulations of the problem, and $\sigma = d^*u$, $p = P_{\mathfrak{H}}f$.*

Theorem 1.3 (Well-posedness of Mixed Formulation, [Arn18]). *For each $f \in W^k$ there exists a unique solution $u \in D(L^k)$ such that $u \perp \mathfrak{H}^k$ and $Lu = f - P_{\mathfrak{H}}f$. Moreover*

$$\|u\| + \|du\| + \|d^*u\| + \|dd^*u\| + \|d^*du\| + \|p\| \leq c\|f\|. \quad (1.6)$$

The constant c entering this estimate depends only on the constant c_P entering the Poincaré inequality.

Theorem 1.4 (Bilinear form inf-sup Satisfaction, [Arn18]). *The bilinear form B defined by*

$$B(\sigma, u, p; \tau, v, q) = \langle \sigma, \tau \rangle - \langle u, d\tau \rangle - \langle d\sigma, v \rangle - \langle du, dv \rangle - \langle p, v \rangle - \langle u, q \rangle, \quad (1.7)$$

satisfies the inf-sup condition

$$\gamma := \inf_{0 \neq w \in X} \sup_{0 \neq y \in X} \frac{B(w, y)}{\|w\|_X \|y\|_X} > 0, \quad (1.8)$$

with γ bounded below by a constant depending only on the Poincaré constant c_P .

1.3.3 A Priori Error Analysis

In this section, we introduce error estimates for the mixed method. In the numerical analysis of differential equations, error estimates are obtained from the stability and consistency of the discretization scheme. The following theorem gives the resulting error estimates in which the spaces V^k and $\mathfrak{H}^k$ are the same spaces defined as in previous section.

Theorem 1.5 (Basic error estimate for the mixed method, [Arn18]). *Assume that the complex*

$$V_h^{k-1} \xrightarrow{d} V_h^k \xrightarrow{d} V_h^{k+1}$$

is a finite dimensional subcomplex of

$$V^{k-1} \xrightarrow{d} V^k \xrightarrow{d} V^{k+1}$$

which admits a V -bounded cochain projection π_h . Let $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ solve the Hodge Laplace problem (37) and let $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ solve the discrete problem

$$\langle \sigma_h, \tau \rangle - \langle u_h, d\tau \rangle = 0, \quad \tau \in V_h^{k-1}, \quad (1.9)$$

$$\langle d\sigma_h, v \rangle + \langle du_h, dv \rangle + \langle p_h, v \rangle = 0, \quad v \in V_h^k, \quad (1.10)$$

$$\langle u_h, q \rangle = 0, \quad q \in \mathfrak{H}_h^k. \quad (1.11)$$

Then

$$\begin{aligned} \|\sigma - \sigma_h\|_V + \|u - u_h\|_V + \|p - p_h\| &\leq C \left(\inf_{\tau \in V_h^{k-1}} \|\sigma - \tau\|_V + \inf_{v \in V_h^k} \|u - v\|_V \right. \\ &\quad \left. + \inf_{q \in V_h^k} \|p - q\|_V + \mu \inf_{v \in V_h^k} \|P_{\text{im}} du - v\|_V \right), \end{aligned} \quad (1.12)$$

where

$$\mu = \mu_h^k = \sup_{\substack{r \in \mathfrak{H}^k \\ \|r\|=1}} \|(I - \pi_h^k)r\| \quad (1.13)$$

and the constant C depends only on c_P and $\|\pi_h^k\|$.

Now for the rates of error approximations we have the following table below for 3d in which we provide the rates of approximation for the choices of the various finite dimensional polynomial subspaces.

V_h^{k-1}	V_h^k	σ	u
$\mathcal{P}_r \cap H^1$	$\mathcal{P}_r \cap H(\text{curl})$	$r+1$	r
$\mathcal{P}_r \cap H(\text{curl})$	$\mathcal{P}_r \cap H(\text{div})$	r	r
$\mathcal{P}_r \cap H(\text{div})$	$\mathcal{P}_r \cap L^2$	r	r

1.3.4 Adjoints: A Finite Dimensional Vector Space Discussion

Suppose $T : U \longrightarrow V$ is a linear map between finite dimensional vector spaces U and V . Let $\langle \cdot, \cdot \rangle_U$ be the inner product on U and $\langle \cdot, \cdot \rangle_V$ be the inner product on V . Then for any $u \in U, v \in V$, we can write $\langle T(u), v \rangle_V \in \mathbb{R}$. The *adjoint* of T is a linear map $T^* : V \longrightarrow U$ such that

$$\langle T(u), v \rangle_V = \langle u, T^*(v) \rangle_U, \quad u \in U, v \in V.$$

For instance, if $U = \mathbb{R}^n$ and $V = \mathbb{R}^m$, then T can be realized as a matrix $A \in \mathbb{R}^{m \times n}$ using the standard basis and the adjoint T^* simply as A^T the transpose of A . This can be readily seen as the following and by using the standard inner product on Euclidean spaces:

$$\langle u, T^*v \rangle = \langle Au, v \rangle, \implies \sum_j \sum_i a_{ij}^* u_j v_i = \sum_j \sum_i a_{ij} u_j v_i,$$

By comparing terms of the above sum we get $a_{ij}^* = a_{ji}$, and further it implies $A^* = A^T$. This notion can also be extended for infinite-dimensional vector spaces. Let us consider this via the example of $\text{grad} : \text{Scalar functions} \longrightarrow \text{Vector fields}$ which is a linear map. It turns out for $\text{grad } u$ of a scalar function u to be well defined, we need the function space of scalar functions to be $L^2(\Omega)$, that is L^2 functions on the domain $\Omega \subseteq \mathbb{R}^n$. Thus, $\text{grad} : H^1(\Omega) \longrightarrow [L^2(\Omega)]^n$ and $u \in H^1(\Omega)$ which implies that $\int_{\Omega} u^2 dx < \infty$ which in turn means that $\text{grad } u \in [L^2(\Omega)]^n$ and finally $\int_{\Omega} (\text{grad } u) \cdot (\text{grad } u) dx < \infty$. The space of L^2 -vector fields $[L^2(\Omega)]^n$ is simply one in which every component of a vector field is a L^2 -function. Now, the adjoint of $\text{grad} : L^2(\Omega) \longrightarrow [L^2(\Omega)]^n$, by definition grad^* (adjoint of grad) satisfies:

$$\langle \text{grad } u, \tau \rangle_{[L^2(\Omega)]^n} = \langle u, \text{grad}^* \tau \rangle_{H^1(\Omega)} \iff \int_{\Omega} \text{grad } u \cdot \tau dx = \int_{\Omega} u \cdot \text{grad}^* \tau dx.$$

To determine grad^* , we use integration by parts. We know that for $\Omega \subseteq \mathbb{R}^n$ (open), $u \in H^1(\Omega)$, $v \in [L^2(\Omega)]^n$ satisfies:

$$\int_{\Omega} (\text{grad } u) \cdot v \, d\Omega = - \int_{\Omega} u(\text{div } v) \, d\Omega + \int_{\Gamma} u(v \cdot \hat{n}) \, d\Gamma, \quad \Gamma = \partial\Omega,$$

where $\hat{n}$ is unit normal to $\partial\Omega$. Comparing with what we have, we get:

$$\begin{aligned} \int_{\Omega} \text{grad } u \cdot \tau \, dx &= \int_{\Omega} u \cdot \text{grad}^* \tau \, dx, \\ \Rightarrow - \int_{\Omega} u(\text{div } \tau) \, dx + \int_{\Gamma} u(v \cdot \hat{n}) \, d\Gamma &= \int_{\Omega} u \cdot \text{grad}^* \tau \, dx, \quad (\text{By integration by parts}) \\ \Rightarrow - \int_{\Omega} u(\text{div } \tau) \, dx + \int_{\Gamma} u(v \cdot \hat{n}) \, d\Gamma &= \int_{\Omega} u \cdot \text{grad}^* \tau \, dx. \end{aligned}$$

If $\int_{\Gamma} u(\tau \cdot \hat{n}) \, d\Gamma = 0$ either (i) $u = 0$ on $\partial\Omega$ or (ii) $\tau \cdot \hat{n} = 0$ on $\partial\Omega$. This will ensure $\int_{\Gamma} u(\tau \cdot \hat{n}) \, d\Gamma$ vanishes and (ii) means that τ on the boundary is tangential to the boundary $\partial\Omega$. Choosing (i) $u = 0$ on $\partial\Omega$ is much easier! For grad to be well defined and have a well defined adjoint, pick that subset of $L^2(\partial\Omega)$ such that their value on $\partial\Omega$ is zero. We gently redefine the domain of grad , therefore $\text{grad} : V \rightarrow [L^2(\Omega)]^n$, where $V = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega\}$. Further by choice, $\text{grad } u \in [L^2(\Omega)]^n$. In one dimensions (1d), $u \in L^2, u' \in L^2 \iff u \in H^1$, similarly, $V = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega\}$, and it is then denoted by $\dot{H}^1(\Omega)$. Thus, $\text{grad} : \dot{H}^1(\Omega) \rightarrow [L^2(\Omega)]^n$ has a adjoint and its adjoint is simply $-\text{div}$. This can be written as:

$$\langle \text{grad } u, \tau \rangle = \langle u, \text{div } \tau \rangle \quad \forall u \in \dot{H}^1(\Omega), \tau \in [L^2(\Omega)]^n.$$

Now, going back to the FEM problem and using Equations (1.3) to (1.5), we can write our problem as:

Strong form PDE: $\sigma - \text{grad } u = 0$,

Weak form PDE: $\langle \sigma - \text{grad } u, \tau \rangle = 0$,

Adjoint argument: $\langle \sigma, \tau \rangle - \langle \text{grad } u, \tau \rangle = 0 \implies \langle \sigma, \tau \rangle - \langle u, \text{div } \tau \rangle = 0$.

Now, it is enough for $u \in \dot{H}^1(\Omega)$ and $\tau \in [L^2(\Omega)]^n$.

1.3.5 Coercivity of Bilinear Form

Let $a(\cdot, \cdot)$ be a bilinear form then $a(\cdot, \cdot)$ is coercive if $a(u, u) \geq \alpha \|u\|^2 \forall u$ and $\alpha > 0$ is *coercivity constant*. If u is in a finite dimensional vector space, $a(\cdot, \cdot)$ has a matrix representation, call it A , then $a(u, v)$ would be $v^T A u$. Coercivity in a finite dimensional setting then means; $u^T A u \geq \alpha u^T W u$, where W would be a SPD matrix. Then the L^2 norm $\|u\|_{L^2} \implies W = I$, and $u^T A u \geq \alpha u^T u > 0 \iff \text{Rayleigh Quotient} := \frac{u^T A u}{u^T u} \geq \alpha > 0$. Further, the minimum eigenvalue is strictly nonzero: “well-separated” from zero. For the Standard formulation, FEM coercivity would provide existence and uniqueness of discrete solution, whereas for the Mixed formulation, the bilinear

form will not be coercive. The existence and uniqueness of the discrete problem will generalize to singular values of the matrix being non-zero. This is captured in what's called the *inf-sup theorem* sometimes also called *LBB inf-sup theorem*. To tie things up, $\sigma \in [L^2(\Omega)]^d$ and $u \in H^1(\Omega)$ will not satisfy discrete inf-sup, it turns out that $\sigma \in H(\text{div}, \Omega)$, i.e., σ is a L^2 vector field and its divergence is a L^2 function:

$$\int_{\Omega} \sigma \cdot \sigma < \infty \quad \text{and} \quad \int_{\Omega} (\text{div } \sigma)^2 < \infty,$$

and $u \in L^2(\Omega)$ will yield a stable finite dimensional inf-sup condition. What this means is that we don't move the " $-\text{div}$ " away from σ in the variational formulation

$$\begin{cases} \sigma = \text{grad } u, \\ -\text{div } \sigma = f, \end{cases} \xRightarrow{\text{V.F.}} \begin{cases} \langle \sigma, \tau \rangle = \langle \text{grad } u, \tau \rangle, \\ \langle -\text{div } \sigma, v \rangle = \langle f, v \rangle, \end{cases}$$

$$\Rightarrow \text{Seek } (u, \sigma) \text{ such that: } \begin{cases} \langle \sigma, \tau \rangle = \langle u, -\text{div } \tau \rangle, \\ \langle -\text{div } \sigma, v \rangle = \langle f, v \rangle, \end{cases} \quad \forall (\tau, v)$$

$$\begin{cases} \langle \sigma, \tau \rangle + \langle u, \text{div } \tau \rangle = 0, \\ \langle -\text{div } \sigma, v \rangle = \langle f, v \rangle. \end{cases} \quad \forall (\tau, v)$$

Fix function spaces: $u, v \in L^2(\Omega); \sigma, \tau \in H(\text{div}, \Omega)$.

Discrete spaces: finite dimensional subspaces of these function spaces.

1.3.6 The Function Spaces

Our setup is like that we consider our domain $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega = \Gamma_D + \Gamma_N$ ($\Gamma_D \cap \Gamma_N = \emptyset$), where Γ_D denotes Dirichlet and Γ_N denotes Neumann boundaries, and following are the definitions of function spaces:

$$L^2(\Omega) := \{u \mid \int_{\Omega} u^2 dx < \infty\},$$

$$H^1(\Omega) := \{u \in L^2(\Omega) \mid \int_{\Omega} \|\nabla u\|^2 dx < \infty\},$$

$$\dot{H}^1(\Omega) := \{u \in H^1(\Omega) \mid u|_{\partial\Omega} = 0\} \quad (\partial\Omega = \Gamma_D),$$

$$H^1(\Omega) := \{u \in L^2(\Omega) \mid \text{All partial derivatives up to order } r \text{ are in } L^2(\Omega)\},$$

$$[L^2(\Omega)]^n := \{u \in \mathbb{R}^n \mid u_i \in L^2(\Omega), i = 1, \dots, n\},$$

$$= \{u \in \mathbb{R}^n \mid \int_{\Omega} u \cdot u \, dx < \infty\},$$

$$H(\text{div}; \Omega) := \{u \in [L^2(\Omega)]^n \mid \text{div } u \in L^2(\Omega)\},$$

$$\dot{H}(\text{div}; \Omega) := \{u \in H(\text{div}; \Omega) \mid u \cdot n = 0 \text{ on } \partial\Omega, \, n \text{ is a unit normal on } \partial\Omega\},$$

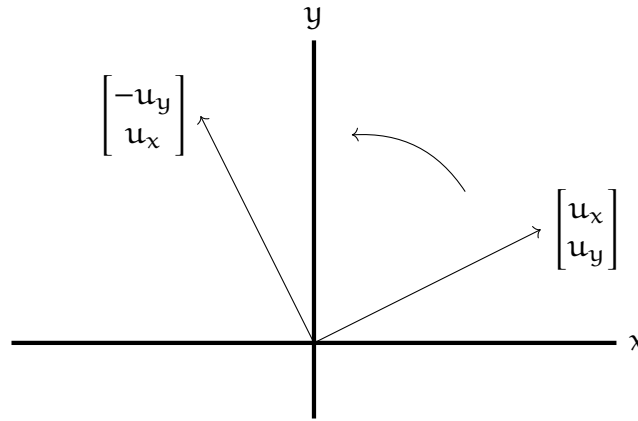
$$H(\text{curl}; \Omega) := \{u \in [L^2(\Omega)]^n \mid \text{curl } u \in [L^2(\Omega)]^n\}, \quad (n \geq 3), \text{ where}$$

$$\text{curl } u = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{bmatrix}.$$

If $n = 2$, $\text{curl } \mathbf{u}$ is obtained by setting $u_z = 0$ and $\frac{\partial}{\partial z}$ is taken to be 0 for u_x and u_y . Thus, in $n = 2$, for $\mathbf{u} = u_x \hat{x} + u_y \hat{y}$,

$$\text{curl } \mathbf{u} = \left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right) \hat{z}.$$

This is taken to be a scalar function by dropping the $\hat{z}$. Comparing it with $\text{div } \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}$, $\text{curl } \mathbf{u}$ is the divergence of the vector field $\mathbf{u}$ rotated by $\frac{\pi}{2}$ counterclockwise.



That is, rotation matrix is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ for 90° counterclockwise.

Remark 1.6. 1. All the function spaces listed above are infinite dimensional ones.

2. For $L^2(\Omega)$, $H^1(\Omega)$, $\dots$, $H^r(\Omega)$ finite dimensional subspaces are appropriate polynomial spaces.

3. There are infinitely many basis choices but some are named bases and are used more commonly.

1.3.7 $H(\text{div}; \Omega)$ and its Finite Dimensional Subspaces: Lowest Order

$$H(\text{div}; \Omega) := \{ \mathbf{u} \in [L^2(\Omega)]^n \mid \text{div } \mathbf{u} \in L^2(\Omega) \}.$$

Note that this is an infinite dimensional vector space consisting of vector fields which are geometrically vectors at each point in Ω . Let $V(\Omega) \subseteq H(\text{div}; \Omega)$ be a finite dimensional subspace of $H(\text{div}; \Omega)$. In analogy as with functions, a finite dimensional subspace would consist of polynomial vector fields. That is each component of the vector field is a polynomial on Ω . For example,

1. Constant vector fields:

$$V^0(\Omega) = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \mid u_i \in \mathbb{R}, i = 1, \dots, n \right\}$$

Thus, $\dim V^0(\Omega) = n$.

2. Linear vector fields:

$$V^1(\Omega) = \left\{ u = \begin{bmatrix} u_1 + v_1 x_1 \\ \vdots \\ u_n + v_n x_n \end{bmatrix} \mid u_i, v_i \in \mathbb{R}, i = 1, \dots, n \right\}$$

Thus, $\dim V^1(\Omega) = 2n$.

Note that this is already a big simplification! A general linear vector field on $\Omega \subseteq \mathbb{R}^n$ would really be:

$$\begin{bmatrix} u_1^0 + u_1^1 x_1 + u_1^2 x_2 + \dots + u_1^n x_n \\ u_2^0 + u_2^1 x_1 + u_2^2 x_2 + \dots + u_2^n x_n \\ \vdots \\ u_n^0 + u_n^1 x_1 + u_n^2 x_2 + \dots + u_n^n x_n \end{bmatrix}$$

The dimension of this space is: $n(n+1)$, so for $n=2$, this is: $2 \cdot 3 = 6$.

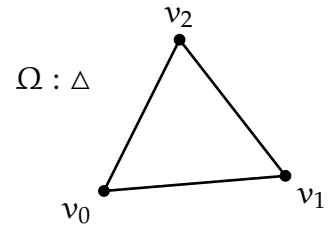
Finite dimensional subspace of $H(\text{div}; \Omega)$ on a 2-simplex

1. lowest-order case: Constant vector fields:

$$V^0(\Delta) = \left\{ u = \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}; \quad \dim V^0(\Delta) = 2.$$

To simplify, we could have:

$$\tilde{V}^0(\Delta) = \left\{ u = \begin{bmatrix} a \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}; \quad \dim \tilde{V}^0(\Delta) = 1.$$



2. Linear vector fields and variations:

(a)

$$V^1(\Delta) = \left\{ u = \begin{bmatrix} a_1 + b_1 x \\ a_2 + b_2 y \end{bmatrix} \mid a_1, b_1, a_2, b_2 \in \mathbb{R} \right\}; \quad \dim V^1(\Delta) = 4.$$

A basis may be fixed (that is coefficients determined) by demanding the basis values at vertices of Δ and the barycenter (for example) as specified, that is, let $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ be a basis for $V^1(\Delta)$ then we'll say that:

$$\phi_i(v_j) = \begin{cases} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \text{if } i = j \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{if } i \neq j \end{cases} \quad i, j = 1, 2, 3; \quad \phi_4\left(\frac{v_1 + v_2 + v_3}{3}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \phi_4(v_i) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad i = 1, 2, 3.$$

(b)

$$V^1(\Delta) = \left\{ u = \begin{bmatrix} a + bx \\ c + by \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}; \quad \dim V^1(\Delta) = 3.$$

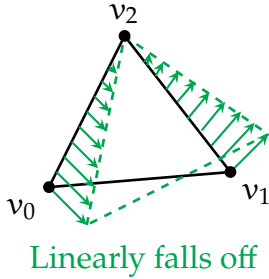
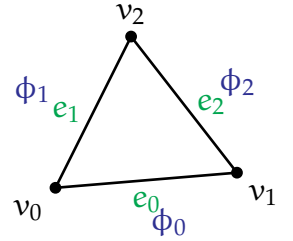
(c)

$$V^1(\Delta) = \left\{ u = \begin{bmatrix} a + bx \\ a + cy \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}; \quad \dim V^1(\Delta) = 3.$$

The space in (b) has a special name *Raviart-Thomas basis of lowest-order*. It is denoted by $\mathcal{RT}_0(\mathcal{RT}_0)$. And we equivalently write it as:

$$\mathcal{RT}_0 = [\mathbb{P}_0]^n \bigoplus \mathbf{xP}_n = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} c \mid a, b, c \in \mathbb{R} \right\} \quad (n = 2); \quad \dim \mathcal{RT}_0 = 3.$$

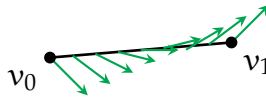
Now, we will put some light on the geometrical interpretation of $\mathcal{RT}_0(\Delta)$ and its basis on the triangle Δ with vertices $\{v_0, v_1, v_2\}$. Let $\{\phi_1, \phi_2, \phi_3\}$ be as basis corresponding to the edges $\{e_0, e_1, e_2\}$ of Δ respectively as depicted in the figure on the right. The basis is defined as $\int_{e_j} \phi_i \cdot t = \delta_{ij}$, where t is the tangential component.



For instance, for basis ϕ_0 , we have:

$$\int_{e_0} \phi_0 \cdot t = 1, \quad \int_{e_1} \phi_0 \cdot t = 0, \quad \int_{e_2} \phi_0 \cdot t = 0,$$

and ϕ_0 will look like as in the figure on the left.



On edge e_0 , it will look like as:

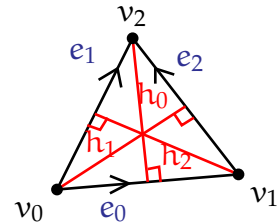
(The pictures here are technically for Whitney basis W_0 . The Whitney basis is a 90° counterclockwise rotation of the $\mathcal{RT}_0$ basis.)

3. Whitney basis for finite dimensional subspace of $H(\text{div})$:

$$W_0(\Delta) := 90^\circ / \frac{\pi}{2} \text{ rotation} \left\{ \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} c \mid a, b, c \in \mathbb{R} \right\},$$

or

$$:= \text{span}\{\phi_0, \phi_1, \phi_2\}.$$



Remark 1.7. The space W_0 is a subspace of all linear vector fields.

We shall write $\{\phi_0, \phi_1, \phi_2\}$ using barycentric functions $\{\lambda_0, \lambda_1, \lambda_2\}$ and will denote it as $\{W_{01}, W_{02}, W_{12}\}$ as represented in the table below:

Associated with edge	Basis	Expression in terms of $\{\lambda_0, \lambda_1, \lambda_2\}$
e_0	W_{01}	$\lambda_0 \nabla \lambda_1 - \lambda_1 \nabla \lambda_0$
e_1	W_{02}	$\lambda_0 \nabla \lambda_2 - \lambda_2 \nabla \lambda_0$
e_2	W_{12}	$\lambda_1 \nabla \lambda_2 - \lambda_2 \nabla \lambda_1$

We can verify its defining property for W_{01} :

$$\int_{e_0} W_{01} \cdot \mathbf{t} = 1; \int_{e_1} W_{01} \cdot \mathbf{t} = 0; \int_{e_2} W_{01} \cdot \mathbf{t} = 0$$

Note that $\int_{e_1} W_{01} \cdot \mathbf{t} = \int_{e_1} (\lambda_0 \nabla \lambda_1 - \lambda_1 \nabla \lambda_0) \cdot \mathbf{t}_{e_1}$, where $\nabla \lambda_1$ is a constant vector of magnitude equal to the altitude h_1 and in the direction of this altitude. This altitude is perpendicular to edge v_0, v_2 , i.e., $\nabla \lambda_1 \perp \mathbf{t}_{e_1}$. Thus, $\int_{e_1} W_{01} \cdot \mathbf{t}_{e_1} = 0$. Likewise, $\nabla \lambda_0 \perp \mathbf{t}_{e_2}$ and thus,

$$\int_{e_2} W_{01} \cdot \mathbf{t}_{e_2} = \int_{e_2} (\lambda_0 \nabla \lambda_1 - \lambda_1 \nabla \lambda_0) \cdot \mathbf{t}_{e_2} = 0.$$

We shall leave the verification of $\int_{e_0} W_{01} \cdot \mathbf{t}_{e_0} = 1$ for the reader. We'll need computations like the following:

$$\int_{\Delta} \phi_i \cdot \phi_j$$

This would be the “Mass Matrix” for these vector basis functions.

1.3.8 Degrees of Freedom

A finite element space [Arn18] is determined by the specification of a triangulation $\mathcal{T}_h$ and for each element $T \in \mathcal{T}_h$, a finite-dimensional space of scalar- or vector-valued polynomials $\mathcal{P}$ on T , and a basis of the dual space $\mathcal{P}'$ denoted, say, $\Sigma = \{\sigma_1, \dots, \sigma_n\}$. The dual space basis σ_i are functionals and are usually called the *degrees of freedom*. That Σ is a basis of $\mathcal{P}'$ is precisely the classical *unisolvence* condition, and it furnishes a unique dual basis $\{\varphi_j\}$ of $\mathcal{P}$ with $\sigma_i(\varphi_j) = \delta_{ij}$. For elements of finite element spaces forming a de Rham complex, the degrees of freedom can take the form of moments over the faces of T . Thus, the pairing $\sigma_i(\varphi_j)$ can be vertex evaluations together with moments of the function on the higher-dimensional faces for the H^1 -conforming Lagrange elements; moments of tangential traces on edges and, in three dimensions, on faces, together with interior moments, for the $H(\text{curl})$ -conforming Nédélec elements; moments of normal traces on the facets (edges in two dimensions, faces in three) together with interior moments for the $H(\text{div})$ -conforming Raviart-Thomas elements; and interior moments alone for L^2 discontinuous Galerkin elements. Not every lower-dimensional face carries degrees of freedom at every order: the moments on higher-dimensional faces appear only once the polynomial degree is sufficiently large. We illustrate these in Figures 2.18 and 2.23 in the Chapter 2 for the linear, quadratic, and cubic cases in two and three dimensions. The assignment of each σ_i to a subsimplex encodes the inter-element continuity which may be full continuity, or tangential or normal- trace continuity

as required in the conforming global space. The degrees of freedom also underlie the canonical interpolation operators of the interpolation theory and, through them, the commuting-diagram properties of the corresponding discrete de Rham complex.

Having served this purpose in the definition of a finite element space as originally envisioned by [Cia02], in our view and an approach that we adopt, the degrees of freedom themselves play a comparatively quieter role in the remainder of this thesis. Throughout, we simply regard the discrete solution as an element of the polynomial subspace $V_h \subseteq \mathcal{P}(\Omega)$ (where Ω is the domain of the problem). Therefore, we simply choose to write the unknown solution field, whether a scalar function or a vector field, as $u_h = \sum_j c_j \varphi_j$ in its appropriate basis. We use the variational formulation as posed directly on V_h to determine the coefficients c_j . The dual functionals are never evaluated in the analysis or in the implementation; they act implicitly, through the identification $c_j = \sigma_j(u_h)$ and, in the assembly through the association of each degree of freedom with a subsimplex of the mesh. This supplies the so-called local-to-global numbering by which coefficients on shared faces are identified across element boundaries. This identification is what enforces, in the assembled global space, the continuity of tangential traces for the Whitney elements and of the normal traces for the Raviart-Thomas elements across inter-element boundaries. In our point of view, therefore, the degrees of freedom serve to complete the definition of the finite element by fixing the dimension of the space, establishing unisolvence, and guaranteeing the conformity of the assembled global space. We have adopted this perspective throughout in this work and the actual theoretical developments and computations take place simply on the primal finite-dimensional vector space spanned by the polynomial basis. We emphasize that this is a matter of our preference and taste rather than substance and the dual perspective is indeed essential where it is genuinely needed, for example, in the construction of interpolants and the approximation theory built on those. Thus, apart from this description and the examples of the degrees of freedom in Chapter 2, we largely omit them from our further discussions.

1.4 Maxwell's Equations

1.4.1 Introduction to Maxwell's Equations

Maxwell's Equations describe the behavior of four electromagnetic quantities which are expressed by vector-valued functions of space $\Omega \subseteq \mathbb{R}^3$ and time $\mathcal{T} \subseteq \mathbb{R}$. These quantities are the *electric* and *magnetic flux densities* $\mathbf{D}, \mathbf{B} : \Omega \times \mathcal{T} \rightarrow \mathbb{R}^3$ and the *electric* and *magnetic field strengths* $\mathbf{E}, \mathbf{H} : \Omega \times \mathcal{T} \rightarrow \mathbb{R}^3$ depending on the *electric charge density* $\rho : \Omega \times \mathcal{T} \rightarrow \mathbb{R}$ and *electric current density* $\mathbf{J} : \Omega \times \mathcal{T} \rightarrow \mathbb{R}^3$ which are assumed to be continuous.

Name	Equation
Gauss's law	$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho(\mathbf{r}, t)$
Gauss's law for magnetism	$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0$
Maxwell-Faraday's law	$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{B}(\mathbf{r}, t)$
Maxwell-Ampère's law	$\nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}, t) + \frac{\partial}{\partial t} \mathbf{D}(\mathbf{r}, t)$

1.4.2 Computational Solution of Maxwell's equations

Computational electromagnetics entails numerical solution of Maxwell's equations and has been one of the foundational pillars of modern electrical engineering. Given the ubiquity of electrical and electronic devices at all power levels, from microwatts to gigawatts in our everyday world, the importance of a provably correct yet accurate finite element discretization of this PDE cannot be understated. In this work, we demonstrate higher-order, structure preserving FEM for the solution of problems modeled using Maxwell's equations. Our model problems include eigenvalue problems and circuit coupled problems. Our demonstration shows how these modern finite elements, incorporating ideas from differential geometry and algebraic topology, can lead to preservation of both mathematical and physical structures such as, for example, exactly divergence free discrete solutions, discrete preservation of topological invariants, and discrete spectra free of spurious modes.

The whole set of Maxwell's equations has only sometimes been subjected to finite element analysis without being rewritten in some fashion. The goal is to reduce the complexity or expense of the numerical approach, or to modify the equations to the specific application at hand. In light of these assumptions, we now consider the following set of two coupled PDEs obtained by a suitable combination of the four equations which represent Maxwell's equations. [Räb14] [Che20]

$$\begin{aligned}
\frac{\partial p}{\partial t} + \nabla \cdot \varepsilon \mathbf{E} &= f_p \text{ in } \Omega \times (0, T], \\
\nabla p + \varepsilon \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{H} &= f_E \text{ in } \Omega \times (0, T], \\
\mu \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} &= f_H \text{ in } \Omega \times (0, T],
\end{aligned} \tag{1.14a}$$

where $\Omega \subseteq \mathbb{R}^2/\mathbb{R}^3$ is a domain with Lipschitz boundary $\partial\Omega$ and with the following homogeneous boundary conditions :

$$p = 0, \mathbf{E} \times \mathbf{n} = 0, \mathbf{H} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \times (0, T], \tag{1.14b}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$, and with the following initial conditions :

$$p(\mathbf{x}, 0) = p_0(\mathbf{x}), \mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0(\mathbf{x}), \text{ and } \mathbf{H}(\mathbf{x}, 0) = \mathbf{H}_0(\mathbf{x}) \text{ for } \mathbf{x} \in \Omega. \tag{1.14c}$$

In these equations, $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{H}(\mathbf{x}, t)$ denote the electric and magnetic fields , respectively, and

$p(x, t)$ is a physically fictitious term that is related to the time varying divergence of the electric field. We refer to all variables without the x and t descriptors when this is clear to avoid clutter. The additional electric variable facilitates a mixed formulation for the Maxwell's equations in the sense of finite element exterior calculus (FEEC) and allows for a direct approximation of $\nabla \cdot \varepsilon E$. The boundary conditions correspond to a perfectly electric conductor at the boundary and this is usually a good abstraction for the study of Maxwell's equations. This formulation was first introduced in [APZ13] and further used in [AHZ17; Adl+21], and the authors there attribute it to Douglas Arnold who is one of the authors of several works in FEEC [AFW10; Arn18]. We slightly generalize their (p, E, H) formulation to allow for arbitrary right hand side functions $f_p(x, t)$, $f_E(x, t)$ and $f_H(x, t)$ as shown in Equation (1.14a). Clearly, if $\nabla \cdot \varepsilon E = 0$, then with $p_0(x) = 0$, $f_p = 0$, $f_E = J(x, t)$ (which is the current density) and $f_H = 0$, we recover the more standard form of the Maxwell's equations. In our setup, we assume that $t \in [0, T]$ for some finite maximum time T . The material parameters ε (electric permittivity) and μ (magnetic permeability) are assumed to be time independent and positive bounded piecewise constant functions, that is, there exist positive constants: $\varepsilon_{\min}, \varepsilon_{\max}, \mu_{\min}, \mu_{\max}$ satisfying $0 < \varepsilon_{\min} \leq \varepsilon(x) \leq \varepsilon_{\max} < \infty$ and $0 < \mu_{\min} \leq \mu(x) \leq \mu_{\max} < \infty$ for all $x \in \Omega$. Although this minor technical condition is necessary, for all intents and purposes, we treat these physical parameters as simply constants in our work here. These constants can then be taken to be either the inverse of the minimum or maximum values as appropriate in all our proofs if these parameters are spatially varying. Finally, we shall assume that the initial conditions provided satisfy:

$$\nabla \cdot (\varepsilon E_0) = p_0, \text{ and } \nabla \cdot (\mu H_0) = 0 \text{ in } \Omega,$$

to be consistent with our motivation for introducing p in conjunction with E , and because magnetic fields are divergence free due to there being no magnetic charges.

The variational formulation for Equation (1.14a) with boundary conditions in Equation (1.14b) is given by: find $(p, E, H) \in \dot{H}_{\varepsilon^{-1}}^1(\Omega) \times \dot{H}_{\varepsilon}(\text{curl}; \Omega) \times \dot{H}_{\mu}(\text{div}; \Omega)$ such that:

$$\left\langle \frac{\partial p}{\partial t}, \tilde{p} \right\rangle - \langle \varepsilon E, \nabla \tilde{p} \rangle = \langle f_p, \tilde{p} \rangle, \quad \tilde{p} \in \dot{H}_{\varepsilon^{-1}}^1(\Omega), \quad (1.15a)$$

$$\langle \nabla p, \tilde{E} \rangle + \left\langle \varepsilon \frac{\partial E}{\partial t}, \tilde{E} \right\rangle - \langle H, \nabla \times \tilde{E} \rangle = \langle f_E, \tilde{E} \rangle, \quad \tilde{E} \in \dot{H}_{\varepsilon}(\text{curl}; \Omega), \quad (1.15b)$$

$$\left\langle \mu \frac{\partial H}{\partial t}, \tilde{H} \right\rangle + \langle \nabla \times E, \tilde{H} \rangle = \langle f_H, \tilde{H} \rangle, \quad \tilde{H} \in \dot{H}_{\mu}(\text{div}; \Omega), \quad (1.15c)$$

for $t \in (0, T]$ with initial conditions as in Equation (1.14c).

1.4.3 Time Discretization

We perform a time discretization using the θ -method which reduces to the explicit, first-order forward Euler scheme for $\theta = 0$, the implicit, first-order backward Euler scheme for $\theta = 1$, and the implicit, second-order Crank-Nicolson scheme for the choice of $\theta = 1/2$. The last scheme has a further advantage of being energy conserving. We provide the equations for each of these schemes below and diagrammatic depiction of the computation at times indexed by $\mathbb{N}$ starting from the initial condition. In each of these depictions, the time dependence of a variable at a time index $i \in \mathbb{N}$ on the variables in the previous time index $i - 1 \in \mathbb{N}$ is shown. Note that for the explicit scheme, the dependence of $E(i)$ or $H(i)$ at time index i is strictly only on variables in the previous

time index $E(i-1)$ and $H(i-1)$. On the contrary, for the implicit schemes, the variables at time index i are dependent on other variables at the same time index i necessitating the solution of these variables at that time using a linear (in these instances) or nonlinear (more generally) solvers.

θ -Method

For a given PDE of the form

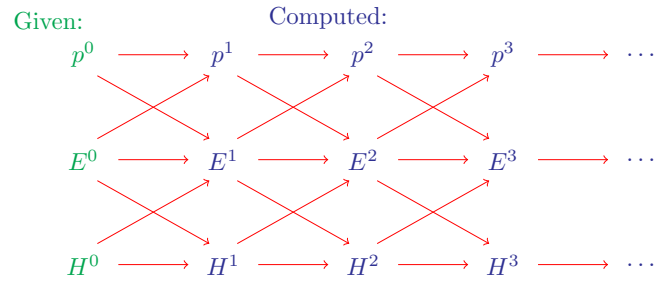
$$\frac{\partial v}{\partial t} + \Lambda v = f, \quad (1.16)$$

for some operator Λ , the time discretization is given as follows:

$$\frac{1}{\Delta t}(v^{m+1} - v^m) + \theta \Lambda v^{m+1} + (1 - \theta) \Lambda v^m = \theta f^{m+1} + (1 - \theta) f^m. \quad (1.17)$$

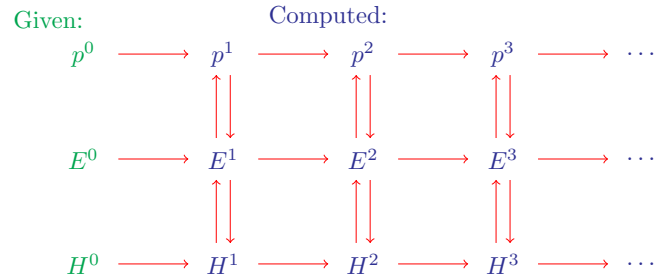
Forward-Euler: $\theta = 0$

$$\begin{aligned} \frac{p^{n+1} - p^n}{\Delta t} + \nabla \cdot E^n &= f_p^n, \\ \nabla p^n + \epsilon \frac{E^{n+1} - E^n}{\Delta t} - \mu \nabla \times H^n &= f_E^n, \\ \nabla \times E^n + \mu \frac{H^{n+1} - H^n}{\Delta t} &= f_H^n. \end{aligned}$$



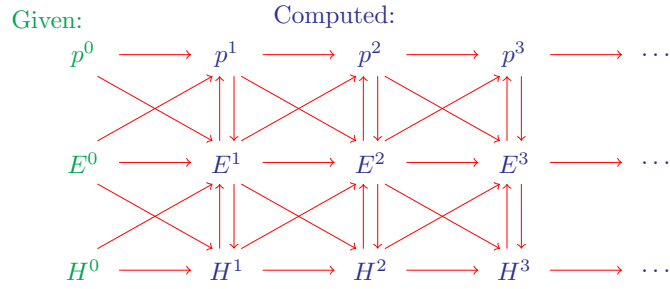
Backward-Euler: $\theta = 1$

$$\begin{aligned} \frac{p^n - p^{n-1}}{\Delta t} + \nabla \cdot E^n &= f_p^n, \\ \nabla p^n + \epsilon \frac{E^n - E^{n-1}}{\Delta t} - \mu \nabla \times H^n &= f_E^n, \\ \nabla \times E^n + \mu \frac{H^n - H^{n-1}}{\Delta t} &= f_H^n. \end{aligned}$$



Crank-Nicolson: $\theta = \frac{1}{2}$

$$\begin{aligned} \frac{p^n - p^{n-1}}{\Delta t} + \frac{1}{2} (\nabla \cdot E^n + \nabla \cdot E^{n-1}) &= \frac{1}{2} (f_p^n + f_p^{n-1}), \\ \frac{1}{2} (\nabla p^n + \nabla p^{n-1}) + \epsilon \frac{E^n - E^{n-1}}{\Delta t} - \frac{1}{2} \mu (\nabla \times H^n + \nabla \times H^{n-1}) &= \frac{1}{2} (f_E^n + f_E^{n-1}), \\ \frac{1}{2} (\nabla \times E^n + \nabla \times E^{n-1}) + \mu \frac{H^n - H^{n-1}}{\Delta t} &= \frac{1}{2} (f_H^n + f_H^{n-1}). \end{aligned}$$



1.5 de Rham Finite Elements

We present here bare minimum basics needed for a ready reference and discussion of the various theorems and results in our work. We are indebted to those foundations and have tried to liberally borrow their overall proof methodologies and general proof strategies for presenting our contributions here.

We begin by first stating the various Hilbert spaces that we shall use as our function spaces in this work. To begin to do so, we first state that the inner product on our Hilbert space $(U, \langle \cdot, \cdot \rangle_\alpha)$ is weighted in the following way: for any $u, v \in U$ and a positive real number α , $\langle u, v \rangle_\alpha := \langle \alpha u, v \rangle$. Consequently, for the same positive real number we have that $\|u\|_\alpha^2 := \langle u, u \rangle_\alpha$. Next, we note that we do not distinguish between scalar- and vector-valued points or functions by their typeface and leave it to the context to make it clear as we find this choice visually friendly. Therefore, instead of $x \in \mathbb{R}$ or $x \in \mathbb{R}^n$, $d = 2, 3$, we simply write $x \in \Omega$ and likewise f may represent either a scalar- or vector-valued function. Consequently, we also do not distinguish in notation function spaces defined over regular (or smooth) scalar or vector functions and simply use $C^r(\Omega)$ where r is the regularity to denote such functions. Thus, $L^2(\Omega)$ can mean the space of either scalar- or vector-valued functions with a bounded L^2 norm. With these reading provisos, for a positive number α we denote by L_α^2 the set of functions which have a bounded L^2 norm $\|\cdot\|_\alpha$. We next define for our purposes the various function spaces which we need in our work here.

$$\begin{aligned}
 H_{\varepsilon^{-1}}^1(\Omega) &:= \{u \in L_{\varepsilon^{-1}}^2(\Omega) \mid \nabla u \in L_{\varepsilon^{-1}}^2(\Omega)\}, \\
 \mathring{H}_{\varepsilon^{-1}}^1(\Omega) &:= \{u \in H_{\varepsilon^{-1}}^1(\Omega) \mid u = 0 \text{ on } \partial\Omega\}, \\
 H_\varepsilon(\text{curl}; \Omega) &:= \{u \in L_\varepsilon^2(\Omega) \mid \nabla \times u \in L_\varepsilon^2(\Omega)\}, \\
 \mathring{H}_\varepsilon(\text{curl}; \Omega) &:= \{u \in H_\varepsilon(\text{curl}; \Omega) \mid u \times n = 0 \text{ on } \partial\Omega\}, \\
 H_\mu(\text{div}; \Omega) &:= \{u \in L_\mu^2(\Omega) \mid \nabla \cdot u \in L_\mu^2(\Omega)\}, \\
 \mathring{H}_\mu(\text{div}; \Omega) &:= \{u \in H_\mu(\text{div}; \Omega) \mid u \cdot n = 0 \text{ on } \partial\Omega\}.
 \end{aligned}$$

1.5.1 de Rham Complex and Approximation using Polynomials

To begin with, we abstract objects and operators of vector calculus, namely, scalar functions and vector fields, and the operators – divergence, gradient and curl – and the scalar and vector Laplacian

operators using the *de Rham complex*. The de Rham complex is an algebraic representation of these objects and the corresponding linear operators as an *exact sequence*. The sequence is shown below in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

$$\mathbb{R}^2: \quad \text{Scalar functions} \xrightleftharpoons[-\text{div}]{\text{grad}} \text{Vector fields} \xrightleftharpoons[\text{rot}]{\text{curl}} \text{Density functions.}$$

$$\mathbb{R}^3: \quad \text{Scalar functions} \xrightleftharpoons[-\text{div}]{\text{grad}} \text{Vector fields} \xrightleftharpoons[\text{curl}]{\text{curl}} \text{Vector fields} \xrightleftharpoons[-\text{grad}]{\text{div}} \text{Density functions.}$$

In this diagram, the scalar functions and vector fields are defined on some region $\Omega \subseteq \mathbb{R}^2/\mathbb{R}^3$. Between the function spaces of such scalar functions and vector fields on Ω , there are linear mappings between successive spaces which are differential operators. Exactness of such a sequence means that composition of two of these operators is identically zero, and this encapsulates the vector identities, namely, $\text{curl grad} = 0$ and $\text{div curl} = 0$. We can also write the de Rham complex with function space notations as follows:

de Rham complex with function space notations:

$$C^\infty(\mathbb{R}^3) \xrightleftharpoons[-\text{div}]{\text{grad}} \chi^\infty(\mathbb{R}^3) \xrightleftharpoons[\text{curl}]{\text{curl}} \chi^\infty(\mathbb{R}^3) \xrightleftharpoons[-\text{grad}]{\text{div}} C^\infty(\mathbb{R}^3),$$

where $C^\infty(\mathbb{R}^3)$ and $\chi^\infty(\mathbb{R}^3)$ are the smooth scalar function spaces and vector field spaces in $\mathbb{R}^3$ respectively. The difference here is that all the adjoints require boundary conditions. In a *Galerkin FEM*, the first step is a weak or variational formulation of the PDE, and we can therefore write a functional analysis version of the above de Rham complex both with boundary and without boundary conditions which we depict below next.

$$\mathbb{R}^2 \text{ Boundary de Rham complex:} \quad \mathring{H}^1(\Omega) \xrightleftharpoons[\text{curl}]{\text{rot}} \mathring{H}(\text{div}, \Omega) \xrightleftharpoons[\text{grad}]{-\text{div}} L^2(\Omega).$$

$$\mathbb{R}^2 \text{ Standard de Rham complex :} \quad H^1(\Omega) \xrightleftharpoons[-\text{div}]{\text{grad}} H(\text{curl}, \Omega) \xrightleftharpoons[\text{rot}]{\text{curl}} L^2(\Omega).$$

$$\mathbb{R}^3 \text{ Boundary de Rham complex:} \quad \mathring{H}^1(\Omega) \xrightleftharpoons[-\text{div}]{\text{grad}} \mathring{H}(\text{curl}, \Omega) \xrightleftharpoons[\text{curl}]{\text{curl}} \mathring{H}(\text{div}, \Omega) \xrightleftharpoons[\text{grad}]{-\text{div}} L^2(\Omega).$$

$$\mathbb{R}^3 \text{ Standard de Rham complex :} \quad H^1(\Omega) \xrightleftharpoons[-\text{div}]{\text{grad}} H(\text{curl}, \Omega) \xrightleftharpoons[\text{curl}]{\text{curl}} H(\text{div}, \Omega) \xrightleftharpoons[\text{grad}]{-\text{div}} L^2(\Omega).$$

We shall use a discretization of the Poisson's equation with appropriate boundary conditions that make the problem well posed using one of these de Rham complexes. In particular, we shall work with finite-dimensional, higher-order polynomial subspaces of the functions spaces of scalar

functions and vector fields. Such discretized de Rham complexes expressed using polynomial spaces is given by:

$\mathbb{R}^2$ Finite dimensional boundary de Rham complex:

$$\mathcal{P}_r^-(\Omega) \cap \mathring{H}^1(\Omega) \xrightleftharpoons[\text{curl}]{\text{rot}} \mathcal{P}_r^-(\Omega) \cap \mathring{H}(\text{div}, \Omega) \xrightleftharpoons[\text{grad}]{-\text{div}} \mathcal{P}_r^-(\Omega) \cap L^2(\Omega).$$

$\mathbb{R}^2$ Finite dimensional standard de Rham complex:

$$\mathcal{P}_r^-(\Omega) \cap H^1(\Omega) \xrightleftharpoons[-\text{div}]{\text{grad}} \mathcal{P}_r^-(\Omega) \cap H(\text{curl}, \Omega) \xrightleftharpoons[\text{rot}]{\text{curl}} \mathcal{P}_r^-(\Omega) \cap L^2(\Omega).$$

$\mathbb{R}^3$ Finite dimensional boundary de Rham complex:

$$\mathcal{P}_r^-(\Omega) \cap \mathring{H}^1(\Omega) \xrightleftharpoons[-\text{div}]{\text{grad}} \mathcal{P}_r^-(\Omega) \cap \mathring{H}(\text{curl}, \Omega) \xrightleftharpoons[\text{curl}]{\text{curl}} \mathcal{P}_r^-(\Omega) \cap \mathring{H}(\text{div}, \Omega) \xrightleftharpoons[\text{grad}]{-\text{div}} \mathcal{P}_r^-(\Omega) \cap L^2(\Omega).$$

$\mathbb{R}^3$ Finite dimensional standard de Rham complex:

$$\mathcal{P}_r^-(\Omega) \cap H^1(\Omega) \xrightleftharpoons[-\text{div}]{\text{grad}} \mathcal{P}_r^-(\Omega) \cap H(\text{curl}, \Omega) \xrightleftharpoons[\text{curl}]{\text{curl}} \mathcal{P}_r^-(\Omega) \cap H(\text{div}, \Omega) \xrightleftharpoons[\text{grad}]{-\text{div}} \mathcal{P}_r^-(\Omega) \cap L^2(\Omega).$$

The following two tables summarizes the finite dimensional subspaces of the de Rham complexes and their basis.

Boundary de Rham Complex in $\mathbb{R}^3$:

Function Space	Finite Dimensional Subspace	Basis
$W = \mathring{H}^1(\Omega)$	W_h	Lagrange of order r
$\mathcal{T} = \mathring{H}(\text{curl}; \Omega)$	$\mathcal{T}_h$	Raviart-Thomas on edges of order $r + 1$
$\Sigma = \mathring{H}(\text{div}; \Omega)$	Σ_h	Raviart-Thomas on faces of order $r + 1$
$V = L^2(\Omega)$	V_h	Discontinuous Galerkin of order r

Standard de Rham Complex in $\mathbb{R}^3$:

Function Space	Finite Dimensional Subspace	Basis
$V = H^1(\Omega)$	V_h	Lagrange of order r
$\Sigma = H(\text{curl}; \Omega)$	Σ_h	Whitney vector fields on edges of order $r + 1$
$\mathcal{T} = H(\text{div}; \Omega)$	$\mathcal{T}_h$	Whitney vector fields on faces of order $r + 1$
$W = L^2(\Omega)$	W_h	Discontinuous Galerkin of order r

where $r \geq 0$ is the order of basis. The basis are related in the de Rham and boundary de Rham complexes since the spaces k and $n - k$ are isomorphic in the simplicial complex.

Boundary conditions	Homogeneous equations
Neumann	$\nabla \mathbf{u} \cdot \hat{\mathbf{n}} = 0$
Magnetic	$\mathbf{u} \cdot \hat{\mathbf{n}} = 0, \nabla \times \mathbf{u} \times \hat{\mathbf{n}} = 0$
Electric	$\mathbf{u} \times \hat{\mathbf{n}} = 0, \nabla \cdot \mathbf{u} = 0$
Dirichlet	$\mathbf{u} = 0$

where $\hat{\mathbf{n}}$ is outward unit normal to $\partial\Omega$.

1.5.2 Finite Elements that form a de Rham Complex

We refer to [AFW06; AFW10; Arn18] for conforming finite element spaces forming a de Rham complex as also to [CMO11; CR16] for various details pertaining to the construction of finite element systems, the Whitney forms basis, and the canonical and smoothed projection operators. We wish to simply summarize what we need by stating the following two commuting diagrams of de Rham complexes of the function spaces and their discretizations in $\mathbb{R}^2$ and $\mathbb{R}^3$. We wish to note that $\mathcal{P}_r^-$ denotes the degree r Whitney forms which may then be interpreted to mean scalar Lagrange basis or vector Nedelec or Raviart-Thomas elements, or discontinuous Galerkin basis elements. Also, in these diagrams Π_h represents the appropriate canonical or smoothed projection and their approximability results are presented in [see AFW06, Theorems 5.3, 5.6]. For our later analysis however, we shall use only the smoothed projection operators.

$$\begin{array}{ccccc}
\mathring{H}_{\varepsilon^{-1}}^1(\Omega) & \xrightleftharpoons[\text{curl}]{\text{rot}} & \mathring{H}_{\varepsilon}(\text{div}, \Omega) & \xrightleftharpoons[\text{grad}]{-\text{div}} & L_{\mu}^2(\Omega) \\
\downarrow \Pi_h^0 & & \downarrow \Pi_h^1 & & \downarrow \Pi_h^2 \\
\mathring{H}_{\varepsilon^{-1}}^1(\Omega) & \xrightleftharpoons[\text{curl}]{\text{rot}} & \mathring{H}_{\varepsilon}(\text{div}, \Omega) & \xrightleftharpoons[\text{grad}]{-\text{div}} & L_{\mu}^2(\Omega) \\
\cap \mathcal{P}_r^-(\Omega) & & \cap \mathcal{P}_r^-(\Omega) & & \cap \mathcal{P}_r^-(\Omega)
\end{array}$$

$$\begin{array}{ccccccc}
\mathring{H}_{\varepsilon^{-1}}^1(\Omega) & \xrightleftharpoons[-\text{div}]{\text{grad}} & \mathring{H}_{\varepsilon}(\text{curl}, \Omega) & \xrightleftharpoons[\text{curl}]{\text{curl}} & \mathring{H}_{\mu}(\text{div}, \Omega) & \xrightleftharpoons[\text{grad}]{-\text{div}} & L^2(\Omega) \\
\downarrow \Pi_h^0 & & \downarrow \Pi_h^1 & & \downarrow \Pi_h^2 & & \downarrow \Pi_h^3 \\
\mathring{H}_{\varepsilon^{-1}}^1(\Omega) & \xrightleftharpoons[-\text{div}]{\text{grad}} & \mathring{H}_{\varepsilon}(\text{curl}, \Omega) & \xrightleftharpoons[\text{curl}]{\text{curl}} & \mathring{H}_{\mu}(\text{div}, \Omega) & \xrightleftharpoons[\text{grad}]{-\text{div}} & L^2(\Omega) \\
\cap \mathcal{P}_r^-(\Omega) & & \cap \mathcal{P}_r^-(\Omega) & & \cap \mathcal{P}_r^-(\Omega) & & \cap \mathcal{P}_r^-(\Omega)
\end{array}$$

1.6 Closing Set of Preliminaries

1.6.1 Algebraic Identities and Inequalities

In the following, we shall take u, v to be in a Hilbert space $(U, \langle \cdot, \cdot \rangle_{\alpha})$ for a fixed positive scalar α , and a and b to be nonnegative real numbers.

Triangle $\|u + v\|_\alpha \leq \|u\|_\alpha + \|v\|_\alpha$.

Cauchy-Schwarz $\langle u, v \rangle_\alpha \leq \|u\|_{\beta^{-1}\alpha} \|v\|_{\beta\alpha}$ for a nonnegative scalar β .

Polarization identity $2\langle u - v, u \rangle_\alpha = \|u\|_\alpha^2 + \|u - v\|_\alpha^2 - \|v\|_\alpha^2$.

Polarization inequality $2\langle u, v \rangle_\alpha \leq \beta^{-1}\|u\|_\alpha^2 + \beta\|v\|_\alpha^2$ for a nonnegative scalar β .

Arithmetic mean-geometric mean (AM-GM) $\sqrt{ab} \leq \frac{a+b}{2} \implies (a+b)^2 \leq 2(a^2 + b^2)$.

1.6.2 Gronwall-OuLang Inequalities

A recurring theme in the various proofs for establishment of stability and error convergence for time discretization of the Maxwell's system is Gronwall's inequality. We state below the versions of it most pertinent to our usage in both the smooth and discrete settings. We wish to also note the relation of Gronwall's inequality in the smooth setting to the inequality in [Ou-57], and therefore we refer to these results in our work here together as Gronwall-OuLang inequality.

Lemma 1.8 (OuLang, [Ou-57]). *Let u and f be real-valued nonnegative continuous functions defined for all $t \geq 0$. If:*

$$u^2(t) \leq c^2 + 2 \int_0^t f(s)u(s)ds \text{ for all } t \geq 0,$$

where $c \geq 0$ is a constant, then:

$$u(t) \leq c + \int_0^t f(s)ds \text{ for all } t \geq 0.$$

Lemma 1.9 (Gronwall, [QV94, Lemma 1.4.1]). *Let $f \in L^1(t_0, T)$ be a nonnegative function, g and u be continuous functions on $[t_0, T]$. If u satisfies:*

$$u(t) \leq g(t) + \int_{t_0}^t f(s)u(s)ds \text{ for all } t \in [t_0, T],$$

then:

$$u(t) \leq g(t) + \int_{t_0}^t f(s)g(s) \exp\left(\int_s^t f(\tau)d\tau\right) \text{ for all } t \in [t_0, T].$$

Moreover, if g is nondecreasing, then:

$$u(t) \leq g(t) \exp \left(\int_{t_0}^t f(\tau) d\tau \right) \text{ for all } t \in [t_0, T].$$

Lemma 1.10 (Discrete Gronwall , [AA19, Lemma 2]). Let $\delta \geq 0$, $g_0 \geq 0$ and (a_n) , (b_n) , (c_n) and (γ_n) be sequences of nonnegative numbers such that:

$$a_N + \delta \sum_{n=0}^N b_n \leq \delta \sum_{n=0}^N (\gamma_n a_n + c_n) + g_0 \text{ for all } N = 0, 1, \dots$$

Assuming that $\gamma_n \delta < 1$ for all n and setting $\sigma_n := (1 - \gamma_n \delta)^{-1}$, for all $N \geq 0$ we have that:

$$a_N + \delta \sum_{n=0}^N b_n \leq \left(\delta \sum_{n=0}^N c_n + g_0 \right) \exp \left(\delta \sum_{n=0}^N \sigma_n \gamma_n \right).$$

Chapter 2

Implementation of de Rham Finite Elements in 2d and 3d for Linear PDEs

2.1 Finite Element Method in 2d and 3d for Linear PDEs

In this chapter, we discuss various aspects pertaining to computations with the finite element method (FEM) for a de Rham sequence of finite element spaces. Throughout the chapter, we shall use the Poisson's equation, in both 2d and 3d, and for both the scalar and vector Laplacian, as our model problems. We also discuss the numerical solution using various kinds of boundary conditions. We believe that this chapter will help serve as a tutorial-style overview for using the finite element method beyond what is generally discussed in the various standard textbooks such as [BS08; LT09; BBF13; LB13; Arn18; EG21a; EG21b; EG21c]. This chapter should complement anyone trying to read from any of these sources with our own attempt to clearly explain some of these implementation details.

2.1.1 Boundary Conditions

We wish to remark that in a boundary de Rham complex, the function spaces are endowed with boundary conditions in the function spaces themselves. We typically assume that the differential operators that extend rightward in these complexes are all strong but all our notions readily extend also to the case when they are, on the other hand, only weakly defined. The operators in these diagrams that map leftward are formal adjoints in case of the standard de Rham complex but actual adjoints with the appropriate boundary conditions in case of the boundary de Rham complex. As an example, thus the adjoint grad^* is indeed $-\text{div}$ in both cases but for the boundary complex, the boundary conditions are actually used by setting the integral term on the boundary to 0 in the definition of this adjointness. Likewise, $\text{curl}^* = \text{curl}$. We wish to reiterate that these boundary conditions in the boundary de Rham complex are those which make the corresponding Poisson problem well defined.

Natural and Essential Boundary Conditions

We explain the difference between natural and essential boundary conditions by giving an example of 1d Poisson's equation with non-homogeneous Dirichlet boundary condition. Consider the

following 1d model problem:

$$-\frac{d^2u}{dx^2} = f \text{ on } \Omega = [0, 1], \quad u(x=0) = g_0, \quad u(x=1) = g_1. \quad (2.1)$$

The variational formulation for the above problem is given by: find $u \in H^1([0, 1])$ such that

$$\begin{aligned} \left\langle -\frac{d^2u}{dx^2}, v \right\rangle &= \langle f, v \rangle, \quad v \in H^1([0, 1]), \\ \iff \int_0^1 \frac{du}{dx} \frac{dv}{dx} dx + \left[-\frac{du}{dx} v \right]_0^1 &= \int_0^1 f v dx, \quad v \in H^1([0, 1]). \end{aligned}$$

After choosing an appropriate function space that makes this problem well posed, say V , we pick a finite-dimensional subspace $V_h \subseteq V$ with dimension $N + 1$ and a basis $\{\phi_i\}_{i=1}^N$ which results into the following linear system:

$$\left[\int_0^1 \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx \right] [u_i] = \left[\int_0^1 f \phi_j dx \right].$$

Natural Boundary Conditions: Natural boundary conditions are then not satisfied exactly, that is, by requiring it to be so in the function space but only in a “weak” sense through its appearance in a boundary term in the variational formulation. For instance, for the above example we have:

$$V = H^1(\Omega) := \left\{ u \in L^2(\Omega) \mid \frac{du}{dx} \in L^2(\Omega) \right\}.$$

The boundary term of the variational formulation will look like below for the “natural boundary conditions” case:

$$\begin{aligned} \left[\frac{du}{dx} v \right]_0^1 &= \left[\frac{du}{dx} \right]_0^1 [v]_0^1, \\ &= u_N \frac{d\phi_N}{dx} \Big|_{x=1} v|_{x=1} - u_0 \frac{d\phi_0}{dx} \Big|_{x=1} v|_{x=0}, \quad \left(\text{because } u = \sum_{i=0}^N u_i \phi_i \right) \\ &= u_N \frac{d\phi_N}{dx} \Big|_{x=1} g_1 - u_0 \frac{d\phi_0}{dx} \Big|_{x=1} g_0. \end{aligned}$$

Essential Boundary Conditions: Essential boundary conditions are pointwise satisfied by functions or the vector fields through requiring them to be so in the definition of the function space itself. For instance, for the above example we have:

$$V = \mathring{H}^1(\Omega) := \{ u \in L^2(\Omega) \mid \text{grad } u \in [L^2(\Omega)]^2 \text{ and } u|_{\partial\Omega} = 0 \}.$$

And for the essential boundary case, in the boundary term we set:

$$u_0 = g_0 \quad \text{and} \quad u_N = g_1.$$

And thus the linear system is given by:

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ \int \phi_1 \phi_0 & \int \phi_1 \phi_1 & \dots & \int \phi_1 \phi_{N-1} & \int \phi_1 \phi_N \\ \vdots & \vdots & & \vdots & \vdots \\ \int \phi_{N-1} \phi_0 & \int \phi_{N-1} \phi_1 & \dots & \int \phi_{N-1} \phi_{N-1} & \int \phi_{N-1} \phi_N \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} g_0 \\ \int f \phi_1 \\ \vdots \\ \int f \phi_{N-1} \\ g_N \end{bmatrix}.$$

Homogenous and Nonhomogenous Boundary Conditions

A boundary condition is said to be *homogenous* if the boundary data vanishes on the boundary of the given domain, otherwise it is called a *non-homogenous* boundary condition. The following tables provide a summary of these various classification of the boundary conditions in the context of the various function spaces that we have used in the different de Rham complexes.

Function Space	Essential Boundary conditions	Function Space	Natural Boundary conditions
$\dot{H}^1(\Omega)$	Dirichlet	$H^1(\Omega)$	Neumann
$\dot{H}(\text{curl}; \Omega)$	Electric	$H(\text{curl}; \Omega)$	Magnetic
$\dot{H}(\text{div}; \Omega)$	Magnetic	$H(\text{div}; \Omega)$	Electric
$L^2(\Omega)$	Neumann	$L^2(\Omega)$	Dirichlet

2.2 FEM in 2d

For the 2d case, we first consider the scalar Laplacian using the standard formulation, and the Poisson's equation in Cartesian coordinates is given by:

$$-\text{div grad } u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f \text{ on } [0, 1] \times [0, 1], \quad (2.2)$$

$$u|_{\partial\Omega} = 0.$$

We consider Dirichlet boundary conditions here and our domain Ω is unit square which can be discretized as in Figure 2.1. If N_0 is the number of vertices in a given triangulation of the unit square, then looking ahead, in the discretized problem, $u \in \mathbb{R}^{N_0}$. The comparison of linear 1d and 2d Lagrange basis we have shown in Table 2.1. We can define these basis ϕ_i 's in terms of barycentric coordinates as well. The barycentric coordinates of $v = \begin{pmatrix} x \\ y \end{pmatrix}$ with respect to v_0, v_1, v_2 are $\{\lambda_0, \lambda_1, \lambda_2\}$. Thus, we can have the value of these λ_i 's in terms of triangle vertices as represented in Figure 2.2. Each $\lambda_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by $\lambda_i \begin{pmatrix} x \\ y \end{pmatrix} = \lambda_i(x, y)$, and each $\phi_i = \lambda_i$ where ϕ_i are linear Lagrange basis on this triangle. Pictorially in 1d we can represent it as in Figure 2.4.

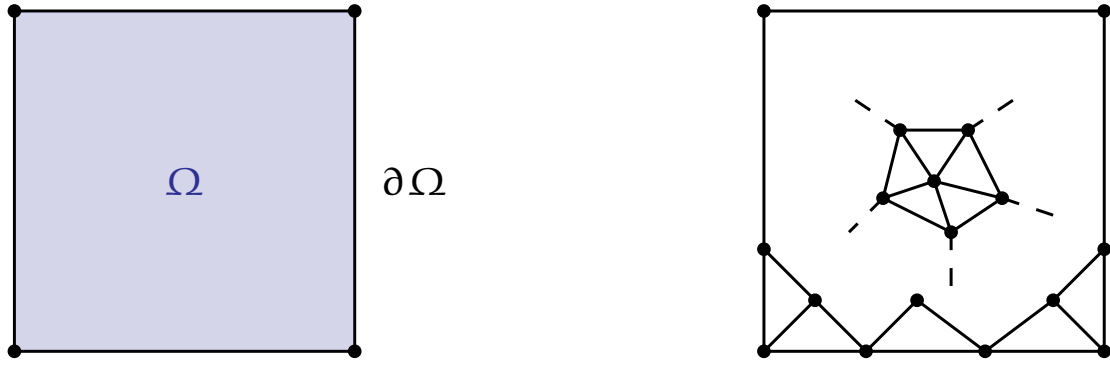


FIGURE 2.1: Unit square (left) and an illustration of its discretization (right).

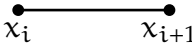
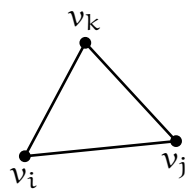
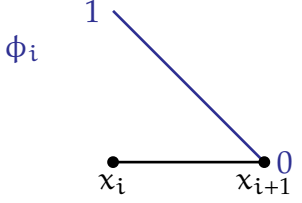
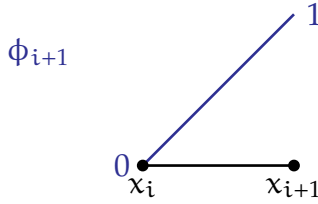
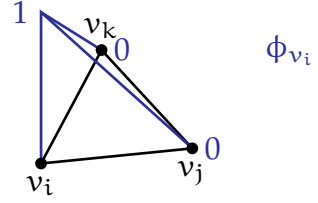
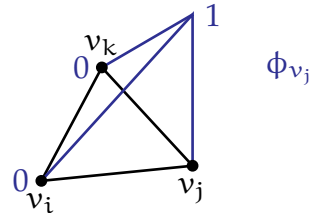
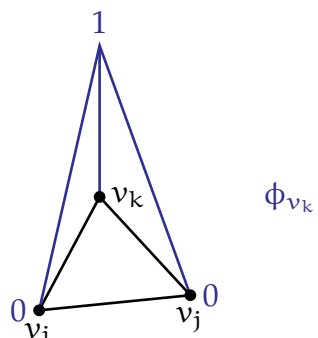
	1D $N_0 = 2$	2D $N_0 = 3$
Simplex		
Basis	 ϕ_i  ϕ_{i+1}	 ϕ_{v_i}  ϕ_{v_j}  ϕ_{v_k}

TABLE 2.1: Comparison of 1d and 2d Lagrange basis

And this can be carried over to 2d as global view under which whole mesh is considered as in

$$\lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 = v.$$

$$\Updownarrow$$

$$\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 = x,$$

$$\lambda_0 y_0 + \lambda_1 y_1 + \lambda_2 y_2 = y,$$

$$\lambda_0 + \lambda_1 + \lambda_2 = 1.$$

$$\Updownarrow$$

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

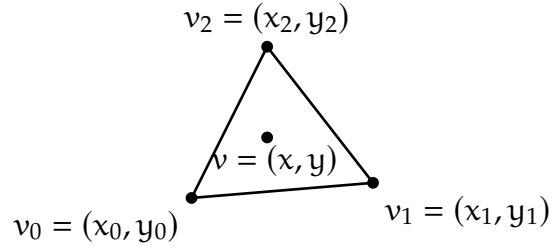


FIGURE 2.2: Illustration of barycentric coordinates in term of triangle's coordinates

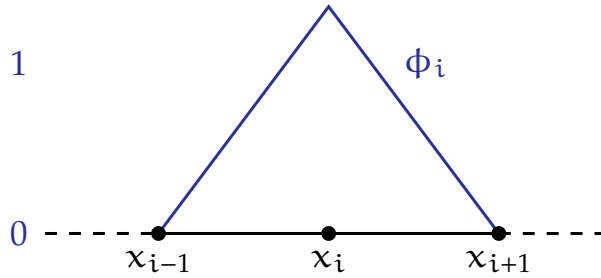


FIGURE 2.3: Linear basis for node x_i .

Figure 2.5 and elemental view which is considered over a particular simplex of the mesh as in Figure 2.6.

$$\underbrace{\int_{\Omega} \phi_i \phi_j \text{ and } \int_{\Omega} \nabla \phi_i \nabla \phi_j}_{\text{Globally}} \Rightarrow \underbrace{\sum_e \int_t \phi_i^e \phi_j^e \text{ and } \int_t \nabla \phi_i^e \nabla \phi_j^e}_{\text{Locally}} \text{ since } \phi_i = \sum_e \lambda^e.$$

Now we make matrices arrangement first locally and then globally. For instance, we can pick two triangles from a mesh having a common vertex and have them arranged locally as in Figure 2.7 which are 3×3 symmetric matrices. Next we insert these local matrices into global matrix as we are doing in Figure 2.8 which is a $N_0 \times N_0$ symmetric matrix where N_0 is the number of vertices in the mesh. Similarly we can assemble the global right hand side vector $b \in \mathbb{R}^{N_0}$ from elementwise right hand side vectors $b^e \in \mathbb{R}^3$.

We can also express quadratic Lagrange basis in terms of barycentric functions and these basis are $\phi_0 = \lambda_0(2\lambda_0 - 1)$, $\phi_1 = 4\lambda_0\lambda_1$, $\phi_2 = \lambda_1(2\lambda_1 - 1)$. At x_i , $\lambda_0 = 1$ which implies $\phi_0 = 1$, similarly at $\frac{x_{i+1} + x_i}{2}$, $\lambda_0 = \frac{1}{2}$ gives $\phi_0 = 0$ and at x_{i+1} , $\lambda_0 = 0$ gives $\phi_0 = 0$. Pictorially we have represented these basis in Figure 2.9.

These higher-order basis can in general be written as a combination of terms involving barycentric



FIGURE 2.4: Elemental view of linear Lagrange basis in 1d.

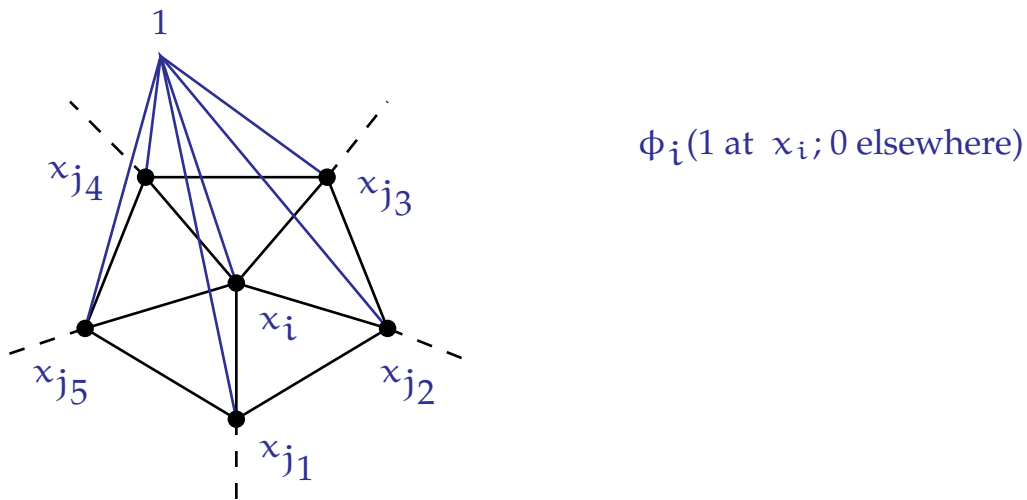


FIGURE 2.5: Global view of 2d linear Lagrange basis.

functions of the form: $\lambda_0^a \lambda_1^b$ for some $a \geq 0, b \geq 0$, technically, it will look like: $\sum_{i \in I} \alpha_i \lambda_0^{a_i} \lambda_1^{b_i}$.

Generalizing to 2d, a general polynomial basis for scalar functions on a simplex is going to be of the form: $\lambda_0^a \lambda_1^b \lambda_2^c$ for some $a \geq 0, b \geq 0, c \geq 0$. If $\{\lambda_0, \lambda_1, \lambda_2\}$ are the barycentric functions on the 2-simplex, then the gradient of $\nabla(\lambda_0^a \lambda_1^b \lambda_2^c)$ is going to be of the form:

$$\nabla(\lambda_0^a \lambda_1^b \lambda_2^c) = a \lambda_0^{a-1} \lambda_1^b \lambda_2^c \nabla \lambda_0 + b \lambda_0^a \lambda_1^{b-1} \lambda_2^c \nabla \lambda_1 + c \lambda_0^a \lambda_1^b \lambda_2^{c-1} \nabla \lambda_2. \quad (2.3)$$

To be precise, a basis for degree p polynomials on the simplex would look like:

$$\phi_i = \sum_{j \in J} \alpha_j^i \lambda_0^{a_j} \lambda_1^{b_j} \lambda_2^{c_j},$$

where $a_j \geq 0, b_j \geq 0, c_j \geq 0$ with $0 \leq a_j + b_j + c_j \leq p$. Thus, at the core of $\nabla \phi_i$ will be gradients of the form as in Equation (2.3), at the heart of Equation (2.3) are $\{\nabla \lambda_0, \nabla \lambda_1, \nabla \lambda_2\}$. Therefore, for the linear case and triangle $\Delta = \{v_0, v_1, v_2\}$, the basis for $\mathcal{P}_1(\Delta) = \{\phi_0, \phi_1, \phi_2\}$ and the basis functions are indeed the barycentric functions themselves i.e. $\phi_0 = \lambda_0, \phi_1 = \lambda_1, \phi_2 = \lambda_2$, which can be represented as shown in Figure 2.10.

We can also have quadratic Lagrange basis on a triangle given by: $\{\lambda_i(2\lambda_i - 1), 4\lambda_i \lambda_j\}, i, j =$

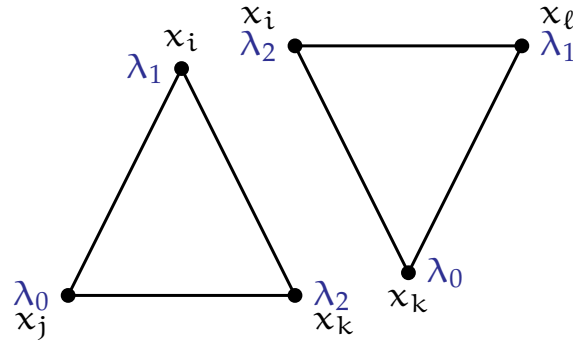


FIGURE 2.6: Elemental view of 2d Lagrange basis.

l: local index g: global index

g		x_i	x_j	x_k
	l	0	1	2
x_i	0	00	01	02
x_j	1	10	11	12
x_k	2	20	21	22
g		x_i	x_k	x_l
	l	0	1	2
x_i	0	00	01	02
x_k	1	10	11	12
x_l	2	20	21	22

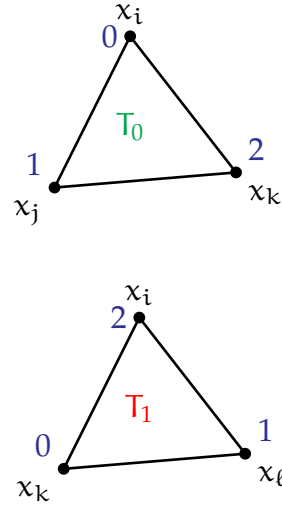


FIGURE 2.7: Local arrangement of basis in matrix form

0, 1, 2 and $\dim(\mathcal{P}_2(\Delta)) = 6$, where $\mathcal{P}_2(\Delta)$ is the space of polynomials (quadratic) on a triangle and $\{\phi_j\}_{j=0}^5$ is a basis for $\mathcal{P}_2(\Delta)$ i.e., $\{\phi_0 : \lambda_0(2\lambda_0 - 1), \phi_1 : \lambda_1(2\lambda_1 - 1), \phi_2 : \lambda_2(2\lambda_2 - 1), \phi_3 : 4\lambda_0\lambda_1, \phi_4 : 4\lambda_1\lambda_2, \phi_5 : 4\lambda_2\lambda_0\}$. We can also compute gradients of these quadratic basis functions as follows:

$$\begin{aligned}
 \nabla \phi_0 &= \nabla(\lambda_0(2\lambda_0 - 1)), \\
 &= \nabla \lambda_0(2\lambda_0 - 1) + \lambda_0(2\nabla \lambda_0), \\
 &= \nabla \lambda_0(2\lambda_0 - 1 + 2\lambda_0), \\
 &= \underbrace{(4\lambda_0 - 1)}_{\text{Scalar function}} \underbrace{\nabla \lambda_0}_{\text{vector}}.
 \end{aligned}$$

Similarly, $\nabla \phi_1 = (4\lambda_1 - 1)\nabla \lambda_1$, $\nabla \phi_2 = (4\lambda_2 - 1)\nabla \lambda_2$. And $\nabla \phi_3 = \nabla(4\lambda_0\lambda_1) = 4\nabla \lambda_0\lambda_1 + 4\lambda_0\nabla \lambda_1$, $\nabla \phi_4 = 4\nabla \lambda_1\lambda_2 + 4\lambda_1\nabla \lambda_2$, $\nabla \phi_5 = 4\nabla \lambda_0\lambda_2 + 4\lambda_0\nabla \lambda_2$.

For quadratic case, the global stiffness matrix $S_g \in \mathbb{R}^{(N_0+N_1) \times (N_0+N_1)}$ and the load vector (the right-hand side vector) $b \in \mathbb{R}^{(N_0+N_1)}$, where N_0 and N_1 are the number of vertices and edges, respectively, in a given triangulation (also referred to as the mesh) of Ω . On the other hand, the elemental stiffness matrix $S_e \in \mathbb{R}^{6 \times 6}$ and the elemental load vector $b_e \in \mathbb{R}^6$. Hence, the solution vector $u \in \mathbb{R}^{(N_0+N_1)}$. These can be represented locally and globally for quadratic case as follows:

$$\begin{array}{c}
 \vdots \\
 x_k \\
 \vdots \\
 x_i \\
 \vdots \\
 x_\ell \\
 \vdots \\
 x_j \\
 \vdots
 \end{array}
 \begin{bmatrix}
 \dots & x_k & \dots & x_i & \dots & x_\ell & \dots & x_j & \dots \\
 & 22 + 00 & & 20 + 02 & & & & & \\
 & & & & & & & & \\
 & 02 + 20 & & 00 + 22 & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & &
 \end{bmatrix}$$

FIGURE 2.8: Global arrangement of basis in matrix form

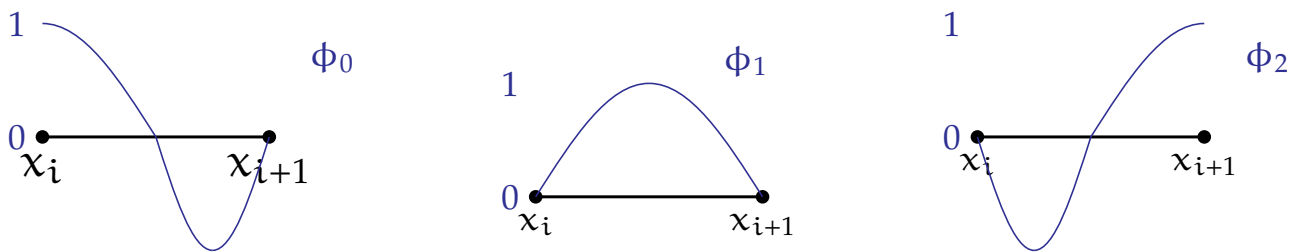


FIGURE 2.9: Elemental view of quadratic Lagrange basis in 1d.

	Global Index	
Edges: $e_0 = [v_0, v_1]$	0	
0 1		
$e_1 = [v_0, v_2]$	1	
0 4		
$e_2 = [v_1, v_2]$	4	
1 4		

Global u	Local u
$ \begin{bmatrix} 0 \\ \vdots \\ N_0 - 1 \\ 0 \\ \vdots \\ N_1 - 1 \end{bmatrix} $	$ \begin{bmatrix} u_{v_0} \\ u_{v_1} \\ u_{v_2} \\ u_{e_0} \\ u_{e_1} \\ u_{e_2} \end{bmatrix} $
$ \left. \begin{array}{l} \text{u on vertices} \\ \text{u on midpoints of edges} \end{array} \right\} $	$ \left. \begin{array}{l} \text{vertices } v_0, v_1, v_2 \\ \text{midpoints on edges } e_0, e_1, e_2 \end{array} \right\} $
$(N_0 + N_1) \times 1$	6×1

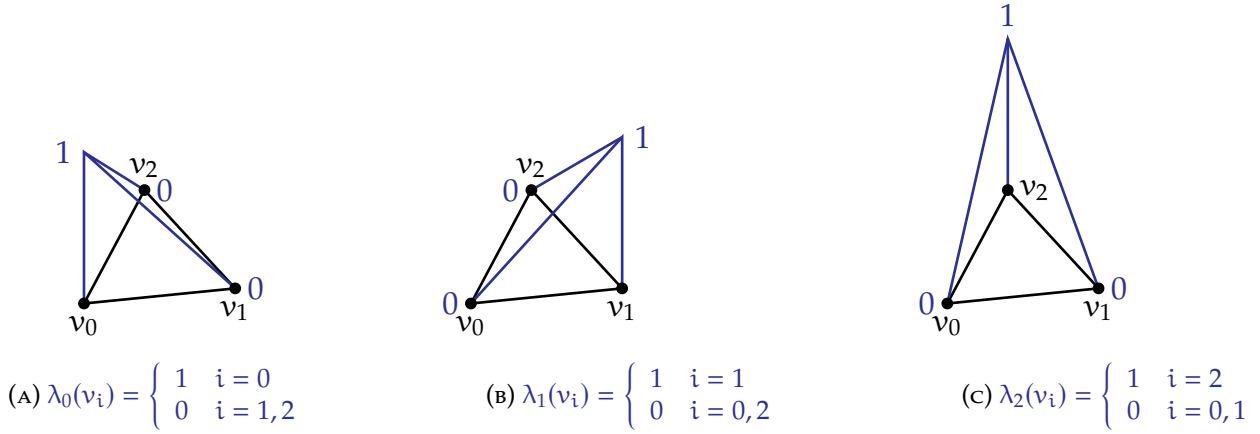


FIGURE 2.10: Lagrange linear basis in 2d on a triangle.

Local Stiffness:

$$\begin{array}{c}
 0 \quad u_{v_0} \\
 1 \quad u_{v_1} \\
 4 \quad u_{v_2} \\
 0 \quad u_{e_0} \\
 1 \quad u_{e_1} \\
 4 \quad u_{e_2}
 \end{array}
 \begin{bmatrix}
 0 & 1 & 4 & 0 & 1 & 4 \\
 u_{v_0} & u_{v_1} & u_{v_2} & u_{e_0} & u_{e_1} & u_{e_2}
 \end{bmatrix}$$

Global Stiffness:

$$\begin{array}{c}
 N_0 \left\{ \begin{array}{c|c} \text{Vertex-Vertex} & \text{Vertex-Edge} \\ \hline \text{Edge-Vertex} & \text{Edge-Edge} \end{array} \right\} \\
 \underbrace{\hspace{1.5cm}}_{N_0} \quad \underbrace{\hspace{1.5cm}}_{N_1}
 \end{array}$$

Here, “Vertex-Vertex” denotes the inner product of basis corresponding to the vertices of mesh with themselves, similarly “Edge-Edge” denotes the inner product of basis corresponding to the edges of mesh with themselves. And “Vertex-Edge” denotes the inner product of basis corresponding to the vertices of mesh with basis corresponding to the edges of mesh, whereas “Edge-Vertex” denotes its transpose. Now, we can perform mixed formulation for the following problem:

$$\begin{aligned}
 -\operatorname{div} \operatorname{grad} u &= -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = f \quad \text{on } [0, 1] \times [0, 1], \\
 u|_{\partial\Omega} &= 0.
 \end{aligned} \tag{2.4}$$

Its first order form is given as:

$$\left. \begin{array}{l} \sigma = \operatorname{grad} u \\ -\operatorname{div} \sigma = f \end{array} \right\} \text{ on } \Omega, \quad u|_{\partial\Omega} = 0, \tag{2.5}$$

then we perform weak formulation and finite element formulation and seek (u, σ) such that:

$$\left\{ \begin{array}{l} \langle \sigma, \tau \rangle = \langle u, \operatorname{div} \tau \rangle \\ \langle \operatorname{div} \sigma, v \rangle = \langle f, v \rangle \end{array} \right\} \iff \left\{ \begin{array}{l} -\langle \sigma, \tau \rangle + \langle u, \operatorname{div} \tau \rangle = 0 \\ \langle \operatorname{div} \sigma, v \rangle = \langle f, v \rangle \end{array} \right. \quad \forall (\tau, v) \in \Sigma_h \times V_h,$$

where function spaces are $u, v \in L^2(\Omega)$, $\sigma, \tau \in H(\operatorname{div}, \Omega)$ and $\Sigma_h \subset H(\operatorname{div}, \Omega)$, $V_h \subset L^2(\Omega)$ are finite dimensional subspaces of these function spaces. Then the resulting finite dimensional problem has

matrix form: find (σ, u) such that:

$$\begin{bmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{bmatrix} \begin{bmatrix} \sigma \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ f_h \end{bmatrix}, \quad (2.6)$$

where S_{00} denotes 00^{th} block of the stiffness matrix, and consists of the inner product of basis for Σ_h , S_{01} and S_{10} denotes 01^{th} and 10^{th} block of the stiffness matrix respectively, and consists of the inner product of divergence of basis for Σ_h with basis for V_h , and S_{11} denotes 11^{th} block of the stiffness matrix, and consists of the inner product of basis for V_h . These basis are defined as follows for $\mathbb{R}^2$:

Order	$H_0^1(\Omega)$ basis	$H(\text{curl}; \Omega)$ basis
Linear	λ_i	$W_{ij} := \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i$
Quadratic	$4\lambda_i \lambda_j, \lambda_i(2\lambda_i - 1)$	$\lambda_i W_{ij}, \lambda_j W_{ij}$
Edge _{ij}		$\lambda_k W_{ij}, \lambda_j W_{ik}$
Triangle _{ijk}		

Some of them are illustrated in Figures 2.11 to 2.15 and 2.17 which we have adapted from [Scr+25].

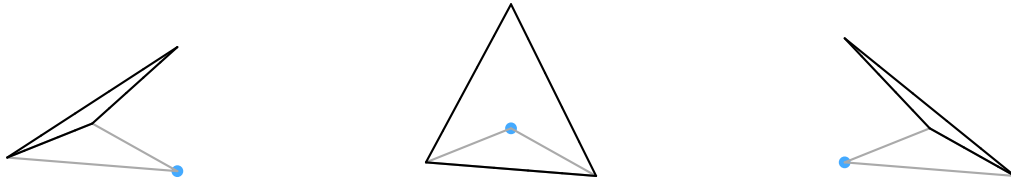


FIGURE 2.11: Linear Lagrange Basis in 2d adapted from [Scr+25].

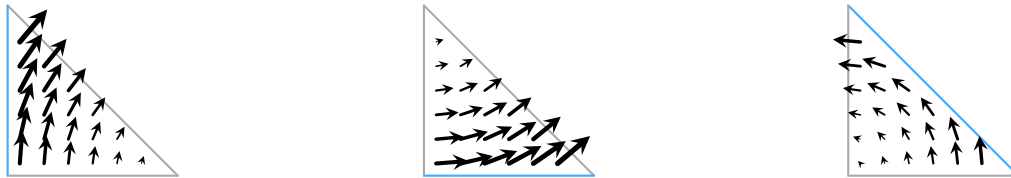


FIGURE 2.12: Linear Whitney Vector Fields in 2d adapted from [Scr+25].

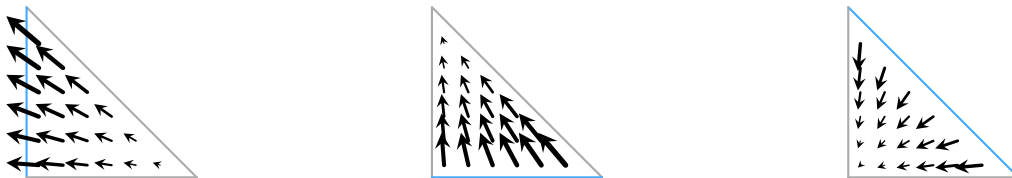


FIGURE 2.13: Linear Raviart-Thomas Fields in 2d adapted from [Scr+25].

Above we have seen that how can we arrange local stiffness matrix into global stiffness matrix corresponding to quadratic Lagrange basis, now we discuss how can we arrange those corresponding to quadratic Whitney basis. Note that for a particular triangle we have 8 bases in total (2 per edge and 2 corresponding to triangle itself), so size of local mass matrix is 8×8 and global mass matrix is of size $2N_1 + 2N_2$.

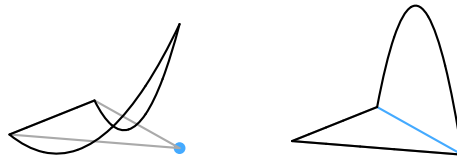


FIGURE 2.14: Quadratic Lagrange Basis in 2d corresponding to a vertex (left) and corresponding to an edge (right) adapted from [Scr+25].

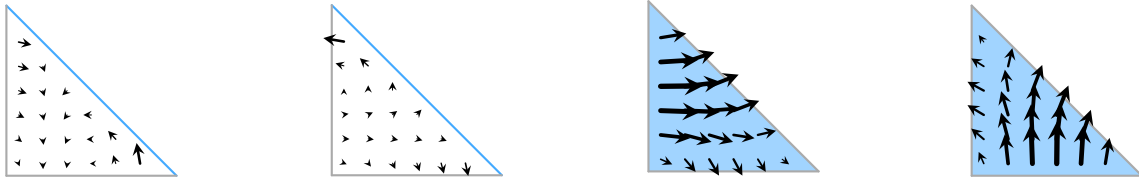


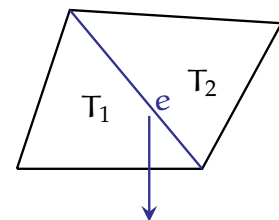
FIGURE 2.15: Quadratic Whitney Vector Fields in 2d corresponding to an edge (leftmost two) and corresponding to triangle (rightmost two) adapted from [Scr+25]. Similarly, we can refer and elaborate for the remaining edges of the triangle.

$$\begin{array}{ccc}
 \text{Local} & & \text{Global} \\
 & & \begin{array}{cc} 2N_1 & 2N_2 \end{array} \\
 \left[\begin{array}{c} \int \phi_i \phi_j \end{array} \right]_{8 \times 8} & & \begin{array}{cc} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{array} \begin{array}{l} 2N_1 \\ 2N_2 \end{array} \begin{array}{l} \\ \end{array} \begin{array}{l} \\ \end{array} \\
 & & \phantom{\left[\int \phi_i \phi_j \right]_{8 \times 8}} \phantom{\left[\int \phi_i \phi_j \right]_{8 \times 8}} \phantom{\left[\int \phi_i \phi_j \right]_{8 \times 8}} \phantom{\left[\int \phi_i \phi_j \right]_{8 \times 8}} 2N_1 + 2N_2 \times 2N_1 + 2N_2
 \end{array}$$

We can arrange our global matrix in the following two ways:

1. One option is to first list all first-basis followed by all second-basis associated with the edges; then list all first-basis followed by all second-basis associated with the triangles.
2. Second option is to list the first basis followed by the second basis for each edge, and repeat this process for all edges; then we apply the same procedure to the triangles.

We will use second way to build the global matrix and addition of local entries happens at common edges. Now to make things a bit simpler and clear, we consider a small mesh consisting of two triangles having a common edge like the way represented in the figure on the right. Since we are using second option, so we start with the edge basis (first and then second for each edge) of T_1 and then similarly for T_2 and for the common edge 'e' we need to add its particular local entries at its global index. Then we put the triangle basis again first for T_1 and then for T_2 in the increasing order.



Addition happens
at common edges

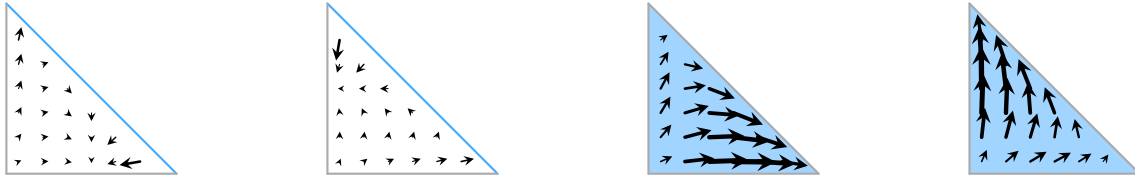


FIGURE 2.16: Quadratic Raviart-Thomas Fields in 2d corresponding to an edge (leftmost two) and corresponding to triangle (rightmost two) adapted from [Scr+25]. Similarly, we can show for the remaining edges of the triangle.

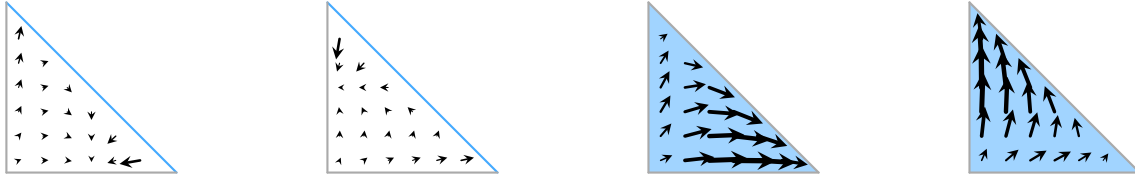
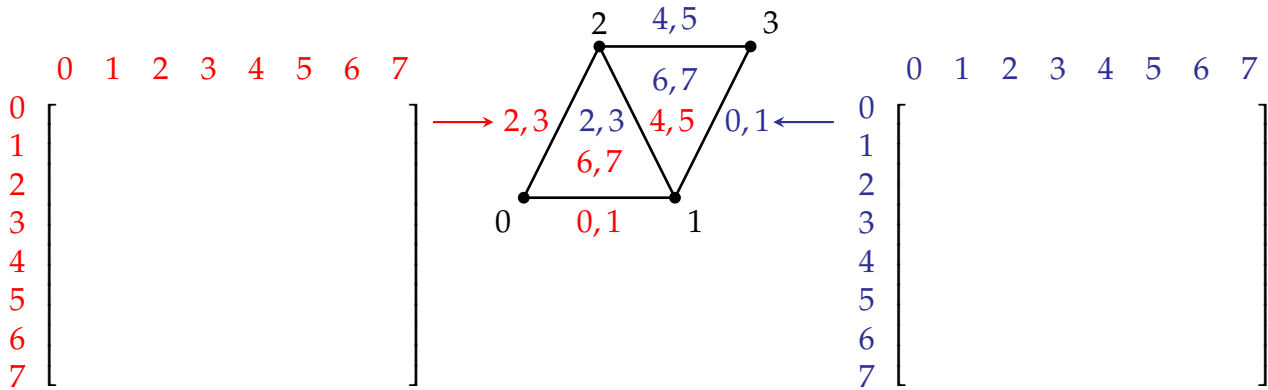


FIGURE 2.17: Quadratic Raviart-Thomas Fields in 2d corresponding to an edge (leftmost two) and corresponding to triangle (rightmost two) adapted from [Scr+25]. Similarly, we can show for the remaining edges of the triangle.



Note that both local mass matrices are *dense matrices* of size 8×8 consisting of entries $\langle \phi_i, \phi_j \rangle$ for $i, j = 0, \dots, 7$. These local mass matrices have to be assembled into a global matrix and globally, there are 5 edges and 2 triangles in the mesh. Since each edge has two bases associated with it and each triangle has two bases associated with it, so there are $\underbrace{5 \cdot 2}_{\text{(edges)}} + \underbrace{2 \cdot 2}_{\text{(triangles)}} = 14$ bases in total.

Diagram illustrating a 14x14 grid of points. The columns are labeled 0 to 13. The rows are labeled 01, 02, 12, 13, 23, 6, 7, 6, 7. Brackets group the first five rows into pairs (01, 02), (12, 13), and (23, 23). The first five rows have a vertical line at column 9. The last four rows have a vertical line at column 10. A horizontal dashed line is at row 6.

Indices 0,1: Associated with edge {01}; contribution only from red triangle,

Indices 2,3: Associated with edge {02}; contribution only from red triangle,

Indices 4,5: Associated with edge {12}; contribution only from red triangle and blue triangle,

Indices 6,7: Associated with edge {13}; contribution only from blue triangle.

Indices 8,9: Associated with edge {23}; contribution only from blue triangle.

More generally, Index I: Associated with edge $\left[\frac{1}{2}\right]$; so total indices corresponding to edge are $\{0, 1, \dots, 2N_1 - 2, 2N_1 - 1\}$. Indices 10, 11, 12, 13 all denoted bases associated with triangles. There will contribution from only one triangle for each of these; so total indices corresponding to edge are $\{0, 1, \dots, 2N_2 - 2, 2N_2 - 1\}$. Hence in total, we have $2N_1 + 2N_2$ quadratic Whitney basis in total.

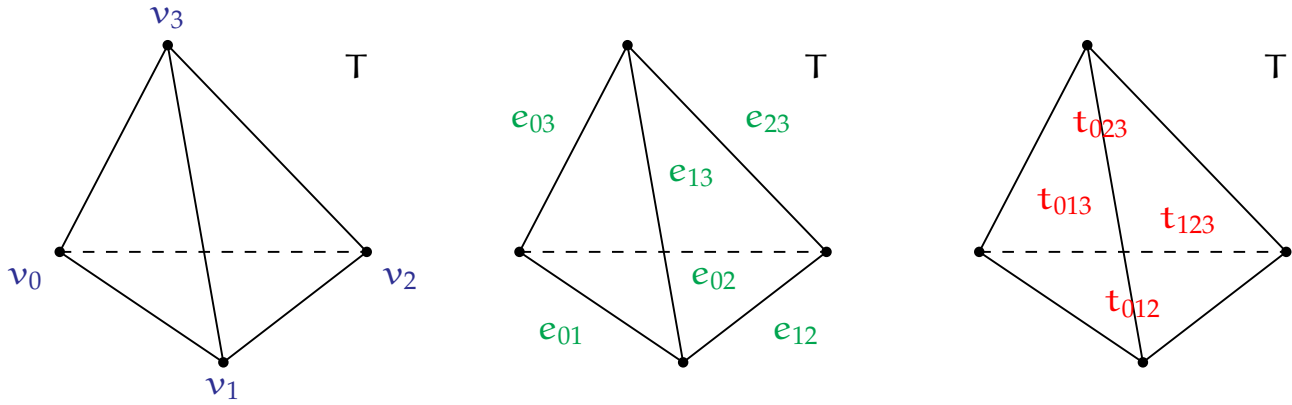
2.3 FEM in 3d

In this part, we create finite element techniques for the numerical resolution of three-dimensional partial differential equations by following the same footsteps two-dimensional partial differential equations. Similar to above as in 2d, first we use barycentric functions to define the basis which precisely represented in below table:

Order	$H_0^1(\Omega)$ basis	$H(\text{curl}; \Omega)$ basis	$H(\text{div}; \Omega)$ basis
Linear	λ_i	$W_{ij} := \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i$	$W_{ijk} := \lambda_i \nabla \lambda_j \times \nabla \lambda_k - \lambda_j \nabla \lambda_i \times \nabla \lambda_k + \lambda_k \nabla \lambda_i \times \nabla \lambda_j$
Quadratic Edge _{ij} Triangle _{ijk} Tetrahedron _{ijkl}	$4\lambda_i \lambda_j, \lambda_i(2\lambda_i - 1)$	$\lambda_i W_{ij}, \lambda_j W_{ij}$ $\lambda_k W_{ij}, \lambda_j W_{ik}$	$\lambda_i W_{ijk}, \lambda_j W_{ijk}, \lambda_k W_{ijk}$ $\lambda_l W_{ijk}, \lambda_k W_{ijl}, \lambda_j W_{ikl}$

Some of them are illustrated in Figures 2.19 to 2.22 which we have adapted from [Scr+25].

For a given mesh of the domain $\Omega \subset \mathbb{R}^3$, let N_0 be number of vertices, N_1 be number of edges, N_2 be number of triangles and N_3 be number of tetrahedrons. Let 'T' be the particular tetrahedron in the mesh, then it can be represented as in the figure below.



Here v_i denotes vertices for $i = 0, 1, 2, 3$, e_{ij} denotes edges for $i = 0, 1, 2, j = 1, 2, 3, i < j$ and t_{ijk} are triangles for $i = 0, 1, j = 1, 2, k = 2, 3, i < j < k$. Now we discuss some different problems in 3d for this domain for natural and essential boundary conditions. This demonstration involves four classifications for both set of boundary conditions consisting of Dirichlet, Neumann, magnetic, and electric boundary conditions. We provide mixed formulation, variational formulation and construction of linear system with a visual of it for each problem. Using the variational formulation, all of these boundary conditions for the corresponding problems are all natural.

2.3.1 Natural Boundary Conditions

Dirichlet Boundary Conditions

$$\begin{aligned} -\operatorname{div} \operatorname{grad} u &= f \text{ on } \Omega \subset \mathbb{R}^3, \\ u &= g \text{ on } \partial\Omega. \end{aligned} \quad \begin{aligned} (2.7a) \\ (2.7b) \end{aligned}$$

$$H^1(\Omega) \xrightleftharpoons[-\operatorname{div}]{\operatorname{grad}} H(\operatorname{curl}, \Omega) \xrightleftharpoons[\operatorname{curl}]{\operatorname{curl}} H(\operatorname{div}, \Omega) \xrightleftharpoons[\operatorname{grad}]{-\operatorname{div}} L^2(\Omega)$$

In the context of finite element exterior calculus, this Dirichlet boundary value problem is solved using a mixed formulation .

$$\begin{aligned}\sigma &= \text{grad } u, -\text{div } \sigma = f \text{ on } \Omega, \\ u &= g \text{ on } \partial\Omega.\end{aligned}$$

Variational Formulation: Seek (u, σ) such that:

$$\begin{aligned}\langle \sigma, \tau \rangle - \langle \text{grad } u, \tau \rangle &= 0, \\ -\langle \text{div } \sigma, v \rangle &= \langle f, v \rangle.\end{aligned}$$

for all (τ, v) in appropriate function spaces. After taking proper adjoints and using Integration by parts we have following final version for above set of equations:

$$-\langle \sigma, \tau \rangle + \langle \text{grad } u, \tau \rangle = 0, \quad (2.10a)$$

$$\langle \sigma, \text{grad } v \rangle = \langle f, v \rangle. \quad (2.10b)$$

Linear System:

$$\begin{bmatrix} -M_{22} & S_{23} \\ S_{23}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \sigma \\ u \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ b \end{bmatrix} \quad (2.11)$$

where $M_{22} = [\langle FW_i, FW_j \rangle]$, $S_{23} = [\langle \text{div } FW_i, \phi_j \rangle]$, and $b = \langle f, \phi_i \rangle$.

FW_i 's are *Whitney* basis corresponding to faces of tetrahedron, and ϕ_i 's are *Discontinuous Galerkin* basis for the finite dimensional polynomial subspaces of vector fields $\mathring{H}(\text{div}, \Omega)$ and density functions $L^2(\Omega)$, respectively. For linear case, $M_{22} \in \mathbb{R}^{N_2 \times N_2}$, $S_{23} \in \mathbb{R}^{N_2 \times N_3}$, and $b \in \mathbb{R}^{N_3 \times 1}$; which can be arranged locally as:

$$M_{22}^e = \begin{matrix} & \begin{matrix} t_{012} & t_{013} & t_{023} & t_{123} \end{matrix} \\ \begin{matrix} t_{012} \\ t_{013} \\ t_{023} \\ t_{123} \end{matrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{matrix} \quad S_{23}^e = \begin{matrix} & \begin{matrix} t_{012} \\ t_{013} \\ t_{023} \\ t_{123} \end{matrix} \\ \begin{matrix} t_{012} \\ t_{013} \\ t_{023} \\ t_{123} \end{matrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{matrix}^T$$

where $S_{00} = [\langle \text{grad } \lambda_i, \text{grad } \lambda_j \rangle]$, $C = [\langle \mathfrak{H}, \lambda_i \rangle]$, and $b = [\langle f, \lambda_i \rangle]$. λ_i 's are *Lagrange* basis for the finite dimensional polynomial subspace of functions $\mathring{H}^1(\Omega)$. For linear case, $S_{00} \in \mathbb{R}^{N_0 \times N_0}$, $C \in \mathbb{R}^{N_0 \times 1}$,

and $b \in \mathbb{R}^{N_0 \times 1}$; which can be arranged locally as:

$$S_{00}^e = \begin{matrix} & v_0 & v_1 & v_2 & v_3 \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{matrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{matrix}$$

For quadratic case, $S_{00} \in \mathbb{R}^{N_0 + N_1 \times N_0 + N_1}$; which can be arranged locally as:

$$S_{00}^e = \begin{matrix} & v_0 & v_1 & v_2 & v_3 & & & & & & \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ \hline e_{01} \\ e_{02} \\ e_{03} \\ e_{12} \\ e_{13} \\ e_{23} \end{matrix} & \begin{bmatrix} & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ \hline & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \end{bmatrix} \end{matrix}$$

Magnetic Boundary Conditions

$$\text{curl curl } u - \text{grad div } u = f \text{ on } \Omega, \quad (2.14a)$$

$$u \cdot n = m_1, \text{ curl } u \times n = m_2 \text{ on } \partial\Omega, \quad (2.14b)$$

$$u \perp \mathfrak{H} \text{ harmonic vector fields for well-posedness.} \quad (2.14c)$$

$$\boxed{H^1(\Omega) \xrightleftharpoons[-\text{div}]{\text{grad}} H(\text{curl}, \Omega) \xrightleftharpoons[\text{curl}]{\text{curl}} H(\text{div}, \Omega) \xrightleftharpoons[\text{grad}]{-\text{div}} L^2(\Omega)}$$

Mixed Formulation: Seek (u, σ) such that:

$$\begin{aligned} \sigma &= -\text{div } u, \\ \text{grad } \sigma + \text{curl curl } u &= f, \\ u &\perp \mathfrak{H}. \end{aligned}$$

Variational Formulation: Seek (u, σ) such that:

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle -\text{div } u, \tau \rangle &= 0, \\ \langle \text{grad } \sigma, v \rangle + \langle \text{curl curl } u, v \rangle &= \langle f, v \rangle, \\ u &\perp \mathfrak{H}. \end{aligned}$$

$$M_{00}^e = \begin{bmatrix} v_0 & v_1 & v_2 & v_2 & e_{01} & e_{02} & e_{03} & e_{12} & e_{13} & e_{23} \\ v_0 & & & & & & & & & \\ v_1 & & & & & & & & & \\ v_2 & & & & & & & & & \\ v_3 & & & & & & & & & \\ \hline e_{01} & - & - & - & - & - & - & - & - & - \\ e_{02} & & & & & & & & & \\ e_{03} & & & & & & & & & \\ e_{12} & & & & & & & & & \\ e_{13} & & & & & & & & & \\ e_{23} & & & & & & & & & \end{bmatrix}$$

$$S_{01}^e = \begin{array}{c} \begin{array}{cccccc} \overbrace{e_{01}} & \overbrace{e_{02}} & \overbrace{e_{03}} & \overbrace{e_{12}} & \overbrace{e_{13}} & \overbrace{e_{23}} \\ 0\ 1 & 0\ 1 & 0\ 1 & 0\ 1 & 0\ 1 & 0\ 1 \end{array} \\ \begin{array}{c} v_0 \\ v_1 \\ v_2 \\ v_3 \end{array} \end{array} \left[\begin{array}{c} \begin{array}{cccccc} \overbrace{t_{012}} & \overbrace{t_{013}} & \overbrace{t_{023}} & \overbrace{t_{123}} \\ 0\ 1 & 0\ 1 & 0\ 1 & 0\ 1 \end{array} \\ \begin{array}{c} e_{01} \\ e_{02} \\ e_{03} \\ e_{12} \\ e_{13} \\ e_{23} \end{array} \end{array} \right]$$

$$S_{11}^e = \begin{array}{c} \begin{array}{cccccc} \overbrace{e_{01}} & \overbrace{e_{02}} & \overbrace{e_{03}} & \overbrace{e_{12}} & \overbrace{e_{13}} & \overbrace{e_{23}} \\ 0\ 1 & 0\ 1 & 0\ 1 & 0\ 1 & 0\ 1 & 0\ 1 \end{array} \\ \begin{array}{c} e_{01} \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right. \\ e_{02} \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right. \\ e_{03} \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right. \\ e_{12} \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right. \\ e_{13} \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right. \\ e_{23} \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right. \end{array} \end{array} \left[\begin{array}{c} \begin{array}{cccccc} \overbrace{t_{012}} & \overbrace{t_{013}} & \overbrace{t_{023}} & \overbrace{t_{123}} \\ 0\ 1 & 0\ 1 & 0\ 1 & 0\ 1 \end{array} \\ \begin{array}{c} t_{012} \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right. \\ t_{013} \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right. \\ t_{023} \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right. \\ t_{123} \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right. \end{array} \end{array} \right]$$

Electric Boundary Conditions

$$\text{curl curl } u - \text{grad div } u = f \text{ on } \Omega, \quad (2.19a)$$

$$u \times n = e_1, \text{ div } u = e_2 \text{ on } \partial\Omega, \quad (2.19b)$$

$$\mathbf{u} \perp \mathfrak{H} \text{ harmonic vector fields for well-posedness.} \quad (2.19c)$$

$$\mathbf{H}^1(\Omega) \begin{array}{c} \xrightarrow{\text{grad}} \\ \xleftarrow{-\text{div}} \end{array} \mathbf{H}(\text{curl}, \Omega) \begin{array}{c} \xrightarrow{\text{curl}} \\ \xleftarrow{\text{curl}} \end{array} \mathbf{H}(\text{div}, \Omega) \begin{array}{c} \xrightarrow{-\text{div}} \\ \xleftarrow{\text{grad}} \end{array} \mathbf{L}^2(\Omega)$$

Mixed Formulation: Seek (u, σ) such that:

$$\begin{aligned} \sigma &= \operatorname{curl} u, \\ -\operatorname{grad} \operatorname{div} u + \operatorname{curl} \sigma &= f, \\ u &\perp \mathfrak{H}. \end{aligned}$$

Variational Formulation: Seek (u, σ) such that:

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle \operatorname{curl} u, \tau \rangle &= 0, \\ \langle -\operatorname{grad} \operatorname{div} u, v \rangle + \langle \operatorname{curl} \sigma, v \rangle &= \langle f, v \rangle, \\ u &\perp \mathfrak{H}. \end{aligned}$$

for all (τ, ν) in appropriate function spaces. After taking proper adjoints and using Integration by parts we have following final version for above set of equations:

$$\begin{aligned} -\langle \sigma, \tau \rangle + \langle u, \operatorname{curl} \tau \rangle &= 0, \\ \langle \operatorname{curl} \sigma, v \rangle + \langle \operatorname{div} u, \operatorname{div} v \rangle &= \langle f, v \rangle, \\ u &\perp \mathfrak{H}. \end{aligned}$$

Linear System:

$$\begin{bmatrix} -M_{11} & S_{12} & & \\ & S_{12}^T & S_{22} & \\ & & C & \\ & & & 0 \end{bmatrix} \begin{bmatrix} \sigma \\ u \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \\ 0 \end{bmatrix} \quad (2.23)$$

where $M_{11} = [\langle EW_i, EW_j \rangle]$, $S_{12} = [\langle \text{curl } EW_i, FW_j \rangle]$, $S_{11} = [\langle \text{div } FW_i, \text{div } FW_j \rangle]$, $C = [\langle h, FW_i \rangle]$, and $b = [\langle f, FW_i \rangle]$.

EW_i's are *Whitney* basis corresponding to edges of tetrahedron, and FW_i's are *Whitney* basis corresponding to faces of tetrahedron for their respective function spaces. For linear case, $M_{11} \in \mathbb{R}^{N_1 \times N_1}$, $S_{12} \in \mathbb{R}^{N_1 \times N_2}$, $S_{22} \in \mathbb{R}^{N_2 \times N_2}$, $C \in \mathbb{R}^{N_2 \times 1}$, and $b \in \mathbb{R}^{N_2 \times 1}$; which can be arranged locally as:

$$M_{11}^e = \begin{bmatrix} e_{01} & e_{02} & e_{03} & e_{12} & e_{13} & e_{23} \\ e_{01} & & & & & \\ e_{02} & & & & & \\ e_{03} & & & & & \\ e_{12} & & & & & \\ e_{13} & & & & & \\ e_{23} & & & & & \end{bmatrix}$$

$$S_{12}^e = \begin{bmatrix} e_{01} & t_{012} & t_{013} & t_{023} & t_{123} \\ e_{02} & & & & \\ e_{03} & & & & \\ e_{12} & & & & \\ e_{13} & & & & \\ e_{23} & & & & \end{bmatrix} \quad S_{22}^e = \begin{bmatrix} t_{012} & t_{013} & t_{023} & t_{123} \\ t_{012} & & & \\ t_{013} & & & \\ t_{023} & & & \\ t_{123} & & & \end{bmatrix}$$

For quadratic case, $M_{11} \in \mathbb{R}^{2N_1+2N_2 \times 2N_1+2N_2}$, $S_{12} \in \mathbb{R}^{2N_1+2N_2 \times 3N_2+3N_3}$, $S_{22} \in \mathbb{R}^{3N_2+3N_3 \times 3N_2+3N_3}$, $C \in \mathbb{R}^{3N_2+3N_3 \times 1}$, and $b \in \mathbb{R}^{3N_2+3N_3 \times 1}$, which can be arranged locally as:

$$M_{11}^e = \begin{array}{c} \begin{array}{l} e_{01} \left\{ \begin{array}{l} 0 \\ 1 \end{array} \right. \\ e_{02} \left\{ \begin{array}{l} 0 \\ 1 \end{array} \right. \\ e_{03} \left\{ \begin{array}{l} 0 \\ 1 \end{array} \right. \\ e_{12} \left\{ \begin{array}{l} 0 \\ 1 \end{array} \right. \\ e_{13} \left\{ \begin{array}{l} 0 \\ 1 \end{array} \right. \\ e_{23} \left\{ \begin{array}{l} 0 \\ 1 \end{array} \right. \end{array} \quad \begin{array}{c} \overbrace{01}^{e_{01}} \quad \overbrace{01}^{e_{02}} \quad \overbrace{01}^{e_{03}} \quad \overbrace{01}^{e_{12}} \quad \overbrace{01}^{e_{13}} \quad \overbrace{01}^{e_{23}} \\ \hline \overbrace{01}^{t_{012}} \quad \overbrace{01}^{t_{013}} \quad \overbrace{01}^{t_{023}} \quad \overbrace{01}^{t_{123}} \end{array} \end{array}$$

$$S_{12}^e =$$

$$S_{22}^e =$$

2.3.2 Essential Boundary Conditions

Dirichlet Boundary Conditions

$$\begin{aligned} -\operatorname{div} \operatorname{grad} u &= f \text{ on } \Omega \subset \mathbb{R}^3, \\ u &= g \text{ on } \partial\Omega. \end{aligned} \quad \begin{aligned} (2.24a) \\ (2.24b) \end{aligned}$$

$$\boxed{\mathring{H}^1(\Omega) \xrightleftharpoons[-\operatorname{div}]{\operatorname{grad}} \mathring{H}(\operatorname{curl}, \Omega)} \xrightleftharpoons[\operatorname{curl}]{\operatorname{curl}} \mathring{H}(\operatorname{div}, \Omega) \xrightleftharpoons[\operatorname{grad}]{-\operatorname{div}} L^2(\Omega)$$

In the context of finite element exterior calculus, this Dirichlet boundary value problem is solved using a mixed formulation.

$$\begin{aligned} \sigma &= \operatorname{grad} u, -\operatorname{div} \sigma = f \text{ on } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{aligned}$$

Variational Formulation: Seek (u, σ) such that:

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle \operatorname{grad} u, \tau \rangle &= 0, \\ -\langle \operatorname{div} \sigma, v \rangle &= \langle f, v \rangle. \end{aligned}$$

for all (τ, v) in appropriate function spaces. After taking proper adjoints and using Integration by parts we have following final version for above set of equations:

$$-\langle \sigma, \tau \rangle + \langle \operatorname{grad} u, \tau \rangle = 0, \quad (2.27a)$$

$$\langle \sigma, \operatorname{grad} v \rangle = \langle f, v \rangle. \quad (2.27b)$$

Linear System:

$$\begin{bmatrix} -M_{00} & S_{01} \\ S_{01}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \sigma \\ u \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ b \end{bmatrix} \quad (2.28)$$

where $M_{00} = [\langle \lambda_i, \lambda_j \rangle]$, $S_{01} = [\langle \lambda_i, \operatorname{div} EW_j \rangle]$, and $b = [\langle f, EW_i \rangle]$.

λ_i 's are *Lagrange* basis and EW_i 's are *Whitney* basis corresponding to edges of tetrahedron for the finite dimensional polynomial subspace of functions $\mathring{H}^1(\Omega)$ and $\mathring{H}(\operatorname{curl}, \Omega)$ respectively. For linear case, $M_{00} \in \mathbb{R}^{N_0 \times N_0}$, $S_{01} \in \mathbb{R}^{N_0 \times N_1}$, and $b \in \mathbb{R}^{N_1 \times 1}$; which can be arranged locally as:

$$M_{00}^e = \begin{matrix} & v_0 & v_1 & v_2 & v_3 \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{matrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{matrix} \quad S_{01}^e = \begin{matrix} & e_{01} & e_{02} & e_{03} & e_{12} & e_{13} & e_{23} \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{matrix} & \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{matrix}$$

For quadratic case, $M_{00} \in \mathbb{R}^{N_0+N_1 \times N_0+N_1}$, $S_{01} \in \mathbb{R}^{N_0+N_1 \times 2N_1+2N_2}$, and $b \in \mathbb{R}^{2N_1+2N_2 \times 1}$; which can be arranged locally as:

$$M_{00}^e = \begin{array}{c} \begin{array}{c} v_0 \\ v_1 \\ v_2 \\ v_3 \end{array} \\ \begin{array}{c} e_{01} \\ e_{02} \\ e_{03} \\ e_{12} \\ e_{13} \\ e_{23} \end{array} \end{array} \left[\begin{array}{cccc|cccc} v_0 & v_1 & v_2 & v_2 & e_{01} & e_{02} & e_{03} & e_{12} & e_{13} & e_{23} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right]$$

$$S_{01}^e = \begin{array}{c} \begin{array}{c} v_0 \\ v_1 \\ v_2 \\ v_3 \end{array} \\ \begin{array}{c} e_{01} \\ e_{02} \\ e_{03} \\ e_{12} \\ e_{13} \\ e_{23} \end{array} \end{array} \left[\begin{array}{cccccc|cccc} \overbrace{e_{01}}^{01} & \overbrace{e_{02}}^{01} & \overbrace{e_{03}}^{01} & \overbrace{e_{12}}^{01} & \overbrace{e_{13}}^{01} & \overbrace{e_{23}}^{01} & \overbrace{t_{012}}^{01} & \overbrace{t_{013}}^{01} & \overbrace{t_{023}}^{01} & \overbrace{t_{123}}^{01} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right]$$

Neumann Boundary Conditions

$$-\operatorname{div} \operatorname{grad} u = f \text{ on } \Omega \subset \mathbb{R}^3, \quad (2.29a)$$

$$\operatorname{grad} u \cdot n = h \text{ on } \partial\Omega, \quad (2.29b)$$

$$u \perp \mathfrak{H} \text{ harmonics (basis: constant functions) for well posedness.} \quad (2.29c)$$

$$\mathring{H}^1(\Omega) \xrightleftharpoons[-\operatorname{div}]{\operatorname{grad}} \mathring{H}(\operatorname{curl}, \Omega) \xrightleftharpoons[\operatorname{curl}]{\operatorname{curl}} \mathring{H}(\operatorname{div}, \Omega) \xrightleftharpoons[\operatorname{grad}]{-\operatorname{div}} L^2(\Omega)$$

Neumann boundary value problem for the scalar Laplacian is implemented via the standard formulation .

Variational formulation: Seek u such that:

$$\left\{ \begin{array}{l} \langle -\operatorname{div} \operatorname{grad} u, v \rangle = \langle f, v \rangle, \\ u \perp \mathfrak{H}. \end{array} \right\} \implies \left\{ \begin{array}{l} \langle \operatorname{grad} u, \operatorname{grad} v \rangle = \langle f, v \rangle, \\ u \perp \mathfrak{H}. \end{array} \right.$$

Linear system:

$$\begin{bmatrix} S_{22} & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} \quad (2.30)$$

where $S_{22} = [\langle \text{grad FRT}_i, \text{grad FRT}_j \rangle]$, $C = [\langle \mathfrak{H}, \text{FRT}_i \rangle]$, and $b = [\langle f, \text{FRT}_i \rangle]$. FRT_i 's are *Raviart-Thomas* basis corresponding to faces of tetrahedron for the finite dimensional polynomial subspaces of vector fields $\mathring{H}(\text{div}, \Omega)$. For linear case, $S_{22} \in \mathbb{R}^{N_2 \times N_2}$, $C \in \mathbb{R}^{N_2 \times 1}$, and $b \in \mathbb{R}^{N_2 \times 1}$; which can be arranged locally as:

$$S_{22}^e = \begin{bmatrix} t_{012} & t_{013} & t_{023} & t_{123} \\ t_{012} \\ t_{013} \\ t_{023} \\ t_{123} \end{bmatrix}$$

For quadratic case, $S_{22} \in \mathbb{R}^{3N_2+3N_3 \times 3N_2+3N_3}$, $C \in \mathbb{R}^{3N_2+3N_3 \times 1}$, and $b \in \mathbb{R}^{3N_2+3N_3 \times 1}$, which can be arranged locally as:

$$S_{22}^e = \begin{array}{c} \begin{array}{c} \textcolor{red}{t_{012}} \left\{ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \right. \\ \textcolor{red}{t_{013}} \left\{ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \right. \\ \textcolor{red}{t_{023}} \left\{ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \right. \\ \textcolor{red}{t_{123}} \left\{ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \right. \end{array} \left[\begin{array}{cccc|c} \overbrace{0 \ 1 \ 2}^{\textcolor{red}{t_{012}}} & \overbrace{0 \ 1 \ 2}^{\textcolor{red}{t_{013}}} & \overbrace{0 \ 1 \ 2}^{\textcolor{red}{t_{023}}} & \overbrace{0 \ 1 \ 2}^{\textcolor{red}{t_{123}}} & \overbrace{0 \ 1 \ 2}^{\textcolor{red}{T}} \\ \hline \end{array} \right] \end{array}$$

Magnetic Boundary Conditions

$$\operatorname{curl} \operatorname{curl} \mathbf{u} - \operatorname{grad} \operatorname{div} \mathbf{u} = \mathbf{f} \text{ on } \Omega, \quad (2.31a)$$

$$u \cdot n = m_1, \operatorname{curl} u \times n = m_2 \text{ on } \partial\Omega, \quad (2.31b)$$

$$u \perp \mathfrak{H} \text{ harmonic vector fields for well-posedness.} \quad (2.31c)$$

$$\dot{H}^1(\Omega) \xrightleftharpoons[-\text{div}]{\text{grad}} \boxed{\dot{H}(\text{curl}, \Omega) \xrightleftharpoons[\text{curl}]{\text{curl}} \dot{H}(\text{div}, \Omega) \xrightleftharpoons[\text{grad}]{-\text{div}} L^2(\Omega)}$$

Mixed Formulation: Seek (u, σ) such that:

$$\begin{aligned} \sigma &= \text{curl } u, \\ -\text{grad div } u + \text{curl } \sigma &= f, \\ u &\perp \mathfrak{H}. \end{aligned}$$

Variational Formulation: Seek (u, σ) such that:

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle \text{curl } u, \tau \rangle &= 0, \\ \langle -\text{grad div } u, v \rangle + \langle \text{curl } \sigma, v \rangle &= \langle f, v \rangle, \\ u &\perp \mathfrak{H}. \end{aligned}$$

for all (τ, v) in appropriate function spaces. After taking proper adjoints and using Integration by parts we have following final version for above set of equations:

$$-\langle \sigma, \tau \rangle + \langle u, \text{curl } \tau \rangle = 0, \quad (2.34a)$$

$$\langle \text{curl } \sigma, v \rangle + \langle \text{div } u, \text{div } v \rangle = \langle f, v \rangle, \quad (2.34b)$$

$$u \perp \mathfrak{H}. \quad (2.34c)$$

Linear System:

$$\begin{bmatrix} -M_{11} & S_{12} & C^T \\ S_{12}^T & S_{22} & 0 \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma \\ u \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} \quad (2.35)$$

where $M_{11} = [\langle EW_i, EW_j \rangle]$, $S_{12} = [\langle \text{curl } EW_i, \text{FRT}_j \rangle]$, $S_{22} = [\langle \text{div } FW_i, \text{div } \text{FRT}_j \rangle]$, $C = [\langle h, \text{FRT}_i \rangle]$, and $b = [\langle f, \text{FRT}_i \rangle]$.

EW_i 's are *Whitney* basis corresponding to edges of tetrahedron, and FRT_i 's are *Raviart-Thomas* basis corresponding to faces of tetrahedron for their respective function spaces. For linear case, $M_{11} \in \mathbb{R}^{N_1 \times N_1}$, $S_{12} \in \mathbb{R}^{N_1 \times N_2}$, $S_{22} \in \mathbb{R}^{N_2 \times N_2}$, $C \in \mathbb{R}^{N_2 \times 1}$, and $b \in \mathbb{R}^{N_2 \times 1}$; which can be arranged locally as:

$$M_{11}^e = \begin{matrix} & e_{01} & e_{02} & e_{03} & e_{12} & e_{13} & e_{23} \\ \begin{matrix} e_{01} \\ e_{02} \\ e_{03} \\ e_{12} \\ e_{13} \\ e_{23} \end{matrix} & \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{matrix}$$

$$S_{12}^e = \begin{matrix} & t_{012} & t_{013} & t_{023} & t_{123} \\ \begin{matrix} e_{01} \\ e_{02} \\ e_{03} \\ e_{12} \\ e_{13} \\ e_{23} \end{matrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{matrix} \quad S_{22}^e = \begin{matrix} & t_{012} & t_{013} & t_{023} & t_{123} \\ \begin{matrix} t_{012} \\ t_{013} \\ t_{023} \\ t_{123} \end{matrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{matrix}$$

For quadratic case, $M_{11} \in \mathbb{R}^{2N_1+2N_2 \times 2N_1+2N_2}$, $S_{12} \in \mathbb{R}^{2N_1+2N_2 \times 3N_2+3N_3}$, $S_{22} \in \mathbb{R}^{3N_2+3N_3 \times 3N_2+3N_3}$, $C \in \mathbb{R}^{3N_2+3N_3 \times 1}$, and $b \in \mathbb{R}^{3N_2+3N_3 \times 1}$; which can be arranged locally as:

$$M_{11}^e = \begin{array}{c} \begin{array}{l} e_{01} \\ e_{02} \\ e_{03} \\ e_{12} \\ e_{13} \\ e_{23} \end{array} \left\{ \begin{array}{l} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{array} \right. \end{array} \left[\begin{array}{cccccc|cccc} \overbrace{0 \ 1}^{e_{01}} & \overbrace{0 \ 1}^{e_{02}} & \overbrace{0 \ 1}^{e_{03}} & \overbrace{0 \ 1}^{e_{12}} & \overbrace{0 \ 1}^{e_{13}} & \overbrace{0 \ 1}^{e_{23}} & \overbrace{0 \ 1}^{t_{012}} & \overbrace{0 \ 1}^{t_{013}} & \overbrace{0 \ 1}^{t_{023}} & \overbrace{0 \ 1}^{t_{123}} \\ \hline & & & & & & & & & \end{array} \right]$$

$$S_{12}^e =$$

$$S_{22}^e =$$

Electric Boundary Conditions

$$\text{curl curl } \mathbf{u} - \text{grad div } \mathbf{u} = \mathbf{f} \text{ on } \Omega, \quad (2.36a)$$

$$\mathbf{u} \times \mathbf{n} = \mathbf{e}_1, \text{ div } \mathbf{u} = \mathbf{e}_2 \text{ on } \partial\Omega, \quad (2.36b)$$

$$\mathbf{u} \perp \mathfrak{H} \text{ harmonic vector fields for well-posedness.} \quad (2.36c)$$

$$\boxed{\mathring{H}^1(\Omega) \xrightleftharpoons[-\text{div}]{\text{grad}} \mathring{H}(\text{curl}, \Omega) \xrightleftharpoons[\text{curl}]{\text{curl}} \mathring{H}(\text{div}, \Omega)} \xrightleftharpoons[\text{grad}]{-\text{div}} L^2(\Omega)$$

Mixed Formulation: Seek $(\mathbf{u}, \sigma)$ such that:

$$\sigma = -\text{div } \mathbf{u},$$

$$\text{grad } \sigma + \text{curl curl } \mathbf{u} = \mathbf{f},$$

$$\mathbf{u} \perp \mathfrak{H}.$$

Variational Formulation: Seek $(\mathbf{u}, \sigma)$ such that:

$$\langle \sigma, \tau \rangle - \langle -\text{div } \mathbf{u}, \tau \rangle = 0,$$

$$\langle \text{grad } \sigma, \mathbf{v} \rangle + \langle \text{curl curl } \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle,$$

$$\mathbf{u} \perp \mathfrak{H}.$$

for all $(\tau, \mathbf{v})$ in appropriate function spaces. After taking proper adjoints and using Integration by parts we have following final version for above set of equations:

$$-\langle \sigma, \tau \rangle + \langle \mathbf{u}, \text{grad } \tau \rangle = 0, \quad (2.39a)$$

$$\langle \text{grad } \sigma, \mathbf{v} \rangle + \langle \text{curl } \mathbf{u}, \text{curl } \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle, \quad (2.39b)$$

$$\mathbf{u} \perp \mathfrak{H}. \quad (2.39c)$$

Linear System:

$$\begin{bmatrix} -M_{00} & S_{01} & C^T \\ S_{01}^T & S_{11} & 0 \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma \\ \mathbf{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \\ 0 \end{bmatrix} \quad (2.40)$$

where $M_{00} = [\langle \lambda_i, \lambda_j \rangle]$, $S_{01} = [\langle \text{grad } \lambda_i, \text{EW}_j \rangle]$, $S_{11} = [\langle \text{curl } \text{EW}_i, \text{curl } \text{EW}_j \rangle]$, $C = [\langle \mathbf{h}, \text{EW}_i \rangle]$, and $\mathbf{b} = [\langle \mathbf{f}, \text{EW}_i \rangle]$.

EW_i 's are *Whitney* basis corresponding to edges of tetrahedron and λ_i 's are *Lagrange* basis for the finite dimensional subspaces of $\mathring{H}(\text{curl}, \Omega)$ and $\mathring{H}^1(\Omega)$, respectively. For linear case, $M_{00} \in \mathbb{R}^{N_0 \times N_0}$, $S_{01} \in \mathbb{R}^{N_0 \times N_1}$, $S_{11} \in \mathbb{R}^{N_1 \times N_1}$, $C \in \mathbb{R}^{N_1 \times 1}$, and $\mathbf{b} \in \mathbb{R}^{N_1 \times 1}$; which can be arranged locally as:

$$M_{00}^e = \begin{matrix} & \overset{v_0}{\color{blue}} & \overset{v_1}{\color{blue}} & \overset{v_2}{\color{blue}} & \overset{v_3}{\color{blue}} \\ \overset{v_0}{\color{blue}} & & & & \\ \overset{v_1}{\color{blue}} & & & & \\ \overset{v_2}{\color{blue}} & & & & \\ \overset{v_3}{\color{blue}} & & & & \end{matrix} \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \quad S_{01}^e = \begin{matrix} & \overset{e_{01}}{\color{green}} & \overset{e_{02}}{\color{green}} & \overset{e_{03}}{\color{green}} & \overset{e_{12}}{\color{green}} & \overset{e_{13}}{\color{green}} & \overset{e_{23}}{\color{green}} \\ \overset{v_0}{\color{blue}} & & & & & & \\ \overset{v_1}{\color{blue}} & & & & & & \\ \overset{v_2}{\color{blue}} & & & & & & \\ \overset{v_3}{\color{blue}} & & & & & & \end{matrix} \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

$$S_{11}^e = \begin{bmatrix} e_{01} & e_{02} & e_{03} & e_{12} & e_{13} & e_{23} \\ e_{01} & & & & & \\ e_{02} & & & & & \\ e_{03} & & & & & \\ e_{12} & & & & & \\ e_{13} & & & & & \\ e_{23} & & & & & \end{bmatrix}$$

For quadratic case, $M_{00} \in \mathbb{R}^{N_0+N_1 \times N_0+N_1}$, $S_{01} \in \mathbb{R}^{N_0+N_1 \times 2N_1+2N_2}$, $S_{11} \in \mathbb{R}^{2N_1+2N_2 \times 2N_1+2N_2}$, $C \in \mathbb{R}^{2N_1+2N_2 \times 1}$, and $b \in \mathbb{R}^{2N_1+2N_2 \times 1}$; which can be arranged locally as:

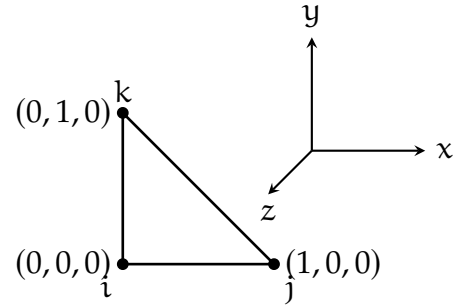
$$M_{00}^e = \begin{bmatrix} v_0 & v_1 & v_2 & v_2 & e_{01} & e_{02} & e_{03} & e_{12} & e_{13} & e_{23} \\ v_0 & & & & & & & & & \\ v_1 & & & & & & & & & \\ v_2 & & & & & & & & & \\ v_3 & & & & & & & & & \\ \vdots & & & & & & & & & \\ e_{01} & & & & & & & & & \\ e_{02} & & & & & & & & & \\ e_{03} & & & & & & & & & \\ e_{12} & & & & & & & & & \\ e_{13} & & & & & & & & & \\ e_{23} & & & & & & & & & \end{bmatrix}$$

$$S_{01}^e = \left[\begin{array}{cccccc|cccc} \overbrace{e_{01}} & \overbrace{e_{02}} & \overbrace{e_{03}} & \overbrace{e_{12}} & \overbrace{e_{13}} & \overbrace{e_{23}} & \overbrace{t_{012}} & \overbrace{t_{013}} & \overbrace{t_{023}} & \overbrace{t_{123}} \\ 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 \\ v_0 & & & & & & & & & \\ v_1 & & & & & & & & & \\ v_2 & & & & & & & & & \\ v_3 & & & & & & & & & \\ \hline e_{01} & & & & & & & & & \\ e_{02} & & & & & & & & & \\ e_{03} & & & & & & & & & \\ e_{12} & & & & & & & & & \\ e_{13} & & & & & & & & & \\ e_{23} & & & & & & & & & \end{array} \right]$$

$$S_{11}^e = \begin{array}{c} \begin{array}{l} e_{01} \\ e_{02} \\ e_{03} \\ e_{12} \\ e_{13} \\ e_{23} \end{array} \left\{ \begin{array}{l} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{array} \right. \end{array} \left[\begin{array}{cccccc|cccc} \overbrace{e_{01}} & \overbrace{e_{02}} & \overbrace{e_{03}} & \overbrace{e_{12}} & \overbrace{e_{13}} & \overbrace{e_{23}} & \overbrace{t_{012}} & \overbrace{t_{013}} & \overbrace{t_{023}} & \overbrace{t_{123}} \\ 0\ 1 & 0\ 1 & 0\ 1 & 0\ 1 & 0\ 1 & 0\ 1 & 0\ 1 & 0\ 1 & 0\ 1 & 0\ 1 \\ \hline & & & & & & & & & \end{array} \right]$$

2.4 Implementation of Boundary conditions

Standard Triangle: First we discuss implementation of boundary conditions on a standard triangle say $\{i, j, k\}$, where $i = (0, 0, 0)$, $j = (1, 0, 0)$, $k = (0, 1, 0)$ as represented in the figure on the right hand side. Let W_{ij} and W_{ijk} denotes quadratic Whitney bases corresponding to edge $\{ij\}$ and face $\{ijk\}$ respectively, and λ_i 's denote the Lagrange basis or barycentric functions.



Normal Trace: First, we find the normal trace of W_{ijk} on face $\{ijk\}$ of standard triangle as follows:

$$\lambda_i = 1 - x - y, \lambda_j = x, \lambda_k = y. \implies \nabla \lambda_i = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \nabla \lambda_j = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \nabla \lambda_k = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$\implies \nabla \lambda_j \times \nabla \lambda_k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, -\nabla \lambda_i \times \nabla \lambda_k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \nabla \lambda_i \times \nabla \lambda_j = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since the value of normal component is $\hat{n}_{ijk} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \hat{z}$, thus $W_{ijk} \cdot \hat{n}_{ijk} = (\lambda_i + \lambda_j + \lambda_k) = 1$,

and, hence $\int_{\Delta_{ijk}} W_{ijk} \cdot \hat{n}_{ijk} = |\Delta_{ijk}| \left(= \frac{1}{2} \text{ for this standard triangle} \right)$.

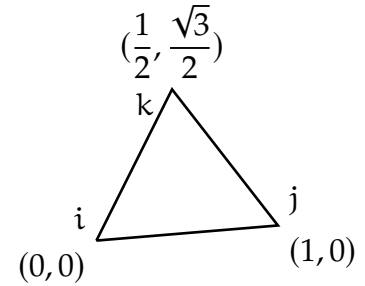
Tangential Trace: Now we compute the tangential trace of W_{ij} on edge $\{ij\}$ of the same standard triangle as follows:

$$W_{ij} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i = (1 - x - y) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - x \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - y \\ x \\ 0 \end{bmatrix}.$$

Since the value of tangential component is $\hat{t}_{ij} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \hat{x}$, thus $W_{ij} \cdot \hat{t}_{ij} = 1 - y$, which

implies $\int_{e_{ij}} W_{ij} \cdot \hat{t}_{ij} = \int_{x=0}^{x=1} (1 - y)|_{y=0, z=0} dx = 1$ ($= 1$ for this standard triangle edge).

Not the Standard Triangle: Next we will discuss the boundary conditions implementation on a regular triangle, say $\{i, j, k\}$, where $i = (0, 0, 0)$, $j = (1, 0, 0)$, $k = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$ as represented in the figure on the right hand side. Again W_{ij} , W_{ijk} , λ_i denote the same basis respectively.



Normal Trace of W_{ijk} on Face $\{ijk\}$: Here we also first find the normal trace of W_{ijk} on face $\{ijk\}$ of regular triangle as follows:

$$\lambda_i = 1 - x - \frac{y}{\sqrt{3}}, \lambda_j = x - \frac{y}{\sqrt{3}}, \lambda_k = \frac{2}{\sqrt{3}}y \implies \nabla \lambda_i = \begin{bmatrix} -1 \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}, \nabla \lambda_j = \begin{bmatrix} 1 \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}, \nabla \lambda_k = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{3}} \\ 0 \end{bmatrix}.$$

$$\implies \nabla \lambda_j \times \nabla \lambda_k = \begin{bmatrix} 0 \\ 0 \\ \frac{2}{\sqrt{3}} \end{bmatrix}, -\nabla \lambda_i \times \nabla \lambda_k = \begin{bmatrix} 0 \\ 0 \\ \frac{2}{\sqrt{3}} \end{bmatrix}, \nabla \lambda_i \times \nabla \lambda_j = \begin{bmatrix} 0 \\ 0 \\ \frac{2}{\sqrt{3}} \end{bmatrix}.$$

Since again $\hat{n}_{ijk} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \hat{z}$, thus $W_{ijk} \cdot \hat{n}_{ijk} = (\lambda_i + \lambda_j + \lambda_k) \cdot \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}$, and hence

$$\int_{\Delta_{ijk}} W_{ijk} \cdot \hat{n}_{ijk} = \frac{2}{\sqrt{3}} \cdot |\Delta_{ijk}| = \frac{2}{\sqrt{3}} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{1}{2}.$$

Tangential Trace of W_{ij} on Edge $\{ij\}$: Then we compute the tangential trace of W_{ij} on edge $\{ij\}$ of the same regular triangle as follows:

$$W_{ij} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i = \left(1 - x - \frac{y}{\sqrt{3}}\right) \begin{bmatrix} 1 \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} - \left(x - \frac{y}{\sqrt{3}}\right) \begin{bmatrix} -1 \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - \frac{2}{\sqrt{3}}y \\ -\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}x \\ 0 \end{bmatrix}.$$

Since the value of tangential component is $\hat{t}_{ij} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \hat{x}$, thus $W_{ij} \cdot \hat{t}_{ij}|_{y=0, z=0} = 1$, which implies $\int_{e_{ij}} W_{ij} \cdot \hat{t}_{ij}|_{y=0, z=0} = 1$.

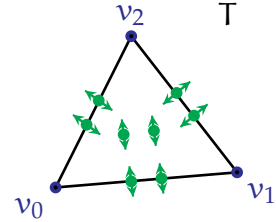
2.5 Interpolation of Computed Solution on a Triangle T

Let u be the computed solution on the edges of a triangle with respect to quadratic Whitney basis, then the size of u is $(2N_0 + 2N_1) \times 1$ since there are two quadratic basis for every edge and two quadratic basis for every triangle.

Now we will interpolate of u on T , so on T we need u_T of size 8×1 extracted from u . Then for some point $x \in T$, we have:

$$u_{I,T} = \sum_{i=1}^8 u_{T,i} \phi_i(x). \rightarrow \text{Append to } u_arrows$$

For visualization, we shall use x in barycentric coordinates as $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.



Compute barycenter as $\frac{(v_0 + v_1 + v_2)}{3}$ and append to u_bases .

This is for quadratic case. For linear case, we can directly extract from pydec:

`pydec.simplex_quivers(sc, u)`

Additionally, note that computed solution u is $N_1 \times 1$ since the linear basis are corresponding to edges only (one on each edge). And the interpolant of computed solution on T is given by:

$$u_{I,T} = \sum_{i=1}^3 u_{T,i} \phi_i(x).$$

2.6 L^2 Error Computation

In this section, we discuss the computation of L^2 error. Let u_A be the analytical solution and u_I be the piecewise polynomial solution, then on each triangle $T \in \mathcal{T}$, the discrete solution can be written as:

$$u_{I,T} = u_I|_T(x) = \sum_{i=1}^n u_{T,i} \phi_{T,i}(x), \quad x = (x, y),$$

where n is the number of local basis per triangle, $u_{T,i}$ are the coefficients obtained from the linear system, and $\phi_{T,i}$ are the local basis functions. Let N be the total number of global basis functions over the triangulation $\mathcal{T}$.

Now we can define L^2 error to be

$$e_h = \|u_I - u_A\| = \langle u_I - u_A, u_I - u_A \rangle,$$

where u_I is the interpolated computed solution, u_A is the analytical solution, and depending on L^2 norm inner product will be appropriately different for scalar and vector case. The error e_h is the only piecewise definition as u_I is piecewise globally. To get a single number for the triangulation $\mathcal{T}$, we should compute the L^2 norm of this $e_h(x)$ as follows:

$$\begin{aligned} \|e_h(x)\|_{L^2} &= \langle e_h(x), e_h(x) \rangle^{\frac{1}{2}}, \\ &= \left[\int_{\mathcal{T}} (u_I(x) - u_A(x))^2 dx \right]^{\frac{1}{2}}, \\ &= \left[\sum_{T \in \mathcal{T}} \int_T (u_I(x) - u_A(x))^2 dx \right]^{\frac{1}{2}}, \end{aligned}$$

For a given $\mathcal{T}$, $e_h \in \mathbb{R}$ and we need to compute the error on each $T \in \mathcal{T}$ using quadrature, so we have:

$$e_T = \int_T (u(x) - u_h(x))^2 dx \in \mathbb{R}$$

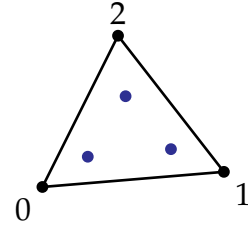
We will use the numerical quadrature for computing the integrals as in [XG10]. The global L^2 -error is defined to be the inner product $\langle u_I - u_A, u_I - u_A \rangle$, where u_I is the interpolated computed solution and u_A is the analytical solution. $\|u\|_{L^2,h}$ is the rotation for global L^2 -error of $u = \left(\sum_{T \in \mathcal{T}} \|u\|_{L^2,T}^2 \right)^{\frac{1}{2}}$, where

$$\|u\|_{L^2,T}^2 = \int_T (u_{I,T} - u_A) \cdot (u_{I,T} - u_A).$$

Evaluate this at quadrature points, say there are r in count, then we have

$$u_{I,T} := \sum_{i=1}^r u_{T,i} \phi_{T,i},$$

where $u_{T,i}$ are the coefficients extracted from the linear system solution.



2.7 Solving Mixed Poisson with Non-trivial Kernel for Laplacian Operator

We solve our problem on domain Ω where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. So consider the Laplacian problem:

$$\Delta u = f \text{ on } \Omega,$$

where Δ can be scalar or vector Laplacian.

- If Ω has multiple (m_0) connected components, the scalar Laplacian Δ_0 has a m_0 dimensional kernel.
- If Ω has m_1 holes in it, the vector Laplacian Δ_1 has a m_1 dimensional kernel.

Lets first discuss how the dimensions of the homology groups H_k reflect the topology of a domain with the help of some examples. In particular, H_0 measures connected components, so for a connected space, $\dim H_0 = 1$. The dimension of H_1 counts independent loops that cannot be continuously contracted to a point, while $\dim H_2$ counts enclosed two-dimensional cavities. The following table Table 2.2 illustrate these examples in $\mathbb{R}^2$ and $\mathbb{R}^3$.

The elements of the kernel (for Δ_0 or Δ_1) are called *harmonics* (harmonic functions or vector fields). We shall denote harmonics by $\mathfrak{H}$. Similar to linear algebra, if in solving $Ax = b$, A is singular (or has a kernel that is nontrivial), we can solve for x by being orthogonal to $\ker A$. That is the problem can be made well posed: Find x such that $Ax = b$ and $x \perp \ker A$. Likewise, the Poisson equation can be made well posed by solving:

$$\Delta u = f \text{ such that } u \perp \mathfrak{H}.$$

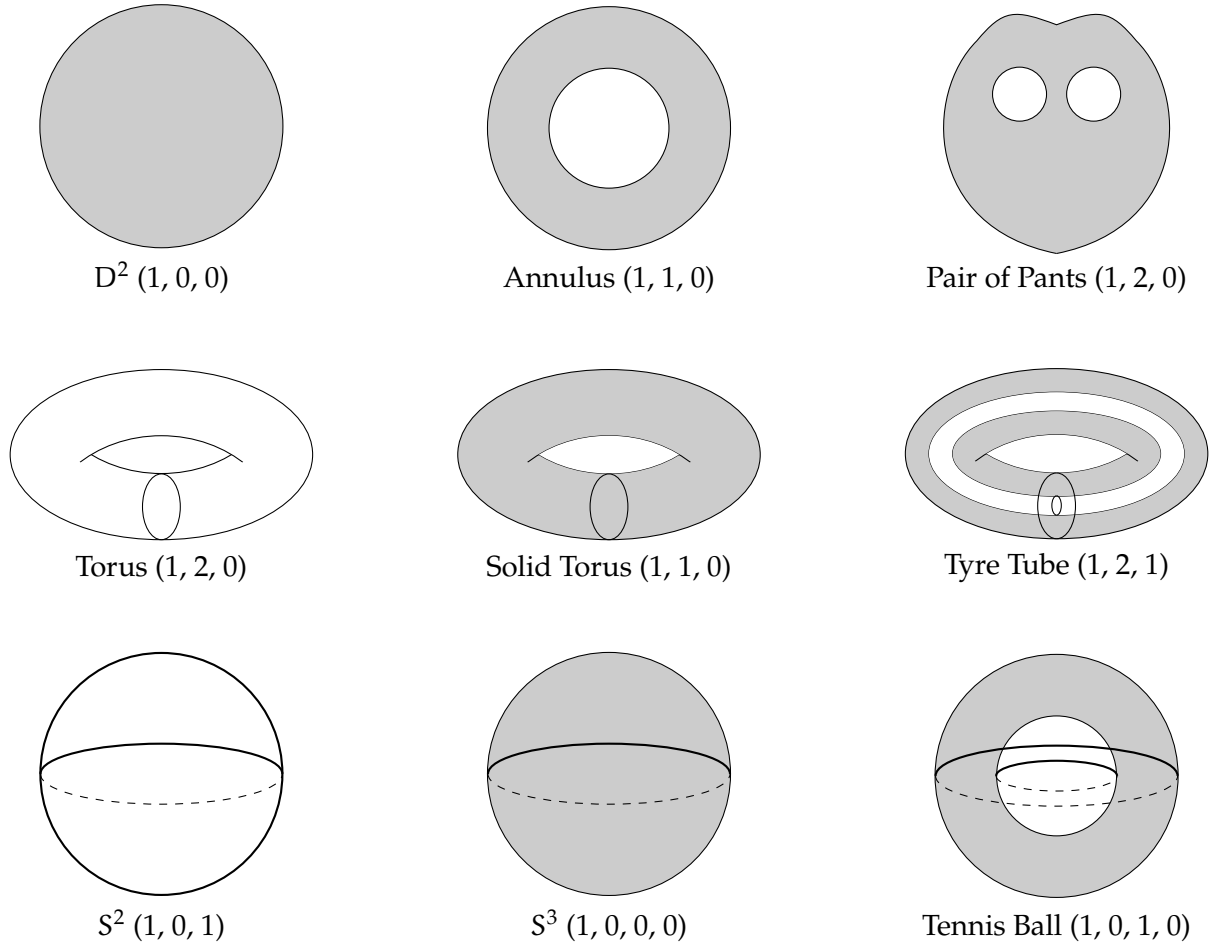
Mixed formulation will also have the same condition for well posedness:

$$\begin{cases} \sigma & = \operatorname{div} u \\ -\operatorname{grad} \sigma + \operatorname{curl} \operatorname{curl} u & = f \end{cases} \text{ such that } u \perp \mathfrak{H}$$

which gives the following linear system

$$\begin{bmatrix} -S_{00} & -S_{01} \\ -S_{01}^T & S_{11} \end{bmatrix} \begin{bmatrix} \sigma \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} \text{ such that } u \perp \mathfrak{H}$$

Let $h_1, h_2, \dots, h_p$ be the basis for p -dimensional $\mathfrak{H}$.
 $\Rightarrow u \perp h_1, u \perp h_2, \dots, u \perp h_p.$

TABLE 2.2: Homology examples in $\mathbb{R}^2$ and $\mathbb{R}^3$

$$\Rightarrow \text{Set of } p \text{ equations: } \left. \begin{array}{l} \langle u, h_1 \rangle = 0 \\ \langle u, h_2 \rangle = 0 \\ \vdots \\ \langle u, h_p \rangle = 0 \end{array} \right\} \Rightarrow u^T \begin{bmatrix} | & | & \dots & | \\ h_1 & h_2 & & h_p \\ | & | & & | \end{bmatrix} = 0.$$

$$\Downarrow$$

$$u^T H = 0. \iff H^T u = 0.$$

Thus, our linear system is now a constrained one:

$$[S] \begin{bmatrix} \sigma \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} \text{ such that } H^T u = 0.$$

can solve this by posing it as a constrained optimization problem because solving

$$Ax = b \iff \min_x \frac{1}{2}(x^T A x - x^T b).$$

So, solving $Ax = b$ such that $Cx = d$ is then:

$$\min_x \frac{1}{2}(x^T A x - x^T b) \text{ such that } Cx = d$$

This can be solved using Lagrange multipliers and we can write the Lagrangian as:

$$\begin{aligned} \mathcal{L}(x) &= \frac{1}{2}(x^T A x - x^T b) + \lambda^T (Cx - d), \text{ if } C \in \mathbb{R}^{p \times n}, \lambda \in \mathbb{R}^p, d \in \mathbb{R}^p \\ \left. \begin{aligned} \nabla \mathcal{L} &= 0 \implies Ax + C^T \lambda = b \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= 0 \implies Cx = d \end{aligned} \right\} & \text{Necessary Conditions for Optimality} \\ \implies \begin{bmatrix} A & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} &= \begin{bmatrix} b \\ d \end{bmatrix} \end{aligned}$$

If we solve this bigger linear system, our x would satisfy the constraint $Cx = d$.

2.7.1 Vector Laplacian with Magnetic Boundary Conditions on an Annulus

Consider the vector Laplacian:

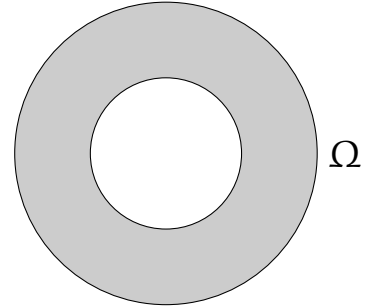
$$\Delta_1 := -\text{grad} \cdot \text{div} + \text{curl} \cdot \text{curl}.$$

Let Ω be a 2-dimensional annulus, then Ω is a bounded domain with one topological hole, therefore $\dim \ker \Delta_1 = 1$ and this is the same as $\dim H_1$ (1-homology) $= 1 = \beta_1$ (1-Betti). Moreover by Lefschetz-Poincare duality, the first homology and cohomology groups are isomorphic, i.e., " $H_1 \simeq H^1$ " (1-cohomology). Thus, there is a 1-dimensional space of nontrivial harmonic vector fields.

These harmonic vector fields belong to the kernel of Δ_1 and will have to have the same boundary conditions as for $\Delta_1 u = f$. In particular, they satisfy the homogeneous boundary conditions:

$$u \cdot n = 0, \text{ curl } u \times n = 0,$$

where n denotes the outward unit normal vector to $\partial\Omega$.



Theorem 2.1. u is harmonic $\iff \Delta_1 u = 0 \iff \text{div } u = 0$ and $\text{curl } u = 0$.

Proof. ($\Leftarrow$) Trivial.

$$\left. \begin{aligned} \text{div } u &= 0 \implies -\text{grad} \cdot \text{div } u = 0 \\ \text{curl } u &= 0 \implies \text{curl} \cdot \text{curl } u = 0 \end{aligned} \right\} \implies \Delta_1 u = 0.$$

$$\begin{aligned} (\Rightarrow) \Delta_1 u &= 0 \implies \langle \Delta_1 u, u \rangle = \langle 0, u \rangle = 0. \\ \implies \langle -\text{grad} \text{ div } u, u \rangle &+ \langle \text{curl} \text{ curl } u, u \rangle = 0. \\ \implies \langle \text{div } u, \text{div } u \rangle &+ \langle \text{curl } u, \text{curl } u \rangle = 0. \text{ (via adjoints)} \end{aligned}$$

And this is why the boundary conditions for harmonic vector fields have to be the same.

$$\implies \|\operatorname{div} \mathbf{u}\| + \|\operatorname{curl} \mathbf{u}\| = 0.$$

$$\implies \operatorname{div} \mathbf{u} = 0 \text{ \& } \operatorname{curl} \mathbf{u} = 0.$$

□

On the annulus, the harmonic vector fields have the following for basis:

$$\mathfrak{H} = \hat{\theta}. \text{ in polar coordinates}$$

For polar coordinates:

$$\left. \begin{array}{l} x = r \cos \theta \\ y = r \sin \theta \end{array} \right\} \implies \begin{array}{l} x^2 + y^2 = r^2 \\ \tan \theta = \frac{y}{x} \end{array}$$

$$\text{For vector fields: } F(x, y) = f_1(x, y)\hat{x} + f_2(x, y)\hat{y} \implies F(r, \theta) = f_1(r, \theta)\hat{x} + f_2(r, \theta)\hat{y}.$$

$$\text{Now, } \hat{x} = \frac{\partial}{\partial r}(r \cos \theta)\hat{r} + \frac{\partial}{\partial \theta}(r \cos \theta)\hat{\theta} \implies \hat{x} = \cos \theta \hat{r} - r \sin \theta \hat{\theta}.$$

$$\text{Likewise, } \hat{y} = \sin \theta \hat{r} + r \cos \theta \hat{\theta}.$$

$$\implies \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}}_{\substack{\text{Jacobian of} \\ (x, y) \mapsto (r, \theta) \\ \mathbb{R}^2 \rightarrow \mathbb{R}^2}} \begin{bmatrix} \hat{r} \\ \hat{\theta} \end{bmatrix}$$

Our harmonic on the annulus is thus:

$$h(x, y) = \hat{\theta}$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}}_{\text{Orthogonal matrix}} \begin{bmatrix} \hat{r} \\ r\hat{\theta} \end{bmatrix} \implies \begin{bmatrix} \hat{r} \\ r\hat{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}}_{\text{Inverse is transpose}} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$$

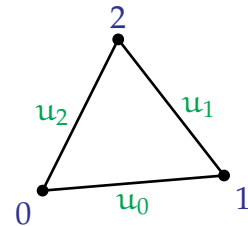
$$\text{Therefore, } r\hat{\theta} = -\sin \theta \hat{x} + \cos \theta \hat{y} \left(\implies \hat{\theta} = -\frac{\sin \theta}{r} \hat{x} + \frac{\cos \theta}{r} \hat{y} \right) \text{ and } \hat{r} = \cos \theta \hat{x} + \sin \theta \hat{y}.$$

$$\implies h(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \left(\frac{-y}{\sqrt{x^2 + y^2}} \hat{x} + \frac{x}{\sqrt{x^2 + y^2}} \hat{y} \right).$$

$$\implies h(x, y) = \frac{-y}{x^2 + y^2} \hat{x} + \frac{x}{x^2 + y^2} \hat{y}.$$

Now for mixed Poisson, the interpolated solution is perpendicular to the harmonic function, i.e., $u_I \perp h(x, y) \iff \langle u_I, h \rangle = 0$. On a triangle T , for linear case we can write $u_I = u_0 W_{01} + u_1 W_{02} + u_2 W_{12}$.

$$\langle u_I, h \rangle = 0 \implies u_0 \langle W_{01}, h \rangle + u_1 \langle W_{02}, h \rangle + u_2 \langle W_{12}, h \rangle = 0.$$



$$\Rightarrow [\langle W_{01}, h \rangle \quad \langle W_{02}, h \rangle \quad \langle W_{12}, h \rangle] \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = 0.$$

Extending to whole mesh of Ω , $C^T u = 0$, where $u = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{\# \text{Edges}} \end{bmatrix}$ and $C = [\langle \phi_i, h \rangle]$. Moreover, its computation is similar to RHS computation.

$$[S] \begin{bmatrix} \sigma \\ u \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ b \end{bmatrix} \text{ such that } \{C^T u = 0\}$$

$$\begin{array}{c} \updownarrow \\ \begin{bmatrix} \mathbf{0} & C \end{bmatrix}^T \begin{bmatrix} \sigma \\ u \end{bmatrix} = 0. \\ \begin{array}{cc} \uparrow & \uparrow \\ N_0 \times 1 & N_1 \times 1 \end{array} \end{array}$$

We solve the bigger linear system:

$$\begin{array}{c} (N_0 + N_1) \times 1 \\ \downarrow \\ \begin{bmatrix} S & \begin{bmatrix} \mathbf{0} \\ C \end{bmatrix} \\ \begin{bmatrix} \mathbf{0} & C^T \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} \sigma \\ u \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ b \\ 0 \end{bmatrix}. \\ \begin{array}{cc} \uparrow & \uparrow \\ 1 \times (N_0 + N_1) & 1 \times 1 \end{array} \end{array}$$

If S is $(N_0 + N_1) \times (N_0 + N_1)$, the above matrix $\tilde{S}$ is $(N_0 + N_1 + 1) \times (N_0 + N_1 + 1)$ and

$$u = x[N_0 : N_0 + N_1].$$

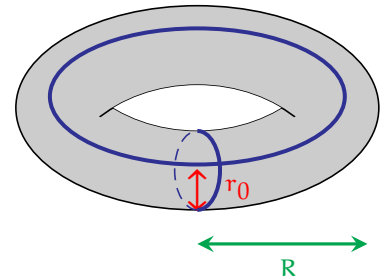
2.7.2 Vector Poisson Problem on Solid Torus

Let Ω be a solid torus in $\mathbb{R}$, thus $\Omega := D^2 \times S^1$ i.e. it is solid of revolution obtained by revolving a unit 2-disc of radius r_0 about a unit circle of radius R . The vector poisson problem on Ω is defined as:

$$\Delta_1 u = f \Rightarrow \nabla \times \nabla \times u - \nabla \nabla \cdot u = f \text{ on } \Omega,$$

such that $u \cdot \hat{n} = 0$ and $\nabla \times u \times \hat{n} = 0$ on $\partial\Omega$, with $u \perp$ harmonic vector fields $:= \mathfrak{H}$. We will consider the coordinate system $\{r, \theta, \phi\}$. The embedding of this torus in $\mathbb{R}^3$ is then given by

$$x = (R + r \cos \theta) \cos \phi, \quad y = (R + r \cos \theta) \sin \phi, \quad z = r \sin \theta$$



and in which the orthonormal basis for 1-forms is $\{\hat{r}, r\hat{\theta}, (R + r \cos \theta)\hat{\phi}\}$.

$$\left. \begin{aligned} x &= (R + r \cos \theta) \cos \phi \\ y &= (R + r \cos \theta) \sin \phi \\ z &= r \sin \theta \end{aligned} \right\} \Rightarrow \begin{aligned} r &= ((-R + \sqrt{x^2 + y^2})^2 + z^2)^{\frac{1}{2}} \\ \theta &= \tan^{-1} \left(\frac{z}{-R + \sqrt{x^2 + y^2}} \right) \\ \phi &= \tan^{-1} \left(\frac{y}{x} \right) \end{aligned}$$

Consider $\hat{x} = \frac{\partial x}{\partial r} \hat{r} + \frac{\partial x}{\partial \theta} \hat{\theta} + \frac{\partial x}{\partial \phi} \hat{\phi} = \cos \theta \cos \phi \hat{r} - r \sin \theta \cos \phi \hat{\theta} - (R + r \cos \theta) \sin \phi \hat{\phi}$. Similarly, $\hat{y} = \cos \theta \sin \phi \hat{r} - r \sin \theta \sin \phi \hat{\theta} + (R + r \cos \theta) \cos \phi \hat{\phi}$ and $\hat{z} = \sin \theta \hat{r} + r \cos \theta \hat{\theta}$.

$$\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos \theta \cos \phi & -r \sin \theta \cos \phi & -(R + r \cos \theta) \sin \phi \\ \cos \theta \sin \phi & -r \sin \theta \sin \phi & (R + r \cos \theta) \cos \phi \\ \sin \theta & r \cos \theta & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix},$$

$$\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos \theta \cos \phi & -\sin \theta \cos \phi & -\sin \phi \\ \cos \theta \sin \phi & -\sin \theta \sin \phi & \cos \phi \\ \sin \theta & \cos \theta & 0 \end{bmatrix}}_{\text{Orthogonal Matrix}} \begin{bmatrix} \hat{r} \\ r\hat{\theta} \\ (R + r \cos \theta)\hat{\phi} \end{bmatrix},$$

$$\Rightarrow \begin{bmatrix} \hat{r} \\ r\hat{\theta} \\ (R + r \cos \theta)\hat{\phi} \end{bmatrix} = \begin{bmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & \sin \theta \\ -\sin \theta \cos \phi & -\sin \theta \sin \phi & \cos \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix},$$

$\Rightarrow \hat{r} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} + \sin \theta \hat{z}$, $\hat{\theta} = \frac{1}{r} (-\sin \theta \cos \phi \hat{x} - \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z})$, and $\hat{\phi} = \frac{1}{(R + r \cos \theta)} (-\sin \phi \hat{x} + \cos \phi \hat{y})$. Now we can solve the problem by following these steps:

Mixed Formulation: $\nabla \cdot \mathbf{u} = \sigma$,

$$\nabla \times \nabla \times \mathbf{u} - \nabla \sigma = \mathbf{f}.$$

Variational Formulation: $\left. \begin{aligned} \langle \sigma, \tau \rangle - \langle \nabla \cdot \mathbf{u}, \tau \rangle &= 0, \\ -\langle \nabla \sigma, \mathbf{v} \rangle + \langle \nabla \times \nabla \times \mathbf{u}, \mathbf{v} \rangle &= \langle \mathbf{f}, \mathbf{v} \rangle. \end{aligned} \right\} \Rightarrow \left. \begin{aligned} -\langle \sigma, \tau \rangle - \langle \mathbf{u}, \nabla \tau \rangle &= 0 \\ -\langle \nabla \sigma, \mathbf{v} \rangle + \langle \nabla \times \mathbf{u}, \nabla \times \mathbf{v} \rangle &= \langle \mathbf{f}, \mathbf{v} \rangle \end{aligned} \right\} \forall \tau, \mathbf{v}.$

$$\text{Matrix Form: } \begin{bmatrix} -\langle \sigma, \tau \rangle & -\langle \mathbf{u}, \nabla \tau \rangle \\ -\langle \nabla \sigma, \mathbf{v} \rangle & \langle \nabla \times \mathbf{u}, \nabla \times \mathbf{v} \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \langle \mathbf{f}, \mathbf{v} \rangle \end{bmatrix}.$$

Spaces: $\sigma, \tau \in H^1(\Omega)$; $\mathbf{u}, \mathbf{v} \in H(\text{curl}; \Omega)$.

Finite dimensional subspaces: $\Sigma_h \subset H^1(\Omega)$; $V_h \subset H(\text{curl}; \Omega)$.

Basis for these subspaces:

$r = 1$: *Linear*

Σ_h : $\{\lambda_i\}$ on each vertex i of Tet $\{0, 1, 2, 3\}$,

V_h : $\{\phi_{ij}\}$ on each edge ij of Tet $\{0, 1, 2, 3\}$,

where $\phi_{ij} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i$ is the Whitney basis.

$r = 2$: *Quadratic*

Σ_h : $\{\lambda_i(2\lambda_i - 1), 4\lambda_i\lambda_j\}$ on each vertex pairs $\{i, j\}$ of Tet $\{0, 1, 2, 3\}$,

V_h : $\{\lambda_i\phi_{ij}, \lambda_j\phi_{ij}\}$ on each edge ij of Tet $\{0, 1, 2, 3\}$,

$\{\lambda_k\phi_{ij}, \lambda_j\phi_{ik}\}$ on each face ijk of Tet $\{0, 1, 2, 3\}$.

Linear System: $\begin{bmatrix} -S_{00} & -S_{01} \\ -S_{01}^T & S_{11} \end{bmatrix} \begin{bmatrix} \sigma \\ u \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ b \end{bmatrix}$ such that $u \perp$ basis for $\mathfrak{H} \implies \begin{bmatrix} S & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \sigma \\ u \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ b \\ 0 \end{bmatrix}$.

S_{00} : Inner product between basis for Σ_h

On a Tet T:

r = 1: $S_{00,T}$ is 4×4

r = 2: $S_{00,T}$ is 10×10

$$\begin{array}{l} \text{Vertices} \\ \text{Edges} \end{array} \left[\begin{array}{c|c} 4 \times 4 & 4 \times 6 \\ \hline 6 \times 4 & 6 \times 6 \end{array} \right]$$

10×10

On a mesh, S_{00} is:

r = 1: $N_0 \times N_0$

r = 2: $(N_0 + N_1) \times (N_0 + N_1)$

S_{01} : Inner product between gradients of basis for Σ_h and basis for V_h

On a Tet T:

r = 1: $S_{01,T}$ is 4×6

r = 2: $S_{01,T}$ is 10×20

On a mesh, S_{01} is:

r = 1: $N_0 \times N_1$

r = 2: $(N_0 + N_1) \times (2N_1 + 2N_2)$

Gradients of basis for Σ_h :

r = 1: $\{\nabla \lambda_i\}$

r = 2: $\{\nabla(\lambda_i(2\lambda_i - 1)), \nabla(4\lambda_i\lambda_j)\} = \{\nabla\lambda_i(2\lambda_i - 1) + 2\lambda_i\nabla\lambda_i, 4\nabla\lambda_i\lambda_j + 4\lambda_i\nabla\lambda_j\}$

S_{11} : Inner product between curl of the basis for V_h

On a Tet T:

r = 1: $S_{11,T}$ is 6×6

r = 2: $S_{11,T}$ is 20×20

$$\begin{array}{l} \text{Edges} \\ \text{Faces} \end{array} \left[\begin{array}{c|c} 12 \times 12 & 12 \times 8 \\ \hline 8 \times 12 & 8 \times 8 \end{array} \right]$$

20×20

On a mesh, S_{11} is:

r = 1: $N_1 \times N_1$

r = 2: $(2N_1 + 2N_2) \times (2N_1 + 2N_2)$

Curl of ϕ_{ij} and higher-order basis are computed before.

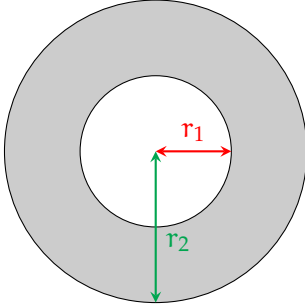
Global Stiffness Matrix size is:

r = 1: $(N_0 + N_1) \times (N_0 + N_1)$

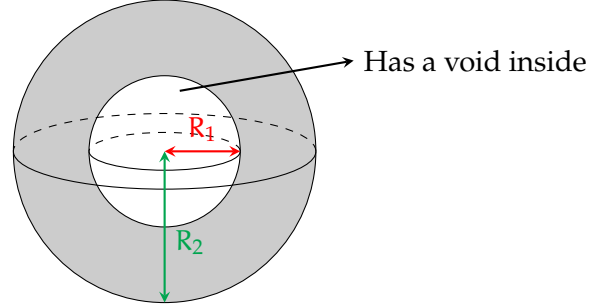
r = 2: $((N_0 + N_1) + (2N_1 + 2N_2)) \times ((N_0 + N_1) + (2N_1 + 2N_2))$

2.7.3 Vector Laplacian Poisson with Electric Boundary Conditions on 3d Annulus

$\mathbb{R}^2$: Annulus



$\mathbb{R}^3$: Solid Annulus



Ω : Solid Annulus := Annulus $\times S^1$ ($S^2 \times I$) [$\beta_2 = 1 \iff \dim H_2(\Omega, \mathbb{R}) = 1$]

$$\Delta_1 u = f \implies \nabla \times \nabla \times u - \nabla \nabla \cdot u = f \text{ on } \Omega,$$

such that $u \times \hat{n} = 0$ and $\nabla \cdot u = 0$ on $\partial\Omega$ with $u \perp$ harmonic vector fields := $\mathcal{H}$. We can use spherical coordinates $\{r, \theta, \phi\}$ for embedding and solution. For $R_1 \leq r \leq R, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$, the embedding of this sphere in $\mathbb{R}^3$ is then given by

$$x = r \cos \theta \sin \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \phi,$$

and in which the orthonormal basis for 1-forms is $\{\hat{r}, r \sin \phi \hat{\theta}, r \hat{\phi}\}$. We will consider the coordinate system

$$\left. \begin{aligned} x &= r \cos \theta \sin \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \phi \end{aligned} \right\} \implies \begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \tan^{-1} \left(\frac{y}{x} \right) \\ \phi &= \tan^{-1} \left(\frac{\sqrt{x^2 + y^2}}{z} \right) \end{aligned}$$

Consider $\hat{x} = \frac{\partial x}{\partial r} \hat{r} + \frac{\partial x}{\partial \theta} \hat{\theta} + \frac{\partial x}{\partial \phi} \hat{\phi} = \cos \theta \sin \phi \hat{r} - r \sin \theta \sin \phi \hat{\theta} + r \cos \theta \cos \phi \hat{\phi}$. Similarly, $\hat{y} = \sin \theta \sin \phi \hat{r} + r \cos \theta \sin \phi \hat{\theta} + r \sin \theta \cos \phi \hat{\phi}$, and $\hat{z} = \cos \phi \hat{r} - r \sin \phi \hat{\phi}$.

$$\implies \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos \theta \sin \phi & -r \sin \theta \sin \phi & r \cos \theta \cos \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \phi & 0 & -r \sin \phi \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix},$$

$$\implies \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos \theta \sin \phi & -\sin \theta & \cos \theta \cos \phi \\ \sin \theta \sin \phi & \cos \theta & \sin \theta \cos \phi \\ \cos \phi & 0 & -\sin \phi \end{bmatrix}}_{\text{Orthogonal Matrix}} \begin{bmatrix} \hat{r} \\ r \sin \phi \hat{\theta} \\ r \hat{\phi} \end{bmatrix},$$

$$\implies \begin{bmatrix} \hat{r} \\ r \sin \phi \hat{\theta} \\ r \hat{\phi} \end{bmatrix} = \begin{bmatrix} \cos \theta \sin \phi & \sin \theta \sin \phi & \cos \phi \\ -\sin \theta & \cos \theta & 0 \\ \cos \theta \cos \phi & \sin \theta \cos \phi & -\sin \phi \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}.$$

$$\implies \hat{r} = \cos \theta \sin \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \phi \hat{z}, \hat{\theta} = \frac{1}{r \sin \phi} (-\sin \theta \hat{x} + \cos \theta \hat{y}), \text{ and}$$

$$\hat{\phi} = \frac{1}{r} (\cos \theta \cos \phi \hat{x} + \sin \theta \cos \phi \hat{y} - \sin \phi \hat{z})$$

Mixed Formulation: (u & σ are vector fields)

$$\nabla \cdot u = \sigma,$$

$$\nabla \times \nabla \times u - \nabla \sigma = f.$$

Variational Formulation: $\left\{ \begin{array}{l} \langle \sigma, \tau \rangle - \langle \nabla \times u, \tau \rangle = 0, \\ \langle \nabla \times \sigma, v \rangle - \langle \nabla \nabla \cdot u, v \rangle = \langle f, v \rangle. \end{array} \right\} \implies \left\{ \begin{array}{l} -\langle \sigma, \tau \rangle + \langle u, \nabla \times \tau \rangle = 0, \\ \langle \nabla \times \sigma, v \rangle + \langle \nabla \cdot u, \nabla \cdot v \rangle = \langle f, v \rangle, \end{array} \right\} \forall \tau, v$

with (Integration by parts and B.C.s) $\iff$ (Operator Adjoints)

Matrix Form: $\begin{bmatrix} -\langle \sigma, \tau \rangle & \langle u, \nabla \times \tau \rangle \\ \langle \nabla \times \sigma, v \rangle & \langle \nabla \cdot u, \nabla \cdot v \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \langle f, v \rangle \end{bmatrix}.$

Spaces: $\sigma, \tau \in \dot{H}(\text{curl}; \Omega)$; $u, v \in \dot{H}(\text{div}; \Omega)$, where

$$\dot{H}(\text{curl}; \Omega) = \{u \in [L^2(\Omega)]^3 \mid \text{curl } u \in [L^2(\Omega)]^3 \text{ \& } u \cdot n = 0 \text{ on } \partial\Omega\},$$

$$\dot{H}(\text{div}; \Omega) = \{u \in [L^2(\Omega)]^3 \mid \text{div } u \in L^2(\Omega) \text{ \& } \text{div } u|_{\partial\Omega} = 0\}.$$

Finite dimensional subspaces: $\Sigma_h \subset \dot{H}(\text{curl}; \Omega)$; $V_h \subset \dot{H}(\text{div}; \Omega)$.

Basis for these subspaces:

$\dot{H}(\text{curl}; \Omega)$: “Whitney” Vector Fields (Dimension: $N_1 = \# \text{ Edges}$)

$$W_{ij} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i \text{ on Edge } ij.$$

$\dot{H}(\text{div}; \Omega)$: “Nedelec” Vector Fields (Dimension: $N_2 = \# \text{ Faces}$)

$$W_{ijk} = \lambda_i \nabla \lambda_j \times \nabla \lambda_k - \lambda_j \nabla \lambda_i \times \nabla \lambda_k + \lambda_k \nabla \lambda_i \times \nabla \lambda_j \text{ on Face } ijk.$$

Linear System: $\begin{bmatrix} -S_{00} & -S_{01} \\ -S_{01}^T & S_{11} \end{bmatrix} \begin{bmatrix} \sigma \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$ such that $u \perp \text{Basis for } \mathfrak{H} \implies \begin{bmatrix} S & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \sigma \\ u \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}.$

Stiffness Matrix:

Linear:

$$S_g = \begin{matrix} N_1 & N_2 \\ N_2 & \end{matrix} \left[\begin{array}{c|c} S_{00} & S_{01} \\ \hline -S_{01}^T & S_{11} \end{array} \right]$$

On a Tet T:

$$S_{00,T} : 6 \times 6 \text{ (6 Edges)}$$

$$S_{00}[e_i, e_j] = \int_T W_{e_i} \cdot W_{e_j}$$

$$S_{01,T} : 6 \times 4 \text{ (Edges} \times \text{Faces)}$$

$$S_{01}[e_i, f_j] = \int_T \nabla \times W_{e_i} \cdot W_{f_j}$$

$$S_{11,T} : 4 \times 4 \text{ (4 Faces)}$$

$$S_{11}[f_i, f_j] = \int_T \nabla \cdot W_{f_i} \nabla \cdot W_{f_j}$$

R.H.S.: $b = \begin{bmatrix} 0 \\ f_h \end{bmatrix}$, $f_h \in \mathbb{R}^{N_2}$ where $f_h[i] = \int_T f(x, y, z) \cdot W_{f_i}.$

Quadratic:

$$S_g = \begin{matrix} 2N_1 + 2N_2 & 3N_2 + 3N_3 \\ 3N_2 + 3N_3 & \end{matrix} \left[\begin{array}{c|c} S_{00} & S_{01} \\ \hline -S_{01}^T & S_{11} \end{array} \right]$$

On a Tet T:

$$S_{00,T} : 20 \times 20 \text{ (2} \times \text{6 Edges + 2} \times \text{4 Faces)}$$

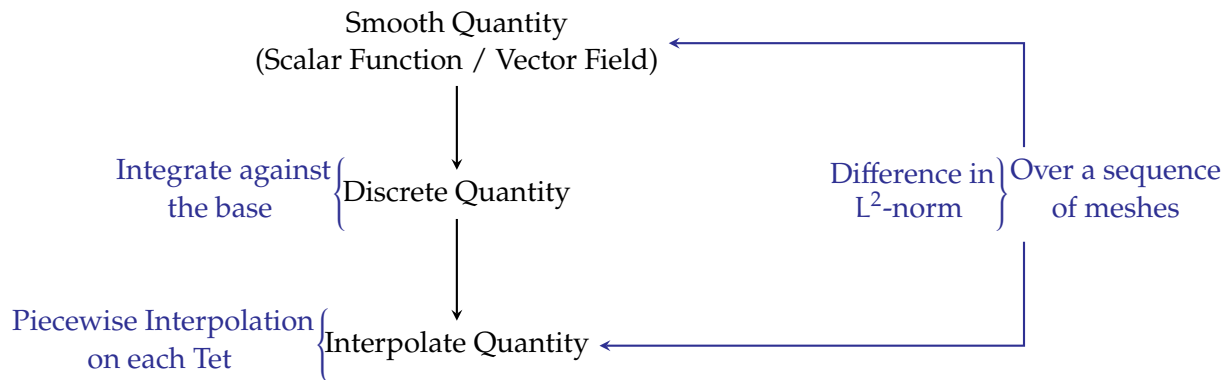
$$S_{01,T} : 20 \times 15 \text{ (2} \times \text{6 Edges + 2} \times \text{4 Faces)} \times \text{(3} \times \text{4 Faces + 3} \times \text{1 Tet)}$$

$$S_{11,T} : 15 \times 15 \text{ (3} \times \text{4 Faces + 3} \times \text{1 Tet)}$$

R.H.S.: $b = \begin{bmatrix} 0 \\ f_h \end{bmatrix}$, $f_h \in \mathbb{R}^{3N_2+3N_3}$ where $f_h[i] = \int_T f(x, y, z) \cdot W_{f_i}.$

2.8 Approximability

We want to check if:



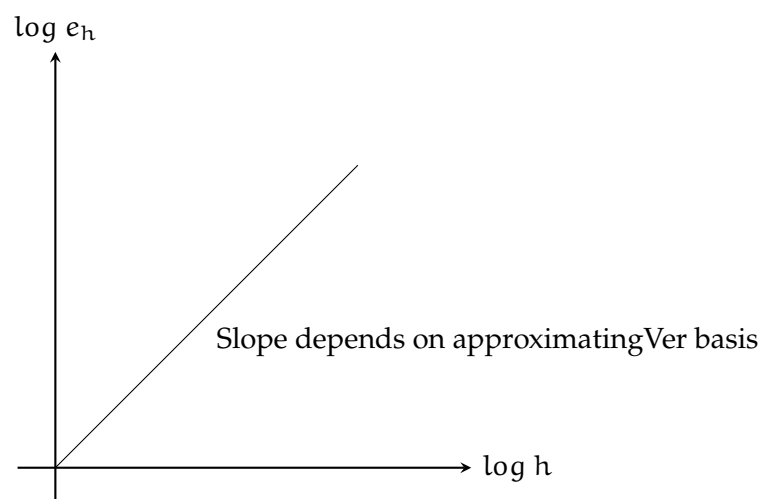
Error slope: If u_a is an analytical vector field and u_h is the discretization, then we can say:

$$u_h = \sum_{T \in \mathfrak{T}} \left(\sum_{i=1}^n u_{i,T} \phi_{i,T} \right) := u_{I,T},$$

(All tetrahedra)

$$e_h = \|u_a - u_h\| = \left[\sum_{T \in \mathfrak{T}} \int_T \left(\left(\sum_{i=1}^n u_{i,T} \phi_{i,T} \right) - u_a \right)^2 \right]^{\frac{1}{2}}.$$

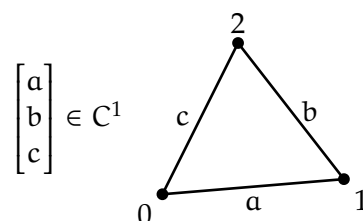
Expect to see:



A Problem!

$$W_{01} = \lambda_0 \nabla \lambda_1 - \lambda_1 \nabla \lambda_0.$$

Take $u = W_{01}$.



Let's discretize u on the triangle $\{0, 1, 2\}$. Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the coefficients in the discretization of u .

What we expect: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

The way we're computing $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is as follows:

$$a = \int_T \langle u, W_{01} \rangle, \quad b = \int_T \langle u, W_{02} \rangle, \quad c = \int_T \langle u, W_{12} \rangle.$$

Now, mass matrix $M = \left[\int_T \langle W_{ij}, W_{kl} \rangle \right]_{3 \times 3}$, $i < j < k < l$, $i, j, k, l \in \{0, 1, 2\}$,

$$\begin{bmatrix} \overbrace{\langle W_{01}, W_{01} \rangle}^a & \overbrace{\langle W_{01}, W_{02} \rangle}^b & \overbrace{\langle W_{01}, W_{12} \rangle}^c \\ \overbrace{\langle W_{02}, W_{01} \rangle}^b & \overbrace{\langle W_{02}, W_{02} \rangle}^b & \overbrace{\langle W_{02}, W_{12} \rangle}^c \\ \overbrace{\langle W_{12}, W_{01} \rangle}^c & \overbrace{\langle W_{12}, W_{02} \rangle}^c & \overbrace{\langle W_{12}, W_{12} \rangle}^c \end{bmatrix}$$

Vector field f : For each edge E of the mesh, we want to compute

$$f_h[E] = \int_E f \cdot \hat{t}_E; \quad \hat{t}_E : \text{Unit tangent along } E,$$

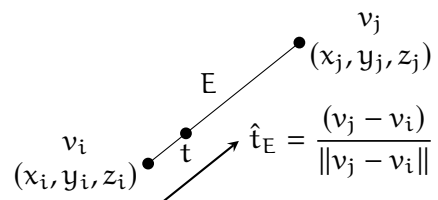
where $f(x, y, z)$ is a vector field.

$$\int_E f \cdot \hat{t}_E$$

$$E(t) = v_i + t(v_j - v_i), \quad t \in [0, 1]$$

↑
Points on edge E

$$g(t) = f(E(t)) \cdot \hat{t}_E$$



$$v_i, v_j = \text{sc}[1].\text{simplices}[E]$$

Python:

```
e_vector = Vertices[v_i] - Vertices[v_j]
t_E = (e_vector/np.linalg.norm(e_vector))
g(t) = lambda t: np.dot(f(v_i + t * e_vector), t_E)
f_h[E] = scipy.integrate.quad(g, 0, 1)
```

L^2 projection: Given f , we want to approximate f in some basis $\{\phi_i\}_{i=1}^n$,

$$f_h := \sum_{i=1}^n f_i \phi_i = Pf, \quad f_i \in \mathbb{R},$$

is an approximation of f . Then we want to check that as $h \rightarrow 0$, $\|f - f_h\| \rightarrow 0$. Essentially, we want to compute L^2 -projection of f into the space spanned by $\{\phi_i\}_{i=1}^n$. So, computing the L^2 -projection is the

following problem: $\min \|f - \sum_{i=1}^n f_i \phi_i\|_{L^2}$. This is like solving: $Ax \simeq b$

$$L(f_1, \dots, f_n) := \|f - \sum_{i=1}^n f_i \phi_i\|_{L^2}^2, \text{ where } \|u\|_{L^2} = \langle u, u \rangle = \int_{\Omega} u^2 dx.$$

$$\begin{aligned} L &:= \int_{\Omega} \left(f - \sum_{i=1}^n f_i \phi_i \right) \cdot \left(f - \sum_{i=1}^n f_i \phi_i \right) dx, \\ &= \left\langle f - \sum_{i=1}^n f_i \phi_i, f - \sum_{i=1}^n f_i \phi_i \right\rangle \\ &= \langle f, f \rangle - \left\langle f, \sum_{i=1}^n f_i \phi_i \right\rangle - \left\langle \sum_{i=1}^n f_i \phi_i, f \right\rangle + \left\langle \sum_{i=1}^n f_i \phi_i, \sum_{i=1}^n f_i \phi_i \right\rangle. \end{aligned}$$

$$\Rightarrow L = \langle f, f \rangle - \left\langle f, \sum_{i=1}^n f_i \phi_i \right\rangle - \left\langle \sum_{i=1}^n f_i \phi_i, f \right\rangle + \left\langle \sum_{i=1}^n f_i \phi_i, \sum_{i=1}^n f_i \phi_i \right\rangle.$$

Since $\min L \Rightarrow \frac{\partial L}{\partial f_i} = 0$, so we have

$$\begin{aligned} \frac{\partial L}{\partial f_i} &= -2\langle \phi_i, f \rangle + \left\langle \phi_i, \sum_{j=1}^n f_j \phi_j \right\rangle + \left\langle \sum_{j=1}^n f_j \phi_j, \phi_i \right\rangle = -2\langle \phi_i, f \rangle + 2\left\langle \phi_i, \sum_{j=1}^n f_j \phi_j \right\rangle = 0, \\ \Rightarrow \left\langle \phi_i, \sum_{j=1}^n f_j \phi_j \right\rangle &= \langle \phi_i, f \rangle \forall i = 1, \dots, n. \end{aligned}$$

Thus, we have:

$$\begin{aligned} \left\langle \phi_1, \sum_{j=1}^n f_j \phi_j \right\rangle &= \langle \phi_1, f \rangle, \\ \left\langle \phi_2, \sum_{j=1}^n f_j \phi_j \right\rangle &= \langle \phi_2, f \rangle, \\ &\vdots \\ \left\langle \phi_n, \sum_{j=1}^n f_j \phi_j \right\rangle &= \langle \phi_n, f \rangle \\ &\Downarrow \\ M\tilde{f} &= b, \quad b_j = [\langle \phi_j, f \rangle], \quad \tilde{f} = [f_j] \quad \left(\text{because } \int \phi_i \sum_{j=1}^n f_j \phi_j = \sum_{j=1}^n f_j \langle \phi_i, \phi_j \rangle \right). \end{aligned}$$

Scalar function Approximability: Given f , a scalar function we want to check:

$$\|f - f_h\| \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

Case 1: Linear basis

We want to find $\{f_i\}_{i=1}^{n_0}$, n_0 = number of linear basis such that:

$$f_h = P_{L^2} f = \sum_{i=1}^{n_0} f_i \phi_i.$$

Previously, when we need to find f_i , we just did $f_i = f(v_i)$, $i = 1, \dots, n_0$, where v_i is the i^{th} vertex. But now, instead we shall find $\{f_i\}_{i=1}^{n_0}$ by solving:

$$\begin{array}{ccc} M_{\text{linear}}^{\text{scalar}} & \tilde{f} & = \tilde{b}, \\ \uparrow & \uparrow & \uparrow \\ n_0 \times n_0 & n_0 \times 1 & n_0 \times 1 \end{array}$$

where $M_{\text{linear}}^{\text{scalar}}$ is the mass matrix of linear basis for scalar functions, and $\tilde{f} = [f_1, f_2, \dots, f_{n_0}]^T$, $\tilde{b} = [\tilde{b}_j]$, $\tilde{b}_j = \langle f, \phi_j \rangle$. Once we obtain $\tilde{f}$, we can check

$$f_j \stackrel{?}{=} f(v_j), \quad j = 1, \dots, n_0.$$

Case 2: Quadratic basis

In this case, we simply use the quadratic basis: $\{\phi_i\}_{i=1}^{N_0}$ and solve:

$$\begin{array}{ccc} M_{\text{quadratic}}^{\text{scalar}} & \tilde{f} & = \tilde{b}, \quad N_0 = n_0 + n_1 \\ \uparrow & \uparrow & \uparrow \\ N_0 \times N_0 & N_0 \times 1 & N_0 \times 1 \end{array} \quad \begin{array}{cc} \uparrow & \uparrow \\ \# \text{ Vertices} & \# \text{ Edges} \end{array}$$

We now need to check,

$$f_j \stackrel{?}{=} f(v_j), \quad j = 1, \dots, n_0 \quad \text{and} \quad f_j \stackrel{?}{=} f\left(\frac{v_i + v_j}{2}\right), \quad j = n_0 + 1, \dots, n_0 + n_1.$$

Vector function Approximability: Given F , a vector field we want to check:

$$\|F - F_h\| \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

Case 1: Linear basis

We want to find $\{F_i\}_{i=1}^{n_1}$, n_1 = number of linear basis such that:

$$F_h = P_{L^2} F = \sum_{i=1}^{n_1} F_i W_i^{\text{linear}}.$$

Thus, setup will look like:

$$\begin{array}{ccc} M_{\text{linear}}^{\text{Whitney}} & \tilde{F} & = \tilde{B}, \\ \uparrow & \uparrow & \uparrow \\ n_1 \times n_1 & n_1 \times 1 & n_1 \times 1 \end{array}$$

where $M_{\text{linear}}^{\text{scalar}}$ is the mass matrix of linear basis for Whitney vector fields,

$$\tilde{F} = [F_1, F_2, \dots, F_{n_1}]^T, \quad \tilde{B} = [\tilde{B}_j], \quad \tilde{B}_j = \int_{\Omega} F \cdot \phi_j.$$

We need to check $B_i \stackrel{?}{=} \text{Integral of tangential component of } F \text{ over the } i^{\text{th}} \text{ edge of the mesh.}$

Case 2: Quadratic basis

Similar to above cases, we need to solve:

$$\begin{array}{ccc} M_{\text{quadratic}}^{\text{Whitney}} & \tilde{F} & = \tilde{B}, \quad N_1 = 2n_1 + 2n_2. \\ \uparrow & \uparrow & \uparrow \\ N_1 \times N_1 & N_1 \times 1 & N_1 \times 1 \end{array} \quad \begin{array}{cc} \uparrow & \uparrow \\ \# \text{ Edges} & \# \text{ Faces} \end{array}$$

2.9 Some Notes on Computations with Maxwell's Equations

1. On $\Omega = [0, 1]^3$, consider the following set of equations:

$$\begin{aligned}\sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{H} &= \mathbf{f}_E, \\ \mathbf{H} + \mu \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} &= \mathbf{f}_H.\end{aligned}$$

We'll set $\sigma = 1, \mu = 1, \epsilon = 1$, and take our analytical solutions to be:

$$\mathbf{E}(x, t) = \begin{bmatrix} 0 \\ \cos 2\pi(x - t) \\ \sin 2\pi(x - t) \end{bmatrix}, \quad \mathbf{H}(x, t) = \begin{bmatrix} 0 \\ -\sin 2\pi(x - t) \\ \cos 2\pi(x - t) \end{bmatrix}.$$

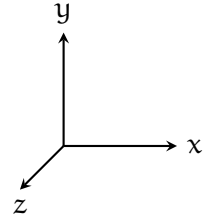
$$\mathbf{f}_E = \mathbf{E} + \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{H} = \begin{bmatrix} 0 \\ \cos 2\pi(x - t) \\ \sin 2\pi(x - t) \end{bmatrix} - 2\pi \begin{bmatrix} 0 \\ -\sin 2\pi(x - t) \\ \cos 2\pi(x - t) \end{bmatrix} + 2\pi \begin{bmatrix} 0 \\ -\sin 2\pi(x - t) \\ \cos 2\pi(x - t) \end{bmatrix} = \begin{bmatrix} 0 \\ \cos 2\pi(x - t) \\ \sin 2\pi(x - t) \end{bmatrix},$$

$$\mathbf{f}_H = \mathbf{H} + \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = \begin{bmatrix} 0 \\ -\sin 2\pi(x - t) \\ \cos 2\pi(x - t) \end{bmatrix} - 2\pi \begin{bmatrix} 0 \\ -\cos 2\pi(x - t) \\ -\sin 2\pi(x - t) \end{bmatrix} + 2\pi \begin{bmatrix} 0 \\ -\cos 2\pi(x - t) \\ -\sin 2\pi(x - t) \end{bmatrix} = \begin{bmatrix} 0 \\ -\sin 2\pi(x - t) \\ \cos 2\pi(x - t) \end{bmatrix}.$$

Boundary conditions are:

$$\mathbf{E}(x, t) = \begin{bmatrix} 0 \\ \cos 2\pi(x - t) \\ \sin 2\pi(x - t) \end{bmatrix}, \quad \mathbf{H}(x, t) = \begin{bmatrix} 0 \\ -\sin 2\pi(x - t) \\ \cos 2\pi(x - t) \end{bmatrix},$$

$$\nabla \times \mathbf{E}(x, t) = -2\pi \begin{bmatrix} 0 \\ \cos 2\pi(x - t) \\ \sin 2\pi(x - t) \end{bmatrix}, \quad \nabla \times \mathbf{H}(x, t) = 2\pi \begin{bmatrix} 0 \\ \sin 2\pi(x - t) \\ -\cos 2\pi(x - t) \end{bmatrix}.$$



Boundaries are:

$$x = 0: \mathbf{n} = -\hat{x} \implies \mathbf{H} \cdot \mathbf{n} = 0, \quad \mathbf{E} \cdot \mathbf{n} = 0,$$

$$\nabla \times \mathbf{H}(x, t) = 2\pi \begin{bmatrix} 0 \\ -\sin 2\pi t \\ -\cos 2\pi t \end{bmatrix}, \quad \nabla \times \mathbf{E}(x, t) = -2\pi \begin{bmatrix} 0 \\ \cos 2\pi t \\ -\sin 2\pi t \end{bmatrix}.$$

$$x = 1: \mathbf{n} = \hat{x} \implies \mathbf{H} \cdot \mathbf{n} = 0, \quad \mathbf{E} \cdot \mathbf{n} = 0,$$

$$\nabla \times \mathbf{H}(x, t) = 2\pi \begin{bmatrix} 0 \\ -\sin 2\pi t \\ -\cos 2\pi t \end{bmatrix}, \quad \nabla \times \mathbf{E}(x, t) = -2\pi \begin{bmatrix} 0 \\ \cos 2\pi t \\ -\sin 2\pi t \end{bmatrix}.$$

$$y = 0: \mathbf{n} = -\hat{y} \implies \mathbf{H} \cdot \mathbf{n} = \sin 2\pi(x - t), \quad \mathbf{E} \cdot \mathbf{n} = -\cos 2\pi(x - t),$$

$$\nabla \times \mathbf{H}(x, t) \neq 0, \quad \nabla \times \mathbf{E}(x, t) \neq 0.$$

$$y = 1, z = 0, z = 1 \text{ are similar.}$$

	$\mathbf{E}$	$\mathbf{H}$
Basis	Edge Elements	Face Elements
B.C.s		
(Homogeneous)	$\mathbf{E} \times \mathbf{n} = 0; \operatorname{div} \mathbf{E} = 0$	$\mathbf{H} \cdot \mathbf{n} = 0; \operatorname{curl} \mathbf{H} \times \mathbf{n} = 0$
For us (on $\partial\Omega$)	$\mathbf{E} \times \mathbf{n} \neq 0; \operatorname{div} \mathbf{E} = 0$	$\mathbf{H} \cdot \mathbf{n} \neq 0; \operatorname{curl} \mathbf{H} \times \mathbf{n} \neq 0$

Variational formulation is given by:
$$\begin{cases} \langle \sigma E, \tilde{E} \rangle + \langle \epsilon \frac{\partial E}{\partial t}, \tilde{E} \rangle - \langle \nabla \times H, \tilde{E} \rangle = \langle f_E, \tilde{E} \rangle, \\ \langle H, \tilde{H} \rangle + \langle \mu \frac{\partial H}{\partial t}, \tilde{H} \rangle + \langle \nabla \times E, \tilde{H} \rangle = \langle f_H, \tilde{H} \rangle. \end{cases}$$

Time discretizations are:

Backward Euler:

$$\begin{cases} \langle \sigma E^{n+1}, \tilde{E} \rangle + \langle \epsilon \frac{E^{n+1} - E^n}{\Delta t}, \tilde{E} \rangle - \langle \nabla \times H^{n+1}, \tilde{E} \rangle = \langle f_E, \tilde{E} \rangle, \\ \langle H^{n+1}, \tilde{H} \rangle + \langle \mu \frac{H^{n+1} - H^n}{\Delta t}, \tilde{H} \rangle + \langle \nabla \times E^{n+1}, \tilde{H} \rangle = \langle f_H, \tilde{H} \rangle. \end{cases}$$

$$\Rightarrow \begin{cases} -\langle (\sigma + \frac{\epsilon}{\Delta t}) E^{n+1}, \tilde{E} \rangle + \langle H^{n+1}, \nabla \times \tilde{E} \rangle = -\langle f_E, \tilde{E} \rangle - \langle \frac{\epsilon}{\Delta t} E^n, \tilde{E} \rangle, \\ \langle \nabla \times E^{n+1}, \tilde{H} \rangle + \langle (1 + \frac{\mu}{\Delta t}) H^{n+1}, \tilde{H} \rangle = \langle f_H, \tilde{H} \rangle + \langle \frac{\mu}{\Delta t} H^n, \tilde{H} \rangle. \end{cases}$$

Then we can form the linear system as:

$$\begin{bmatrix} -(\sigma + \frac{\epsilon}{\Delta t}) M_{11} & S_{12} \\ S_{12}^T & (1 + \frac{\mu}{\Delta t}) M_{22} \end{bmatrix} \begin{bmatrix} E^{n+1} \\ H^{n+1} \end{bmatrix} = \begin{bmatrix} -\langle f_E, \tilde{E} \rangle \\ \langle f_H, \tilde{H} \rangle \end{bmatrix} + \begin{bmatrix} -\frac{\epsilon}{\Delta t} M_{11} E^n \\ \frac{\mu}{\Delta t} M_{22} H^n \end{bmatrix}.$$

Forward Euler:

$$\begin{cases} \epsilon \frac{E^{n+1} - E^n}{\Delta t} = f_E(t^n) - \sigma E^n + \nabla \times H^n, \\ \mu \frac{H^{n+1} - H^n}{\Delta t} = f_H(t^n) - H^n - \nabla \times E^n. \end{cases}$$

With variational formulation it gives:

$$\begin{cases} \langle E^{n+1}, \tilde{E} \rangle = \langle E^n, \tilde{E} \rangle + \frac{\Delta t}{\epsilon} (\langle f_E(t^n), \tilde{E} \rangle - \sigma \langle E^n, \tilde{E} \rangle + \langle H^n, \nabla \times \tilde{E} \rangle), \\ \langle H^{n+1}, \tilde{H} \rangle = \langle H^n, \tilde{H} \rangle + \frac{\Delta t}{\mu} (\langle f_H(t^n), \tilde{H} \rangle - \langle H^n, \tilde{H} \rangle + \langle \nabla \times E^n, \tilde{H} \rangle). \end{cases}$$

$$\Rightarrow \begin{cases} \langle E^{n+1}, \tilde{E} \rangle = \left(1 - \sigma \frac{\Delta t}{\epsilon}\right) \langle E^n, \tilde{E} \rangle + \frac{\Delta t}{\epsilon} (\langle H^n, \nabla \times \tilde{E} \rangle + \langle f_E(t^n), \tilde{E} \rangle), \\ \langle H^{n+1}, \tilde{H} \rangle = \left(1 - \frac{\Delta t}{\mu}\right) \langle H^n, \tilde{H} \rangle + \frac{\Delta t}{\mu} (-\langle \nabla \times E^n, \tilde{H} \rangle + \langle f_H(t^n), \tilde{H} \rangle). \end{cases}$$

$$\Rightarrow \begin{cases} M_{11} E^{n+1} = \left(1 - \sigma \frac{\Delta t}{\epsilon}\right) M_{11} E^n + \frac{\Delta t}{\epsilon} (S_{12} H^n + b_E^n), \\ M_{22} H^{n+1} = \left(1 - \frac{\Delta t}{\mu}\right) M_{22} H^n + \frac{\Delta t}{\mu} (-S_{12}^T E^n + b_H^n), \end{cases}$$

where $M_{11} \in \mathbb{R}^{N_1 \times N_1}$, $M_{22} \in \mathbb{R}^{N_2 \times N_2}$, $S_{12} \in \mathbb{R}^{N_1 \times N_2}$, $b_E^n \in \mathbb{R}^{N_1}$, and $b_H^n \in \mathbb{R}^{N_2}$. Rearranging the linear solver,

(a) $M_{11} \tilde{b}_E^n = (S_{12} H^n + b_E^n),$
 $E^{n+1} = \left(1 - \sigma \frac{\Delta t}{\epsilon}\right) E^n + \frac{\Delta t}{\epsilon} \tilde{b}_E^n,$

(b) $M_{22} \tilde{b}_H^n = (-S_{12}^T E^n + b_H^n),$
 $H^{n+1} = \left(1 - \frac{\Delta t}{\mu}\right) H^n + \frac{\Delta t}{\mu} \tilde{b}_H^n.$

Initial conditions are $E^0 = [0, \cos 2\pi x, \sin 2\pi x]$, $\frac{dE_a}{dt} = f_a$, $a = x, y, z$, $f = [0, 2\pi \sin 2\pi(x - t), -2\pi \cos 2\pi(x - t)]$. And

$$\int_0^t \frac{dE_a}{dt} dt = \int_0^t f_a dt,$$

where if

$$a = x : E_x(t) - E_x(0) = [1]_0^t = 0 \implies E_x(t) = 0,$$

$$a = y : E_y(t) - E_y(0) = \int_0^t 2\pi \sin 2\pi(x - t) - \cos 2\pi(x - t)|_0^t \implies E_y(t) - E_y(0) = \cos 2\pi(x - t) - \cos 2\pi x,$$

$$a = z : E_z(t) - E_z(0) = \int_0^t -2\pi \cos 2\pi(x - t) - \sin 2\pi(x - t)|_0^t \implies E_z(t) - E_z(0) = \sin 2\pi(x - t) - \sin 2\pi x.$$

2. On $\Omega = [0, 1]^3$,

$$\begin{aligned} \epsilon \frac{\partial E}{\partial t} - \nabla \times H &= -j, \\ \mu \frac{\partial H}{\partial t} + \nabla \times E &= 0. \end{aligned}$$

We'll set $\mu = 1$, $\epsilon = 1$, and take analytical solutions to be:

$$E(x, t) = \frac{e^{-t}}{\pi} \begin{bmatrix} -\cos \pi x & \sin \pi y & \sin \pi z \\ \sin \pi x & \cos \pi y & \sin \pi z \\ 0 \end{bmatrix}, \quad H(x, t) = e^{-t} \begin{bmatrix} -\sin \pi x & \cos \pi y & \cos \pi z \\ -\cos \pi x & \sin \pi y & \cos \pi z \\ 2 \cos \pi x & \cos \pi y & \sin \pi z \end{bmatrix},$$

$$j(x, t) = -e^{-t} \left(\frac{1}{\pi} + 3\pi \right) \begin{bmatrix} \cos \pi x & \sin \pi y & \sin \pi z \\ -\sin \pi x & \cos \pi y & \sin \pi z \\ 0 \end{bmatrix}.$$

Boundary conditions on E are given by:

- $x = 0, x = 1 : \hat{n} = \pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \implies E \times \hat{n} = 0,$
- $y = 0, y = 1 : \hat{n} = \pm \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \implies E \times \hat{n} = 0,$
- $z = 0, z = 1 : \hat{n} = \pm \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \implies E \times \hat{n} = 0.$

And boundary conditions on H are given by:

- $x = 0, x = 1 : \hat{n} = \pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \implies H \cdot \hat{n} = 0,$
- $y = 0, y = 1 : \hat{n} = \pm \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \implies H \cdot \hat{n} = 0,$

- $z = 0, z = 1 : \hat{n} = \pm \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \implies \mathbf{H} \cdot \hat{n} = 0.$

Variational formulation is given by:
$$\begin{cases} \langle \epsilon \frac{\partial \mathbf{E}}{\partial t}, \tilde{\mathbf{E}} \rangle - \langle \nabla \times \mathbf{H}, \tilde{\mathbf{E}} \rangle = \langle -\mathbf{j}, \tilde{\mathbf{E}} \rangle, \\ \langle \mu \frac{\partial \mathbf{H}}{\partial t}, \tilde{\mathbf{H}} \rangle + \langle \nabla \times \mathbf{E}, \tilde{\mathbf{H}} \rangle = 0. \end{cases}$$

Time discretization which we will discuss here is only:

- Backward Euler:
$$\begin{cases} \langle \epsilon \frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{\Delta t}, \tilde{\mathbf{E}} \rangle - \langle \nabla \times \mathbf{H}^{n+1}, \tilde{\mathbf{E}} \rangle = \langle -\mathbf{j}, \tilde{\mathbf{E}} \rangle, \\ \langle \mu \frac{\mathbf{H}^{n+1} - \mathbf{H}^n}{\Delta t}, \tilde{\mathbf{H}} \rangle + \langle \nabla \times \mathbf{E}^{n+1}, \tilde{\mathbf{H}} \rangle = 0. \end{cases}$$

$$\implies \begin{cases} -\frac{\epsilon}{\Delta t} \langle \mathbf{E}^{n+1}, \tilde{\mathbf{E}} \rangle + \langle \mathbf{H}^{n+1}, \nabla \times \tilde{\mathbf{E}} \rangle = -\langle -\mathbf{j}, \tilde{\mathbf{E}} \rangle - \frac{\epsilon}{\Delta t} \langle \mathbf{E}^n, \tilde{\mathbf{E}} \rangle, \\ \langle \nabla \times \mathbf{E}^{n+1}, \tilde{\mathbf{H}} \rangle + \frac{\mu}{\Delta t} \langle \mathbf{H}^{n+1}, \tilde{\mathbf{H}} \rangle = \frac{\mu}{\Delta t} \langle \mathbf{H}^n, \tilde{\mathbf{H}} \rangle. \end{cases}$$

Linear system looks alike:

$$\begin{bmatrix} -\frac{\epsilon}{\Delta t} M_{11} & S_{12} \\ S_{12}^T & \frac{\mu}{\Delta t} M_{22} \end{bmatrix} \begin{bmatrix} \mathbf{E}^{n+1} \\ \mathbf{H}^{n+1} \end{bmatrix} = \begin{bmatrix} -\langle -\mathbf{j}, \tilde{\mathbf{E}} \rangle - \frac{\epsilon}{\Delta t} M_{11} \mathbf{E}^n \\ \frac{\mu}{\Delta t} M_{22} \mathbf{H}^n \end{bmatrix}.$$

3. After introducing pressure variable p , we have

$$\begin{aligned} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{E} &= f_p, \\ \nabla p + \epsilon \frac{\partial \mathbf{E}}{\partial t} - \mu \nabla \times \mathbf{H} &= f_E, \\ \mu \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} &= f_H. \end{aligned}$$

We can follow the similar steps for this version as well like the previous ones.

2.10 Computational Results in $\mathbb{R}^2$ and $\mathbb{R}^3$

2.10.1 Poisson's Equation

Scalar Poisson ($\mathbb{R}^2$): We demonstrate results on a mesh of the 2d unit square for Dirichlet and Neumann boundary conditions. On unit square we consider

Example 2.1: $u = x(x-1)y(y-1)$ for Dirichlet boundary conditions, and

Example 2.2: $u = \cos \pi x \cos \pi y$ for Neumann boundary conditions and thus $\sigma = \text{grad } u = -\pi(\sin \pi x \cos \pi y \hat{x} + \cos \pi x \sin \pi y \hat{y})$.

We provide plots of the computed solutions with linear and quadratic basis and analytical solutions as well in order to provide a visible comparison of accuracy of computed solutions for both examples as represented in Figures 2.24 and 2.26. Moreover we also provide error plots for both examples and are shown in Figures 2.25 and 2.27.

Vector Poisson ($\mathbb{R}^2$): We demonstrate results on a mesh of the 2d unit square and 2d annulus for Magnetic and only on a mesh of the 2d unit square for Electric boundary conditions. For Magnetic boundary conditions, on unit square we consider

Example 2.3: $u = (x(1-x)y(1-y))^2 - y - 1 \hat{x} + (x(1-x)y(1-y))^2 + x + 1 \hat{y}$, so $\sigma = \text{rot } u = (-x^2y^2(1-x)^2(2y-2) + x^2y^2(1-y)^2(2x-2) - 2x^2y(1-x)^2(1-y)^2 + 2xy^2(1-x)^2(1-y)^2) + 2$, and

Example 2.4: $u = \sin \pi y \hat{x} + \sin \pi x \hat{y}$, so again $\sigma = \text{rot } u = \pi(\cos \pi x - \cos \pi y)$, and on annulus, we take

Example 2.5: $u = r \cos \theta \hat{r}$, so $\sigma = \text{rot } u = \sin \theta$.

For Electric boundary conditions, on unit square we consider

Example 2.6: $u = (x(1-x)y(1-y))^2 + x + 1 \hat{x} + (x(1-x)y(1-y))^2 + y + 1 \hat{y}$, so $\sigma = \text{div } u = 2 - (-1+x)x(-1+y)^2 - 2(-1+x)x(-1+y)y + 2(-1+x)^2x(-1+y)^2y^2 + 2(-1+x)x^2(-1+y)^2y^2$, and

Example 2.7: $u = \sin \pi x \hat{x} + \sin \pi y \hat{y}$, gives $\sigma = \text{div } u = \pi(\cos \pi x + \cos \pi y)$.

We provide plots of the computed solutions with linear and quadratic basis and analytical solutions as well in order to provide a visible comparison of accuracy of computed solutions for both examples as represented in Figures 2.28, 2.30, 2.32, 2.34 and 2.36. Moreover we also provide error plots for both examples and are shown in Figures 2.29, 2.31, 2.33, 2.35 and 2.37.

Vector Poisson ($\mathbb{R}^3$): We demonstrate results on two meshes of the 3d unit cube for Magnetic and Electric boundary conditions, we will use coarser mesh for quadratic basis and finer mesh for linear basis. We provide plots of the computed solutions with linear and quadratic basis and two views for each case of

Example 2.8: $u = (x(1-x)y(1-y)z(1-z))^2 (\hat{x} + \hat{y} + \hat{z}) + 0.01y \hat{x}$ for Electric boundary conditions, and

Example 2.9: $u = (\sin \pi x + 0.01y) \hat{x} + \sin \pi y \hat{y} + \sin \pi z \hat{z}$ for Magnetic boundary conditions.

We represent solutions in Figures 2.38 and 2.41. Moreover we also provide error plots for both values of u with both boundary conditions which are shown in Figures 2.39, 2.40, 2.42 and 2.43.

2.10.2 Eigenvalue Problem

Here we discuss the results for the *resonant cavity* problem as discussed in [AAG97]. For the rectangular box $\Omega = [0, a\pi] \times [0, b\pi] \times [0, c\pi]$, the analytical solutions have the form:

$$e(x) = \begin{bmatrix} \eta_1 \cos \frac{k_1 x_1}{a} \sin \frac{k_2 x_2}{b} \sin \frac{k_3 x_3}{c} \\ \eta_2 \sin \frac{k_1 x_1}{a} \cos \frac{k_2 x_2}{b} \sin \frac{k_3 x_3}{c} \\ \eta_3 \sin \frac{k_1 x_1}{a} \sin \frac{k_2 x_2}{b} \cos \frac{k_3 x_3}{c} \end{bmatrix}, \quad k_i \in \mathbb{Z}^+ \quad (2.41)$$

and all eigenvalues are positive and have the form:

$$\lambda_k = \left(\frac{k_1}{a} \right)^2 + \left(\frac{k_2}{b} \right)^2 + \left(\frac{k_3}{c} \right)^2, \quad k_1 k_2 + k_2 k_3 + k_1 k_3 > 0.$$

If $k_1 k_2 k_3 > 0$ then there are two linearly independent eigenmodes of the form Equation (2.41) corresponding to one k triplet. If one of the k_i is zero then there is only one such function given by $\eta_j = \delta_{ij}$. Notice that this argument does not take into account that different k triplets may yield the same eigenvalue. For instance, consider the following problem in which we seek the eigenvalues λ and eigenmodes u for:

Example 2.10: $\text{curl curl } \mathbf{u} = \lambda \mathbf{u}$ on the cubical region $\Omega = [0, \pi]^3$ with boundary conditions $\mathbf{u} \times \mathbf{n} = 0$ on $\partial\Omega$, where $\mathbf{n}$ is the outward unit normal vector on the boundary of the problem region.

Table 2.3 shows the first 30 eigenvalues for this problem, and its computation using two different tetrahedral meshes of this region. Mesh 1 is “coarse” and consists of 50 vertices and 560 tetrahedra for this region; mesh 2 is “refined”, and consists of 7690 vertices and 36745 tetrahedra. The vector fields are approximated using spaces of piecewise linear and piecewise quadratic vector fields. As can be seen, no spurious eigenvalues are computed. Furthermore, the computations using the quadratic basis are already quite accurate even on the coarser mesh 1. This demonstrates the superiority of this structure preserving, higher-order FEM discretization for the resonant cavity problem.

Exact Eigenvalue	Linear Mesh 2	Quadratic Mesh 1 Mesh 2		Exact Eigenvalue	Linear Mesh 2	Quadratic Mesh 1 Mesh 2	
2	1.9967	2.0002	2.0000	6	7.9886	6.0277	6.0000
2	2.9845	2.0003	2.0000	6	7.9895	6.0310	6.0000
2	2.9857	2.0004	2.0000	8	8.0037	8.0184	8.0001
3	4.9868	3.0017	3.0000	8	8.9580	8.0379	8.0001
3	4.9874	3.0018	3.0000	8	8.9601	8.0624	8.0002
5	4.9886	4.9941	5.0000	9	8.9686	8.9673	9.0001
5	4.9889	4.9946	5.0000	9	8.9778	8.9951	9.0001
5	4.9903	4.9958	5.0000	9	8.9975	9.0104	9.0001
5	4.9920	5.0038	5.0000	9	9.0004	9.0215	9.0002
5	5.9584	5.0046	5.0000	9	9.9198	9.0253	9.0002
5	5.9614	5.0080	5.0000	9	9.9291	9.0267	9.0003
6	5.9624	6.0087	6.0000	10	9.9361	10.0179	10.0005
6	5.9714	6.0118	6.0000	10	9.9600	10.0308	10.0005
6	5.9740	6.0129	6.0000	10	9.9619	10.0367	10.0005
6	5.9770	6.0249	6.0000	10	9.9679	10.0750	10.0005

TABLE 2.3: Comparison of computed eigenvalues, using linear basis over mesh 2 and quadratic basis over mesh 1 and mesh 2, with exact eigenvalues corresponding to Example 2.10.

2.10.3 Maxwell's Equation and its Discretizations

We solve for the the Maxwell's equations on a unit square $\Omega = [0, 1]^3$ with a tetrahedral discretization consisting of 560 tetrahedra, 1312 faces, 966 edges and 215 vertices, for a time range of $[0, 0.1]$.

Example 2.11: We choose an analytical solution given by:

$$\mathbf{E} = \frac{e^{-t}}{\pi} \begin{bmatrix} -\cos(\pi x) \sin(\pi y) \sin(\pi z) \\ \sin(\pi x) \cos(\pi y) \sin(\pi z) \\ 0 \end{bmatrix}, \quad \mathbf{H} = e^{-t} \begin{bmatrix} -\sin(\pi x) \cos(\pi y) \cos(\pi z) \\ -\cos(\pi x) \sin(\pi y) \cos(\pi z) \\ 2 \cos(\pi x) \cos(\pi y) \sin(\pi z) \end{bmatrix}, \quad f_p = 0, \quad (2.42)$$

and with f_E and f_H being deduced from the analytical solution. We verify the energy conservation property of Crank Nicolson as in the Table 2.4. We present our computed solutions at a final time in Figure 2.44.

	t = 0	t = 0.03	t = 0.06	t = 0.09	t = 0.1
Analytical E_{L^2}	0.159	0.154	0.150	0.145	0.144
Computed E_{L^2} (Linear)	0.147	0.141	0.137	0.133	0.133
Computed E_{L^2} (Quadratic)	0.159	0.154	0.150	0.145	0.144
Analytical H_{L^2}	0.866	0.840	0.816	0.791	0.784
Computed H_{L^2} (Linear)	0.830	0.807	0.785	0.765	0.758
Computed H_{L^2} (Quadratic)	0.865	0.840	0.815	0.791	0.783

TABLE 2.4: Energy conservation of Crank Nicolson for **Example 2.11**.

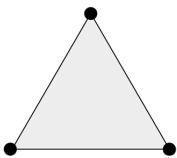
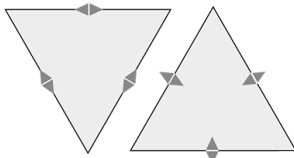
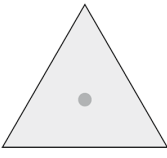
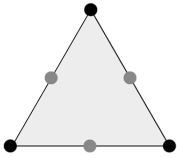
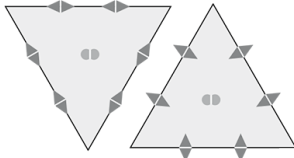
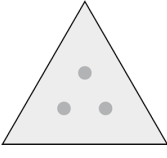
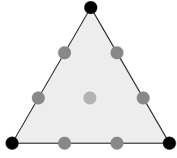
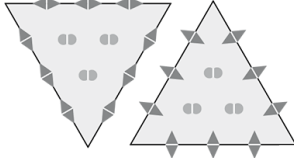
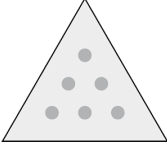
$n = 2$			
	$k = 0$	$k = 1$	$k = 2$
$r = 1$	$\dot{H}_{\varepsilon^{-1}}^1(\Omega) \cap \mathcal{P}_1^-(\Omega)$  $3 \times (1 \times V) = 3$	$\dot{H}_{\varepsilon}(\text{div}, \Omega) \cap \mathcal{P}_1^-(\Omega)$  $3 \times (1 \times E) = 3$	$L_{\mu}^2(\Omega) \cap \mathcal{P}_1^-(\Omega)$  $1 \times (1 \times F) = 1$
$r = 2$	$\dot{H}_{\varepsilon^{-1}}^1(\Omega) \cap \mathcal{P}_2^-(\Omega)$  $3 \times (1 \times V) + 3 \times (1 \times E) = 6$	$\dot{H}_{\varepsilon}(\text{div}, \Omega) \cap \mathcal{P}_2^-(\Omega)$  $3 \times (2 \times E) + 1 \times (2 \times F) = 8$	$L_{\mu}^2(\Omega) \cap \mathcal{P}_2^-(\Omega)$  $1 \times (3 \times F) = 3$
$r = 3$	$\dot{H}_{\varepsilon^{-1}}^1(\Omega) \cap \mathcal{P}_3^-(\Omega)$  $3 \times (1 \times V) + 3 \times (2 \times E) + 1 \times (1 \times F) = 10$	$\dot{H}_{\varepsilon}(\text{div}, \Omega) \cap \mathcal{P}_3^-(\Omega)$  $3 \times (3 \times E) + 1 \times (6 \times F) = 15$	$L_{\mu}^2(\Omega) \cap \mathcal{P}_3^-(\Omega)$  $1 \times (6 \times F) = 6$

FIGURE 2.18: The degrees of freedom for various r^{th} -order de Rham basis elements in two dimensions. Here, V denotes the degrees of freedom attach to the vertices, E to the edges, and F to the face of a 2-simplex in a 2d mesh. The figures are reproduced from [AL14].

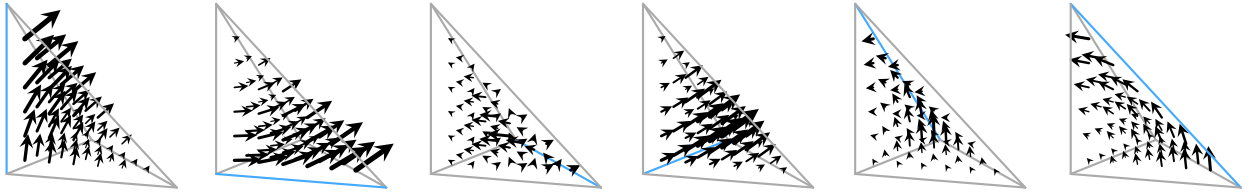


FIGURE 2.19: Linear Whitney Vector Fields in 3d adapted from [Scr+25].

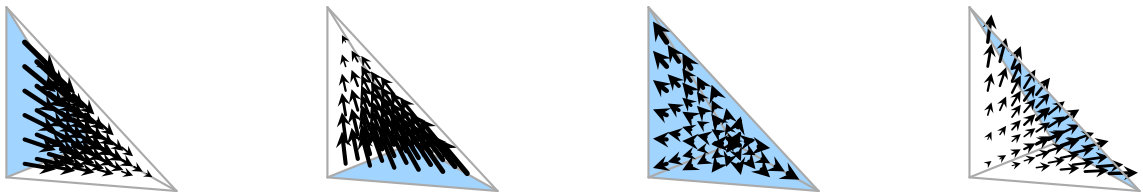


FIGURE 2.20: Linear Raviart-Thomas Fields in 3d adapted from [Scr+25].

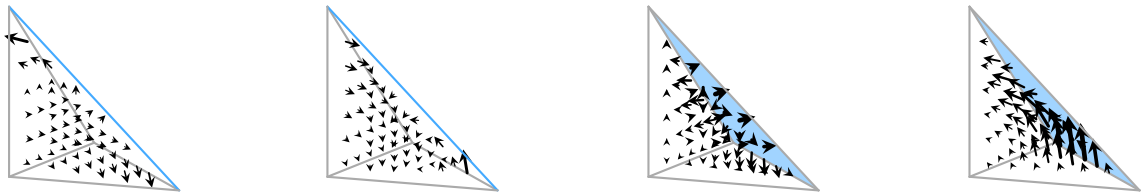


FIGURE 2.21: Quadratic Whitney Vector Fields in 3d corresponding to an edge (leftmost two) and corresponding to a triangle (rightmost two) adapted from [Scr+25]. Similarly, we can show for the remaining edges and faces of the tetrahedron.

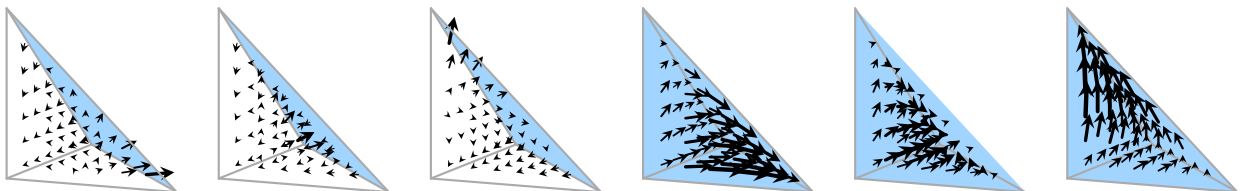


FIGURE 2.22: Quadratic Raviart-Thomas Fields in 3d corresponding to a triangle (leftmost two) and corresponding to tetrahedron (rightmost two) adapted from [Scr+25]. Similarly, we can show for the remaining faces of the tetrahedron.

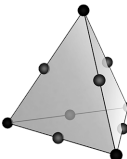
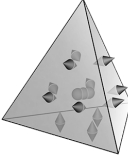
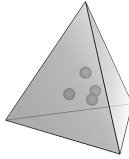
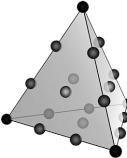
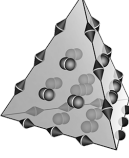
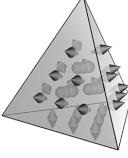
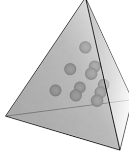
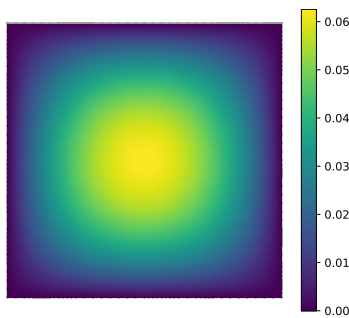
$n = 3$				
	$k = 0$	$k = 1$	$k = 2$	$k = 3$
$r = 1$	$\hat{H}_{\varepsilon-1}^1(\Omega) \cap \mathcal{P}_1^-(\Omega)$  $4 \times (1 \times V) = 4$	$\hat{H}_{\varepsilon}(\text{curl}, \Omega) \cap \mathcal{P}_3^-(\Omega)$  $6 \times (1 \times E) = 6$	$\hat{H}_{\mu}(\text{div}, \Omega) \cap \mathcal{P}_3^-(\Omega)$  $4 \times (1 \times F) = 4$	$L^2(\Omega) \cap \mathcal{P}_1^-(\Omega)$  $1 \times (1 \times T) = 1$
$r = 2$	$\hat{H}_{\varepsilon-1}^1(\Omega) \cap \mathcal{P}_2^-(\Omega)$  $4 \times (1 \times V) + 6 \times (1 \times E) = 10$	$\hat{H}_{\varepsilon}(\text{curl}, \Omega) \cap \mathcal{P}_3^-(\Omega)$  $6 \times (2 \times E) + 4 \times (2 \times F) = 20$	$\hat{H}_{\mu}(\text{div}, \Omega) \cap \mathcal{P}_3^-(\Omega)$  $4 \times (3 \times F) + 1 \times (3 \times T) = 15$	$L^2(\Omega) \cap \mathcal{P}_2^-(\Omega)$  $1 \times (4 \times T) = 4$
$r = 3$	$\hat{H}_{\varepsilon-1}^1(\Omega) \cap \mathcal{P}_3^-(\Omega)$  $4 \times (1 \times V) + 6 \times (2 \times E) + 4 \times (1 \times F) = 20$	$\hat{H}_{\varepsilon}(\text{curl}, \Omega) \cap \mathcal{P}_3^-(\Omega)$  $6 \times (3 \times E) + 4 \times (6 \times F) + 1 \times (3 \times T) = 45$	$\hat{H}_{\mu}(\text{div}, \Omega) \cap \mathcal{P}_3^-(\Omega)$  $4 \times (6 \times F) + 1 \times (12 \times T) = 36$	$L^2(\Omega) \cap \mathcal{P}_3^-(\Omega)$  $1 \times (10 \times T) = 10$

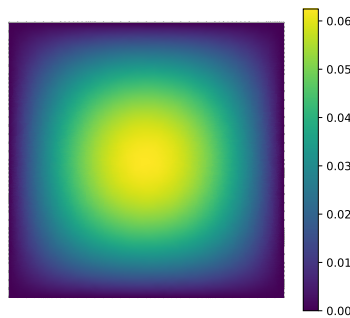
FIGURE 2.23: The degrees of freedom for various r^{th} -order de Rham basis elements in three dimensions. Here, V denotes the degrees of freedom attach to the vertices, E to the edges, F to the faces, and T to the tetra of a 3-simplex in a 3d mesh. The figures are reproduced from [AL14].

Analytical Solution u

Computed Solution u using



Linear Basis



Quadratic Basis

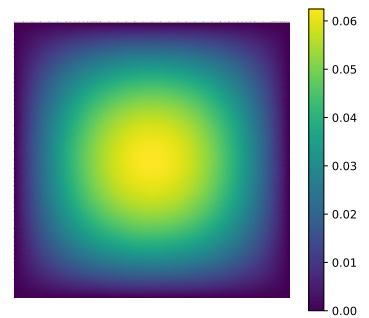


FIGURE 2.24: **Scalar Poisson with Dirichlet boundary conditions:** Plots of Example 2.1 on unit square using linear and quadratic basis.

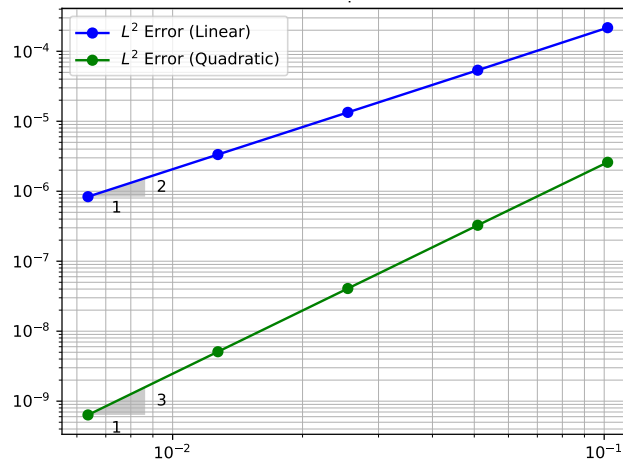
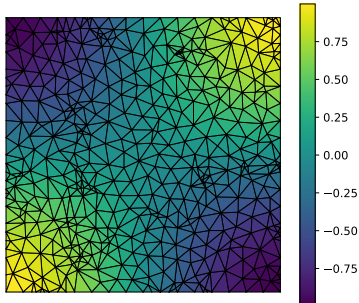


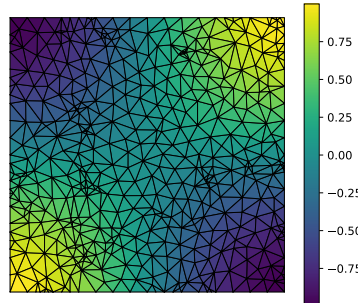
FIGURE 2.25: **Scalar Poisson with Dirichlet boundary conditions:** Error plots of Example 2.1 on unit square using linear and quadratic basis.

Analytical Solution u

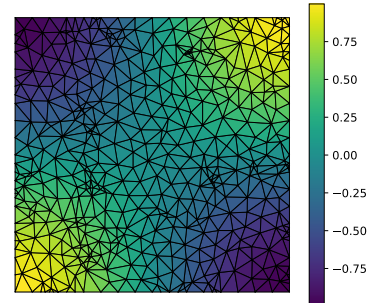


Computed Solution u using

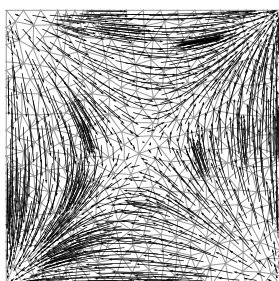
Linear Basis



Quadratic Basis

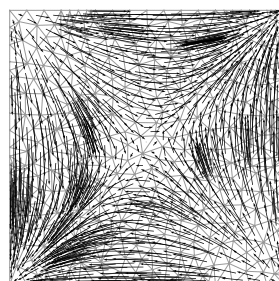


Analytical Solution σ



Computed Solution σ using

Linear Basis



Quadratic Basis

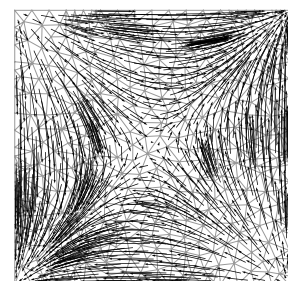


FIGURE 2.26: **Scalar Poisson with Neumann boundary conditions:** Plots of Example 2.2 on unit square using linear and quadratic basis.

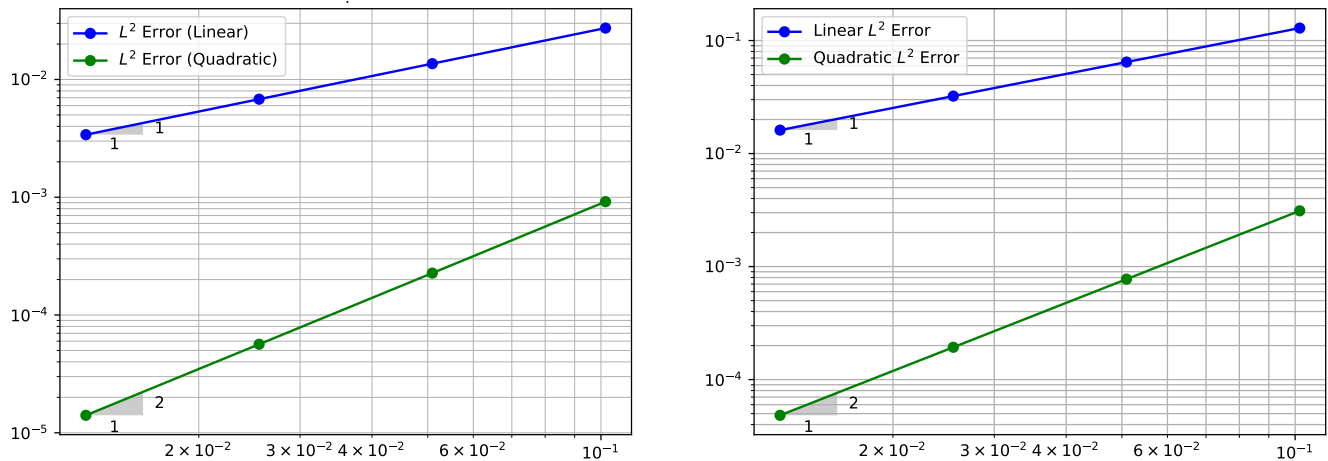
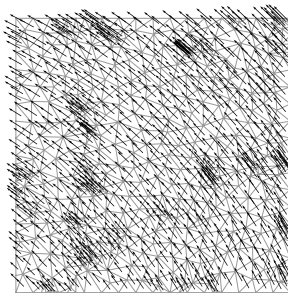


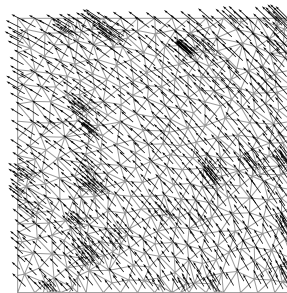
FIGURE 2.27: **Scalar Poisson with Neumann boundary conditions:** Error plots of Example 2.2 on unit square using linear and quadratic basis (left) and for $\sigma = \text{grad } u$ using linear and quadratic basis (right).

Analytical Solution u

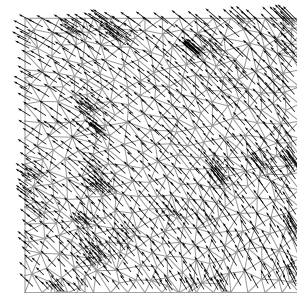


Computed Solution u using

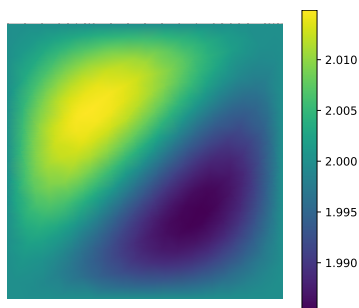
Linear Basis



Quadratic Basis

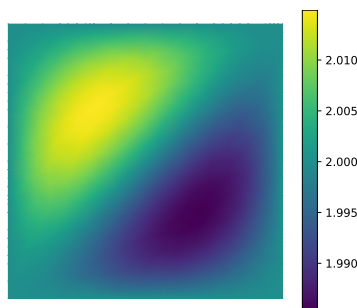


Analytical Solution σ



Computed Solution σ using

Linear Basis



Quadratic Basis

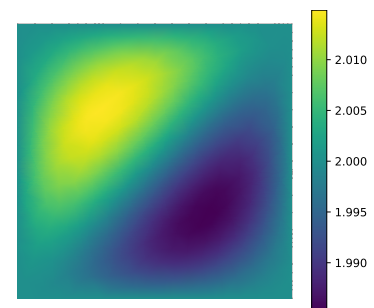


FIGURE 2.28: **Vector Poisson with Magnetic boundary conditions:** Plots of Example 2.3 on unit square using linear and quadratic basis.

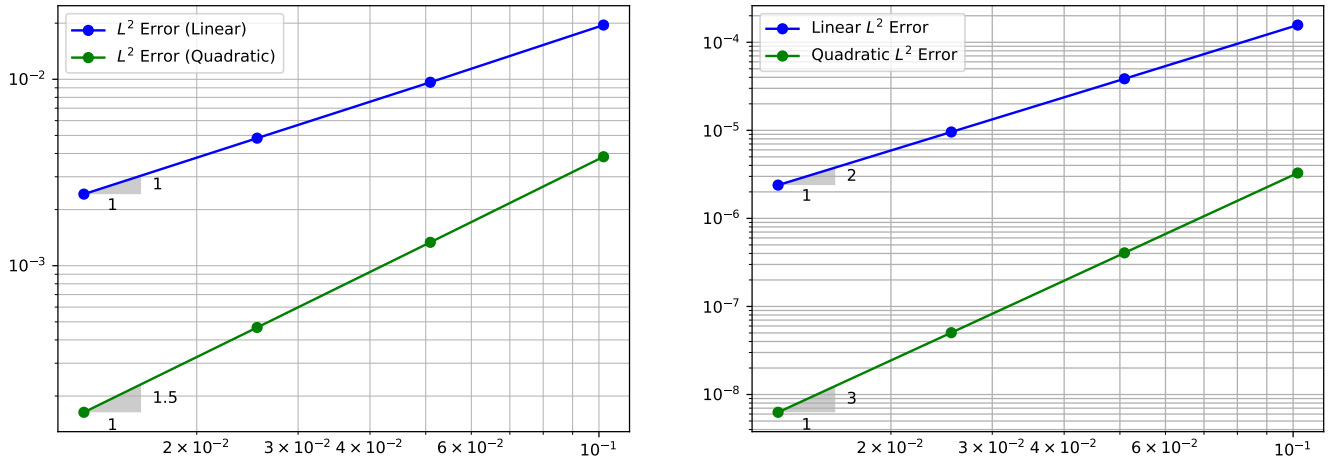
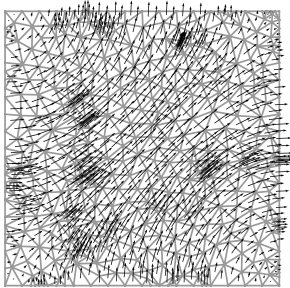


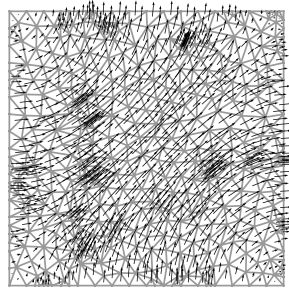
FIGURE 2.29: **Vector Poisson with Magnetic boundary conditions:** Error plots of **Example 2.3** on unit square using linear and quadratic basis (left) and for $\sigma = \text{rot } u$ using linear and quadratic basis (right).

Analytical Solution u

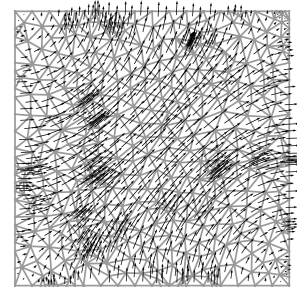


Computed Solution u using

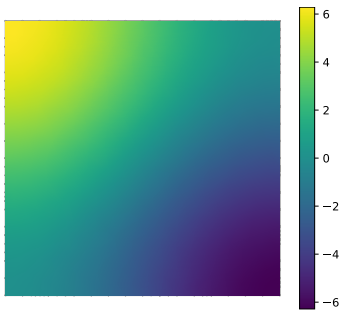
Linear Basis



Quadratic Basis

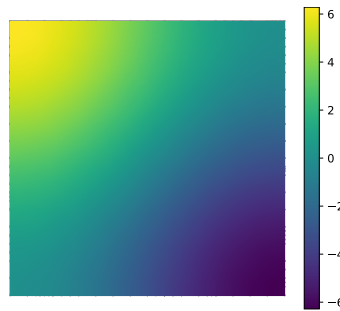


Analytical Solution σ



Computed Solution σ using

Linear Basis



Quadratic Basis

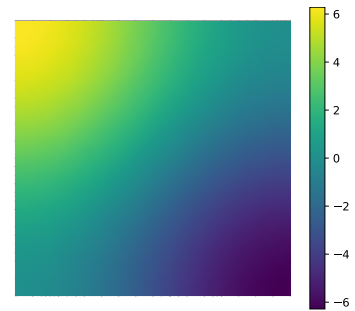


FIGURE 2.30: **Vector Poisson with Magnetic boundary conditions:** Plots of **Example 2.4** on unit square using linear and quadratic basis.

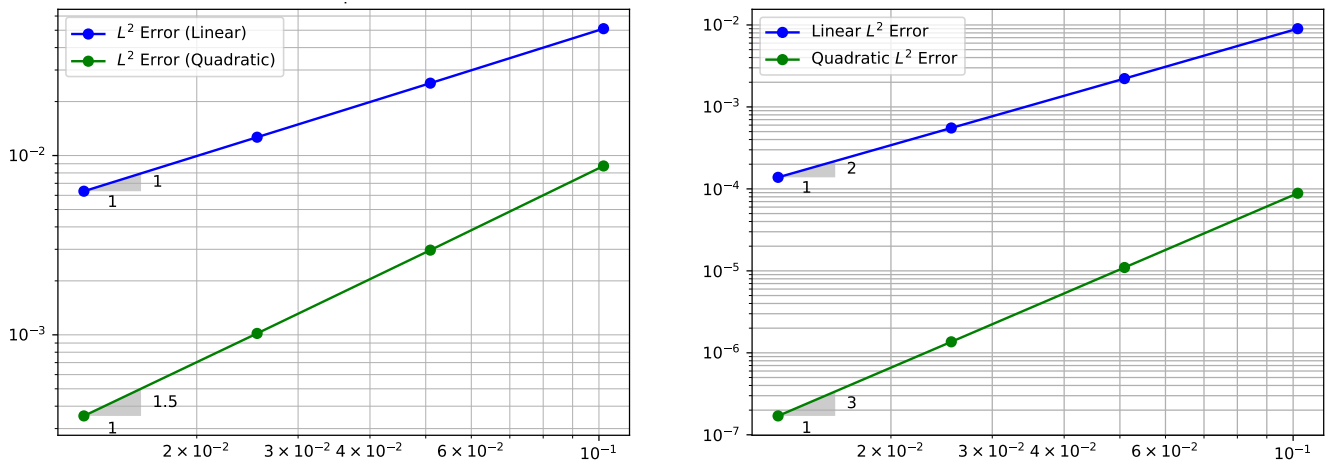
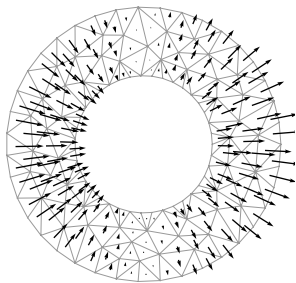


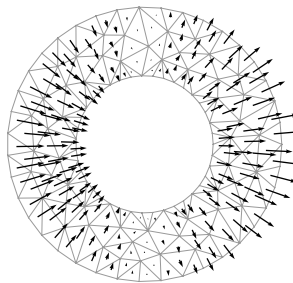
FIGURE 2.31: **Vector Poisson with Magnetic boundary conditions:** Error plots of **Example 2.4** on unit square using linear and quadratic basis (left) and for $\sigma = \text{rot } u$ using linear and quadratic basis.

Analytical Solution u

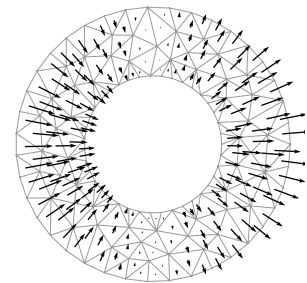


Computed Solution u using

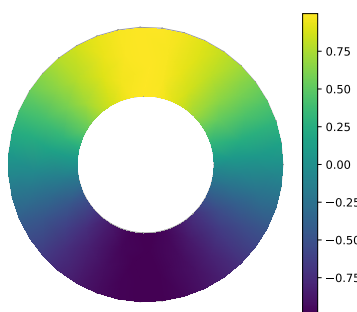
Linear Basis



Quadratic Basis

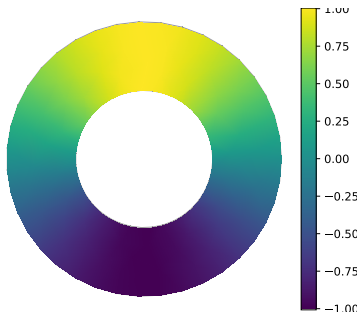


Analytical Solution σ



Computed Solution σ using

Linear Basis



Quadratic Basis

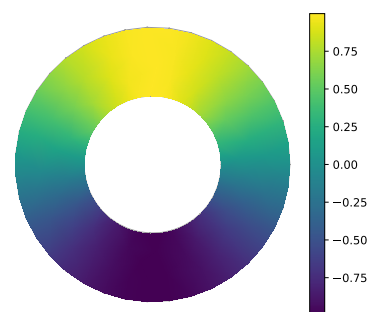


FIGURE 2.32: **Vector Poisson with Magnetic boundary conditions:** Plots of **Example 2.5** on annulus using linear and quadratic basis.

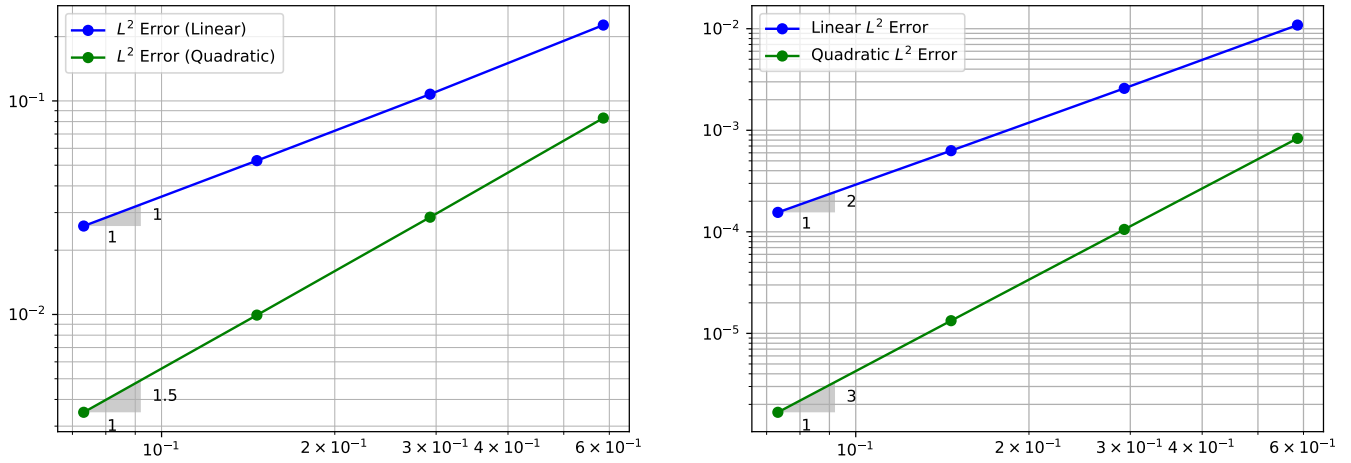
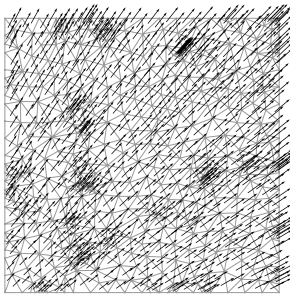


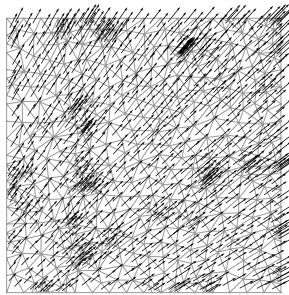
FIGURE 2.33: **Vector Poisson with Magnetic boundary conditions:** Error plots of **Example 2.5** on annulus using linear and quadratic basis (left) and for $\sigma = \text{rot } u$ using linear and quadratic basis (right).

Analytical Solution u

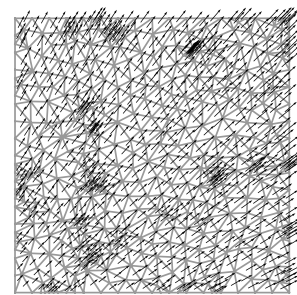


Computed Solution u using

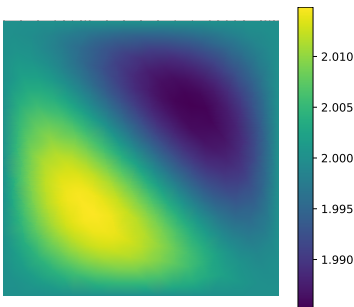
Linear Basis



Quadratic Basis

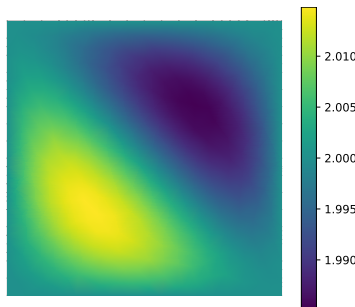


Analytical Solution σ



Computed Solution σ using

Linear Basis



Quadratic Basis

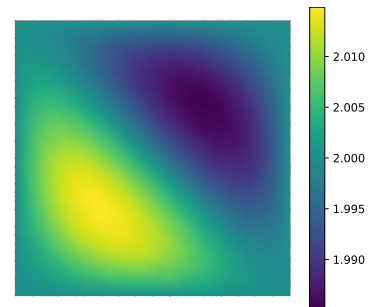


FIGURE 2.34: **Vector Poisson with Electric boundary conditions:** Plots of **Example 2.6** on unit square using linear and quadratic basis.

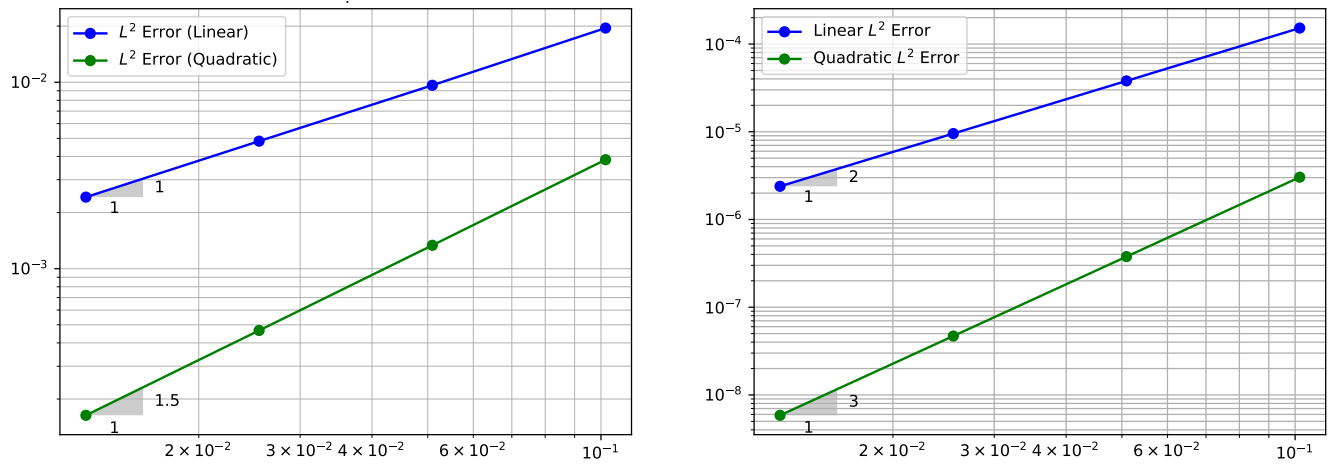
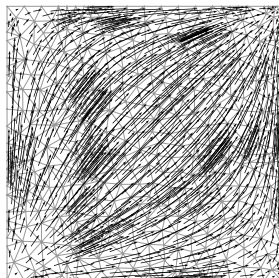


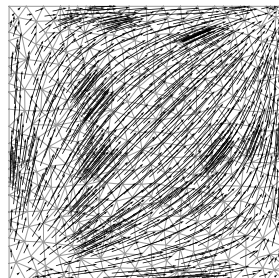
FIGURE 2.35: **Vector Poisson with Electric boundary conditions:** Error plots of **Example 2.6** on unit square using linear and quadratic basis (left) and for $\sigma = \text{div } u$ using linear and quadratic basis (right).

Analytical Solution u

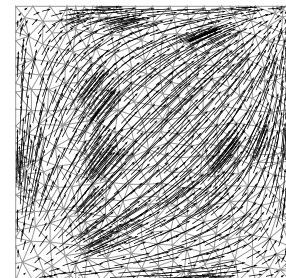


Computed Solution u using

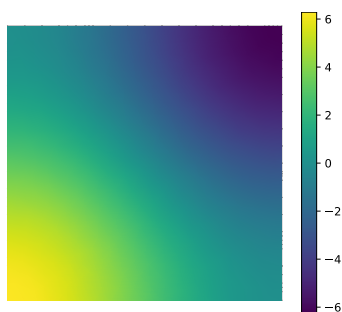
Linear Basis



Quadratic Basis

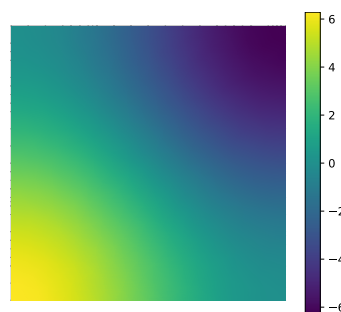


Analytical Solution σ



Computed Solution σ using

Linear Basis



Quadratic Basis

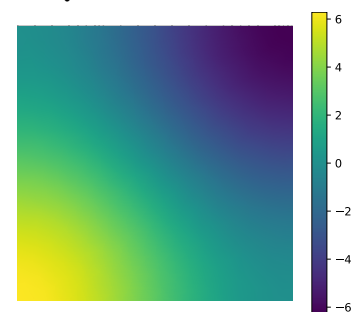


FIGURE 2.36: **Vector Poisson with Electric boundary conditions:** Plots of **Example 2.7** on unit square using linear and quadratic basis.

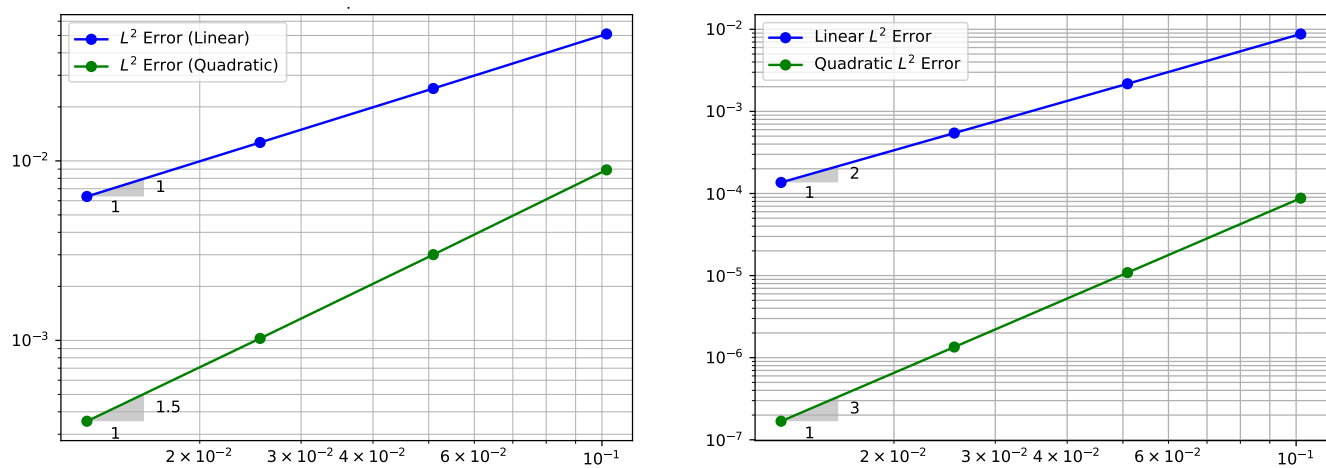
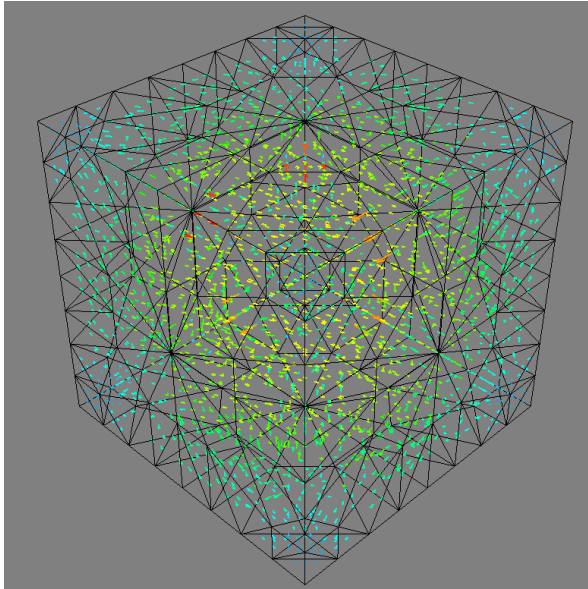
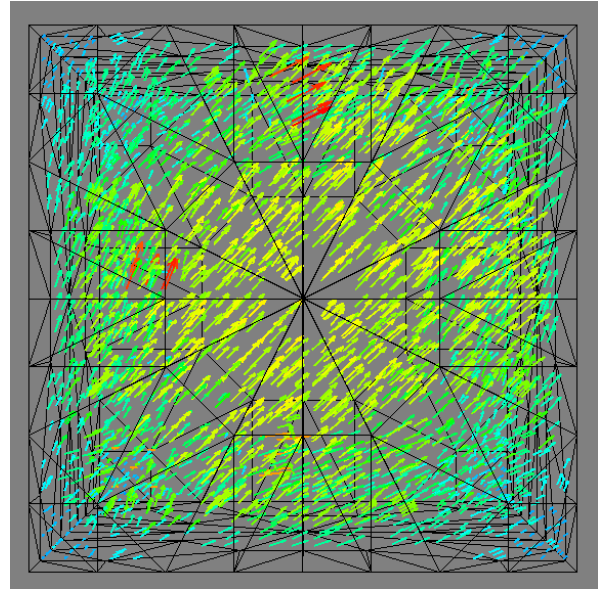


FIGURE 2.37: **Vector Poisson with Electric boundary conditions:** Error plots of **Example 2.7** on unit square using linear and quadratic basis (left) and $\sigma = \operatorname{div} u$ using linear and quadratic basis (right).

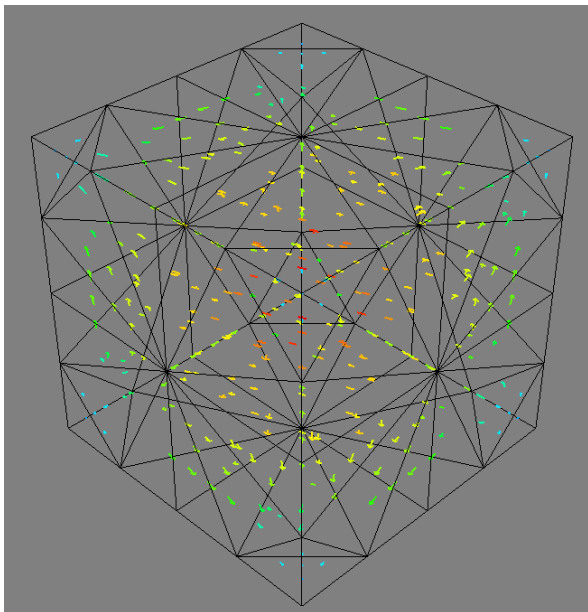
Isometric View using Linear Basis



X-View using Linear Basis



Isometric View using Quadratic Basis



X-View using Quadratic Basis

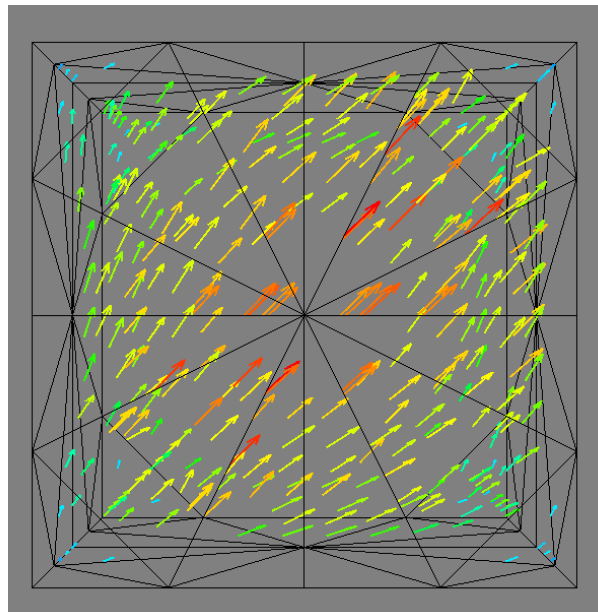


FIGURE 2.38: **Vector Poisson with Electric boundary conditions:** Isometric View (left) and X-View (right) of plots of **Example 2.8** on unit cube using linear basis over finer mesh and using quadratic basis over coarser mesh.

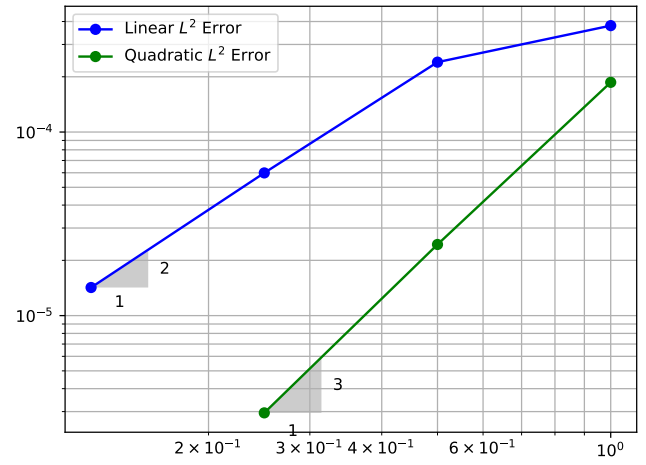
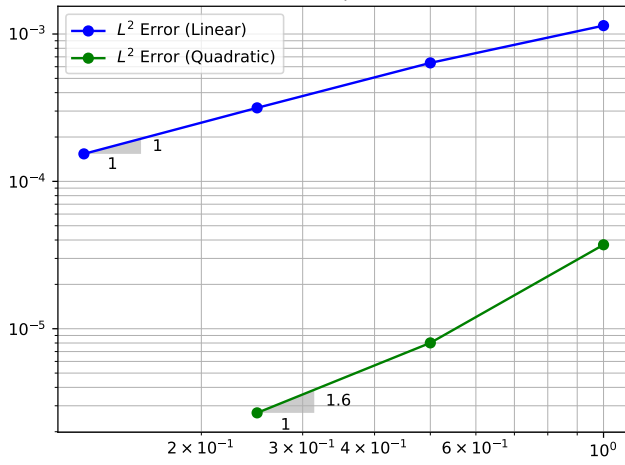


FIGURE 2.39: **Vector Poisson with Electric boundary conditions:** Error plots of **Example 2.8** on unit square using linear and quadratic basis (left) and $\sigma = \text{curl } u$ using linear and quadratic basis (right).

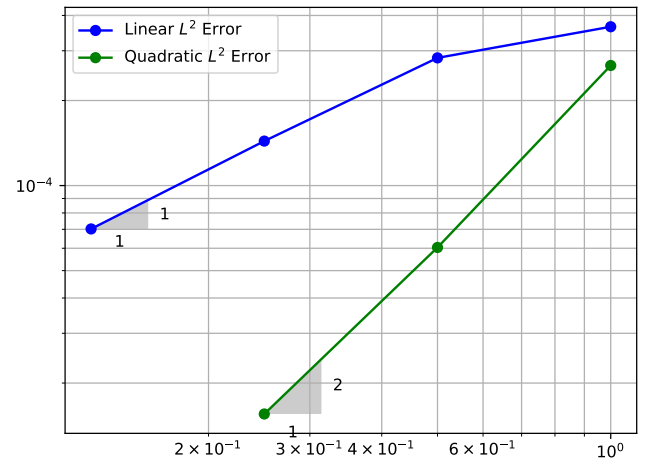
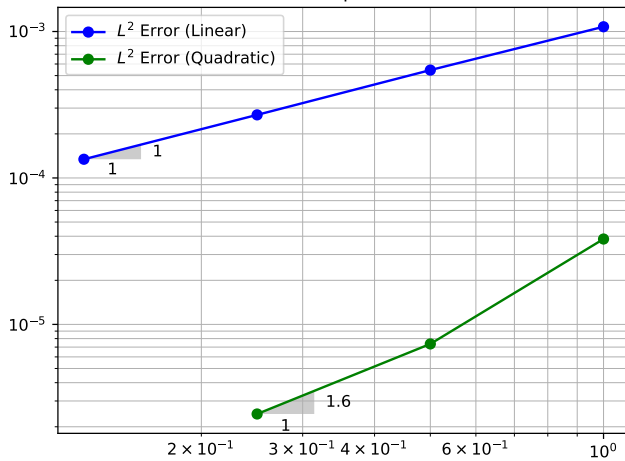
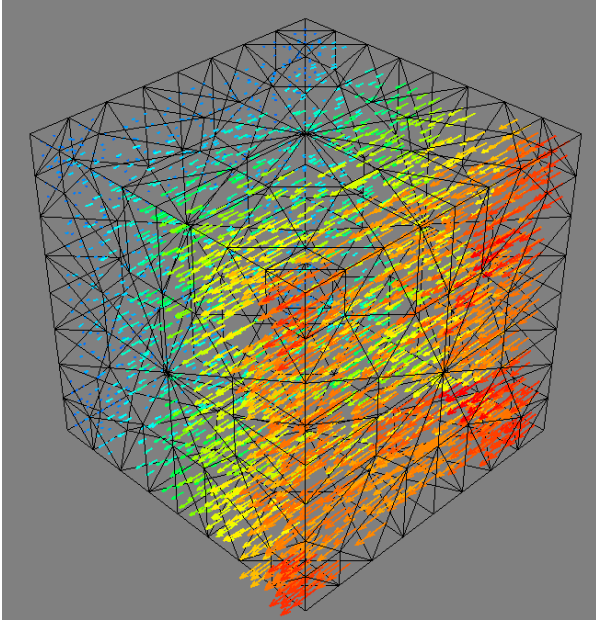
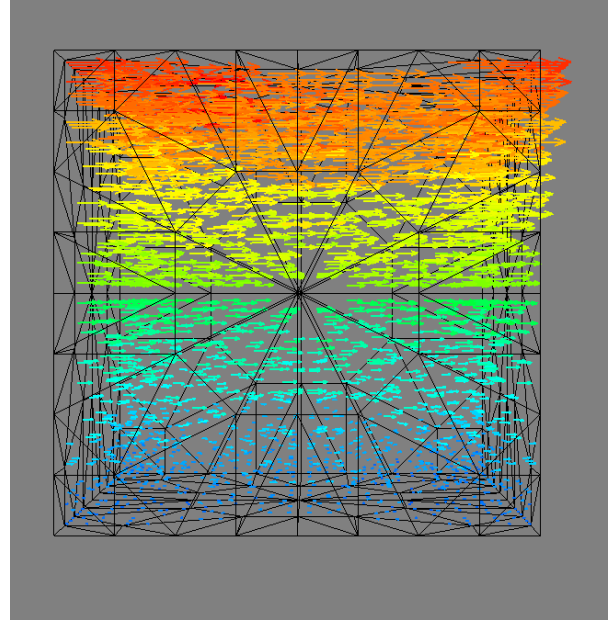


FIGURE 2.40: **Vector Poisson with Magnetic boundary conditions:** Error plots of **Example 2.8** on unit square using linear and quadratic basis (left) and $\sigma = \text{curl } u$ using linear and quadratic basis (right).

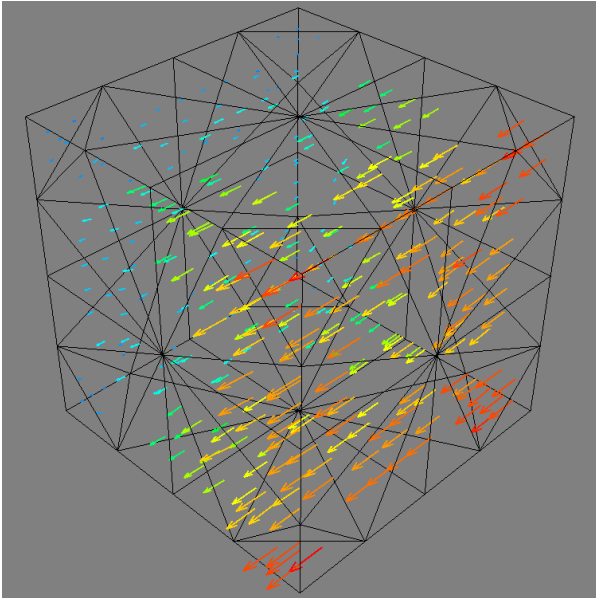
Isometric View using Linear Basis



Z-View using Linear Basis



Isometric View using Quadratic Basis



Z-View using Quadratic Basis

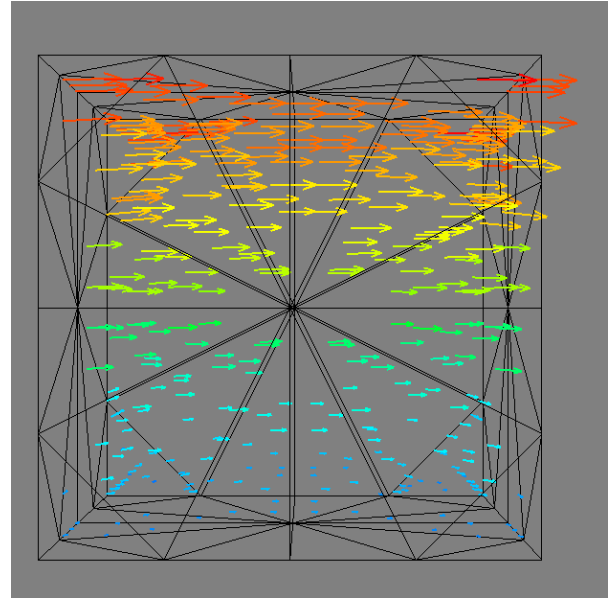


FIGURE 2.41: **Vector Poisson with Magnetic boundary conditions:** Isometric View (left) and Z-View (right) of plots of **Example 2.9** on unit cube using linear basis over finer mesh and using quadratic basis over coarser mesh.

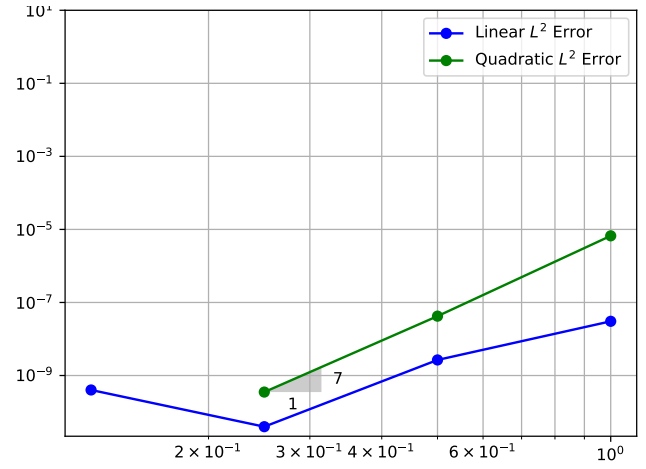
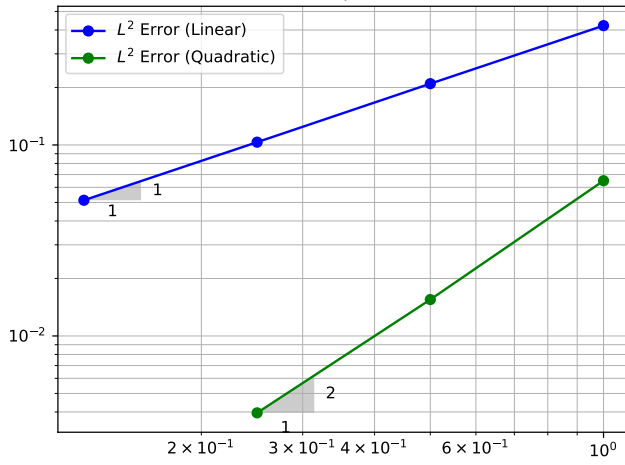


FIGURE 2.42: **Vector Poisson with Magnetic boundary conditions:** Error plots of **Example 2.9** on unit square using linear and quadratic basis (left) and $\sigma = \text{curl } u$ using linear and quadratic basis (right).

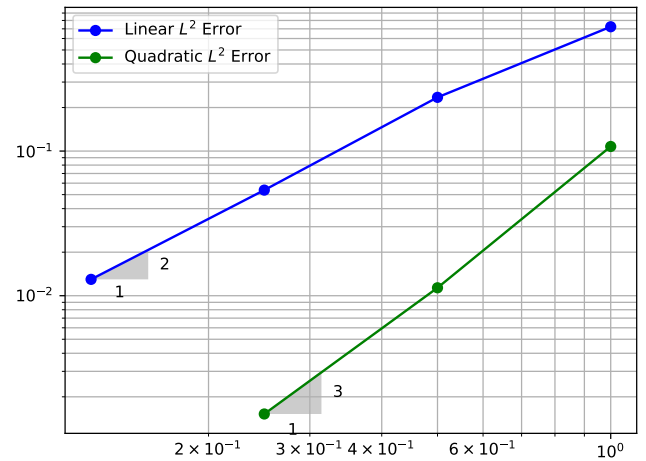
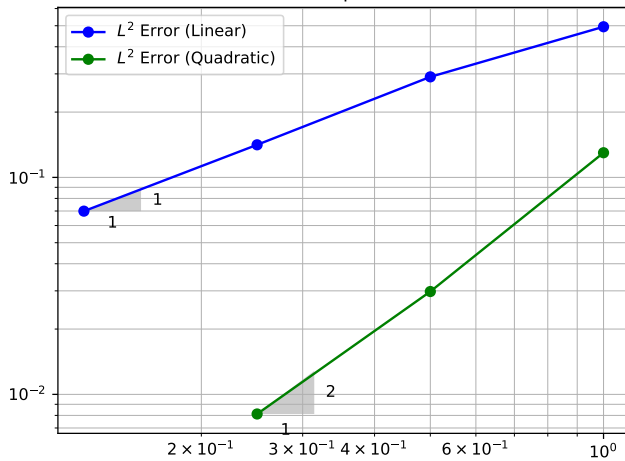


FIGURE 2.43: **Vector Poisson with Electric boundary conditions:** Error plots of **Example 2.9** on unit square using linear and quadratic basis (left) and $\sigma = \text{curl } u$ using linear and quadratic basis (right).

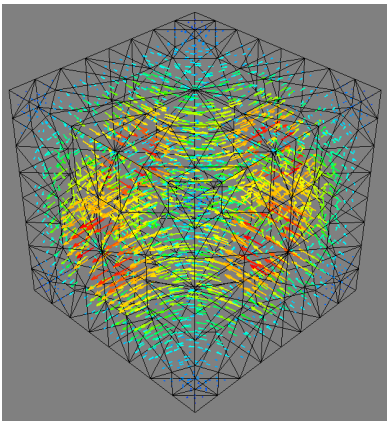
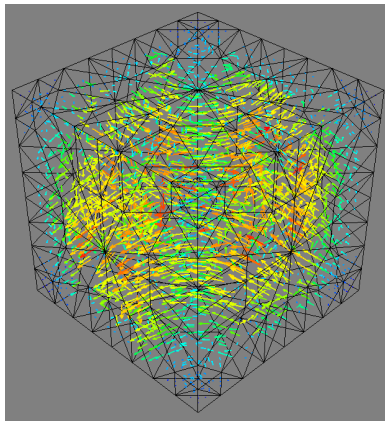
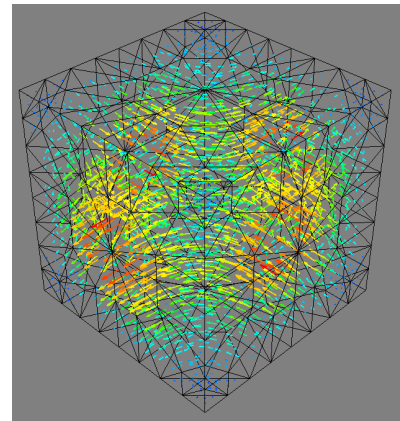
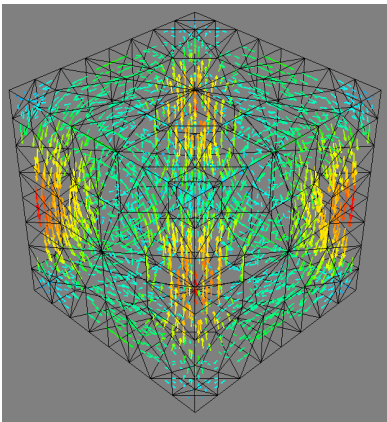
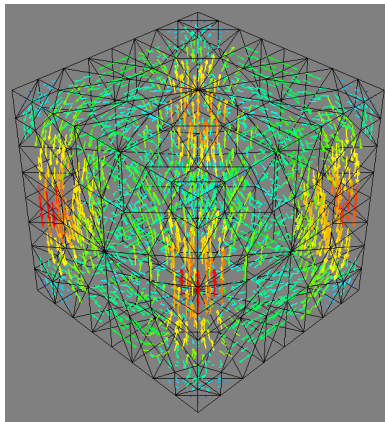
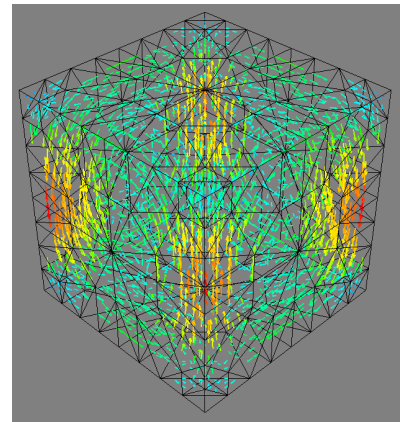
Analytical E at $t = 0.1$ Computed Linear E at $t = 0.1$ Computed Quadratic E at $t = 0.1$ Analytical H at $t = 0.1$ Computed Linear H at $t = 0.1$ Computed Quadratic H at $t = 0.1$ 

FIGURE 2.44: **Maxwell's Equation using Crank Nicolson:** Plots of **Example 2.11** on unit cube using linear and quadratic basis for spatial discretization and Crank Nicolson for temporal discretization.

Chapter 3

Energy Conserving and Second-order Time Implicit Mixed Finite Element Discretizations of Maxwell's Equations

3.1 Introduction

Consider the following set of Maxwell's equations

$$\begin{aligned} \frac{\partial p}{\partial t} + \nabla \cdot \varepsilon E &= f_p \text{ in } \Omega \times (0, T], \\ \nabla p + \varepsilon \frac{\partial E}{\partial t} - \nabla \times H &= f_E \text{ in } \Omega \times (0, T], \\ \mu \frac{\partial H}{\partial t} + \nabla \times E &= f_H \text{ in } \Omega \times (0, T], \end{aligned} \quad (3.1a)$$

where $\Omega \subset \mathbb{R}^2/\mathbb{R}^3$ is a domain with Lipschitz boundary $\partial\Omega$ and with the following homogeneous boundary conditions :

$$p = 0, E \times n = 0, H \cdot n = 0 \text{ on } \partial\Omega \times (0, T], \quad (3.1b)$$

where n is the unit outward normal to $\partial\Omega$, and with the following initial conditions :

$$p(x, 0) = p_0(x), E(x, 0) = E_0(x), \text{ and } H(x, 0) = H_0(x) \text{ for } x \in \Omega. \quad (3.1c)$$

In these equations, $E(x, t)$ and $H(x, t)$ denote the electric and magnetic fields, respectively, and $p(x, t)$ is a physically fictitious term that is related to the time varying divergence of the electric field. We refer to all variables without the x and t descriptors when this is clear to avoid clutter. The additional electric variable facilitates a mixed formulation for the Maxwell's equations in the sense of finite element exterior calculus (FEEC) and allows for a direct approximation of $\nabla \cdot \varepsilon E$. The boundary conditions correspond to a perfectly electric conductor at the boundary and this is usually a good abstraction for the study of Maxwell's equations. This formulation was first introduced in [APZ13] and further used in [AHZ17; Adl+21], and the authors there attribute it to Douglas Arnold who is one of the authors of several works in FEEC [AFW10; Arn18]. We slightly generalize their (p, E, H) formulation to allow for arbitrary right hand side functions $f_p(x, t)$, $f_E(x, t)$ and $f_H(x, t)$ as shown in Equation (3.1a). Clearly, if $\nabla \cdot \varepsilon E = 0$, then with $p_0(x) = 0$, $f_p = 0$, $f_E = J(x, t)$ (which is the current density) and $f_H = 0$, we recover the more standard form of the Maxwell's equations. In our setup, we assume that $t \in [0, T]$ for some finite maximum time T . The material parameters ε (electric permittivity) and μ (magnetic permeability) are assumed to be time independent and positive bounded piecewise constant functions, that is, there exist positive constants: $\varepsilon_{\min}, \varepsilon_{\max}, \mu_{\min}, \mu_{\max}$ satisfying $0 < \varepsilon_{\min} \leq \varepsilon(x) \leq \varepsilon_{\max} < \infty$ and $0 < \mu_{\min} \leq \mu(x) \leq \mu_{\max} < \infty$ for all $x \in \Omega$. Although this minor technical

condition is necessary, for all intents and purposes, we treat these physical parameters as simply constants in our work here. These constants can then be taken to be either the inverse of the minimum or maximum values as appropriate in all our proofs if these parameters are spatially varying. Finally, we shall assume that the initial conditions provided satisfy:

$$\nabla \cdot (\varepsilon E_0) = p_0, \text{ and } \nabla \cdot (\mu H_0) = 0 \text{ in } \Omega,$$

to be consistent with our motivation for introducing p in conjunction with E , and because magnetic fields are divergence free due to there being no magnetic charges.

The variational formulation for Equation (3.1a) with boundary conditions in Equation (3.1b) is given by: find $(p, E, H) \in \dot{H}_{\varepsilon^{-1}}^1(\Omega) \times \dot{H}_\varepsilon(\text{curl}; \Omega) \times \dot{H}_\mu(\text{div}; \Omega)$ such that:

$$\left\langle \frac{\partial p}{\partial t}, \tilde{p} \right\rangle - \langle \varepsilon E, \nabla \tilde{p} \rangle = \langle f_p, \tilde{p} \rangle, \quad \tilde{p} \in \dot{H}_{\varepsilon^{-1}}^1(\Omega), \quad (3.2a)$$

$$\langle \nabla p, \tilde{E} \rangle + \left\langle \varepsilon \frac{\partial E}{\partial t}, \tilde{E} \right\rangle - \langle H, \nabla \times \tilde{E} \rangle = \langle f_E, \tilde{E} \rangle, \quad \tilde{E} \in \dot{H}_\varepsilon(\text{curl}; \Omega), \quad (3.2b)$$

$$\left\langle \mu \frac{\partial H}{\partial t}, \tilde{H} \right\rangle + \langle \nabla \times E, \tilde{H} \rangle = \langle f_H, \tilde{H} \rangle, \quad \tilde{H} \in \dot{H}_\mu(\text{div}; \Omega), \quad (3.2c)$$

for $t \in (0, T]$ with initial conditions as in Equation (3.1c). For this problem, we shall use the following two time discretizations .

Crank-Nicolson scheme: Find $(p(t^n), E(t^n), H(t^n)) \in \dot{H}_{\varepsilon^{-1}}^1(\Omega) \times \dot{H}_\varepsilon(\text{curl}; \Omega) \times \dot{H}_\mu(\text{div}; \Omega)$ such that:

$$\left\langle \frac{p^n - p^{n-1}}{\Delta t}, \tilde{p} \right\rangle - \left\langle \frac{\varepsilon}{2} (E^n + E^{n-1}), \nabla \tilde{p} \right\rangle = \left\langle \frac{1}{2} (f_p^n + f_p^{n-1}), \tilde{p} \right\rangle, \quad (3.3a)$$

$$\left\langle \frac{1}{2} \nabla (p^n + p^{n-1}), \tilde{E} \right\rangle + \left\langle \varepsilon \frac{E^n - E^{n-1}}{\Delta t}, \tilde{E} \right\rangle - \left\langle \frac{1}{2} (H^n + H^{n-1}), \nabla \times \tilde{E} \right\rangle = \left\langle \frac{1}{2} (f_E^n + f_E^{n-1}), \tilde{E} \right\rangle, \quad (3.3b)$$

$$\left\langle \mu \frac{H^n - H^{n-1}}{\Delta t}, \tilde{H} \right\rangle + \left\langle \frac{1}{2} \nabla \times (E^n + E^{n-1}), \tilde{H} \right\rangle = \left\langle \frac{1}{2} (f_H^n + f_H^{n-1}), \tilde{H} \right\rangle, \quad (3.3c)$$

for all $\tilde{p} \in \dot{H}_{\varepsilon^{-1}}^1(\Omega)$, $\tilde{E} \in \dot{H}_\varepsilon(\text{curl}; \Omega)$ and $\tilde{H} \in \dot{H}_\mu(\text{div}; \Omega)$, and with uniform discretization of $[0, T]$ as $t^n := n\Delta t$, $n = 0, 1, \dots, N$ for some $\Delta t > 0$ being the fixed time step size such that $N\Delta t = T$.

Implicit leapfrog scheme: Find $(p(t^{n+1/2}), E(t^{n+1/2}), H(t^n)) \in \dot{H}_{\varepsilon^{-1}}^1(\Omega) \times \dot{H}_\varepsilon(\text{curl}; \Omega) \times \dot{H}_\mu(\text{div}; \Omega)$ such that:

$$\left\langle \frac{p^{n+\frac{1}{2}} - p^{n-\frac{1}{2}}}{\Delta t}, \tilde{p} \right\rangle - \left\langle \frac{\varepsilon}{2} (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla \tilde{p} \right\rangle = \langle f_p^n, \tilde{p} \rangle, \quad (3.4a)$$

$$\left\langle \frac{1}{2} \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), \tilde{E} \right\rangle + \left\langle \varepsilon \frac{E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}}{\Delta t}, \tilde{E} \right\rangle - \left\langle \frac{1}{2} (H^{n+1} + H^n), \nabla \times \tilde{E} \right\rangle = \langle f_E^n, \tilde{E} \rangle, \quad (3.4b)$$

$$\left\langle \mu \frac{H^{n+1} - H^n}{\Delta t}, \tilde{H} \right\rangle + \left\langle \frac{1}{2} \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \tilde{H} \right\rangle = \langle f_H^{n+\frac{1}{2}}, \tilde{H} \rangle, \quad (3.4c)$$

for all $\tilde{p} \in \dot{H}_{\varepsilon^{-1}}^1(\Omega)$, $\tilde{E} \in \dot{H}_\varepsilon(\text{curl}; \Omega)$ and $\tilde{H} \in \dot{H}_\mu(\text{div}; \Omega)$, and with two uniform staggered discretizations of $[0, T]$: for the variables p and E as $t^{n+\frac{1}{2}} := (n + 1/2)\Delta t$, $n = 0, 1, \dots, N - 1$ for some $\Delta t > 0$ being the fixed time step size such that $N\Delta t = T$, and for the variable H as $t^n := n\Delta t$, $n = 0, 1, \dots, N$. Clearly, our method is inspired by the classical (but explicit) leapfrog scheme, and we obtain p, E at half time indices and H at integer time indices. We propose this semi discretization by evaluating all inner product terms in Equations (3.2a) to (3.2c) for p and E at t^n and for H at $t^{n+\frac{1}{2}}$. To bootstrap this scheme using the given initial values at $t = 0$ as in Equation (3.1c), we shall use the following discretization for the first half time step for p

and E , and for the first time step for H :

$$\left\langle \frac{p^{\frac{1}{2}} - p_0}{\Delta t/2}, \tilde{p} \right\rangle - \frac{1}{2} \left\langle \frac{\varepsilon}{2} (E^{\frac{1}{2}} + E_0), \nabla \tilde{p} \right\rangle = \langle f_p^0, \tilde{p} \rangle, \quad (3.5a)$$

$$\frac{1}{2} \left\langle \frac{1}{2} \nabla (p^{\frac{1}{2}} + p_0), \tilde{E} \right\rangle + \left\langle \varepsilon \frac{E^{\frac{1}{2}} - E_0}{\Delta t/2}, \tilde{E} \right\rangle - \left\langle \frac{1}{2} (H^1 + H_0), \nabla \times \tilde{E} \right\rangle = \langle f_E^0, \tilde{E} \rangle, \quad (3.5b)$$

$$\left\langle \mu \frac{H^1 - H_0}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \left\langle \frac{1}{2} \nabla \times (E^{\frac{1}{2}} + E_0), \tilde{H} \right\rangle = \langle f_H^0, \tilde{H} \rangle. \quad (3.5c)$$

In all the time discretizations shown in Equations (3.3a) to (3.3c), (3.4a) to (3.4c) and (3.5a) to (3.5c), we use a superscript on our variables of interest to denote their values at the time corresponding to this time index, for example, $p^n := p(t^n, x)$ and so on.

3.2 Stability of Variational Formulation

We first demonstrate the existence and uniqueness of the solution to the variational formulation of the Maxwell's system as presented in Equations (3.2a) to (3.2c). To do so, we have the following energy estimate for it.

Theorem 3.1 (Energy Estimate). *Let $f_p \in L^1[0, T] \times L^2_{\varepsilon^{-1}}(\Omega)$, $f_E \in L^1(0, T; L^2_{\varepsilon^{-1}}(\Omega))$, and $f_H \in L^1(0, T; L^2_{\mu^{-1}}(\Omega))$. Then the solution (p, E, H) of Equations (3.2a) to (3.2c) with initial conditions as in Equation (3.1c) and assuming sufficient regularity with $p \in C^1(0, T; \dot{H}^1_{\varepsilon^{-1}}(\Omega))$, $E \in C^1(0, T; \dot{H}_\varepsilon(\text{curl}; \Omega))$, and $H \in C^1(0, T; \dot{H}_\mu(\text{div}; \Omega))$ satisfies:*

$$\begin{aligned} \|p\|_{\varepsilon^{-1}} + \|E\|_\varepsilon + \|H\|_\mu &\leq \sqrt{3} \left[\|p_0\|_{\varepsilon^{-1}} + \|E_0\|_\varepsilon + \|H_0\|_\mu \right. \\ &\quad \left. + \|f_p\|_{L^1(0, T; L^2_{\varepsilon^{-1}}(\Omega))} + \|f_E\|_{L^1(0, T; L^2_{\varepsilon^{-1}}(\Omega))} + \|f_H\|_{L^1(0, T; L^2_{\mu^{-1}}(\Omega))} \right]. \end{aligned}$$

Proof. Since Equations (3.2a) to (3.2c) are true for all $\tilde{p} \in \dot{H}^1_{\varepsilon^{-1}}(\Omega)$, $\tilde{E} \in \dot{H}_\varepsilon(\text{curl}; \Omega)$, $\tilde{H} \in \dot{H}_\mu(\text{div}; \Omega)$, we choose $\tilde{p} = \varepsilon^{-1}p$, $\tilde{E} = E$ and $\tilde{H} = H$ which then yields the following set of equations:

$$\begin{aligned} \left\langle \frac{\partial p}{\partial t}, \varepsilon^{-1}p \right\rangle - \langle \varepsilon E, \nabla \varepsilon^{-1}p \rangle &= \langle f_p, \varepsilon^{-1}p \rangle, \\ \langle \nabla p, E \rangle + \left\langle \varepsilon \frac{\partial E}{\partial t}, E \right\rangle - \langle H, \nabla \times E \rangle &= \langle f_E, E \rangle, \\ \left\langle \mu \frac{\partial H}{\partial t}, H \right\rangle + \langle \nabla \times E, H \rangle &= \langle f_H, H \rangle. \end{aligned}$$

Adding these together and using the properties of the inner product, we obtain:

$$\left\langle \frac{\partial p}{\partial t}, \varepsilon^{-1}p \right\rangle + \left\langle \varepsilon \frac{\partial E}{\partial t}, E \right\rangle + \left\langle \mu \frac{\partial H}{\partial t}, H \right\rangle = \langle f_p, \varepsilon^{-1}p \rangle + \langle f_E, E \rangle + \langle f_H, H \rangle. \quad (3.6)$$

Now consider that:

$$\frac{d}{dt} \|p\|_{\varepsilon^{-1}}^2 = \frac{d}{dt} \langle \varepsilon^{-1}p, p \rangle = 2 \left\langle \frac{\partial p}{\partial t}, \varepsilon^{-1}p \right\rangle.$$

Similarly, we also have:

$$\frac{d}{dt} \|E\|_\varepsilon^2 = 2 \langle \varepsilon \frac{\partial E}{\partial t}, E \rangle, \text{ and } \frac{d}{dt} \|H\|_\mu^2 = 2 \langle \mu \frac{\partial H}{\partial t}, H \rangle.$$

Using these in Equation (3.6) and the Cauchy-Schwarz inequality, we get:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} [\|p\|_{\varepsilon^{-1}}^2 + \|E\|_\varepsilon^2 + \|H\|_\mu^2] &= \langle f_p, \varepsilon^{-1} p \rangle + \langle \varepsilon^{-\frac{1}{2}} f_E, \varepsilon^{\frac{1}{2}} E \rangle + \langle \mu^{-\frac{1}{2}} f_H, \mu^{\frac{1}{2}} H \rangle, \\ &\leq \|f_p\|_{\varepsilon^{-1}} \|p\|_{\varepsilon^{-1}} + \|f_E\|_{\varepsilon^{-1}} \|E\|_\varepsilon + \|f_H\|_{\mu^{-1}} \|H\|_\mu. \end{aligned}$$

Integrating both sides of this inequality from 0 to t for any $t \in (0, T]$, we next get:

$$\begin{aligned} \|p\|_{\varepsilon^{-1}}^2 + \|E\|_\varepsilon^2 + \|H\|_\mu^2 &\leq \|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_\varepsilon^2 + \|H_0\|_\mu^2 \\ &\quad + 2 \int_0^t (\|f_p\|_{\varepsilon^{-1}} + \|f_E\|_{\varepsilon^{-1}} + \|f_H\|_{\mu^{-1}}) \sqrt{\|p\|_{\varepsilon^{-1}}^2 + \|E\|_\varepsilon^2 + \|H\|_\mu^2} ds. \end{aligned}$$

Then using the Gronwall-OuLang inequality as described in Lemma 1.8, this results in:

$$\sqrt{\|p\|_{\varepsilon^{-1}}^2 + \|E\|_\varepsilon^2 + \|H\|_\mu^2} \leq \sqrt{\|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_\varepsilon^2 + \|H_0\|_\mu^2} + \int_0^t (\|f_p\|_{\varepsilon^{-1}} + \|f_E\|_{\varepsilon^{-1}} + \|f_H\|_{\mu^{-1}}) ds,$$

for any $t \in (0, T]$. Finally, using the equivalence of 1- and 2-norms, we obtain our result:

$$\begin{aligned} \|p\|_{\varepsilon^{-1}} + \|E\|_\varepsilon + \|H\|_\mu &\leq \sqrt{3} \left[\|p_0\|_{\varepsilon^{-1}} + \|E_0\|_\varepsilon + \|H_0\|_\mu \right. \\ &\quad \left. + \|f_p\|_{L^1(0,T;L^2_{\varepsilon^{-1}}(\Omega))} + \|f_E\|_{L^1(0,T;L^2_{\varepsilon^{-1}}(\Omega))} + \|f_H\|_{L^1(0,T;L^2_{\mu^{-1}}(\Omega))} \right]. \quad \square \end{aligned}$$

Corollary 3.2 (Energy Conservation). *If the forcing functions in Equation (3.1a) of the Maxwell's equations are all zero, that is, $f_p = 0$ and $f_E = f_H = 0$ and with initial conditions as in Equation (3.1c) then:*

$$\|p\|_{\varepsilon^{-1}}^2 + \|E\|_\varepsilon^2 + \|H\|_\mu^2 = \|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_\varepsilon^2 + \|H_0\|_\mu^2.$$

Remark 3.3 (Uniqueness of Solution). *As a result of Theorem 3.1, and by a standard argument, for example [Eva10, Theorem 5, Section 2.4, Chapter 2], the solution to the variational formulation of the Maxwell's equations in Equations (3.2a) to (3.2c) with initial conditions as in Equation (3.1c) has a unique solution.*

Remark 3.4 (Improved Energy Estimate). *Our proof for Theorem 3.1 is a bit more involved than the straightforward similar proof in [APZ13, Proposition 3.2] in that we arrive at it using Gronwall's inequality as in Lemma 1.9. One advantage of doing so though is that we can establish a precise energy conservation for the case when initial conditions alone drive the evolution unlike Adler and coauthors.*

3.3 Time Discretization Stability

For each of our time discretization schemes, we next show that these semidiscretizations in time yield stable methods and we provide error estimates which show that these schemes are quadratically convergent in

each of their chosen sufficiently small but fixed time step.

3.3.1 Crank-Nicolson Scheme

Theorem 3.5 (Discrete Energy Estimate). *For the semidiscretization using the Crank-Nicolson scheme as given in Equations (3.3a) to (3.3c), for any fixed time step $\Delta t > 0$ sufficiently small, there exists a positive bounded constant C independent of Δt such that:*

$$\|p^N\|_{\varepsilon^{-1}} + \|E^N\|_{\varepsilon} + \|H^N\|_{\mu} \leq C.$$

Proof. Since Equations (3.3a) to (3.3c) are true for all $\tilde{p} \in \dot{H}_{\varepsilon^{-1}}^1(\Omega)$, $\tilde{E} \in \dot{H}_{\varepsilon}(\text{curl}; \Omega)$, $\tilde{H} \in \dot{H}_{\mu}(\text{div}; \Omega)$, using $\tilde{p} = 2\Delta t \varepsilon^{-1} (p^n + p^{n-1})$, $\tilde{E} = 2\Delta t (E^n + E^{n-1})$ and $\tilde{H} = 2\Delta t (H^n + H^{n-1})$ in them, we obtain the following:

$$\begin{aligned} 2\langle p^n - p^{n-1}, \varepsilon^{-1} (p^n + p^{n-1}) \rangle - \Delta t \langle E^n + E^{n-1}, \nabla (p^n + p^{n-1}) \rangle &= \Delta t \langle f_p^n + f_p^{n-1}, \varepsilon^{-1} (p^n + p^{n-1}) \rangle, \\ \Delta t \langle \nabla (p^n + p^{n-1}), E^n + E^{n-1} \rangle + 2\langle \varepsilon (E^n - E^{n-1}), E^n + E^{n-1} \rangle - \Delta t \langle H^n + H^{n-1}, \nabla \times (E^n + E^{n-1}) \rangle \\ &= \Delta t \langle f_E^n + f_E^{n-1}, E^n + E^{n-1} \rangle, \\ 2\langle \mu (H^n - H^{n-1}), H^n + H^{n-1} \rangle + \Delta t \langle \nabla \times (E^n + E^{n-1}), H^n + H^{n-1} \rangle &= \Delta t \langle f_H^n + f_H^{n-1}, H^n + H^{n-1} \rangle. \end{aligned}$$

Adding these equations together and using the properties of inner product, we get:

$$\begin{aligned} 2\left(\langle \varepsilon^{-1} (p^n - p^{n-1}), p^n + p^{n-1} \rangle + \langle \varepsilon (E^n - E^{n-1}), E^n + E^{n-1} \rangle \right. \\ \left. + \langle \mu (H^n - H^{n-1}), H^n + H^{n-1} \rangle\right) = \Delta t \left(\langle f_p^n + f_p^{n-1}, \varepsilon^{-1} (p^n + p^{n-1}) \rangle \right. \\ \left. + \langle f_E^n + f_E^{n-1}, E^n + E^{n-1} \rangle + \langle f_H^n + f_H^{n-1}, H^n + H^{n-1} \rangle\right). \quad (3.7) \end{aligned}$$

Using the Cauchy-Schwarz, AM-GM, and Triangle inequalities for the first term on the right hand side of the above expression, we derive the following inequality:

$$\begin{aligned} \langle f_p^n + f_p^{n-1}, \varepsilon^{-1} (p^n + p^{n-1}) \rangle &\leq \|f_p^n + f_p^{n-1}\|_{\varepsilon^{-1}} \|p^n + p^{n-1}\|_{\varepsilon^{-1}}, \\ &\leq \frac{1}{2} \left(\|f_p^n + f_p^{n-1}\|_{\varepsilon^{-1}}^2 + \|p^n + p^{n-1}\|_{\varepsilon^{-1}}^2 \right), \\ &\leq \|f_p^n\|_{\varepsilon^{-1}}^2 + \|f_p^{n-1}\|_{\varepsilon^{-1}}^2 + \|p^n\|_{\varepsilon^{-1}}^2 + \|p^{n-1}\|_{\varepsilon^{-1}}^2. \end{aligned}$$

Similarly in Equation (3.7), for the second and third terms on the right hand side, we have:

$$\begin{aligned} \langle f_E^n + f_E^{n-1}, E^n + E^{n-1} \rangle &\leq \|f_E^n\|_{\varepsilon^{-1}}^2 + \|f_E^{n-1}\|_{\varepsilon^{-1}}^2 + \|E^n\|_{\varepsilon}^2 + \|E^{n-1}\|_{\varepsilon}^2, \\ \langle f_H^n + f_H^{n-1}, H^n + H^{n-1} \rangle &\leq \|f_H^n\|_{\mu^{-1}}^2 + \|f_H^{n-1}\|_{\mu^{-1}}^2 + \|H^n\|_{\mu}^2 + \|H^{n-1}\|_{\mu}^2. \end{aligned}$$

Incorporating these inequalities into Equation (3.7), we obtain:

$$\begin{aligned} 2\left[\|p^n\|_{\varepsilon^{-1}}^2 - \|p^{n-1}\|_{\varepsilon^{-1}}^2 + \|E^n\|_{\varepsilon}^2 - \|E^{n-1}\|_{\varepsilon}^2 + \|H^n\|_{\mu}^2 - \|H^{n-1}\|_{\mu}^2\right] \\ \leq \Delta t \left[\|f_p^n\|_{\varepsilon^{-1}}^2 + \|f_p^{n-1}\|_{\varepsilon^{-1}}^2 + \|p^n\|_{\varepsilon^{-1}}^2 + \|p^{n-1}\|_{\varepsilon^{-1}}^2 + \|f_E^n\|_{\varepsilon^{-1}}^2 + \|f_E^{n-1}\|_{\varepsilon^{-1}}^2 \right. \end{aligned}$$

$$+ \|E^n\|_\varepsilon^2 + \|E^{n-1}\|_\varepsilon^2 + \|f_H^n\|_{\mu^{-1}}^2 + \|f_H^{n-1}\|_{\mu^{-1}}^2 + \|H^n\|_\mu^2 + \|H^{n-1}\|_\mu^2 \Big].$$

Next, summing over $n = 1$ to N and using the initial conditions as in Equation (3.1c), we have that:

$$2 \Big[\|p^N\|_{\varepsilon^{-1}}^2 + \|E^N\|_\varepsilon^2 + \|H^N\|_\mu^2 \Big] \leq \Delta t \Big[\sum_{n=0}^N 2 \Big(\|p^n\|_{\varepsilon^{-1}}^2 + \|E^n\|_\varepsilon^2 + \|H^n\|_\mu^2 + \|f_p^n\|_{\varepsilon^{-1}}^2 + \|f_E^n\|_{\varepsilon^{-1}}^2 + \|f_H^n\|_{\mu^{-1}}^2 \Big) \Big] + 2 \Big[\|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_\varepsilon^2 + \|H_0\|_\mu^2 \Big],$$

which in turn provides us with the following estimate:

$$\begin{aligned} \|p^N\|_{\varepsilon^{-1}}^2 + \|E^N\|_\varepsilon^2 + \|H^N\|_\mu^2 &\leq \Delta t \sum_{n=0}^N \Big[\|p^n\|_{\varepsilon^{-1}}^2 + \|E^n\|_\varepsilon^2 + \|H^n\|_\mu^2 \Big] \\ &\quad + \Delta t \sum_{n=0}^N \Big[\|f_p^n\|_{\varepsilon^{-1}}^2 + \|f_E^n\|_{\varepsilon^{-1}}^2 + \|f_H^n\|_{\mu^{-1}}^2 \Big] + \Big[\|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_\varepsilon^2 + \|H_0\|_\mu^2 \Big]. \end{aligned}$$

Now, we are ready to apply the discrete Gronwall inequality as in Lemma 1.10. To do so, we set $\delta := \Delta t$, $g_0 := \|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_\varepsilon^2 + \|H_0\|_\mu^2$, $a_n := \|p^n\|_{\varepsilon^{-1}}^2 + \|E^n\|_\varepsilon^2 + \|H^n\|_\mu^2$, $b_n := 0$, $c_n := \|f_p^n\|_{\varepsilon^{-1}}^2 + \|f_E^n\|_{\varepsilon^{-1}}^2 + \|f_H^n\|_{\mu^{-1}}^2$, and $\gamma_n := 1$. Note that for the condition $\gamma_n \delta < 1$ to hold, we need to have $\Delta t < 1/2$ and therefore we have that $\sigma_n = (1 - \Delta t)^{-1}$. Then, we get:

$$\begin{aligned} \|p^N\|_{\varepsilon^{-1}}^2 + \|E^N\|_\varepsilon^2 + \|H^N\|_\mu^2 &\leq \left[\Delta t \sum_{n=0}^N \left(\|f_p^n\|_{\varepsilon^{-1}}^2 + \|f_E^n\|_{\varepsilon^{-1}}^2 + \|f_H^n\|_{\mu^{-1}}^2 \right) + \|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_\varepsilon^2 + \|H_0\|_\mu^2 \right] \\ &\quad \exp \left[\Delta t \sum_{n=0}^N (1 - \Delta t)^{-1} \right], \\ &\leq \left[\Delta t \sum_{n=0}^N \left(\|f_p^n\|_{\varepsilon^{-1}}^2 + \|f_E^n\|_{\varepsilon^{-1}}^2 + \|f_H^n\|_{\mu^{-1}}^2 \right) + \|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_\varepsilon^2 + \|H_0\|_\mu^2 \right] \exp(2T + 1), \end{aligned}$$

where the second inequality is obtained using that $(N + 1)\Delta t = T + \Delta t < T + 1/2$. We now estimate the term:

$$\sum_{n=0}^N \Delta t \left(\|f_p^n\|_{\varepsilon^{-1}}^2 + \|f_E^n\|_{\varepsilon^{-1}}^2 + \|f_H^n\|_{\mu^{-1}}^2 \right),$$

which is a lower sum approximation of the integral:

$$\int_0^T \left(\|f_p\|_{\varepsilon^{-1}}^2 + \|f_E\|_{\varepsilon^{-1}}^2 + \|f_H\|_{\mu^{-1}}^2 \right) dt,$$

and which in turn is equal to:

$$\|f_p\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \|f_E\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \|f_H\|_{L^2(0,T;L_{\mu^{-1}}^2(\Omega))}^2,$$

and hence is bounded. Set this bound to be a positive constant M . By using the equivalence of 1- and 2-norms, and $C = \sqrt{3(M + \|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_{\varepsilon}^2 + \|H_0\|_{\mu}^2) \exp(2T + 1)}$, we obtain our desired result:

$$\|p^N\|_{\varepsilon^{-1}} + \|E^N\|_{\varepsilon} + \|H^N\|_{\mu} \leq C.$$

□

Corollary 3.6 (discrete energy conservation). *If the forcing functions in Equation (3.1a) are all zero, that is, $f_p = 0$ and $f_E = f_H = 0$ then:*

$$\|p^N\|_{\varepsilon^{-1}}^2 + \|E^N\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 = \|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_{\varepsilon}^2 + \|H_0\|_{\mu}^2.$$

Remark 3.7. *In both Theorem 3.5 and Corollary 3.6, the time index N can be replaced by any $n \in [1, N]$ thereby making these to be discrete analogues of Theorem 3.1 and Corollary 3.2, respectively. Moreover, an argument similar to that in Remark 3.3 would show that the solution for the Crank-Nicolson semidiscretization is also unique.*

Theorem 3.8 (discrete error estimate). *For the semidiscretization using the Crank-Nicolson scheme as given in Equations (3.3a) to (3.3c), for the solution (p, E, H) of Equations (3.2a) to (3.2c) with initial conditions as in Equation (3.1c) and assuming sufficient regularity with $p \in C^3(0, T; \dot{H}_{\varepsilon^{-1}}^1(\Omega))$, $E \in C^3(0, T; \dot{H}_{\varepsilon}(\text{curl}; \Omega))$, and $H \in C^3(0, T; \dot{H}_{\mu}(\text{div}; \Omega))$, and for the time step Δt sufficiently small, there exists a positive bounded constant C independent of Δt such that:*

$$\|e_p^N\|_{\varepsilon^{-1}} + \|e_E^N\|_{\varepsilon} + \|e_H^N\|_{\mu} \leq C [(\Delta t)^2 + \|e_p^0\|_{\varepsilon^{-1}} + \|e_E^0\|_{\varepsilon} + \|e_H^0\|_{\mu}], \quad (3.8)$$

where $e_p^n := p(t^n) - p^n$, $e_E^n := E(t^n) - E^n$ and $e_H^n := H(t^n) - H^n$ are the errors in the time semidiscretization of p , E and H , respectively and at the indicated time indices.

Proof. Using the Taylor remainder theorem, and expressing $p(t)$ about $t = t^{n-1/2}$, we have that:

$$p(t) = p(t^{n-1/2}) + \frac{\partial p}{\partial t}(t^{n-1/2})(t - t^{n-1/2}) + \frac{\partial^2 p}{\partial t^2}(t^{n-1/2}) \frac{(t - t^{n-1/2})^2}{2} + \int_{t^{n-1/2}}^t \frac{(t-s)^2}{2} \frac{\partial^3 p}{\partial t^3}(s) ds,$$

which when evaluated at $t = t^n$ and $t = t^{n-1}$ provides us with:

$$\begin{aligned} p(t^n) &= p(t^{n-1/2}) + \frac{\partial p}{\partial t}(t^{n-1/2})(t^n - t^{n-1/2}) + \frac{\partial^2 p}{\partial t^2}(t^{n-1/2}) \frac{(t^n - t^{n-1/2})^2}{2} + \int_{t^{n-1/2}}^{t^n} \frac{(t^n - s)^2}{2} \frac{\partial^3 p}{\partial t^3}(s) ds, \\ p(t^{n-1}) &= p(t^{n-1/2}) + \frac{\partial p}{\partial t}(t^{n-1/2})(t^{n-1} - t^{n-1/2}) + \frac{\partial^2 p}{\partial t^2}(t^{n-1/2}) \frac{(t^{n-1} - t^{n-1/2})^2}{2} + \int_{t^{n-1/2}}^{t^{n-1}} \frac{(t^{n-1} - s)^2}{2} \frac{\partial^3 p}{\partial t^3}(s) ds. \end{aligned}$$

Subtracting the second of these equations from the first, we get:

$$\left(p(t^n) - p(t^{n-1})\right) = \frac{\partial p}{\partial t}(t^{n-\frac{1}{2}})(t^n - t^{n-1}) + \int_{t^{n-\frac{1}{2}}}^{t^n} \frac{(t^n - s)^2}{2} \frac{\partial^3 p}{\partial t^3}(s) ds - \int_{t^{n-\frac{1}{2}}}^{t^{n-1}} \frac{(t^{n-1} - s)^2}{2} \frac{\partial^3 p}{\partial t^3}(s) ds,$$

which then when cast into the form of the inner product term from the semidiscretization of the variational formulation leads to:

$$\left\langle \frac{p(t^n) - p(t^{n-1})}{\Delta t}, \tilde{p} \right\rangle = \left\langle \frac{\partial p}{\partial t}(t^{n-\frac{1}{2}}), \tilde{p} \right\rangle + \frac{1}{2} \langle R_p^n, \tilde{p} \rangle,$$

and in which we have defined that:

$$R_p^n := \frac{1}{\Delta t} \left[\int_{t^{n-1}}^{t^{n-\frac{1}{2}}} (t^{n-1} - s)^2 \frac{\partial^3 p}{\partial t^3}(s) ds + \int_{t^{n-\frac{1}{2}}}^{t^n} (t^n - s)^2 \frac{\partial^3 p}{\partial t^3}(s) ds \right].$$

Similarly, for E and H, we have the following inner product terms from the variational formulation:

$$\begin{aligned} \left\langle \varepsilon \frac{E(t^n) - E(t^{n-1})}{\Delta t}, \tilde{E} \right\rangle &= \left\langle \varepsilon \frac{\partial E}{\partial t}(t^{n-\frac{1}{2}}), \tilde{E} \right\rangle + \frac{1}{2} \langle \varepsilon R_E^n, \tilde{E} \rangle, \\ \left\langle \mu \frac{H(t^n) - H(t^{n-1})}{\Delta t}, \tilde{H} \right\rangle &= \left\langle \mu \frac{\partial H}{\partial t}(t^{n-\frac{1}{2}}), \tilde{H} \right\rangle + \frac{1}{2} \langle \mu R_H^n, \tilde{H} \rangle, \end{aligned}$$

and in each of which we have defined that:

$$\begin{aligned} R_E^n &:= \frac{1}{\Delta t} \left[\int_{t^{n-1}}^{t^{n-\frac{1}{2}}} (t^{n-1} - s)^2 \frac{\partial^3 E}{\partial t^3}(s) ds + \int_{t^{n-\frac{1}{2}}}^{t^n} (t^n - s)^2 \frac{\partial^3 E}{\partial t^3}(s) ds \right], \\ R_H^n &:= \frac{1}{\Delta t} \left[\int_{t^{n-1}}^{t^{n-\frac{1}{2}}} (t^{n-1} - s)^2 \frac{\partial^3 H}{\partial t^3}(s) ds + \int_{t^{n-\frac{1}{2}}}^{t^n} (t^n - s)^2 \frac{\partial^3 H}{\partial t^3}(s) ds \right]. \end{aligned}$$

Using these terms in Equations (3.2a) to (3.2c) at time $t = t^{n-\frac{1}{2}}$ and with the definition that:

$$f(t^{n-\frac{1}{2}}) := \frac{f(t^n) + f(t^{n-1})}{2},$$

where f is any of f_p , f_E or f_H , and wielding the Taylor remainder theorem to also write:

$$u(t^{n-\frac{1}{2}}) := \frac{u(t^n) + u(t^{n-1})}{2} - r_u^n,$$

for u being p , E or H , and with the definition that:

$$r_u^n := \left[\int_{t^{n-\frac{1}{2}}}^{t^n} (t^n - s) \frac{\partial^2 u}{\partial t^2}(s) ds - \int_{t^{n-1}}^{t^{n-\frac{1}{2}}} (t^{n-1} - s) \frac{\partial^2 u}{\partial t^2}(s) ds \right],$$

leads to the following set of equations:

$$\left\langle \frac{p(t^n) - p(t^{n-1})}{\Delta t}, \tilde{p} \right\rangle - \frac{1}{2} \left\langle \varepsilon \left(E(t^n) + E(t^{n-1}) \right), \nabla \tilde{p} \right\rangle = \frac{1}{2} \left\langle f_p^n + f_p^{n-1} + R_p^n + \varepsilon \nabla \cdot r_E^n, \tilde{p} \right\rangle, \quad (3.9a)$$

$$\begin{aligned} \frac{1}{2} \left\langle \nabla \left(p(t^n) + p(t^{n-1}) \right), \tilde{E} \right\rangle + \left\langle \varepsilon \frac{E(t^n) - E(t^{n-1})}{\Delta t}, \tilde{E} \right\rangle - \frac{1}{2} \left\langle H(t^n) + H(t^{n-1}), \nabla \times \tilde{E} \right\rangle \\ = \frac{1}{2} \left\langle f_E^n + f_E^{n-1} + \varepsilon R_E^n + \nabla r_p^n - \nabla \times r_H^n, \tilde{E} \right\rangle, \end{aligned} \quad (3.9b)$$

$$\left\langle \mu \frac{H(t^n) - H(t^{n-1})}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \left\langle \nabla \times \left(E(t^n) + E(t^{n-1}) \right), \tilde{H} \right\rangle = \frac{1}{2} \left\langle f_H^n + f_H^{n-1} + \mu R_H^n + \nabla \times r_E^n, \tilde{H} \right\rangle. \quad (3.9c)$$

Subtracting the equations defining the Crank-Nicolson semidiscretization as in Equations (3.3a) to (3.3c) from Equations (3.9a) to (3.9c) leads us to:

$$\begin{aligned} \left\langle \frac{e_p^n - e_p^{n-1}}{\Delta t}, \tilde{p} \right\rangle - \frac{1}{2} \left\langle \varepsilon \left(e_E^n + e_E^{n-1} \right), \nabla \tilde{p} \right\rangle &= \frac{1}{2} \left\langle R_p^n + \varepsilon \nabla \cdot r_E^n, \tilde{p} \right\rangle, \\ \frac{1}{2} \left\langle \nabla \left(e_p^n + e_p^{n-1} \right), \tilde{E} \right\rangle + \left\langle \varepsilon \frac{e_E^n - e_E^{n-1}}{\Delta t}, \tilde{E} \right\rangle - \frac{1}{2} \left\langle e_H^n + e_H^{n-1}, \nabla \times \tilde{E} \right\rangle &= \frac{1}{2} \left\langle \varepsilon R_E^n + \nabla r_p^n - \nabla \times r_H^n, \tilde{E} \right\rangle, \\ \left\langle \mu \frac{e_H^n - e_H^{n-1}}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \left\langle \nabla \times \left(e_E^n + e_E^{n-1} \right), \tilde{H} \right\rangle &= \frac{1}{2} \left\langle \mu R_H^n + \nabla \times r_E^n, \tilde{H} \right\rangle. \end{aligned}$$

Now, in this set of weak formulation equations for the errors, we choose the test functions to be $\tilde{p} = 2\Delta t \varepsilon^{-1} (e_p^n + e_p^{n-1})$, $\tilde{E} = 2\Delta t (e_E^n + e_E^{n-1})$ and $\tilde{H} = 2\Delta t (e_H^n + e_H^{n-1})$. Next, by following essentially the same sequence of steps as in Theorem 3.5, we obtain the estimate for these error terms to be:

$$\begin{aligned} 2 \left[\|e_p^n\|_{\varepsilon^{-1}}^2 - \|e_p^{n-1}\|_{\varepsilon^{-1}}^2 + \|e_E^n\|_\varepsilon^2 - \|e_E^{n-1}\|_\varepsilon^2 + \|e_H^n\|_\mu^2 - \|e_H^{n-1}\|_\mu^2 \right] \\ \leq \Delta t \left[\|e_p^n\|_{\varepsilon^{-1}}^2 + \|e_p^{n-1}\|_{\varepsilon^{-1}}^2 + \|e_E^n\|_\varepsilon^2 + \|e_E^{n-1}\|_\varepsilon^2 + \|e_H^n\|_\mu^2 + \|e_H^{n-1}\|_\mu^2 \right] \\ + \Delta t \left[\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_\varepsilon^2 + \|R_H^n\|_\mu^2 + \|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_E^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_H^n\|_\mu^2 \right]. \end{aligned}$$

Now, summing over $n = 1$ to N , using the initial conditions as in Equation (3.1c), and the positivity of all the right hand side terms, we have that:

$$\begin{aligned} \|e_p^N\|_{\varepsilon^{-1}}^2 + \|e_E^N\|_\varepsilon^2 + \|e_H^N\|_\mu^2 \leq \Delta t \sum_{n=0}^N \left[\|e_p^n\|_{\varepsilon^{-1}}^2 + \|e_E^n\|_\varepsilon^2 + \|e_H^n\|_\mu^2 \right] + \Delta t \sum_{n=0}^N \left[\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_\varepsilon^2 + \|R_H^n\|_\mu^2 \right. \\ \left. + \|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_E^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_H^n\|_\mu^2 \right] + \left[\|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^0\|_\varepsilon^2 + \|e_H^0\|_\mu^2 \right]. \end{aligned}$$

Next to apply the discrete Gronwall inequality as in Lemma 1.10, we choose $\delta := \Delta t$, $g_0 := \|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^0\|_\varepsilon^2 + \|e_H^0\|_\mu^2$, $a_n := \|e_p^n\|_{\varepsilon^{-1}}^2 + \|e_E^n\|_\varepsilon^2 + \|e_H^n\|_\mu^2$, $b_n := 0$, $c_n := \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_\varepsilon^2 + \|R_H^n\|_\mu^2 + \|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_E^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_H^n\|_\mu^2$, and $\gamma_n := 1$. Now the required condition $\gamma_n \delta < 1$ holds with $\Delta t < 1/2$ and $\sigma_n = (1 - \Delta t)^{-1}$. Therefore, using that $(N + 1)\Delta t = T + \Delta t \leq T + 1/2$, we get that:

$$\begin{aligned} \|e_p^N\|_{\varepsilon^{-1}}^2 + \|e_E^N\|_\varepsilon^2 + \|e_H^N\|_\mu^2 \leq \left[\Delta t \sum_{n=0}^N \left(\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_\varepsilon^2 + \|R_H^n\|_\mu^2 + \|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_\varepsilon^2 \right. \right. \\ \left. \left. + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_E^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_H^n\|_\mu^2 \right) + \left(\|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^0\|_\varepsilon^2 + \|e_H^0\|_\mu^2 \right) \right] \exp(2T + 1). \end{aligned} \quad (3.10)$$

Now, we need to obtain bounding estimates for each of the Taylor remainder terms and to do so, we first consider the first remainder term corresponding to p and argue as follows:

$$\begin{aligned}
\|R_p^n\|_{\varepsilon^{-1}}^2 &= \frac{1}{(\Delta t)^2} \left\| \int_{t^{n-1}}^{t^{n-\frac{1}{2}}} (t^{n-1} - s)^2 \frac{\partial^3 p}{\partial t^3}(s) ds + \int_{t^{n-\frac{1}{2}}}^{t^n} (t^n - s)^2 \frac{\partial^3 p}{\partial t^3}(s) ds \right\|_{\varepsilon^{-1}}^2, \\
&\leq \frac{1}{(\Delta t)^2} \left\| \int_{t^{n-1}}^{t^n} (t^n - s)^2 \frac{\partial^3 p}{\partial t^3}(s) ds \right\|_{\varepsilon^{-1}}^2, \quad (\text{using } t^{n-1} < t^n) \\
&\leq \frac{1}{(\Delta t)^2} \int_{t^{n-1}}^{t^n} (s - t^n)^4 ds \int_{t^{n-1}}^{t^n} \left\| \frac{\partial^3 p}{\partial t^3}(s) \right\|_{\varepsilon^{-1}}^2 ds, \quad (\text{by Cauchy-Schwarz}) \\
&= \frac{(\Delta t)^3}{5} \int_{t^{n-1}}^{t^n} \left\| \frac{\partial^3 p}{\partial t^3}(s) \right\|_{\varepsilon^{-1}}^2 ds,
\end{aligned}$$

and now summing both sides over $n = 0$ to N , we have that:

$$\sum_{n=0}^N \|R_p^n\|_{\varepsilon^{-1}}^2 \leq \frac{(\Delta t)^3}{5} \int_0^T \left\| \frac{\partial^3 p}{\partial t^3}(s) \right\|_{\varepsilon^{-1}}^2 ds = \frac{(\Delta t)^3}{5} \left\| \frac{\partial^3 p}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2. \quad (3.11)$$

Similarly, for the remaining Taylor remainder terms, we have that:

$$\sum_{n=0}^N \|R_E^n\|_{\varepsilon}^2 \leq \frac{(\Delta t)^3}{5} \left\| \frac{\partial^3 E}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \quad (3.12)$$

$$\sum_{n=0}^N \|R_H^n\|_{\mu}^2 \leq \frac{(\Delta t)^3}{5} \left\| \frac{\partial^3 H}{\partial t^3} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2,$$

$$\sum_{n=0}^N \|\nabla r_p^n\|_{\varepsilon^{-1}}^2 \leq \frac{(\Delta t)^3}{12} \left\| \frac{\partial^2 (\nabla p)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2, \quad (3.13)$$

$$\sum_{n=0}^N \|\nabla \cdot r_E^n\|_{\varepsilon}^2 \leq \frac{(\Delta t)^3}{12} \left\| \frac{\partial^2 (\nabla \cdot E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \quad (3.14)$$

$$\sum_{n=0}^N \|\nabla \times r_E^n\|_{\varepsilon}^2 \leq \frac{(\Delta t)^3}{12} \left\| \frac{\partial^2 (\nabla \times E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \quad (3.15)$$

$$\sum_{n=0}^N \|\nabla \times r_H^n\|_{\mu}^2 \leq \frac{(\Delta t)^3}{12} \left\| \frac{\partial^2 (\nabla \times H)}{\partial t^2} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2.$$

By substituting all these Taylor remainder term estimates into Equation (3.10), we obtain:

$$\|e_p^N\|_{\varepsilon^{-1}}^2 + \|e_E^N\|_{\varepsilon}^2 + \|e_H^N\|_{\mu}^2 \leq \left[\frac{(\Delta t)^4}{5} \left(\left\| \frac{\partial^3 p}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^3 E}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial^3 H}{\partial t^3} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right) \right]$$

$$\begin{aligned}
& + \frac{(\Delta t)^4}{12} \left(\left\| \frac{\partial^2(\nabla p)}{\partial t^2} \right\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + \left\| \frac{\partial^2(\nabla \cdot E)}{\partial t^2} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^2(\nabla \times E)}{\partial t^2} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 \right. \\
& \quad \left. + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^2(\nabla \times H)}{\partial t^2} \right\|_{L^2(0,T;L^2_{\mu}(\Omega))}^2 \right) + \left(\|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^0\|_{\varepsilon}^2 + \|e_H^0\|_{\mu}^2 \right) \exp(2T+1).
\end{aligned}$$

By our assumption on the regularity of p , E and H , there exists a positive bounded constant M such that:

$$\begin{aligned}
& \frac{1}{5} \left(\left\| \frac{\partial^3 p}{\partial t^3} \right\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + \left\| \frac{\partial^3 E}{\partial t^3} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + \left\| \frac{\partial^3 H}{\partial t^3} \right\|_{L^2(0,T;L^2_{\mu}(\Omega))}^2 \right) + \frac{1}{12} \left(\left\| \frac{\partial^2(\nabla p)}{\partial t^2} \right\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 \right. \\
& \quad \left. + \left\| \frac{\partial^2(\nabla \cdot E)}{\partial t^2} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^2(\nabla \times E)}{\partial t^2} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^2(\nabla \times H)}{\partial t^2} \right\|_{L^2(0,T;L^2_{\mu}(\Omega))}^2 \right) \leq M.
\end{aligned}$$

Thus, we have that:

$$\|e_p^N\|_{\varepsilon^{-1}}^2 + \|e_E^N\|_{\varepsilon}^2 + \|e_H^N\|_{\mu}^2 \leq \widetilde{C} [(\Delta t)^4 + \|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^0\|_{\varepsilon}^2 + \|e_H^0\|_{\mu}^2],$$

where $\widetilde{C} = \exp(2T+1) \max\{M, 1\}$. Finally using the equivalence of 1- and 2-norms, our desired result follows with $C = 2\sqrt{\widetilde{C}}$. $\square$

3.3.2 Implicit Leapfrog Scheme

Theorem 3.9 (Discrete Energy Estimate). *For the semidiscretization using the implicit leapfrog scheme as given in Equations (3.4a) to (3.4c) and (3.5a) to (3.5c), for any time step Δt sufficiently small, there exists a positive bounded constant C independent of Δt such that:*

$$\|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|E^{N-\frac{1}{2}}\|_{\varepsilon} + \|H^N\|_{\mu} \leq C.$$

Proof. Since Equations (3.4a) to (3.4c) are true for all $\widetilde{p} \in \dot{H}_{\varepsilon^{-1}}^1(\Omega)$, $\widetilde{E} \in \dot{H}_{\varepsilon}(\text{curl}; \Omega)$, $\widetilde{H} \in \dot{H}_{\mu}(\text{div}; \Omega)$, using $\widetilde{p} = 2\Delta t \varepsilon^{-1} (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}})$, $\widetilde{E} = 2\Delta t (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}})$ and $\widetilde{H} = 2\Delta t (H^{n+1} + H^n)$ in them, we obtain the following:

$$\begin{aligned}
& 2\langle p^{n+\frac{1}{2}} - p^{n-\frac{1}{2}}, \varepsilon^{-1} (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}) \rangle - \Delta t \langle E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}, \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}) \rangle \\
& \quad = 2\Delta t \langle f_p^n, \varepsilon^{-1} (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}) \rangle, \\
& \Delta t \langle \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \rangle + 2\langle \varepsilon (E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}), E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \rangle \\
& \quad - \Delta t \langle (H^{n+1} + H^n), \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}) \rangle = 2\Delta t \langle f_E^n, E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \rangle, \\
& 2\langle \mu (H^{n+1} - H^n), H^{n+1} + H^n \rangle + \Delta t \langle \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), H^{n+1} + H^n \rangle = 2\Delta t \langle f_H^{n+\frac{1}{2}}, H^{n+1} + H^n \rangle.
\end{aligned}$$

Adding these equations together and using properties of the inner product, we get:

$$2\langle \varepsilon^{-1} (p^{n+\frac{1}{2}} - p^{n-\frac{1}{2}}), p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}} \rangle + 2\langle \varepsilon (E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}), E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \rangle$$

$$\begin{aligned}
& + 2\langle \mu (H^{n+1} - H^n), H^{n+1} + H^n \rangle \\
& = \Delta t \left(2\langle f_p^n, \varepsilon^{-1} (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}) \rangle + 2\langle f_E^n, E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \rangle + 2\langle f_H^{n+\frac{1}{2}}, H^{n+1} + H^n \rangle \right).
\end{aligned}$$

Using the same arguments as in Theorem 3.5, we have the following estimates for the right hand side terms of this equation:

$$\begin{aligned}
2\langle f_p^n, \varepsilon^{-1} (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}) \rangle & \leq \|f_p^n\|_{\varepsilon^{-1}}^2 + 2 \left(\|p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|p^{n-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 \right) \\
2\langle f_E^n, E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \rangle & \leq \|f_E^n\|_{\varepsilon^{-1}}^2 + 2 \left(\|E^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|E^{n-\frac{1}{2}}\|_{\varepsilon}^2 \right), \\
2\langle f_H^{n+\frac{1}{2}}, H^{n+1} + H^n \rangle & \leq \|f_H^{n+\frac{1}{2}}\|_{\mu^{-1}}^2 + 2 \left(\|H^{n+1}\|_{\mu}^2 + \|H^n\|_{\mu}^2 \right).
\end{aligned}$$

Using these inequalities into the previous expression, and summing over $n = 1$ to $N - 1$ leads us to:

$$\begin{aligned}
& \|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 - \|E^{\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 - \|H^1\|_{\mu}^2 \\
& \leq \Delta t \sum_{n=1}^{N-1} \left[\|f_p^n\|_{\varepsilon^{-1}}^2 + \|f_E^n\|_{\varepsilon^{-1}}^2 + \|f_H^{n+\frac{1}{2}}\|_{\mu^{-1}}^2 \right] + 2\Delta t \sum_{n=1}^{N-2} \left[\|p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|H^{n+1}\|_{\mu}^2 \right] \\
& \quad + \Delta t \left[\|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|E^{\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 + \|H^1\|_{\mu}^2 \right].
\end{aligned}$$

Now, $p^{\frac{1}{2}}$, $E^{\frac{1}{2}}$ and H^1 satisfy Equations (3.5a) to (3.5c), and so using $\widetilde{p} = 2\Delta t \varepsilon^{-1} (p^{\frac{1}{2}} + p^0)$, $\widetilde{E} = 2\Delta t (E^{\frac{1}{2}} + E^0)$ and $\widetilde{H} = 4\Delta t (H^1 + H^0)$ and repeating the arguments presented above leads us to the following estimate:

$$\begin{aligned}
& \|p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|p^0\|_{\varepsilon^{-1}}^2 + \|E^{\frac{1}{2}}\|_{\varepsilon}^2 - \|E^0\|_{\varepsilon}^2 + \|H^1\|_{\mu}^2 - \|H^0\|_{\mu}^2 \leq \Delta t \left[\|f_p^0\|_{\varepsilon^{-1}}^2 + \|f_E^0\|_{\varepsilon^{-1}}^2 + \|f_H^{\frac{1}{2}}\|_{\mu^{-1}}^2 \right] \\
& \quad + \Delta t \left[\|p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|p^0\|_{\varepsilon^{-1}}^2 + \|E^{\frac{1}{2}}\|_{\varepsilon}^2 + \|E^0\|_{\varepsilon}^2 + \|H^1\|_{\mu}^2 + \|H^0\|_{\mu}^2 \right].
\end{aligned}$$

Adding these last two inequalities, we get:

$$\begin{aligned}
& \|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 \leq \left[\|p^0\|_{\varepsilon^{-1}}^2 + \|E^0\|_{\varepsilon}^2 + \|H^0\|_{\mu}^2 \right] \\
& \quad + \Delta t \sum_{n=0}^{N-1} \left[\|f_p^n\|_{\varepsilon^{-1}}^2 + \|f_E^n\|_{\varepsilon^{-1}}^2 + \|f_H^{n+\frac{1}{2}}\|_{\mu^{-1}}^2 \right] + 2\Delta t \sum_{n=0}^{N-2} \left[\|p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|H^{n+1}\|_{\mu}^2 \right] \\
& \quad + \Delta t \left[\|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|p^0\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|E^0\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 + \|H^0\|_{\mu}^2 \right],
\end{aligned}$$

which in turn leads to:

$$\begin{aligned}
& \|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 \leq \frac{1 + \Delta t}{1 - \Delta t} \left[\|p^0\|_{\varepsilon^{-1}}^2 + \|E^0\|_{\varepsilon}^2 + \|H^0\|_{\mu}^2 \right] \\
& \quad + \frac{2\Delta t}{1 - \Delta t} \sum_{n=0}^{N-1} \left[\|p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|H^{n+1}\|_{\mu}^2 \right] + \frac{\Delta t}{1 - \Delta t} \sum_{n=0}^{N-1} \left[\|f_p^n\|_{\varepsilon^{-1}}^2 + \|f_E^n\|_{\varepsilon^{-1}}^2 + \|f_H^{n+\frac{1}{2}}\|_{\mu^{-1}}^2 \right].
\end{aligned}$$

In order to apply the discrete Gronwall inequality as in Lemma 1.10, we choose $\delta := \frac{\Delta t}{1 - \Delta t}$, $g_0 := \frac{1 + \Delta t}{1 - \Delta t} \left[\|p^0\|_{\varepsilon^{-1}}^2 + \|E^0\|_{\varepsilon}^2 + \|H^0\|_{\mu}^2 \right]$, $a_n := \|p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|H^n\|_{\mu}^2$, $b_n := 0$, $c_n := \|f_p^n\|_{\varepsilon^{-1}}^2 + \|f_E^n\|_{\varepsilon^{-1}}^2 +$

$\|f_H^{n+\frac{1}{2}}\|_{\mu^{-1}}^2$, and $\gamma_n := 2$. Note that for the condition $\gamma_n \delta < 1$ to hold, we need to have $\Delta t < 1/3$ and thus we get $\sigma_n = \left(1 - \frac{2\Delta t}{1 - \Delta t}\right)^{-1}$. Then for $\Delta t < 1/6$, we have that:

$$\begin{aligned} & \|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 \\ & \leq \left[\frac{\Delta t}{1 - \Delta t} \sum_{n=0}^{N-1} \left(\|f_p^n\|_{\varepsilon^{-1}}^2 + \|f_E^n\|_{\varepsilon^{-1}}^2 + \|f_H^{n+\frac{1}{2}}\|_{\mu^{-1}}^2 \right) + \frac{1 + \Delta t}{1 - \Delta t} \left(\|p^0\|_{\varepsilon^{-1}}^2 + \|E^0\|_{\varepsilon}^2 + \|H^0\|_{\mu}^2 \right) \right] \\ & \quad \exp \left[\frac{\Delta t}{1 - \Delta t} \sum_{n=0}^{N-1} 2 \left(1 - \frac{2\Delta t}{1 - \Delta t} \right)^{-1} \right], \\ & \leq \left[\frac{6\Delta t}{5} \sum_{n=0}^{N-1} \left(\|f_p^n\|_{\varepsilon^{-1}}^2 + \|f_E^n\|_{\varepsilon^{-1}}^2 + \|f_H^{n+\frac{1}{2}}\|_{\mu^{-1}}^2 \right) + \frac{7}{5} \left(\|p^0\|_{\varepsilon^{-1}}^2 + \|E^0\|_{\varepsilon}^2 + \|H^0\|_{\mu}^2 \right) \right] \exp(4T), \end{aligned}$$

where the second inequality is obtained by using that $N\Delta t = T$. We conclude by using a similar argument as in the end of Theorem 3.5 by setting $M := \|f_p\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + \|f_E\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + \|f_H\|_{L^2(0,T;L^2_{\mu^{-1}}(\Omega))}^2$ and using $C = \sqrt{6 \left[M + \|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_{\varepsilon}^2 + \|H_0\|_{\mu}^2 \right] \exp(4T)}$, we obtain our desired result:

$$\|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|E^{N-\frac{1}{2}}\|_{\varepsilon} + \|H^N\|_{\mu} \leq C. \quad \square$$

Corollary 3.10 (Discrete Energy Conservation). *If the forcing functions in Equation (3.1a) are all zero, that is, $f_p = 0$ and $f_E = f_H = 0$, then:*

$$\|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 = \|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_{\varepsilon}^2 + \|H_0\|_{\mu}^2.$$

Remark 3.11. *Like Remark 3.7, the time indices in Theorem 3.9 and Corollary 3.10, can correspond to any time in $(0, T]$. Again, uniqueness of the semidiscretization solution follows from Remark 3.3.*

Theorem 3.12 (Discrete Error Estimate). *For the semidiscretization using the implicit leapfrog scheme as given in Equations (3.4a) to (3.4c), and (3.5a) to (3.5c), for the solution (p, E, H) of Equations (3.2a) to (3.2c) with initial conditions as in Equation (3.1c) and assuming sufficient regularity with $p \in C^3(0, T; \dot{H}_{\varepsilon^{-1}}^1(\Omega))$, $E \in C^3(0, T; \dot{H}_{\varepsilon}(\text{curl}; \Omega))$, and $H \in C^3(0, T; \dot{H}_{\mu}(\text{div}; \Omega))$, and for the time step Δt sufficiently small, there exists a positive bounded constant C independent of Δt such that:*

$$\|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon} + \|e_H^N\|_{\mu} \leq C \left[(\Delta t)^2 + \|e_p^0\|_{\varepsilon^{-1}} + \|e_E^0\|_{\varepsilon} + \|e_H^0\|_{\mu} \right],$$

where $e_p^{n+\frac{1}{2}} := p(t^{n+\frac{1}{2}}) - p^{n+\frac{1}{2}}$, $e_E^{n+\frac{1}{2}} := E(t^{n+\frac{1}{2}}) - E^{n+\frac{1}{2}}$ and $e_H^n := H(t^n) - H^n$ are the errors in the time semidiscretization of p , E and H , respectively and at the indicated time indices.

Proof. Using the Taylor remainder theorem, and expressing $p(t)$ about $t = t^n$, we have that:

$$p(t) = p(t^n) + \frac{\partial p}{\partial t}(t^n)(t - t^n) + \frac{\partial^2 p}{\partial t^2}(t^n) \frac{(t - t^n)^2}{2} + \int_{t^n}^t \frac{(t - s)^2}{2} \frac{\partial^3 p}{\partial t^3}(s) ds,$$

which when evaluated at $t = t^{n+\frac{1}{2}}$ and $t = t^{n-\frac{1}{2}}$ yields:

$$\begin{aligned} p(t^{n+\frac{1}{2}}) &= p(t^n) + \frac{\partial p}{\partial t}(t^n)(t^{n+\frac{1}{2}} - t^n) + \frac{\partial^2 p}{\partial t^2}(t^n) \frac{(t^{n+\frac{1}{2}} - t^n)^2}{2} + \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^2}{2} \frac{\partial^3 p}{\partial t^3}(s) ds, \\ p(t^{n-\frac{1}{2}}) &= p(t^n) + \frac{\partial p}{\partial t}(t^n)(t^{n-\frac{1}{2}} - t^n) + \frac{\partial^2 p}{\partial t^2}(t^n) \frac{(t^{n-\frac{1}{2}} - t^n)^2}{2} + \int_{t^n}^{t^{n-\frac{1}{2}}} \frac{(t^{n-\frac{1}{2}} - s)^2}{2} \frac{\partial^3 p}{\partial t^3}(s) ds. \end{aligned}$$

Subtracting these two equations, and using the result in the inner product term from the semidiscretization of the variational formulation leads to:

$$\left\langle \frac{p(t^{n+\frac{1}{2}}) - p(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{p} \right\rangle = \left\langle \frac{\partial p}{\partial t}(t^n), \tilde{p} \right\rangle + \langle R_p^n, \tilde{p} \rangle,$$

in which we have defined that:

$$R_p^n := \frac{1}{\Delta t} \left[\int_{t^{n-\frac{1}{2}}}^{t^n} \frac{(t^{n-\frac{1}{2}} - s)^2}{2} \frac{\partial^3 p}{\partial t^3}(s) ds + \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^2}{2} \frac{\partial^3 p}{\partial t^3}(s) ds \right].$$

Similarly, for E and H, we have the following:

$$\begin{aligned} \left\langle \varepsilon \frac{E(t^{n+\frac{1}{2}}) - E(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{E} \right\rangle &= \left\langle \varepsilon \frac{\partial E}{\partial t}(t^n), \tilde{E} \right\rangle + \langle \varepsilon R_E^n, \tilde{E} \rangle, \\ \left\langle \mu \frac{H(t^{n+1}) - H(t^n)}{\Delta t}, \tilde{H} \right\rangle &= \left\langle \mu \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}), \tilde{H} \right\rangle + \langle \mu R_H^{n+\frac{1}{2}}, \tilde{H} \rangle, \end{aligned}$$

and in each of which we have defined that:

$$\begin{aligned} R_E^n &:= \frac{1}{\Delta t} \left[\int_{t^{n-\frac{1}{2}}}^{t^n} \frac{(t^{n-\frac{1}{2}} - s)^2}{2} \frac{\partial^3 E}{\partial t^3}(s) ds + \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^2}{2} \frac{\partial^3 E}{\partial t^3}(s) ds \right], \\ R_H^{n+\frac{1}{2}} &:= \frac{1}{\Delta t} \left[\int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^n - s)^2}{2} \frac{\partial^3 H}{\partial t^3}(s) ds + \int_{t^{n+\frac{1}{2}}}^{t^{n+1}} \frac{(t^{n+1} - s)^2}{2} \frac{\partial^3 H}{\partial t^3}(s) ds \right]. \end{aligned}$$

Using these terms in the weak formulation as in Equations (3.2a) to (3.2c) at time $t = t^n$ for p and E terms, and at time $t = t^{n+\frac{1}{2}}$ for H and with the definition that:

$$f(t^n) := \frac{u(t^{n+\frac{1}{2}}) + u(t^{n-\frac{1}{2}})}{2}, \quad f_H(t^{n+\frac{1}{2}}) := \frac{f_H(t^{n+1}) + f_H(t^n)}{2},$$

where f is either of f_p or f_E , and using the Taylor remainder theorem again with:

$$u(t^n) := \frac{u(t^{n+\frac{1}{2}}) + u(t^{n-\frac{1}{2}})}{2} - r_u^n, \quad H(t^{n+\frac{1}{2}}) := \frac{H(t^{n+1}) + H(t^n)}{2} - r_H^{n+\frac{1}{2}},$$

where u is either of p or E , and in which we have defined that:

$$r_u^n := \frac{1}{2} \left[\int_{t^n}^{t^{n+\frac{1}{2}}} (t^{n+\frac{1}{2}} - s) \frac{\partial^2 u}{\partial t^2}(s) ds - \int_{t^{n-\frac{1}{2}}}^{t^n} (t^{n-\frac{1}{2}} - s) \frac{\partial^2 u}{\partial t^2}(s) ds \right],$$

$$r_H^{n+\frac{1}{2}} := \frac{1}{2} \left[\int_{t^{n+\frac{1}{2}}}^{t^{n+1}} (t^{n+1} - s) \frac{\partial^2 H}{\partial t^2}(s) ds - \int_{t^n}^{t^{n+\frac{1}{2}}} (t^n - s) \frac{\partial^2 H}{\partial t^2}(s) ds \right],$$

and then subtracting the implicit leapfrog semidiscretization as in Equations (3.4a) to (3.4c) leads us to the following set of equations:

$$\begin{aligned} \left\langle \frac{e_p^{n+\frac{1}{2}} - e_p^{n-\frac{1}{2}}}{\Delta t}, \tilde{p} \right\rangle - \frac{1}{2} \left\langle \varepsilon \left(e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}} \right), \nabla \tilde{p} \right\rangle &= \langle R_p^n + \varepsilon \nabla \cdot r_E^n, \tilde{p} \rangle, \\ \frac{1}{2} \left\langle \nabla \left(e_p^{n+\frac{1}{2}} + e_p^{n-\frac{1}{2}} \right), \tilde{E} \right\rangle + \left\langle \varepsilon \frac{e_E^{n+\frac{1}{2}} - e_E^{n-\frac{1}{2}}}{\Delta t}, \tilde{E} \right\rangle - \frac{1}{2} \left\langle e_H^{n+1} + e_H^n, \nabla \times \tilde{E} \right\rangle &= \langle \varepsilon R_E^n + \nabla r_p^n - \nabla \times r_H^{n+\frac{1}{2}}, \tilde{E} \rangle, \\ \left\langle \mu \frac{e_H^{n+1} - e_H^n}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \left\langle \nabla \times \left(e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}} \right), \tilde{H} \right\rangle &= \langle \mu R_H^{n+\frac{1}{2}} + \nabla \times r_E^n, \tilde{H} \rangle. \end{aligned}$$

Likewise, for the semidiscrete approximation of the initial system as in Equations (3.5a) to (3.5c), we obtain for their errors the following system of equations:

$$\begin{aligned} \left\langle \frac{e_p^{\frac{1}{2}} - e_p^0}{\Delta t/2}, \tilde{p} \right\rangle - \frac{1}{4} \left\langle \varepsilon \left(e_E^{\frac{1}{2}} + e_E^0 \right), \nabla \tilde{p} \right\rangle &= \left\langle \frac{1}{2} R_p^0 + \frac{\varepsilon}{2} \nabla \cdot r_E^0, \tilde{p} \right\rangle, \\ \frac{1}{4} \left\langle \nabla \left(e_p^{\frac{1}{2}} + e_p^0 \right), \tilde{E} \right\rangle + \left\langle \varepsilon \frac{e_E^{\frac{1}{2}} - e_E^0}{\Delta t/2}, \tilde{E} \right\rangle - \frac{1}{2} \left\langle e_H^1 + e_H^0, \nabla \times \tilde{E} \right\rangle &= \left\langle \frac{\varepsilon}{2} R_E^0 + \frac{1}{2} \nabla r_p^0 - \nabla \times r_H^{\frac{1}{2}}, \tilde{E} \right\rangle, \\ \left\langle \mu \frac{e_H^1 - e_H^0}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{4} \left\langle \nabla \times \left(e_E^{\frac{1}{2}} + e_E^0 \right), \tilde{H} \right\rangle &= \left\langle \mu R_H^{\frac{1}{2}} + \frac{1}{2} \nabla \times r_E^0, \tilde{H} \right\rangle, \end{aligned}$$

and we define the initial remainder terms as follows:

$$\begin{aligned} R_u^0 &:= \frac{1}{\Delta t} \left[\int_{t^0}^{t^{\frac{1}{4}}} \frac{(t^0 - s)^2}{2} \frac{\partial^3 u}{\partial t^3}(s) ds + \int_{t^{\frac{1}{4}}}^{t^{\frac{1}{2}}} \frac{(t^{\frac{1}{2}} - s)^2}{2} \frac{\partial^3 u}{\partial t^3}(s) ds \right], \\ R_H^{\frac{1}{2}} &:= \frac{1}{\Delta t} \left[\int_{t^0}^{t^{\frac{1}{2}}} \frac{(t^0 - s)^2}{2} \frac{\partial^3 H}{\partial t^3}(s) ds + \int_{t^{\frac{1}{2}}}^{t^1} \frac{(t^1 - s)^2}{2} \frac{\partial^3 H}{\partial t^3}(s) ds \right], \\ r_u^0 &:= \frac{1}{2} \left[\int_{t^{\frac{1}{4}}}^{t^{\frac{1}{2}}} (t^{\frac{1}{2}} - s) \frac{\partial^2 u}{\partial t^2}(s) ds - \int_{t^0}^{t^{\frac{1}{4}}} (t^0 - s) \frac{\partial^2 u}{\partial t^2}(s) ds \right], \end{aligned}$$

$$r_H^{\frac{1}{2}} := \frac{1}{2} \left[\int_{\frac{1}{2}}^{t^1} (t^1 - s) \frac{\partial^2 H}{\partial t^2}(s) ds - \int_{t^0}^{\frac{1}{2}} (t^0 - s) \frac{\partial^2 H}{\partial t^2}(s) ds \right].$$

Now by choosing appropriate test functions for these two systems of equations as in Theorem 3.8 and by repeating the arguments for summing over $n = 1$ to $N - 1$, we get:

$$\begin{aligned} \|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^N\|_{\mu}^2 &\leq \frac{1+\Delta t}{1-\Delta t} \left[\|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^0\|_{\varepsilon}^2 + \|e_H^0\|_{\mu}^2 \right] + \\ &\frac{\Delta t}{1-\Delta t} \sum_{n=0}^{N-1} \left[2 \left(\|e_p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^{n+1}\|_{\mu}^2 \right) + \left(\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 \right. \right. \\ &\quad \left. \left. + \|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_E^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_H^{n+\frac{1}{2}}\|_{\mu}^2 \right) \right]. \end{aligned}$$

Applying the discrete Gronwall inequality similar to Theorem 3.9, we obtain the estimate:

$$\begin{aligned} \|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^N\|_{\mu}^2 &\leq \left[\frac{6\Delta t}{5} \sum_{n=0}^{N-1} \left(\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 + \|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_{\varepsilon}^2 \right. \right. \\ &\quad \left. \left. + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_E^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_H^{n+\frac{1}{2}}\|_{\mu}^2 \right) + \frac{7}{5} \left(\|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^0\|_{\varepsilon}^2 + \|e_H^0\|_{\mu}^2 \right) \right] \exp(4T). \end{aligned}$$

Now, for the Taylor remainder terms on the right hand side of this equation, using arguments similar to that in Theorem 3.8, we can obtain the following inequalities:

$$\begin{aligned} \sum_{n=0}^{N-1} \|R_p^n\|_{\varepsilon^{-1}}^2 &\leq \frac{(\Delta t)^3}{20} \left\| \frac{\partial^3 p}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2, \\ \sum_{n=0}^{N-1} \|R_E^n\|_{\varepsilon}^2 &\leq \frac{(\Delta t)^3}{20} \left\| \frac{\partial^3 E}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \\ \sum_{n=0}^{N-1} \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 &\leq \frac{(\Delta t)^3}{20} \left\| \frac{\partial^3 H}{\partial t^3} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2, \\ \sum_{n=0}^{N-1} \|\nabla r_p^n\|_{\varepsilon^{-1}}^2 &\leq \frac{(\Delta t)^3}{48} \left\| \frac{\partial^2 (\nabla p)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2, \\ \sum_{n=0}^{N-1} \|\nabla \cdot r_E^n\|_{\varepsilon}^2 &\leq \frac{(\Delta t)^3}{48} \left\| \frac{\partial^2 (\nabla \cdot E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \\ \sum_{n=0}^{N-1} \|\nabla \times r_E^n\|_{\varepsilon}^2 &\leq \frac{(\Delta t)^3}{48} \left\| \frac{\partial^2 (\nabla \times E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \\ \sum_{n=0}^{N-1} \|\nabla \times r_H^n\|_{\mu}^2 &\leq \frac{(\Delta t)^3}{48} \left\| \frac{\partial^2 (\nabla \times H)}{\partial t^2} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2. \end{aligned}$$

As a result, we can finally obtain:

$$\begin{aligned}
& \|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^N\|_{\mu}^2 \\
& \leq \left[\frac{3}{50} (\Delta t)^4 \left(\left\| \frac{\partial^3 p}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^3 E}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial^3 H}{\partial t^3} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right) \right. \\
& + \frac{1}{40} (\Delta t)^4 \left(\left\| \frac{\partial^2 (\nabla p)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^2 (\nabla \cdot E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^2 (\nabla \times E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 \\
& \quad \left. + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^2 (\nabla \times H)}{\partial t^2} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right) + \frac{7}{5} (\|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^0\|_{\varepsilon}^2 + \|e_H^0\|_{\mu}^2) \Big] \exp(4T).
\end{aligned}$$

Finally, using the regularity assumptions for p , E and H , and 1- and 2-norm equivalence, we obtain the required result:

$$\|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon} + \|e_H^N\|_{\mu} \leq C [(\Delta t)^2 + \|e_p^0\|_{\varepsilon^{-1}} + \|e_E^0\|_{\varepsilon} + \|e_H^0\|_{\mu}]. \quad \square$$

3.4 Error Estimates for Full Discretization

We now present the error analysis for the full discretization of the Maxwell's equations using finite elements and with our two time integration methods.

3.4.1 Crank-Nicolson Scheme

For the Crank-Nicolson scheme as in Equations (3.3a) to (3.3c), using a de Rham sequence of finite dimensional subspaces of the corresponding function spaces for the spatial discretization of (p^n, E^n, H^n) , we obtain the following discrete problem: find $(p_h^n, E_h^n, H_h^n) \in U_h \times V_h \times W_h \subseteq \dot{H}_{\varepsilon^{-1}}^1(\Omega) \times \dot{H}_{\varepsilon}(\text{curl}; \Omega) \times \dot{H}_{\mu}(\text{div}; \Omega)$ such that:

$$\left\langle \frac{p_h^n - p_h^{n-1}}{\Delta t}, \tilde{p} \right\rangle - \left\langle \frac{\varepsilon}{2} (E_h^n + E_h^{n-1}), \nabla \tilde{p} \right\rangle = \left\langle \frac{1}{2} (f_p^n + f_p^{n-1}), \tilde{p} \right\rangle, \quad (3.19a)$$

$$\left\langle \frac{1}{2} \nabla (p_h^n + p_h^{n-1}), \tilde{E} \right\rangle + \left\langle \varepsilon \frac{E_h^n - E_h^{n-1}}{\Delta t}, \tilde{E} \right\rangle - \left\langle \frac{1}{2} (H_h^n + H_h^{n-1}), \nabla \times \tilde{E} \right\rangle = \left\langle \frac{1}{2} (f_E^n + f_E^{n-1}), \tilde{E} \right\rangle, \quad (3.19b)$$

$$\left\langle \frac{1}{2} \nabla \times (E_h^n + E_h^{n-1}), \tilde{H} \right\rangle + \left\langle \mu \frac{H_h^n - H_h^{n-1}}{\Delta t}, \tilde{H} \right\rangle = \left\langle \frac{1}{2} (f_H^n + f_H^{n-1}), \tilde{H} \right\rangle. \quad (3.19c)$$

for all $(\tilde{p}, \tilde{E}, \tilde{H}) \in U_h \times V_h \times W_h$, and for $n = 1, \dots, N$ given $(p_h^0, E_h^0, H_h^0) \in U_h \times V_h \times W_h$. Let $\Pi_h^0 : \dot{H}_{\varepsilon^{-1}}^1(\Omega) \rightarrow U_h$, $\Pi_h^1 : \dot{H}_{\varepsilon}(\text{curl}; \Omega) \rightarrow V_h$ and $\Pi_h^2 : \dot{H}_{\mu}(\text{div}; \Omega) \rightarrow W_h$ denote the respective smoothed L^2 projection operators as in the sense of [Sch08; Chr07] and a detailed discussion is available in [AFW06]. We wish to note that in the classical setup, the approximation results similar in flavour to the smoothed projection operator in case of scalar functions would be the Bramble-Hilbert lemma as in, for example [BS08, Lemma 4.3.8], for vector fields in $H(\text{curl})$ as in [EG21a, Theorem 16.10], and for $H(\text{div})$ vector fields as in [EG21a, Theorem 17.5]. However, these results do not apply in our case either because they require stronger conditions and then provide a higher approximation order than in the case of smoothed projection in Bramble-Hilbert or a weaker rate of convergence for $H(\text{div})$ fields. Finally, for the initial conditions, we simply set:

$$p_h^0 := \Pi_h^0 p_0, \quad E_h^0 := \Pi_h^1 E_0, \quad \text{and} \quad H_h^0 := \Pi_h^2 H_0, \quad (3.19d)$$

for purposes of our analysis but computational obtain these using the L^2 projections of the analytical solutions. Now, we define the error for p , E and H at time $n\Delta t$ under the full discretization as:

$$e_{p_h}^n := p(t^n) - p_h^n = \eta^n - \eta_h^n, \quad (3.20)$$

$$e_{E_h}^n := E(t^n) - E_h^n = \zeta^n - \zeta_h^n, \quad (3.21)$$

$$e_{H_h}^n := H(t^n) - H_h^n = \xi^n - \xi_h^n, \quad (3.22)$$

and in which we have the following definitions for the newly introduced terms:

$$\eta^n := p(t^n) - \Pi_h^0 p(t^n), \quad \eta_h^n := p_h^n - \Pi_h^0 p_h^n, \quad (3.23)$$

$$\zeta^n := E(t^n) - \Pi_h^1 E(t^n), \quad \zeta_h^n := E_h^n - \Pi_h^1 E_h^n, \quad (3.24)$$

$$\xi^n := H(t^n) - \Pi_h^2 H(t^n), \quad \xi_h^n := H_h^n - \Pi_h^2 H_h^n. \quad (3.25)$$

With this, we can now state and prove our theorem for convergence of errors in the full discretization of the system of Maxwell's equations using the Crank-Nicolson scheme.

Theorem 3.13 (Full Error Estimate). *Let $p \in C^3(0, T; \dot{H}_{\varepsilon^{-1}}^1(\Omega))$, $E \in C^3(0, T; \dot{H}_\varepsilon(\text{curl}; \Omega))$, and $H \in C^3(0, T; \dot{H}_\mu(\text{div}; \Omega))$ be the solution to the variational formulation of the Maxwell's equations as in Equations (3.2a) to (3.2c), and let (p_h^n, E_h^n, H_h^n) be the solution of the fully discretized Maxwell's equations using the Crank-Nicolson scheme as in Equations (3.19a) to (3.19c). If the time step $\Delta t > 0$ and mesh parameter $h > 0$ are sufficiently small, then there exists a positive bounded constant C independent of both Δt and h such that the following error estimate holds:*

$$\|e_{p_h}^N\|_{\varepsilon^{-1}} + \|e_{E_h}^N\|_\varepsilon + \|e_{H_h}^N\|_\mu \leq C [(\Delta t)^2 + h^r + h^r(\Delta t)^2],$$

where the finite element subspaces U_h , V_h and W_h are each spanned by their respective Whitney form basis of polynomial order $r \geq 1$.

Remark 3.14 (Sufficient Regularity). *The approximation error bounds in [AFW06, Theorem 5.6] for the smoothed projection operators are provided for projections from spaces of smooth differential forms onto polynomial subspaces. Since we rely on these estimates for bounding the full error of our discrete solution, therefore we require sufficient regularity of the solution in the variational posing of the Maxwell system in Equations (3.2a) to (3.2c). The sufficiency will depend on the polynomial subspaces being used. For degree r polynomial subspaces, these can be taken to be $\dot{H}_{\varepsilon^{-1}}^1(\Omega) \cap H^r(\Omega)$ for the scalar field and $\dot{H}_\varepsilon(\text{curl}; \Omega) \cap [H^r(\Omega)]^d$ or $\dot{H}_\mu(\text{div}; \Omega) \cap [H^r(\Omega)]^d$ for the vector fields where $d = 2, 3$ as the case may be, and $[H^r(\Omega)]^d$ denotes component wise $H^r(\Omega)$ vector fields.*

Proof. First, we shall subtract the set of equations for the full discretization as in Equations (3.19a) to (3.19c) from Equations (3.9a) to (3.9c), and then use the error terms in Equations (3.20) to (3.22) and thereby obtain:

$$\begin{aligned} & \left\langle \frac{e_{p_h}^n - e_{p_h}^{n-1}}{\Delta t}, \tilde{p} \right\rangle - \frac{1}{2} \left\langle \varepsilon \left(e_{E_h}^n + e_{E_h}^{n-1} \right), \nabla \tilde{p} \right\rangle = \frac{1}{2} \left\langle R_p^n + \varepsilon \nabla \cdot r_E^n, \tilde{p} \right\rangle, \\ & \frac{1}{2} \left\langle \nabla \left(e_{p_h}^n + \nabla e_{p_h}^{n-1} \right), \tilde{E} \right\rangle + \left\langle \varepsilon \frac{e_{E_h}^n - e_{E_h}^{n-1}}{\Delta t}, \tilde{E} \right\rangle - \frac{1}{2} \left\langle e_{H_h}^n + e_{H_h}^{n-1}, \nabla \times \tilde{E} \right\rangle = \frac{1}{2} \left\langle \varepsilon R_E^n + \nabla r_p^n - \nabla \times r_H^n, \tilde{E} \right\rangle, \\ & \frac{1}{2} \left\langle \nabla \times \left(e_{E_h}^n + e_{E_h}^{n-1} \right), \tilde{H} \right\rangle + \left\langle \mu \frac{e_{H_h}^n - e_{H_h}^{n-1}}{\Delta t}, \tilde{H} \right\rangle = \frac{1}{2} \left\langle \mu R_H^n + \nabla \times r_E^n, \tilde{H} \right\rangle. \end{aligned}$$

Next, using the values of the error terms $e_{p_h}^n$, $e_{E_h}^n$ and $e_{H_h}^n$ as in Equations (3.20) to (3.22) in the above equations, we get:

$$\begin{aligned} & \left\langle \frac{(\eta^n - \eta^{n-1}) - (\eta_h^n - \eta_h^{n-1})}{\Delta t}, \tilde{p} \right\rangle - \frac{1}{2} \left\langle \varepsilon (\zeta^n + \zeta^{n-1}) - \varepsilon (\zeta_h^n + \zeta_h^{n-1}), \nabla \tilde{p} \right\rangle = \frac{1}{2} \langle R_p^n + \varepsilon \nabla \cdot r_E^n, \tilde{p} \rangle, \\ & \frac{1}{2} \left\langle \nabla (\eta^n + \eta^{n-1}) - \nabla (\eta_h^n + \eta_h^{n-1}), \tilde{E} \right\rangle + \left\langle \varepsilon \frac{(\zeta^n - \zeta^{n-1}) - (\zeta_h^n - \zeta_h^{n-1})}{\Delta t}, \tilde{E} \right\rangle \\ & \quad - \frac{1}{2} \left\langle (\xi^n + \xi^{n-1}) - (\xi_h^n + \xi_h^{n-1}), \nabla \times \tilde{E} \right\rangle = \frac{1}{2} \langle \varepsilon R_E^n + \nabla r_p^n - \nabla \times r_H^n, \tilde{E} \rangle, \\ & \left\langle \mu \frac{(\xi^n - \xi^{n-1}) - (\xi_h^n - \xi_h^{n-1})}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \left\langle \nabla \times (\zeta^n + \zeta^{n-1}) - \nabla \times (\zeta_h^n + \zeta_h^{n-1}), \tilde{H} \right\rangle = \frac{1}{2} \langle \mu R_H^n + \nabla \times r_E^n, \tilde{H} \rangle. \end{aligned}$$

Since these equations are true for all $(\tilde{p}, \tilde{E}, \tilde{H}) \in U_h \times V_h \times W_h$, we choose $\tilde{p} = -2\Delta t \varepsilon^{-1} (\eta_h^n + \eta_h^{n-1})$, $\tilde{E} = -2\Delta t (\zeta_h^n + \zeta_h^{n-1})$ and $\tilde{H} = -2\Delta t (\xi_h^n + \xi_h^{n-1})$ and using the fact that $\nabla U_h \subseteq V_h$ and $\nabla \times V_h \subseteq W_h$, we obtain:

$$\begin{aligned} & 2 \langle \varepsilon^{-1} (\eta_h^n - \eta_h^{n-1}), \eta_h^n + \eta_h^{n-1} \rangle + 2 \langle \varepsilon (\zeta_h^n - \zeta_h^{n-1}), \zeta_h^n + \zeta_h^{n-1} \rangle + 2 \langle \mu (\xi_h^n - \xi_h^{n-1}), \xi_h^n + \xi_h^{n-1} \rangle \\ & = 2 \langle \varepsilon^{-1} (\eta^n - \eta^{n-1}), \eta_h^n + \eta_h^{n-1} \rangle + 2 \langle \varepsilon (\zeta^n - \zeta^{n-1}), \zeta_h^n + \zeta_h^{n-1} \rangle + 2 \langle \mu (\xi^n - \xi^{n-1}), \xi_h^n + \xi_h^{n-1} \rangle \\ & + \Delta t \langle -R_p^n - \varepsilon \nabla \cdot r_E^n, \varepsilon^{-1} (\eta_h^n + \eta_h^{n-1}) \rangle + \Delta t \langle -\varepsilon R_E^n - \nabla r_p^n + \nabla \times r_H^n, \zeta_h^n + \zeta_h^{n-1} \rangle + \Delta t \langle -\mu R_H^n - \nabla \times r_E^n, \xi_h^n + \xi_h^{n-1} \rangle. \end{aligned} \quad (3.26)$$

Consider that $\varepsilon^{-1} (\eta^n - \eta^{n-1}) = \varepsilon^{-1} (I - \Pi_h^0) (p(t^n) - p(t^{n-1}))$ by Equation (3.23). Using the Taylor theorem with remainder as in Theorem 3.8, applying the Cauchy-Schwarz, AM-GM, and Triangle inequalities, we have the following resulting inequality:

$$\begin{aligned} \langle \varepsilon^{-1} (\eta^n - \eta^{n-1}), \eta_h^n + \eta_h^{n-1} \rangle & = \Delta t \langle \varepsilon^{-1} (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^{n-\frac{1}{2}}), \eta_h^n + \eta_h^{n-1} \rangle \\ & \quad + \frac{\Delta t}{2} \langle \varepsilon^{-1} (I - \Pi_h^0) R_p^n, \eta_h^n + \eta_h^{n-1} \rangle, \\ & \leq \frac{\Delta t}{2} \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 \right] + 2\Delta t \left[\|\eta_h^n\|_{\varepsilon^{-1}}^2 + \|\eta_h^{n-1}\|_{\varepsilon^{-1}}^2 \right]. \end{aligned}$$

Similarly, using Equations (3.24) and (3.25) for the error terms for E and H, we obtain:

$$\begin{aligned} \langle \varepsilon (\zeta^n - \zeta^{n-1}), \zeta_h^n + \zeta_h^{n-1} \rangle & \leq \frac{\Delta t}{2} \left[\left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 \right] + 2\Delta t \left[\|\zeta_h^n\|_{\varepsilon}^2 + \|\zeta_h^{n-1}\|_{\varepsilon}^2 \right], \\ \langle \mu (\xi^n - \xi^{n-1}), \xi_h^n + \xi_h^{n-1} \rangle & \leq \frac{\Delta t}{2} \left[\left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\mu}^2 + \left\| (I - \Pi_h^2) R_H^n \right\|_{\mu}^2 \right] + 2\Delta t \left[\|\xi_h^n\|_{\mu}^2 + \|\xi_h^{n-1}\|_{\mu}^2 \right]. \end{aligned}$$

Using these inequalities for η , ζ and ξ in Equation (3.26), we thus obtain the following estimate:

$$\begin{aligned} & 2 \left[\|\eta_h^n\|_{\varepsilon^{-1}}^2 - \|\eta_h^{n-1}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^n\|_{\varepsilon}^2 - \|\zeta_h^{n-1}\|_{\varepsilon}^2 + \|\xi_h^n\|_{\mu}^2 - \|\xi_h^{n-1}\|_{\mu}^2 \right] \\ & \leq \Delta t \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 \right. \\ & \quad \left. + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\mu}^2 + \left\| (I - \Pi_h^2) R_H^n \right\|_{\mu}^2 \right]. \end{aligned}$$

$$\begin{aligned}
& + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t} (t^{n-\frac{1}{2}}) \right\|_\mu^2 + \left\| (I - \Pi_h^2) R_H^n \right\|_\mu^2 + \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_\varepsilon^2 + \|R_H^n\|_\mu^2 + \|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_\varepsilon^2 \\
& + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_E^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_H^n\|_\mu^2 + 3(\|\eta_h^n\|_{\varepsilon^{-1}}^2 + \|\eta_h^{n-1}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^n\|_\varepsilon^2 + \|\zeta_h^{n-1}\|_\varepsilon^2 + \|\xi_h^n\|_\mu^2 + \|\xi_h^{n-1}\|_\mu^2) \Big].
\end{aligned}$$

Summing from $n = 1$ to N , we get:

$$\begin{aligned}
\|\eta_h^N\|_{\varepsilon^{-1}}^2 + \|\zeta_h^N\|_\varepsilon^2 + \|\xi_h^N\|_\mu^2 & \leq 3\Delta t \sum_{n=0}^N \left[\|\eta_h^n\|_{\varepsilon^{-1}}^2 + \|\zeta_h^n\|_\varepsilon^2 + \|\xi_h^n\|_\mu^2 \right] + \Delta t \sum_{n=0}^N \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\varepsilon^{-1}}^2 \right. \\
& + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t} (t^{n-\frac{1}{2}}) \right\|_\varepsilon^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_\varepsilon^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t} (t^{n-\frac{1}{2}}) \right\|_\mu^2 \\
& + \left\| (I - \Pi_h^2) R_H^n \right\|_\mu^2 + \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_\varepsilon^2 + \|R_H^n\|_\mu^2 + \|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_E^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_H^n\|_\mu^2 \Big] \\
& + \left[\|\eta_h^0\|_{\varepsilon^{-1}}^2 + \|\zeta_h^0\|_\varepsilon^2 + \|\xi_h^0\|_\mu^2 \right].
\end{aligned}$$

We next apply the discrete Gronwall inequality by setting $\delta := \Delta t$, $g_0 := \|\eta_h^0\|_{\varepsilon^{-1}}^2 + \|\zeta_h^0\|_\varepsilon^2 + \|\xi_h^0\|_\mu^2$, $a_n := \|\eta_h^n\|_{\varepsilon^{-1}}^2 + \|\zeta_h^n\|_\varepsilon^2 + \|\xi_h^n\|_\mu^2$, $b_n := 0$, $c_n := \left\| (I - \Pi_h^0) \frac{\partial p}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t} (t^{n-\frac{1}{2}}) \right\|_\varepsilon^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_\varepsilon^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t} (t^{n-\frac{1}{2}}) \right\|_\mu^2 + \left\| (I - \Pi_h^2) R_H^n \right\|_\mu^2 + \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_\varepsilon^2 + \|R_H^n\|_\mu^2 + \|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_E^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_H^n\|_\mu^2$, and $\gamma_n := 3$, and with $\sigma_n = (1 - 3\Delta t)^{-1}$. Note that for the condition $\gamma_n \delta < 1$ to hold, we that $\Delta t < 1/3$, and using that $(N + 1)\Delta t = T + \Delta t \leq T + 1/6$ for $\Delta t < 1/6$, we get:

$$\begin{aligned}
\|\eta_h^N\|_{\varepsilon^{-1}}^2 + \|\zeta_h^N\|_\varepsilon^2 + \|\xi_h^N\|_\mu^2 & \leq \left[\Delta t \sum_{n=0}^N \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t} (t^{n-\frac{1}{2}}) \right\|_\varepsilon^2 \right. \right. \\
& + \left\| (I - \Pi_h^1) R_E^n \right\|_\varepsilon^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t} (t^{n-\frac{1}{2}}) \right\|_\mu^2 + \left\| (I - \Pi_h^2) R_H^n \right\|_\mu^2 + \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_\varepsilon^2 + \|R_H^n\|_\mu^2 + \|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_\varepsilon^2 \\
& + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_E^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_H^n\|_\mu^2 \Big] + \left[\|\eta_h^0\|_{\varepsilon^{-1}}^2 + \|\zeta_h^0\|_\varepsilon^2 + \|\xi_h^0\|_\mu^2 \right] \exp(6T + 1).
\end{aligned}$$

Using our estimates for the Taylor remainders from Equations (3.11) and (3.14) for these terms on the right hand side of the above inequality, we further get that:

$$\Delta t \sum_{n=0}^N \left[\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_\varepsilon^2 + \|R_H^n\|_\mu^2 \right] \leq \frac{(\Delta t)^4}{5} \left[\left\| \frac{\partial^3 p}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^3 E}{\partial t^3} \right\|_{L^2(0,T;L_\varepsilon^2(\Omega))}^2 + \left\| \frac{\partial^3 H}{\partial t^3} \right\|_{L^2(0,T;L_\mu^2(\Omega))}^2 \right],$$

and that:

$$\begin{aligned}
\Delta t \sum_{n=0}^N \left[\|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_E^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_H^n\|_\mu^2 \right] & \leq \frac{(\Delta t)^4}{12} \left[\left\| \frac{\partial^2 (\nabla p)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 \right. \\
& + \left\| \frac{\partial^2 (\nabla \cdot E)}{\partial t^2} \right\|_{L^2(0,T;L_\varepsilon^2(\Omega))}^2 + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^2 (\nabla \times E)}{\partial t^2} \right\|_{L^2(0,T;L_\varepsilon^2(\Omega))}^2 + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^2 (\nabla \times H)}{\partial t^2} \right\|_{L^2(0,T;L_\mu^2(\Omega))}^2 \Big].
\end{aligned}$$

Now, for $p \in \dot{H}_{\varepsilon^{-1}}^1(\Omega)$, $E \in \dot{H}_\varepsilon(\text{curl}; \Omega)$ and $H \in \dot{H}_\mu(\text{div}; \Omega)$, there exists positive bounded constants $C_{1,p}$, $C_{2,p}$, $C_{1,E}$, $C_{2,E}$, $C_{1,H}$, and $C_{2,H}$ such that we have the following error bounds for the L^2 projections:

$$\begin{aligned} \left\| (I - \Pi_h^0) \frac{\partial p}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\varepsilon^{-1}} &\leq C_{1,p} h^r \left\| \frac{\partial p}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\varepsilon^{-1}}, & \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}} &\leq C_{2,p} h^r \left\| R_p^n \right\|_{\varepsilon^{-1}}, \\ \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\varepsilon} &\leq C_{1,E} h^r \left\| \frac{\partial E}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\varepsilon}, & \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon} &\leq C_{2,E} h^r \left\| R_E^n \right\|_{\varepsilon}, \\ \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\mu} &\leq C_{1,H} h^r \left\| \frac{\partial H}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\mu}, & \left\| (I - \Pi_h^2) R_H^n \right\|_{\mu} &\leq C_{2,H} h^r \left\| R_H^n \right\|_{\mu}. \end{aligned}$$

Set $C_0 := \max\{C_{1,p}, C_{2,p}, C_{1,E}, C_{2,E}, C_{1,H}, C_{2,H}\}$. We therefore have that:

$$\begin{aligned} \Delta t \sum_{n=0}^N &\left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\mu}^2 \right], \\ &\leq C_0 h^{2r} \sum_{n=0}^N \Delta t \left[\left\| \frac{\partial p}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial E}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\varepsilon}^2 + \left\| \frac{\partial H}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\mu}^2 \right], \\ &\leq C_0 h^{2r} \int_0^T \left[\left\| \frac{\partial p}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial E}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\varepsilon}^2 + \left\| \frac{\partial H}{\partial t} (t^{n-\frac{1}{2}}) \right\|_{\mu}^2 \right] dt, \\ &= C_0 h^{2r} \left[\left\| \frac{\partial p}{\partial t} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial E}{\partial t} \right\|_{L^2[0,T] \times L_{\varepsilon}^2(\Omega)}^2 + \left\| \frac{\partial H}{\partial t} \right\|_{L^2[0,T] \times L_{\mu}^2(\Omega)}^2 \right], \end{aligned}$$

and likewise for the L^2 projection of the remainder terms:

$$\begin{aligned} \Delta t \sum_{n=0}^N &\left[\left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}} + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon} + \left\| (I - \Pi_h^2) R_H^n \right\|_{\mu} \right], \\ &\leq C_0 h^{2r} \sum_{n=0}^N \Delta t \left[\|R_p^n\|_{L_{\varepsilon^{-1}}^2(\Omega)} + \|R_E^n\|_{L_{\varepsilon}^2(\Omega)} + \|R_H^n\|_{L_{\mu}^2(\Omega)} \right], \\ &\leq C_0 h^{2r} \frac{(\Delta t)^4}{5} \left[\left\| \frac{\partial^3 p}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^3 E}{\partial t^3} \right\|_{L^2[0,T] \times L_{\varepsilon}^2(\Omega)}^2 + \left\| \frac{\partial^3 H}{\partial t^3} \right\|_{L^2[0,T] \times L_{\mu}^2(\Omega)}^2 \right]. \end{aligned}$$

Now, we take $p_h^0 = \Pi_h^0 p_0$, $E_h^0 = \Pi_h^1 E_0$, and $H_h^0 = \Pi_h^2 H_0$, and note that there exists positive bounded constants M_1 , M_2 and M_3 such that:

$$\begin{aligned} \left\| \frac{\partial p}{\partial t} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial E}{\partial t} \right\|_{L^2[0,T] \times L_{\varepsilon}^2(\Omega)}^2 + \left\| \frac{\partial H}{\partial t} \right\|_{L^2[0,T] \times L_{\mu}^2(\Omega)}^2 &\leq M_1, \\ \left\| \frac{\partial^3 p}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^3 E}{\partial t^3} \right\|_{L^2[0,T] \times L_{\varepsilon}^2(\Omega)}^2 + \left\| \frac{\partial^3 H}{\partial t^3} \right\|_{L^2[0,T] \times L_{\mu}^2(\Omega)}^2 &\leq M_2, \end{aligned}$$

$$\left\| \frac{\partial^2 (\nabla p)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^2 (\nabla \cdot E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 +$$

$$\varepsilon^{-1}\mu^{-1}\left\|\frac{\partial^2(\nabla \times E)}{\partial t^2}\right\|_{L^2(0,T;L_\varepsilon^2(\Omega))}^2 + \varepsilon^{-1}\mu^{-1}\left\|\frac{\partial^2(\nabla \times H)}{\partial t^2}\right\|_{L^2(0,T;L_\mu^2(\Omega))}^2 \leq M_3.$$

Consequently, we have the following estimate:

$$\|\eta_h^N\|_{\varepsilon^{-1}}^2 + \|\zeta_h^N\|_\varepsilon^2 + \|\xi_h^N\|_\mu^2 \leq \widetilde{C} [h^{2r} + h^{2r}(\Delta t)^4 + (\Delta t)^4],$$

where $\widetilde{C} = \max\{C_0M_1, C_0M_2/5, M_2/5, M_3/12\} \exp(6T + 1)$ which then using the equivalence between 1- and 2-norms gives us that:

$$\|\eta_h^N\|_{\varepsilon^{-1}} + \|\zeta_h^N\|_\varepsilon + \|\xi_h^N\|_\mu \leq C_1 [h^r + h^r(\Delta t)^2 + (\Delta t)^2],$$

where $C_1 = 2\sqrt{\widetilde{C}}$. Also, there are positive bounded constants $\widetilde{C}_2$, $\widetilde{C}_3$ and $\widetilde{C}_4$ such that:

$$\begin{aligned} \|\eta^N\|_{\varepsilon^{-1}} &= \|(I - \Pi_h^0)p(t^N)\|_{\varepsilon^{-1}} \leq \widetilde{C}_2 h^r \|p(t^N)\|_{L_{\varepsilon^{-1}}^2(\Omega)} \implies \|\eta^N\|_{\varepsilon^{-1}} \leq C_2 h^r, \\ \|\zeta^N\|_\varepsilon &= \|(I - \Pi_h^1)E(t^N)\|_\varepsilon \leq \widetilde{C}_3 h^r \|E(t^N)\|_{L_\varepsilon^2(\Omega)} \implies \|\zeta^N\|_\varepsilon \leq C_3 h^r, \\ \|\xi^N\|_\mu &= \|(I - \Pi_h^2)H(t^N)\|_\mu \leq \widetilde{C}_4 h^r \|H(t^N)\|_{L_\mu^2(\Omega)} \implies \|\xi^N\|_\mu \leq C_4 h^r, \end{aligned}$$

in which $C_2 = \widetilde{C}_2 \|p(t^N)\|_{L_{\varepsilon^{-1}}^2(\Omega)}$, $C_3 = \widetilde{C}_3 \|E(t^N)\|_{L_\varepsilon^2(\Omega)}$ and $C_4 = \widetilde{C}_4 \|H(t^N)\|_{L_\mu^2(\Omega)}$ are all bounded positive constants due to Theorem 3.5. Finally, this provides us with our desired result by choosing $C = C_1 + C_2 + C_3 + C_4$:

$$\begin{aligned} \|e_{p_h}^{N,h}\|_{\varepsilon^{-1}} + \|e_E^{N,h}\|_\varepsilon + \|e_H^{N,h}\|_\mu &= \|\eta^N - \eta_h^N\|_{\varepsilon^{-1}} + \|\zeta^N - \zeta_h^N\|_\varepsilon + \|\xi^N - \xi_h^N\|_\mu \\ &\leq C [(\Delta t)^2 + h^r + h^r(\Delta t)^2]. \end{aligned} \quad \square$$

3.4.2 Implicit Leapfrog Scheme

For the implicit leapfrog scheme as in Equations (3.4a) to (3.4c), using a de Rham sequence of finite dimensional subspaces of the corresponding function spaces for the spatial discretization of $(p^{n+\frac{1}{2}}, E^{n+\frac{1}{2}}, H^{n+1})$, we obtain the following discrete problem: find $(p_h^{n+\frac{1}{2}}, E_h^{n+\frac{1}{2}}, H_h^{n+1}) \in U_h \times V_h \times W_h \subseteq \dot{H}_{\varepsilon^{-1}}^1 \times \dot{H}_\varepsilon(\text{curl}; \Omega) \times \dot{H}_\mu(\text{div}; \Omega)$ such that:

$$\left\langle \frac{p_h^{n+\frac{1}{2}} - p_h^{n-\frac{1}{2}}}{\Delta t}, \widetilde{p} \right\rangle - \left\langle \frac{\varepsilon}{2} (E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}}), \nabla \widetilde{p} \right\rangle = \langle f_p^n, \widetilde{p} \rangle, \quad (3.27a)$$

$$\left\langle \frac{1}{2} \nabla (p_h^{n+\frac{1}{2}} + p_h^{n-\frac{1}{2}}), \widetilde{E} \right\rangle + \left\langle \varepsilon \frac{E_h^{n+\frac{1}{2}} - E_h^{n-\frac{1}{2}}}{\Delta t}, \widetilde{E} \right\rangle - \left\langle \frac{1}{2} (H_h^{n+1} + H_h^n), \nabla \times \widetilde{E} \right\rangle = \langle f_E^n, \widetilde{E} \rangle, \quad (3.27b)$$

$$\left\langle \mu \frac{H_h^{n+1} - H_h^n}{\Delta t}, \widetilde{H} \right\rangle + \left\langle \frac{1}{2} \nabla \times (E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}}), \widetilde{H} \right\rangle = \langle f_H^{n+\frac{1}{2}}, \widetilde{H} \rangle, \quad (3.27c)$$

for all $(\widetilde{p}, \widetilde{E}, \widetilde{H}) \in U_h \times V_h \times W_h$, and for $n \in \{1, \dots, N-1\}$. The $n=0$ bootstrapping as in Equations (3.5a) to (3.5c) leads to the discrete problem: find $(p_h^{\frac{1}{2}}, E_h^{\frac{1}{2}}, H_h^1) \in U_h \times V_h \times W_h \subseteq \dot{H}_{\varepsilon^{-1}}^1 \times \dot{H}_\varepsilon(\text{curl}; \Omega) \times \dot{H}_\mu(\text{div}; \Omega)$ such that:

$$\left\langle \frac{p_h^{\frac{1}{2}} - p_h^0}{\Delta t/2}, \widetilde{p} \right\rangle - \frac{1}{4} \left\langle \varepsilon (E_h^{\frac{1}{2}} + E_h^0), \nabla \widetilde{p} \right\rangle = \langle f_p^0, \widetilde{p} \rangle, \quad (3.28a)$$

$$\frac{1}{4} \langle \nabla (p_h^{\frac{1}{2}} + p_h^0), \tilde{E} \rangle + \langle \varepsilon \frac{E_h^{\frac{1}{2}} - E_h^0}{\Delta t/2}, \tilde{E} \rangle - \langle \frac{1}{2} (H_h^1 + H_h^0), \nabla \times \tilde{E} \rangle = \langle f_E^0, \tilde{E} \rangle, \quad (3.28b)$$

$$\langle \mu \frac{H_h^1 - H_h^0}{\Delta t}, \tilde{H} \rangle + \frac{1}{4} \langle \nabla \times (E_h^{\frac{1}{2}} + E_h^0), \tilde{H} \rangle = \langle f_H^{\frac{1}{2}}, \tilde{H} \rangle, \quad (3.28c)$$

for all $(\tilde{p}, \tilde{E}, \tilde{H}) \in U_h \times V_h \times W_h$ given $(p_h^0, E_h^0, H_h^0) \in U_h \times V_h \times W_h$.

Using the same projection operators Π_h^0 , Π_h^1 and Π_h^2 for p , E , and H , respectively as described in Section 3.4.1 we again set

$$p_h^0 := \Pi_h^0 p_0, E_h^0 := \Pi_h^1 E_0 \text{ and } H_h^0 := \Pi_h^2 H_0. \quad (3.28d)$$

Similarly, we define the errors for p , E and H here as follows:

$$e_{p_h}^{n+\frac{1}{2}} := p(t^{n+\frac{1}{2}}) - p_h^{n+\frac{1}{2}} = \eta^{n+\frac{1}{2}} - \eta_h^{n+\frac{1}{2}}, \quad (3.29)$$

$$e_{E_h}^{n+\frac{1}{2}} := E(t^{n+\frac{1}{2}}) - E_h^{n+\frac{1}{2}} = \zeta^{n+\frac{1}{2}} - \zeta_h^{n+\frac{1}{2}}, \quad (3.30)$$

$$e_{H_h}^n := H(t^n) - H_h^n = \xi^n - \xi_h^n, \quad (3.31)$$

and in which we have the following definitions for the newly introduced terms:

$$\eta^{n+\frac{1}{2}} := p(t^{n+\frac{1}{2}}) - \Pi_h^0 p(t^{n+\frac{1}{2}}), \quad \eta_h^{n+\frac{1}{2}} := p_h^{n+\frac{1}{2}} - \Pi_h^0 p(t^{n+\frac{1}{2}}), \quad (3.32)$$

$$\zeta^{n+\frac{1}{2}} := E(t^{n+\frac{1}{2}}) - \Pi_h^1 E(t^{n+\frac{1}{2}}), \quad \zeta_h^{n+\frac{1}{2}} := E_h^{n+\frac{1}{2}} - \Pi_h^1 E(t^{n+\frac{1}{2}}), \quad (3.33)$$

$$\xi^n := H(t^n) - \Pi_h^2 H(t^n), \quad \xi_h^n := H_h^n - \Pi_h^2 H(t^n). \quad (3.34)$$

Theorem 3.15 (Full Error Estimate). *Let $p \in C^3(0, T; \dot{H}_{\varepsilon^{-1}}^1(\Omega))$, $E \in C^3(0, T; \dot{H}_\varepsilon(\text{curl}; \Omega))$, and $H \in C^3(0, T; \dot{H}_\mu(\text{div}; \Omega))$ be the solution to the variational formulation of the Maxwell's equations as in Equations (3.2a) to (3.2c), and let $(p_h^{n+\frac{1}{2}}, E_h^{n+\frac{1}{2}}, H_h^{n+1})$ be the solution of the fully discretized Maxwell's equations using the implicit leapfrog scheme as in Equations (3.27a) to (3.27c) and (3.28a) to (3.28c). If the fixed time step $\Delta t > 0$ and the mesh parameter $h > 0$ are sufficiently small, then there exists a positive bounded constant C independent of both Δt and h such that the following error estimate holds:*

$$\|e_{p_h}^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|e_{E_h}^{N-\frac{1}{2}}\|_\varepsilon + \|e_{H_h}^N\|_\mu \leq C [(\Delta t)^2 + h^r + h^r(\Delta t)^2],$$

where the finite element subspaces U_h , V_h and W_h are each spanned by their respective Whitney form basis of polynomial order $r \geq 1$.

Proof. As with all our proofs for theorems corresponding to the implicit leapfrog scheme, our proof here will mimic that for Theorem 3.13 *mutatis mutandis*. Nevertheless, we shall describe it to some detail next.

We start from the variational formulation for the error terms η , ζ , ξ , and their discrete analogues:

$$\left\langle \frac{(\eta^{n+\frac{1}{2}} - \eta^{n-\frac{1}{2}}) - (\eta_h^{n+\frac{1}{2}} - \eta_h^{n-\frac{1}{2}})}{\Delta t}, \tilde{p} \right\rangle - \frac{1}{2} \langle \varepsilon ((\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}}) - (\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}})), \nabla \tilde{p} \rangle = \langle R_p^n + \varepsilon \nabla \cdot r_E^n, \tilde{p} \rangle,$$

$$\frac{1}{2} \langle \nabla ((\eta^{n+\frac{1}{2}} + \eta^{n-\frac{1}{2}}) - (\eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}})), \tilde{E} \rangle + \langle \varepsilon \frac{(\zeta^{n+\frac{1}{2}} - \zeta^{n-\frac{1}{2}}) - (\zeta_h^{n+\frac{1}{2}} - \zeta_h^{n-\frac{1}{2}})}{\Delta t}, \tilde{E} \rangle$$

$$\begin{aligned}
& -\frac{1}{2} \langle \left((\xi^{n+1} + \xi^n) - (\xi_h^{n+1} - \xi_h^n) \right), \nabla \times \tilde{E} \rangle = \langle \varepsilon R_E^n + \nabla r_p^n - r_H^{n+\frac{1}{2}}, \tilde{E} \rangle, \\
& \langle \mu \frac{(\xi^{n+1} - \xi^n) - (\xi_h^{n+1} - \xi_h^n)}{\Delta t}, \tilde{H} \rangle + \frac{1}{2} \langle \nabla \times \left((\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}}) - (\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}}) \right), \tilde{H} \rangle = \langle \mu R_H^{n+\frac{1}{2}} + \nabla \times r_E^n, \tilde{H} \rangle.
\end{aligned}$$

Now, using appropriate test functions similar to their choice as in the proof of Theorem 3.13, the exactness of the discrete de Rham sequence of finite element spaces, Equations (3.32) to (3.34) Taylor's theorem with remainder, Cauchy-Schwarz, AM-GM, and Triangle inequalities, we obtain the following estimates:

$$\begin{aligned}
2 \langle \varepsilon^{-1} (\eta^{n+\frac{1}{2}} - \eta^{n-\frac{1}{2}}), \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle & \leq \Delta t \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 \right] \\
& \quad + 4\Delta t \left[\|\eta_h^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\eta_h^{n-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 \right], \\
2 \langle \varepsilon (\zeta^{n+\frac{1}{2}} - \zeta^{n-\frac{1}{2}}), \zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \rangle & \leq \Delta t \left[\left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 \right] \\
& \quad + 4\Delta t \left[\|\zeta_h^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|\zeta_h^{n-\frac{1}{2}}\|_{\varepsilon}^2 \right], \\
2 \langle \mu (\xi^{n+1} - \xi^n), \xi_h^{n+1} + \xi_h^n \rangle & \leq \Delta t \left[\left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 + \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu}^2 \right] \\
& \quad + 4\Delta t \left[\|\xi_h^{n+1}\|_{\mu}^2 + \|\xi_h^n\|_{\mu}^2 \right].
\end{aligned}$$

Similarly, we will have an estimate for the errors for the initial $p^{\frac{1}{2}}$, $E^{\frac{1}{2}}$ and H^1 . Combining these estimates and invoking the necessary arguments as in the proof of Theorem 3.13, we arrive at the following inequality:

$$\begin{aligned}
& \|\eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^N\|_{\mu}^2 \leq \frac{1+3\Delta t}{1-3\Delta t} \left[\|\eta_h^0\|_{\varepsilon^{-1}}^2 + \|\zeta_h^0\|_{\varepsilon}^2 + \|\xi_h^0\|_{\mu}^2 \right] \\
& + \frac{6\Delta t}{1-3\Delta t} \sum_{n=0}^{N-1} \left[\|\eta_h^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^{n+1}\|_{\mu}^2 \right] + \frac{\Delta t}{1-3\Delta t} \sum_{n=0}^{N-1} \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 \right. \\
& \quad + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 + \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu}^2 \\
& \quad \left. + \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 + \|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_E^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_H^{n+\frac{1}{2}}\|_{\mu}^2 \right].
\end{aligned}$$

Now, applying the discrete Gronwall inequality with $\Delta t < 1/18$, we get that:

$$\begin{aligned}
& \|\eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^N\|_{\mu}^2 \leq \left[\frac{6\Delta t}{5} \sum_{n=0}^{N-1} \left(\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 \right. \right. \\
& \quad \left. \left. + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 + \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu}^2 \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \|\mathbf{R}_p^n\|_{\varepsilon^{-1}}^2 + \|\mathbf{R}_E^n\|_{\varepsilon}^2 + \|\mathbf{R}_H^{n+\frac{1}{2}}\|_{\mu}^2 + \|\nabla \mathbf{r}_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \mathbf{r}_E^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times \mathbf{r}_E^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times \mathbf{r}_H^{n+\frac{1}{2}}\|_{\mu}^2 \\
& + \frac{7}{5} \left(\|\eta_h^0\|_{\varepsilon^{-1}}^2 + \|\zeta_h^0\|_{\varepsilon}^2 + \|\xi_h^0\|_{\mu}^2 \right) \Big] \exp(12T).
\end{aligned}$$

Using appropriate estimates for the right hand side terms in this equation as in the proof of Theorem 3.13, and by setting $p_h^0 := \Pi_h^0 p_0$, $E_h^0 := \Pi_h^1 E_0$, and $H_h^0 := \Pi_h^2 \hat{H}$, we finally obtain our required result:

$$\begin{aligned}
\|e_{p_h}^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|e_{E_h}^{N-\frac{1}{2}}\|_{\varepsilon} + \|e_{H_h}^N\|_{\mu} &= \|\eta^{N-\frac{1}{2}} - \eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|\zeta^{N-\frac{1}{2}} - \zeta_h^{N-\frac{1}{2}}\|_{\varepsilon} + \|\xi^N - \xi_h^N\|_{\mu}, \\
&\leq C \left[(\Delta t)^2 + h^r + h^r (\Delta t)^2 \right].
\end{aligned}$$

□

3.4.3 Nonhomogeneous Boundary Conditions

We wish to close our theoretical discussion by reiterating that we have shown all our stability and error estimates for the homogenous boundary conditions as in Equation (3.1b) for the system of Maxwell's equations in Equation (3.1a). However, standard arguments can be used if these boundary conditions are nonhomogeneous and all our proofs can be suitably updated to obtain essentially the same stability and error convergence results for the full discretization using both time integration schemes.

However, an important caveat needs to be stated with regard to Corollary 3.2 and, therefore, also Corollaries 3.6 and 3.10. A nonhomogeneous time varying boundary condition can also act as a source of energy and drive time evolution of the pressure, electric and magnetic fields even when all right hand side forcing functions are zero in Equation (3.1a). Thus, some additional technical qualifications are necessary to warrant energy conservation for such cases, as otherwise, the energy can be time-varying and periodic. We do not know what this precise technical quantification is and leave it to future work, and we do not need it here for our results.

3.4.4 Discussion

In summary, our main theorems in this chapter are Theorem 3.13 and Theorem 3.15. They each state that the Maxwell's equations posed variationally as in Equations (3.2a) to (3.2c) can be *correctly solved* by the Crank-Nicolson method or our newly defined implicit leapfrog method, respectively, in conjunction with mixed finite elements forming a de Rham sequence. They are correct in the sense that, as we have demonstrated, the L^2 error of the finite element solution computed using an appropriate choice of basis from the polynomial de Rham sequence with either of our considered time discretization schemes converges to zero in the limit of both the spatial mesh parameter and time step going to a zero limit. The two methods are both advantageous in that they are implicit and require no Courant–Friedrichs–Lewy (CFL) condition limiting the time step size in terms of the spatial mesh parameter for a given discretization of the domain Ω .

Our two main tools in obtaining these error estimates are the discrete Gronwall's inequality 1.10 for obtaining the temporal estimate and the smoothed projection operator providing the bounds for the spatial error [AFW10, Theorem 5.9]. The latter only provides L^2 error estimates for the projection assuming sufficient elliptic regularity. In this chapter, we have fixed our spatial choices to be vector calculus analogues of the $H\Lambda^k(\Omega)$ spaces of smooth differential k -forms resulting in our choices of $\hat{H}_{\varepsilon^{-1}}^1(\Omega)$, $\hat{H}_{\varepsilon}(\text{curl}, \Omega)$ and $\hat{H}_{\mu}(\text{div}, \Omega)$ with sufficient regularity. A crucial outcome of our analysis with the smoothed projection operators are that the L^2 errors in the solution only provide a convergence rate of r for the choice of degree r polynomial vector subspaces. A higher convergence order of $r + 1$ is in particular not available in the

approximation theory for the scalar functions. Our numerical evidence in the next section also corroborates this explanation.

Our main results are themselves built on the temporal discretization results in Section 3.3. Here too, we have been able to demonstrate that the L^2 error of the time discretized variational formulation of the Maxwell's equations is convergent under the assumption that the initial conditions have zero error in their spatial prescription. For us, this means that the initial conditions have the same elliptic regularity as the solutions, and it appears that our techniques do not allow us to provide a graph norm or energy norm estimate for these errors, for instance, as in [HR90; HS07] for a Crank-Nicolson finite element solution of the Navier-Stokes equation in two or three spatial dimensions. This is in contrast with the solution of the Poisson problem which does allow for such a ready generalization and one which is very well known [Arn18].

3.5 Numerical Results

We now present some empirical validation for our theoretical results for two model problems each on the unit square in $\mathbb{R}^2$ in Examples 1 and 2, and on the unit cube $\mathbb{R}^3$ in Examples 3 and 4. Both our problem domains are discretely realized as simplicial meshes. Our unit square has 1903 vertices and 3600 triangles while the unit cube has 215 vertices and 560 tetrahedra. We performed all our computational experiments with linear and quadratic finite element spaces of Whitney forms but only provide plots of solutions for the quadratic case in $\mathbb{R}^2$ and for both choices in $\mathbb{R}^3$. For all our example problems, we provide the analytical solutions for p , E and H , the physical parameters ϵ and μ , and the initial and final times $T_{\min}$ and $T_{\max}$, respectively. For all these problems, $\Delta t = 0.01$. All our boundary conditions are essentially applied in our computations and the initial conditions are discretely obtained through L^2 projections of the analytical solutions at $t = 0$. All integrals in our computations are obtained using sufficiently high degree quadrature, and all linear systems are solved with a sparse direct solver. Our experiments are performed using code written in Python with the standard scientific Python stack consisting of NumPy, SciPy and Matplotlib, and also in part thanks to some utility support using the library PyDEC [BH12].

All our problem regions are simply connected domains, that is, with trivial relative homologies in all dimensions. Our successive examples are, in spirit, an attempt to provide an “anti-ablation”-like study, that is, we change only one aspect of the problem while holding others fixed. We have performed computations with other general analytical functions in two and three dimensions as well and on domains with nontrivial 1- or 1- and 2-homologies as the case may be in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, but we do not present any of those here. We believe that our presented examples suffice for our purposes of illustration of feasibility and validation of our theory.

Our last problems in Example 3.5 and Example 3.6 is a pointer towards a slight generalization of the system of Maxwell's equations as in Equations (3.1a) to (3.1c) the underpinnings and analysis of which we have not addressed here.

Example 3.1: This problem consists of Maxwell's equations posed on a unit square in $\mathbb{R}^2$ with analytical solutions, material parameters, initial and final times as shown below:

$$p = 0, \quad E = \begin{bmatrix} \sin \pi y \cos \pi t \\ \sin \pi x \cos \pi t \end{bmatrix}, \quad H = (\cos \pi y - \cos \pi x) \sin \pi t,$$

$$\epsilon = 1, \quad \mu = 1, \quad T_{\min} = 0, \quad T_{\max} = 2.$$

The boundary conditions for this problem are homogeneous for p and E . H here also has a zero boundary condition since the analytical functions shown here are all technically the vector calculus realizations of differential k -forms in $\mathbb{R}^2$. Therefore, for the 2-form H , in particular, the boundary condition is the pullback

of this 2-form under the inclusion of the boundary into the domain and is thus vacuously zero. The results of computations performed using quadratic Whitney elements as bases for each of the finite element spaces and with the two time discretizations are shown in Figures 3.1 and 3.2 and Tables 3.1 and 3.2. We note that the homogenous boundary conditions are all imposed essentially by incorporating them into the finite element spaces. We also performed computations using the backward Euler time discretization but only provide a plot of its energy as in Figure 3.9.

Example 3.2: This problem is posed in $\mathbb{R}^3$ on a unit cube and analytical solutions, material parameters, initial and final times as below:

$$p = 0, \quad E = \begin{bmatrix} \sin \pi y \sin \pi z \cos \pi t \\ \sin \pi x \sin \pi z \cos \pi t \\ \sin \pi x \sin \pi y \cos \pi t \end{bmatrix}, \quad H = \begin{bmatrix} \sin \pi x (\cos \pi z - \cos \pi y) \sin \pi t \\ \sin \pi y (\cos \pi x - \cos \pi z) \sin \pi t \\ \sin \pi z (\cos \pi y - \cos \pi x) \sin \pi t \end{bmatrix},$$

$$\epsilon = 2, \quad \mu = 1, \quad T_{\min} = 0, \quad T_{\max} = 2.$$

Like in Example 3.1, the boundary conditions are homogeneous for p , E and H and are essentially imposed. The results of computations performed using linear and quadratic Whitney elements and with our two time discretizations are shown in Figures 3.3 and 3.4 and Section 3.5. The energy computations are summarized in Figure 3.10.

Example 3.3: The problem here consists of nonhomogeneous boundary conditions for p and E , and homogeneous for H with analytical solutions, material parameters, initial and final times as shown below:

$$p = (\cos \pi x + \cos \pi y) \sin \pi t,$$

$$E = \begin{bmatrix} \sin \pi(\sqrt{2}t - x - y) - \sin \pi x \cos \pi t \\ -\sin \pi(\sqrt{2}t - x - y) - \sin \pi y \cos \pi t \end{bmatrix}, \quad H = -\sqrt{2} \sin \pi(\sqrt{2}t - x - y),$$

$$\epsilon = 1, \quad \mu = 1, \quad T_{\min} = 0, \quad T_{\max} = 2.$$

All boundary conditions for this finite element discretization are again imposed in an essential manner by incorporating them into the appropriate function spaces. The results of computations performed with quadratic Whitney elements are shown in Figures 3.5 and 3.6 and Section 3.5, and the energies are visually summarized in Figure 3.9.

		E_h (Linear)				H_h (Constant)		
		$\ E - E_h\ _\varepsilon, E_h \in P_1^- \cap \mathring{H}_\varepsilon(\text{div})$				$\ H - H_h\ _\mu, H_h \in P_1^- \cap L_\mu^2$		
		t	h	$\frac{h}{2}$	$\frac{h}{4}$	h	$\frac{h}{2}$	$\frac{h}{4}$
Δt	0.24	3.08 $\times 10^{-2}$	1.54 $\times 10^{-2}$	7.68 $\times 10^{-3}$		2.83 $\times 10^{-2}$	1.41 $\times 10^{-2}$	7.10 $\times 10^{-3}$
	0.48	8.80 $\times 10^{-3}$	4.88 $\times 10^{-3}$	3.10 $\times 10^{-3}$		4.04 $\times 10^{-2}$	2.01 $\times 10^{-2}$	1.00 $\times 10^{-2}$
	0.84	3.65 $\times 10^{-2}$	1.82 $\times 10^{-2}$	9.21 $\times 10^{-3}$		2.10 $\times 10^{-2}$	1.09 $\times 10^{-2}$	6.10 $\times 10^{-3}$
	1	4.12 $\times 10^{-2}$	2.05 $\times 10^{-2}$	1.02 $\times 10^{-2}$		9.92 $\times 10^{-3}$	6.24 $\times 10^{-3}$	4.88 $\times 10^{-3}$
$\frac{\Delta t}{2}$	0.24	3.03 $\times 10^{-2}$	1.50 $\times 10^{-2}$	7.48 $\times 10^{-3}$		2.80 $\times 10^{-2}$	1.39 $\times 10^{-2}$	6.91 $\times 10^{-3}$
	0.48	7.31 $\times 10^{-3}$	3.55 $\times 10^{-3}$	1.75 $\times 10^{-3}$		4.01 $\times 10^{-2}$	1.99 $\times 10^{-2}$	9.93 $\times 10^{-3}$
	0.84	3.60 $\times 10^{-2}$	1.80 $\times 10^{-2}$	8.96 $\times 10^{-3}$		2.03 $\times 10^{-2}$	1.01 $\times 10^{-2}$	5.04 $\times 10^{-3}$
	1	4.12 $\times 10^{-2}$	2.04 $\times 10^{-2}$	1.02 $\times 10^{-2}$		7.32 $\times 10^{-3}$	3.51 $\times 10^{-3}$	1.95 $\times 10^{-3}$
$\frac{\Delta t}{4}$	0.24	3.02 $\times 10^{-2}$	1.49 $\times 10^{-2}$	7.44 $\times 10^{-3}$		2.79 $\times 10^{-2}$	1.38 $\times 10^{-2}$	6.84 $\times 10^{-3}$
	0.48	6.63 $\times 10^{-3}$	3.03 $\times 10^{-3}$	1.36 $\times 10^{-3}$		4.01 $\times 10^{-2}$	1.99 $\times 10^{-2}$	1.99 $\times 10^{-2}$
	0.84	3.61 $\times 10^{-2}$	1.79 $\times 10^{-2}$	8.91 $\times 10^{-3}$		2.01 $\times 10^{-2}$	9.88 $\times 10^{-3}$	4.89 $\times 10^{-3}$
	1	4.10 $\times 10^{-2}$	2.04 $\times 10^{-2}$	1.01 $\times 10^{-2}$		6.65 $\times 10^{-3}$	2.75 $\times 10^{-3}$	1.26 $\times 10^{-3}$
		E_h (Quadratic)				H_h (Linear)		
		$\ E - E_h\ _\varepsilon, E_h \in P_2^- \cap \mathring{H}_\varepsilon(\text{div})$				$\ H - H_h\ _\mu, H_h \in P_2^- \cap L_\mu^2$		
		t	h	$\frac{h}{2}$	$\frac{h}{4}$	h	$\frac{h}{2}$	$\frac{h}{4}$
Δt	0.24	9.91 $\times 10^{-4}$	6.97 $\times 10^{-4}$	6.78 $\times 10^{-4}$		8.56 $\times 10^{-4}$	7.40 $\times 10^{-4}$	7.23 $\times 10^{-4}$
	0.48	9.55 $\times 10^{-4}$	1.98 $\times 10^{-3}$	1.98 $\times 10^{-3}$		1.99 $\times 10^{-3}$	2.68 $\times 10^{-4}$	1.39 $\times 10^{-4}$
	0.84	3.07 $\times 10^{-3}$	1.68 $\times 10^{-3}$	1.67 $\times 10^{-3}$		1.80 $\times 10^{-3}$	3.04 $\times 10^{-3}$	3.03 $\times 10^{-3}$
	1	4.13 $\times 10^{-3}$	1.83 $\times 10^{-4}$	4.19 $\times 10^{-5}$		7.14 $\times 10^{-4}$	4.12 $\times 10^{-3}$	4.12 $\times 10^{-3}$
$\frac{\Delta t}{2}$	0.24	5.80 $\times 10^{-4}$	2.24 $\times 10^{-4}$	1.73 $\times 10^{-4}$		6.98 $\times 10^{-4}$	2.45 $\times 10^{-4}$	1.86 $\times 10^{-4}$
	0.48	5.47 $\times 10^{-4}$	4.98 $\times 10^{-4}$	4.95 $\times 10^{-4}$		9.50 $\times 10^{-4}$	2.37 $\times 10^{-4}$	6.63 $\times 10^{-5}$
	0.84	7.70 $\times 10^{-4}$	4.47 $\times 10^{-4}$	4.20 $\times 10^{-4}$		9.10 $\times 10^{-4}$	7.70 $\times 10^{-4}$	7.61 $\times 10^{-4}$
	1	7.20 $\times 10^{-4}$	1.79 $\times 10^{-4}$	4.51 $\times 10^{-5}$		1.06 $\times 10^{-3}$	1.03 $\times 10^{-3}$	1.03 $\times 10^{-3}$
$\frac{\Delta t}{4}$	0.24	5.36 $\times 10^{-4}$	1.46 $\times 10^{-4}$	5.44 $\times 10^{-5}$		6.75 $\times 10^{-4}$	1.73 $\times 10^{-4}$	6.14 $\times 10^{-5}$
	0.48	2.63 $\times 10^{-4}$	1.37 $\times 10^{-4}$	1.25 $\times 10^{-4}$		9.50 $\times 10^{-4}$	2.35 $\times 10^{-4}$	5.90 $\times 10^{-5}$
	0.84	6.45 $\times 10^{-4}$	1.91 $\times 10^{-4}$	1.12 $\times 10^{-4}$		5.35 $\times 10^{-4}$	2.27 $\times 10^{-4}$	1.93 $\times 10^{-4}$
	1	7.25 $\times 10^{-4}$	1.80 $\times 10^{-4}$	4.51 $\times 10^{-5}$		3.44 $\times 10^{-4}$	2.65 $\times 10^{-4}$	2.59 $\times 10^{-4}$

TABLE 3.1: The L^2 errors in the solution for the problem in **Example 3.1** using the Crank Nicolson time discretization for linear and quadratic spatial discretizations of E and H . Note that in the mixed formulation of this Maxwell's system, the choice of approximating E by piecewise linear (respectively, quadratic) polynomial vector field basis in $H(\text{curl})$ requires that H be then approximated using piecewise constant (linear) discontinuous Galerkin basis in L^2 in the two dimensional case for stability .

		E_h (Linear)				H_h (Constant)			
		$\ E - E_h\ _\varepsilon, E_h \in P_1^- \cap \dot{H}_\varepsilon(\text{div})$				$\ H - H_h\ _\mu, H_h \in P_1^- \cap L_\mu^2$			
		t	h	$\frac{h}{2}$	$\frac{h}{4}$	t	h	$\frac{h}{2}$	$\frac{h}{4}$
Δt	0.22		3.22×10^{-2}	1.60×10^{-2}	7.96×10^{-3}	0.24	5.58×10^{-2}	5.00×10^{-2}	4.84×10^{-2}
	0.46		8.76×10^{-3}	4.57×10^{-3}	2.81×10^{-3}	0.48	4.06×10^{-2}	2.09×10^{-2}	1.17×10^{-2}
	0.82		3.50×10^{-2}	1.74×10^{-2}	8.82×10^{-3}	0.84	6.09×10^{-2}	5.80×10^{-2}	5.72×10^{-2}
	0.98		4.10×10^{-2}	2.04×10^{-2}	1.02×10^{-2}	1	6.78×10^{-2}	6.72×10^{-2}	6.70×10^{-2}
$\frac{\Delta t}{2}$	0.23		3.11×10^{-2}	1.54×10^{-2}	7.63×10^{-3}	0.24	3.66×10^{-2}	2.73×10^{-2}	2.44×10^{-2}
	0.47		7.31×10^{-3}	3.40×10^{-3}	1.62×10^{-3}	0.48	4.02×10^{-2}	2.00×10^{-2}	1.02×10^{-2}
	0.83		3.53×10^{-2}	1.76×10^{-2}	8.76×10^{-3}	0.84	3.48×10^{-2}	2.98×10^{-2}	2.85×10^{-2}
	0.99		4.11×10^{-2}	2.03×10^{-2}	1.01×10^{-2}	1	3.35×10^{-2}	3.27×10^{-2}	3.25×10^{-2}
$\frac{\Delta t}{4}$	0.235		3.06×10^{-2}	1.51×10^{-2}	7.52×10^{-3}	0.24	3.02×10^{-2}	1.80×10^{-2}	1.35×10^{-2}
	0.475		6.86×10^{-3}	3.03×10^{-3}	1.31×10^{-3}	0.48	4.01×10^{-2}	1.99×10^{-2}	9.94×10^{-3}
	0.835		3.58×10^{-2}	1.77×10^{-2}	8.82×10^{-3}	0.84	2.45×10^{-2}	1.71×10^{-2}	1.48×10^{-2}
	0.995		4.09×10^{-2}	2.04×10^{-2}	1.01×10^{-2}	1	1.74×10^{-2}	1.63×10^{-2}	1.60×10^{-2}
		E_h (Quadratic)				H_h (Linear)			
		$\ E - E_h\ _\varepsilon, E_h \in P_2^- \cap \dot{H}_\varepsilon(\text{div})$				$\ H - H_h\ _\mu, H_h \in P_2^- \cap L_\mu^2$			
		t	h	$\frac{h}{2}$	$\frac{h}{4}$	t	h	$\frac{h}{2}$	$\frac{h}{4}$
Δt	0.22		7.43×10^{-4}	5.67×10^{-4}	5.40×10^{-4}	0.24	4.78×10^{-2}	4.78×10^{-2}	4.78×10^{-2}
	0.46		1.84×10^{-3}	1.82×10^{-3}	1.82×10^{-3}	0.48	6.22×10^{-3}	6.15×10^{-3}	6.14×10^{-3}
	0.82		1.89×10^{-3}	1.79×10^{-3}	1.78×10^{-3}	0.84	5.69×10^{-2}	5.69×10^{-2}	5.69×10^{-2}
	0.98		7.64×10^{-4}	3.15×10^{-4}	2.61×10^{-4}	1	6.68×10^{-2}	6.68×10^{-2}	6.68×10^{-2}
$\frac{\Delta t}{2}$	0.23		5.79×10^{-4}	2.10×10^{-4}	1.55×10^{-4}	0.24	2.34×10^{-2}	2.34×10^{-2}	2.34×10^{-2}
	0.47		5.33×10^{-4}	4.79×10^{-4}	4.76×10^{-4}	0.48	2.68×10^{-3}	2.52×10^{-3}	2.51×10^{-3}
	0.83		7.67×10^{-4}	4.60×10^{-4}	4.35×10^{-4}	0.84	2.80×10^{-2}	2.80×10^{-2}	2.80×10^{-2}
	0.99		7.20×10^{-4}	1.82×10^{-4}	5.54×10^{-5}	1	3.24×10^{-2}	3.24×10^{-2}	3.24×10^{-2}
$\frac{\Delta t}{4}$	0.235		5.42×10^{-4}	1.47×10^{-4}	5.29×10^{-5}	0.24	1.16×10^{-2}	1.16×10^{-2}	1.16×10^{-2}
	0.475		2.70×10^{-4}	1.35×10^{-4}	1.22×10^{-4}	0.48	1.47×10^{-3}	1.14×10^{-3}	1.12×10^{-3}
	0.835		6.41×10^{-4}	1.91×10^{-4}	1.14×10^{-4}	0.84	1.39×10^{-2}	1.39×10^{-2}	1.39×10^{-2}
	0.995		7.24×10^{-4}	1.79×10^{-4}	4.54×10^{-5}	1	1.60×10^{-2}	1.60×10^{-2}	1.60×10^{-2}

TABLE 3.2: The L^2 errors in the solution for the problem in **Example 3.1** using the Implicit Leapfrog time discretization for linear and quadratic spatial discretizations of E and H . Note that in the mixed formulation of this Maxwell's system, the choice of approximating E by piecewise linear (respectively, quadratic) polynomial vector field basis in $H(\text{curl})$ requires that H be then approximated using piecewise constant (linear) discontinuous Galerkin basis in L^2 in the two dimensional case for stability.

	t	E_h (Linear)			H_h (Linear)		
		$\ E - E_h\ _\varepsilon, E_h \in P_1^- \cap \dot{H}_\varepsilon(\text{curl})$	$\frac{h}{2}$	$\frac{h}{4}$	$\ H - H_h\ _\mu, H_h \in P_1^- \cap \dot{H}_\mu(\text{div})$	$\frac{h}{2}$	$\frac{h}{4}$
Δt	0.24	3.53×10^{-1}	2.46×10^{-1}	1.14×10^{-1}	4.86×10^{-1}	2.53×10^{-1}	1.27×10^{-1}
	0.48	1.27×10^{-1}	9.54×10^{-2}	4.11×10^{-2}	6.99×10^{-1}	3.59×10^{-1}	1.75×10^{-1}
	0.84	4.61×10^{-1}	2.80×10^{-1}	1.32×10^{-1}	3.44×10^{-1}	2.08×10^{-1}	9.74×10^{-2}
	1	4.98×10^{-1}	3.09×10^{-1}	1.49×10^{-1}	1.56×10^{-1}	1.42×10^{-1}	5.81×10^{-2}
$\frac{\Delta t}{2}$	0.24	3.53×10^{-1}	2.45×10^{-1}	1.14×10^{-1}	4.86×10^{-1}	2.54×10^{-1}	1.27×10^{-1}
	0.48	1.27×10^{-1}	9.36×10^{-2}	3.98×10^{-2}	7.00×10^{-1}	3.60×10^{-1}	1.75×10^{-1}
	0.84	4.61×10^{-1}	2.80×10^{-1}	1.33×10^{-1}	3.43×10^{-1}	2.08×10^{-1}	9.76×10^{-2}
	1	4.96×10^{-1}	3.06×10^{-1}	1.48×10^{-1}	1.59×10^{-1}	1.44×10^{-1}	5.73×10^{-2}
$\frac{\Delta t}{4}$	0.24	3.53×10^{-1}	2.44×10^{-1}	1.14×10^{-1}	4.86×10^{-1}	2.54×10^{-1}	1.27×10^{-1}
	0.48	1.27×10^{-1}	9.30×10^{-2}	3.96×10^{-2}	7.00×10^{-1}	3.60×10^{-1}	1.76×10^{-1}
	0.84	4.61×10^{-1}	2.80×10^{-1}	1.32×10^{-1}	3.43×10^{-1}	2.09×10^{-1}	9.83×10^{-2}
	1	4.96×10^{-1}	3.05×10^{-1}	1.48×10^{-1}	1.60×10^{-1}	1.45×10^{-1}	5.69×10^{-2}

TABLE 3.3: The L^2 errors in the solution for the problem in **Example 3.2** using the Crank Nicolson time discretization for linear spatial discretizations of E and H . Note that in the mixed formulation of this Maxwell's system, the choice of approximating E by piecewise linear polynomial vector field basis in $H(\text{curl})$ requires that H be then approximated using piecewise linear vector field basis in $H(\text{div})$ in the three dimensional case for stability.

		E_h (Linear)				H_h (Linear)			
		$\ E - E_h\ _\varepsilon, E_h \in P_1^- \cap \dot{H}_\varepsilon(\text{curl})$				$\ H - H_h\ _\mu, H_h \in P_1^- \cap \dot{H}_\mu(\text{div})$			
		t	h	$\frac{h}{2}$	$\frac{h}{4}$	t	h	$\frac{h}{2}$	$\frac{h}{4}$
Δt	0.22		3.67×10^{-1}	2.56×10^{-1}	1.19×10^{-1}	0.24	4.88×10^{-1}	2.64×10^{-1}	1.42×10^{-1}
	0.46		1.48×10^{-1}	1.04×10^{-1}	4.43×10^{-2}	0.48	6.95×10^{-1}	3.58×10^{-1}	1.75×10^{-1}
	0.82		4.44×10^{-1}	2.70×10^{-1}	1.27×10^{-1}	0.84	3.49×10^{-1}	2.30×10^{-1}	1.24×10^{-1}
	0.98		5.02×10^{-1}	3.10×10^{-1}	1.49×10^{-1}	1	1.41×10^{-1}	1.92×10^{-1}	1.12×10^{-1}
$\frac{\Delta t}{2}$	0.23		3.60×10^{-1}	2.50×10^{-1}	1.17×10^{-1}	0.24	4.87×10^{-1}	2.58×10^{-1}	1.31×10^{-1}
	0.47		1.37×10^{-1}	9.83×10^{-2}	4.15×10^{-2}	0.48	6.98×10^{-1}	3.59×10^{-1}	1.75×10^{-1}
	0.83		4.53×10^{-1}	2.76×10^{-1}	1.30×10^{-1}	0.84	3.44×10^{-1}	2.17×10^{-1}	1.06×10^{-1}
	0.99		4.99×10^{-1}	3.07×10^{-1}	1.48×10^{-1}	1	1.48×10^{-1}	1.65×10^{-1}	7.85×10^{-2}
$\frac{\Delta t}{4}$	0.235		3.57×10^{-1}	2.47×10^{-1}	1.16×10^{-1}	0.24	4.86×10^{-1}	2.55×10^{-1}	1.28×10^{-1}
	0.475		1.32×10^{-1}	9.53×10^{-2}	4.04×10^{-2}	0.48	7.00×10^{-1}	3.60×10^{-1}	1.75×10^{-1}
	0.835		4.57×10^{-1}	2.78×10^{-1}	1.31×10^{-1}	0.84	3.43×10^{-1}	2.12×10^{-1}	1.01×10^{-1}
	0.995		4.97×10^{-1}	3.05×10^{-1}	1.48×10^{-1}	1	1.54×10^{-1}	1.54×10^{-1}	6.57×10^{-2}

TABLE 3.4: The L^2 errors in the solution for the problem in **Example 3.2** using the Implicit Leapfrog time discretization for linear spatial discretizations of E and H . Note that in the mixed formulation of this Maxwell's system, the choice of approximating E by piecewise linear polynomial vector field basis in $H(\text{curl})$ requires that H be then approximated using piecewise linear vector field basis in $H(\text{div})$ in the three dimensional case for stability.

		p_h (Linear)		E_h (Linear)		H_h (Constant)		
		$\ p - p_h\ _{\varepsilon^{-1}}, p_h \in P_1^- \cap \dot{H}_{\varepsilon^{-1}}^1$	$\ E - E_h\ _{\varepsilon}, E_h \in P_1^- \cap \dot{H}_{\varepsilon}(\text{div})$	$\ H - H_h\ _{\mu}, H_h \in P_1^- \cap L_{\mu}^2$	$\frac{h}{2}$	$\frac{h}{4}$		
t	h							
Δt	0.24	1.68×10^{-3}	5.72×10^{-4}	3.74×10^{-4}	6.88×10^{-2}	5.58×10^{-2}	2.78×10^{-2}	1.41×10^{-2}
	0.48	2.83×10^{-3}	4.89×10^{-4}	5.13×10^{-4}	5.87×10^{-2}	5.75×10^{-2}	2.90×10^{-2}	1.52×10^{-2}
	0.84	1.47×10^{-3}	2.62×10^{-4}	4.36×10^{-4}	7.33×10^{-2}	5.70×10^{-2}	2.89×10^{-2}	1.58×10^{-2}
	1	2.62×10^{-4}	2.84×10^{-4}	4.08×10^{-4}	7.78×10^{-2}	5.65×10^{-2}	2.87×10^{-2}	1.56×10^{-2}
$\frac{\Delta t}{2}$	0.24	1.59×10^{-3}	4.16×10^{-4}	1.43×10^{-4}	6.85×10^{-2}	5.59×10^{-2}	2.77×10^{-2}	1.38×10^{-2}
	0.48	3.21×10^{-3}	6.99×10^{-4}	1.23×10^{-4}	5.84×10^{-2}	5.71×10^{-2}	2.84×10^{-2}	1.42×10^{-2}
	0.84	1.77×10^{-3}	3.63×10^{-4}	6.44×10^{-5}	7.27×10^{-2}	5.64×10^{-2}	2.79×10^{-2}	1.40×10^{-2}
	1	6.32×10^{-4}	6.50×10^{-5}	7.28×10^{-5}	7.67×10^{-2}	5.60×10^{-2}	2.78×10^{-2}	1.39×10^{-2}
$\frac{\Delta t}{4}$	0.24	1.58×10^{-3}	3.93×10^{-4}	1.04×10^{-4}	6.84×10^{-2}	5.58×10^{-2}	2.77×10^{-2}	1.38×10^{-2}
	0.48	3.31×10^{-3}	7.94×10^{-4}	1.75×10^{-4}	5.84×10^{-2}	5.73×10^{-2}	2.84×10^{-2}	1.41×10^{-2}
	0.84	1.84×10^{-3}	4.40×10^{-4}	9.12×10^{-5}	7.26×10^{-2}	5.62×10^{-2}	2.78×10^{-2}	1.38×10^{-2}
	1	7.29×10^{-4}	1.52×10^{-4}	1.67×10^{-5}	7.69×10^{-2}	5.59×10^{-2}	2.77×10^{-2}	1.38×10^{-2}
		p_h (Quadratic)		E_h (Quadratic)		H_h (Linear)		
		$\ p - p_h\ _{\varepsilon^{-1}}, p_h \in P_2^- \cap \dot{H}_{\varepsilon^{-1}}^1$	$\ E - E_h\ _{\varepsilon}, E_h \in P_2^- \cap \dot{H}_{\varepsilon}(\text{div})$	$\ H - H_h\ _{\mu}, H_h \in P_2^- \cap L_{\mu}^2$	$\frac{h}{2}$	$\frac{h}{4}$		
t	h							
Δt	0.24	3.39×10^{-4}	3.38×10^{-4}	3.38×10^{-4}	1.17×10^{-2}	5.22×10^{-3}	3.09×10^{-3}	2.91×10^{-3}
	0.48	6.64×10^{-4}	6.65×10^{-4}	6.65×10^{-4}	1.13×10^{-2}	8.23×10^{-3}	5.94×10^{-3}	5.65×10^{-3}
	0.84	5.36×10^{-4}	5.37×10^{-4}	5.37×10^{-4}	1.75×10^{-2}	8.01×10^{-3}	7.38×10^{-3}	7.55×10^{-3}
	1	4.50×10^{-4}	4.50×10^{-4}	4.50×10^{-4}	1.85×10^{-2}	9.57×10^{-3}	7.57×10^{-3}	7.34×10^{-3}
$\frac{\Delta t}{2}$	0.24	8.67×10^{-5}	8.45×10^{-5}	8.44×10^{-5}	1.19×10^{-2}	5.68×10^{-3}	1.49×10^{-3}	7.91×10^{-4}
	0.48	1.69×10^{-4}	1.68×10^{-4}	1.68×10^{-4}	1.07×10^{-2}	6.87×10^{-3}	2.22×10^{-3}	1.50×10^{-3}
	0.84	1.30×10^{-4}	1.30×10^{-4}	1.30×10^{-4}	1.42×10^{-2}	5.98×10^{-3}	2.14×10^{-3}	1.85×10^{-3}
	1	1.16×10^{-4}	1.16×10^{-4}	1.16×10^{-4}	1.49×10^{-2}	6.79×10^{-3}	2.52×10^{-3}	1.91×10^{-3}
$\frac{\Delta t}{4}$	0.24	2.86×10^{-5}	2.12×10^{-5}	2.11×10^{-5}	1.28×10^{-2}	6.30×10^{-3}	1.65×10^{-3}	4.39×10^{-4}
	0.48	4.92×10^{-5}	4.22×10^{-5}	4.21×10^{-5}	1.11×10^{-2}	7.13×10^{-3}	1.87×10^{-3}	5.93×10^{-4}
	0.84	3.42×10^{-5}	3.23×10^{-5}	3.23×10^{-5}	1.40×10^{-2}	6.88×10^{-3}	1.77×10^{-3}	5.75×10^{-4}
	1	2.90×10^{-5}	2.93×10^{-5}	2.94×10^{-5}	1.47×10^{-2}	6.97×10^{-3}	1.96×10^{-3}	6.75×10^{-4}

TABLE 3.5: The L^2 errors in the solution for the problem in **Example 3.3** using the Crank Nicolson time discretization for linear and quadratic spatial discretizations of p , E and H . Note that in the mixed formulation of this Maxwell's system, the choice of approximating p by piecewise linear (respectively, quadratic) polynomial scalar basis in $\dot{H}^1$ and E by piecewise linear (respectively, quadratic) polynomial vector field basis in $H(\text{curl})$ requires that H be then approximated using piecewise constant (linear) discontinuous Galerkin basis in L^2 in the two dimensional case for stability.

	p_h (Linear)				E_h (Linear)				H_h (Constant)			
	t	h	$\ p - p_h\ _{\varepsilon^{-1}}, p_h \in P_1^- \cap \dot{H}_{\varepsilon^{-1}}^1$	$\frac{h}{2}$	$\ E - E_h\ _{\varepsilon}, E_h \in P_1^- \cap \dot{H}_{\varepsilon}(\text{div})$	h	$\frac{h}{2}$	t	h	$\frac{h}{2}$	$\frac{h}{4}$	$\frac{h}{4}$
Δt	0.22	1.49×10^{-2}	7.87×10^{-2}	3.17×10^{-1}	7.48×10^{-2}	4.53×10^{-2}	4.47×10^{-2}	0.24	1.07×10^{-1}	9.52×10^{-2}	9.22×10^{-2}	9.22×10^{-2}
	0.46	2.56×10^{-2}	8.15×10^{-2}	3.17×10^{-1}	6.07×10^{-2}	3.51×10^{-2}	4.20×10^{-2}	0.48	1.09×10^{-1}	9.74×10^{-2}	9.43×10^{-2}	9.43×10^{-2}
	0.82	2.46×10^{-2}	4.01×10^{-2}	1.30×10^{-1}	7.48×10^{-2}	8.22×10^{-2}	2.92×10^{-1}	0.84	1.10×10^{-1}	9.82×10^{-2}	9.52×10^{-2}	9.52×10^{-2}
	0.98	2.57×10^{-2}	8.11×10^{-2}	3.18×10^{-1}	7.97×10^{-2}	4.49×10^{-2}	3.28×10^{-2}	1	1.10×10^{-1}	9.87×10^{-2}	9.57×10^{-2}	9.57×10^{-2}
$\frac{\Delta t}{2}$	0.23	7.06×10^{-3}	8.56×10^{-3}	4.19×10^{-2}	7.03×10^{-2}	3.68×10^{-2}	2.25×10^{-2}	0.24	7.16×10^{-2}	5.29×10^{-2}	4.71×10^{-2}	4.71×10^{-2}
	0.47	1.25×10^{-2}	1.39×10^{-2}	3.86×10^{-2}	5.88×10^{-2}	2.99×10^{-2}	2.62×10^{-2}	0.48	7.29×10^{-2}	5.36×10^{-2}	4.77×10^{-2}	4.77×10^{-2}
	0.83	1.37×10^{-2}	1.30×10^{-2}	2.24×10^{-2}	7.27×10^{-2}	3.73×10^{-2}	4.23×10^{-2}	0.84	7.25×10^{-2}	5.35×10^{-2}	4.78×10^{-2}	4.78×10^{-2}
	0.99	1.12×10^{-2}	1.30×10^{-2}	4.06×10^{-2}	7.74×10^{-2}	3.95×10^{-2}	2.61×10^{-2}	1	7.23×10^{-2}	5.36×10^{-2}	4.79×10^{-2}	4.79×10^{-2}
$\frac{\Delta t}{4}$	0.235	3.50×10^{-3}	3.76×10^{-3}	2.18×10^{-1}	6.89×10^{-2}	3.48×10^{-2}	1.24×10^{-1}	0.24	6.01×10^{-2}	3.56×10^{-2}	2.63×10^{-2}	2.63×10^{-2}
	0.475	5.82×10^{-3}	6.64×10^{-3}	6.27×10^{-2}	5.86×10^{-2}	2.91×10^{-2}	2.43×10^{-1}	0.48	6.14×10^{-2}	3.62×10^{-2}	2.66×10^{-2}	2.66×10^{-2}
	0.835	7.57×10^{-3}	6.70×10^{-3}	2.05×10^{-1}	7.24×10^{-2}	3.63×10^{-2}	1.45×10^{-1}	0.84	6.06×10^{-2}	3.58×10^{-2}	2.64×10^{-2}	2.64×10^{-2}
	0.995	4.92×10^{-3}	5.82×10^{-3}	1.41×10^{-1}	7.71×10^{-2}	3.85×10^{-2}	2.08×10^{-1}	1	6.02×10^{-2}	3.57×10^{-2}	2.65×10^{-2}	2.65×10^{-2}
	p_h (Quadratic)				E_h (Quadratic)				H_h (Linear)			
	t	h	$\ p - p_h\ _{\varepsilon^{-1}}, p_h \in P_2^- \cap \dot{H}_{\varepsilon^{-1}}^1$	$\frac{h}{2}$	$\ E - E_h\ _{\varepsilon}, E_h \in P_2^- \cap \dot{H}_{\varepsilon}(\text{div})$	h	$\frac{h}{2}$	t	h	$\frac{h}{2}$	$\frac{h}{4}$	$\frac{h}{4}$
Δt	0.22	5.44×10^{-2}	3.84×10^{-1}	1.43×10^1	3.11×10^{-2}	4.63×10^{-2}	1.34×10^0	0.24	9.15×10^{-2}	9.12×10^{-2}	9.12×10^{-2}	9.12×10^{-2}
	0.46	5.85×10^{-2}	3.84×10^{-1}	1.43×10^1	2.23×10^{-2}	4.58×10^{-2}	1.54×10^0	0.48	9.42×10^{-2}	9.34×10^{-2}	9.33×10^{-2}	9.33×10^{-2}
	0.82	3.40×10^{-2}	1.56×10^{-1}	5.73×10^0	5.41×10^{-2}	3.54×10^{-1}	1.31×10^1	0.84	9.43×10^{-2}	9.42×10^{-2}	9.42×10^{-2}	9.42×10^{-2}
	0.98	5.75×10^{-2}	3.85×10^{-1}	1.43×10^1	2.83×10^{-2}	2.75×10^{-2}	3.15×10^{-1}	1	9.53×10^{-2}	9.48×10^{-2}	9.48×10^{-2}	9.48×10^{-2}
$\frac{\Delta t}{2}$	0.23	9.00×10^{-3}	2.29×10^{-2}	2.57×10^{-1}	1.85×10^{-2}	1.47×10^{-2}	3.14×10^{-2}	0.24	4.57×10^{-2}	4.51×10^{-2}	4.51×10^{-2}	4.51×10^{-2}
	0.47	1.46×10^{-2}	2.35×10^{-2}	2.26×10^{-1}	1.34×10^{-2}	1.36×10^{-2}	1.27×10^{-1}	0.48	4.70×10^{-2}	4.58×10^{-2}	4.56×10^{-2}	4.56×10^{-2}
	0.83	1.31×10^{-2}	1.65×10^{-2}	1.14×10^{-1}	1.76×10^{-2}	2.18×10^{-2}	2.33×10^{-1}	0.84	4.62×10^{-2}	4.58×10^{-2}	4.57×10^{-2}	4.57×10^{-2}
	0.99	1.31×10^{-2}	2.37×10^{-2}	2.44×10^{-1}	1.81×10^{-2}	1.38×10^{-2}	8.68×10^{-2}	1	4.69×10^{-2}	4.61×10^{-2}	4.59×10^{-2}	4.59×10^{-2}
$\frac{\Delta t}{4}$	0.235	3.90×10^{-3}	7.79×10^{-3}	7.90×10^{-2}	1.46×10^{-2}	8.83×10^{-3}	4.49×10^{-2}	0.24	2.36×10^{-2}	2.25×10^{-2}	2.24×10^{-2}	2.24×10^{-2}
	0.475	6.97×10^{-3}	7.27×10^{-3}	2.37×10^{-2}	1.18×10^{-2}	9.20×10^{-3}	8.77×10^{-2}	0.48	2.46×10^{-2}	2.28×10^{-2}	2.26×10^{-2}	2.26×10^{-2}
	0.835	6.40×10^{-3}	9.26×10^{-3}	7.43×10^{-2}	1.49×10^{-2}	7.66×10^{-3}	5.22×10^{-2}	0.84	2.36×10^{-2}	2.26×10^{-2}	2.25×10^{-2}	2.25×10^{-2}
	0.995	6.22×10^{-3}	7.27×10^{-3}	5.12×10^{-2}	1.55×10^{-2}	9.67×10^{-3}	7.50×10^{-2}	1	2.42×10^{-2}	2.28×10^{-2}	2.26×10^{-2}	2.26×10^{-2}

TABLE 3.6: The L^2 errors in the solution for the problem in **Example 3.3** using the Implicit Leapfrog time discretization for linear and quadratic spatial discretizations of p , E and H . Note that in the mixed formulation of this Maxwell's system, the choice of approximating p by piecewise linear (respectively, quadratic) polynomial scalar basis in $\dot{H}^1$ and E by piecewise linear (respectively, quadratic) polynomial vector field basis in $H(\text{curl})$ requires that H be then approximated using piecewise constant (linear) discontinuous Galerkin basis in L^2 in the two dimensional case for stability.

Example 3.4: Our next example is on the unit cube in $\mathbb{R}^3$ with analytical solutions, material parameters, initial and final times as below:

$$p = (\cos \pi x + \cos \pi y) \sin \pi t,$$

$$E = \begin{bmatrix} \sin \pi(\sqrt{2}t - x - y) - \sin \pi x \cos \pi t \\ -\sin \pi(\sqrt{2}t - x - y) - \sin \pi y \cos \pi t \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ 0 \\ -\sqrt{2} \sin \pi(\sqrt{2}t - x - y) \end{bmatrix},$$

$$\epsilon = 1, \quad \mu = 1, \quad T_{\min} = 0, \quad T_{\max} = 2.$$

Similar to Example 3.2, the boundary conditions for p and E are nonhomogeneous everywhere on the boundary while for H it is nonhomogeneous in some parts of the boundary. The computational results are shown in Figures 3.7, 3.8 and 3.10 and Tables 3.7 and 3.8.

t	p_h (Quadratic)		E_h (Quadratic)		H_h (Quadratic)		
	$\ p - p_h\ _{\varepsilon^{-1}}$	$p_h \in P_2^- \cap \dot{H}_{\varepsilon^{-1}}^1$	$\ E - E_h\ _{\varepsilon}$	$E_h \in P_2^- \cap \dot{H}_{\varepsilon}^1$	$\ H - H_h\ _{\mu}$	$H_h \in P_2^- \cap \dot{H}_{\mu}^1(\text{div})$	
	h	$\frac{h}{2}$	h	$\frac{h}{2}$	h	$\frac{h}{2}$	$\frac{h}{4}$
Δt	0.24	1.18×10^{-2}	2.15×10^{-3}	4.02×10^{-4}	1.23×10^{-1}	3.81×10^{-2}	1.04×10^{-2}
	0.48	1.74×10^{-2}	3.01×10^{-3}	7.06×10^{-4}	9.76×10^{-2}	3.33×10^{-2}	9.56×10^{-3}
	0.84	8.23×10^{-3}	1.49×10^{-3}	2.57×10^{-4}	1.14×10^{-1}	4.04×10^{-2}	1.23×10^{-2}
	1	7.90×10^{-4}	4.11×10^{-4}	5.03×10^{-4}	1.35×10^{-1}	4.07×10^{-2}	1.33×10^{-2}
$\frac{\Delta t}{2}$	0.24	1.18×10^{-2}	2.13×10^{-3}	2.93×10^{-4}	1.22×10^{-1}	3.80×10^{-2}	1.03×10^{-2}
	0.48	1.75×10^{-2}	3.00×10^{-3}	4.18×10^{-4}	9.77×10^{-2}	3.31×10^{-2}	8.62×10^{-3}
	0.84	8.27×10^{-3}	1.49×10^{-3}	2.04×10^{-4}	1.14×10^{-1}	3.98×10^{-2}	1.02×10^{-2}
	1	1.02×10^{-3}	2.21×10^{-4}	1.12×10^{-4}	1.36×10^{-1}	4.02×10^{-2}	1.13×10^{-2}
$\frac{\Delta t}{4}$	0.24	1.18×10^{-2}	2.13×10^{-3}	2.81×10^{-4}	1.22×10^{-1}	3.79×10^{-2}	1.01×10^{-2}
	0.48	1.75×10^{-2}	3.01×10^{-3}	4.06×10^{-4}	9.77×10^{-2}	3.29×10^{-2}	8.55×10^{-3}
	0.84	8.28×10^{-3}	1.49×10^{-3}	2.02×10^{-4}	1.14×10^{-1}	3.99×10^{-2}	1.00×10^{-2}
	1	1.08×10^{-3}	2.59×10^{-4}	2.24×10^{-5}	1.37×10^{-1}	4.03×10^{-2}	1.12×10^{-2}

TABLE 3.7: The L^2 errors in the solution for the problem in **Example 3.4** using the Crank Nicolson time discretization for quadratic spatial discretizations of p , E and H . Note that in the mixed formulation of this Maxwell's system, the choice of approximating p by piecewise quadratic polynomial scalar basis in $\dot{H}^1$ and E by piecewise linear quadratic polynomial vector field basis in $H(\text{curl})$ requires that H be then approximated using piecewise quadratic vector field basis in $H(\text{div})$ in the three dimensional case for stability.

	p_h (Quadratic)			E_h (Quadratic)			H_h (Quadratic)		
	t	h	$\ p - p_h\ _{\varepsilon^{-1}}, p_h \in P_2^- \cap \dot{H}_{\varepsilon^{-1}}^1$ $\frac{h}{2}$	h	$\ E - E_h\ _{\varepsilon}, E_h \in P_2^- \cap \dot{H}_{\varepsilon}(\text{curl})$ $\frac{h}{2}$	$\frac{h}{4}$	t	h	$\ H - H_h\ _{\mu}, H_h \in P_2^- \cap \dot{H}_{\mu}(\text{div})$ $\frac{h}{2}$
Δt	0.22	5.96×10^{-2}	5.69×10^{-2}	5.73×10^{-2}	1.43×10^{-1}	9.33×10^{-2}	0.24	2.54×10^{-1}	1.48×10^{-1}
	0.46	3.56×10^{-2}	3.04×10^{-2}	3.09×10^{-2}	1.41×10^{-1}	1.15×10^{-1}	0.48	3.00×10^{-1}	2.13×10^{-1}
	0.82	6.14×10^{-2}	6.79×10^{-2}	6.79×10^{-2}	1.61×10^{-1}	1.29×10^{-1}	0.84	2.92×10^{-1}	2.37×10^{-1}
	0.98	6.07×10^{-2}	7.26×10^{-2}	7.38×10^{-2}	1.51×10^{-1}	1.03×10^{-1}	1	3.23×10^{-1}	2.31×10^{-1}
$\frac{\Delta t}{2}$	0.23	3.34×10^{-2}	2.88×10^{-2}	2.87×10^{-2}	1.28×10^{-1}	5.82×10^{-2}	0.24	2.33×10^{-1}	1.00×10^{-1}
	0.47	2.39×10^{-2}	1.43×10^{-2}	1.42×10^{-2}	1.10×10^{-1}	6.48×10^{-2}	0.48	2.53×10^{-1}	1.30×10^{-1}
	0.83	2.96×10^{-2}	3.45×10^{-2}	3.46×10^{-2}	1.24×10^{-1}	7.18×10^{-2}	0.84	2.26×10^{-1}	1.37×10^{-1}
	0.99	2.99×10^{-2}	3.62×10^{-2}	3.65×10^{-2}	1.37×10^{-1}	6.14×10^{-2}	1	2.65×10^{-1}	1.35×10^{-1}
$\frac{\Delta t}{4}$	0.235	2.11×10^{-2}	1.48×10^{-2}	1.43×10^{-2}	1.23×10^{-1}	4.41×10^{-2}	0.24	2.28×10^{-1}	8.35×10^{-2}
	0.475	1.99×10^{-2}	7.52×10^{-3}	6.88×10^{-3}	1.01×10^{-1}	4.34×10^{-2}	0.48	2.39×10^{-1}	9.76×10^{-2}
	0.835	1.48×10^{-2}	1.72×10^{-2}	1.74×10^{-2}	1.15×10^{-1}	4.96×10^{-2}	0.84	2.07×10^{-1}	9.58×10^{-2}
	0.995	1.50×10^{-2}	1.80×10^{-2}	1.80×10^{-2}	1.35×10^{-1}	4.60×10^{-2}	1	2.48×10^{-1}	9.83×10^{-2}

TABLE 3.8: The L^2 errors in the solution for the problem in **Example 3.4** using the Implicit Leapfrog time discretization for quadratic spatial discretizations of p , E and H . Note that in the mixed formulation of this Maxwell's system, the choice of approximating p by piecewise quadratic polynomial scalar basis in $\dot{H}^1$ and E by piecewise linear quadratic polynomial vector field basis in $H(\text{curl})$ requires that H be then approximated using piecewise quadratic vector field basis in $H(\text{div})$ in the three dimensional case for stability.

Example 3.5: Our next example is on the unit square in $\mathbb{R}^2$ with analytical solutions, material parameters, initial and final times as below:

$$p = (\cos x + \cos y) \sin t,$$

$$E = \begin{bmatrix} \sin \pi(\sqrt{2}t - x - y) - \sin x \cos t \\ -\sin \pi(\sqrt{2}t - x - y) - \sin y \cos t \end{bmatrix}, \quad H = -\sqrt{2} \sin \pi(\sqrt{2}t - x - y),$$

$$\epsilon = 1, \quad \mu = 1, \quad T_{\min} = 0, \quad T_{\max} = 3.14.$$

The computational results are shown in Figures 3.11, 3.12 and 3.15.

Example 3.6: Our final example is on the unit cube in $\mathbb{R}^3$ with analytical solutions, material parameters, initial and final times as below:

$$p = 0, \quad E = \begin{bmatrix} \sin y \sin z \cos t \\ \sin x \sin z \cos t \\ \sin x \sin y \cos t \end{bmatrix}, \quad H = \begin{bmatrix} \sin x(\cos z - \cos y) \sin t \\ \sin y(\cos x - \cos z) \sin t \\ \sin z(\cos y - \cos x) \sin t \end{bmatrix},$$

$$\epsilon = 2, \quad \mu = 1, \quad T_{\min} = 0, \quad T_{\max} = 3.$$

The computational results are shown in Figures 3.13 to 3.15.

3.5.1 Discussion

For all problems in our examples, the right hand side functions for the stated analytical solutions are all zero in Equation (3.1a). We can readily obtain the analytical energies ($\|p\|_{\epsilon^{-1}}^2 + \|E\|_{\epsilon}^2 + \|H\|_{\mu}^2$) for each of these problems by integration as:

	Example 3.1	Example 3.2	Example 3.3	Example 3.4	Example 3.5	Example 3.6
Energy at time t	1	$\frac{3}{2}$	3	3	$3 + \mathcal{E}_1(t)$	$\frac{3}{2} + \mathcal{E}_2(t)$

$$\mathcal{E}_1(t) = -\frac{1}{2} \sin 2 + (2 \sin 1 + \sin 2) \sin^2 t, \quad \mathcal{E}_2(t) = \frac{3}{8} \sin^2 2 \cos 2t + \frac{3}{2} \sin 2(\sin^2 1 \sin^2 t - \cos^2 t).$$

Consequently, the energies of the discrete solutions computed using the Crank-Nicolson and implicit leapfrog schemes are indeed conserved for the various problems in Examples 1 to 4 and as can be seen in Figures 3.9 and 3.10. For a numerical check on the consistency of our computations we verify that for each of the four examples, the backward Euler time discretization is dissipative.

With respect to the choice of higher-order finite elements, indeed choosing quadratic Whitney finite elements does lead to a better approximation for the various solutions p_h^n , E_h^n and H_h^n at the appropriate time indices and this is also evident from the solution plots which we have discussed. However, this effect is more pronounced in $\mathbb{R}^3$, and therefore we provide the linear and quadratic spatial discretization solutions in Figures 3.3, 3.4, 3.7 and 3.8.

Finally, in Tables 3.9 and 3.10, we provide some numerical validation for Theorems 3.13 and 3.15. We fix a quadratic choice of basis for Examples 3 and 4, and study the behavior of the full error over a sequence of refined meshes and for refined time steps. In particular, for our two dimensional triangulation of the unit square, we choose a sequence of Delaunay meshes with their mesh parameters (for example, the longest edge length) being h , $h/2$ and $h/4$ with $h \approx 0.1$. These correspond to meshes with number of vertices and triangles to be (502, 900), (1903, 3600) and (7405, 14400), respectively. Likewise, in three dimensions, we use a sequence of three Delaunay triangulations of the unit cube with the number of vertices and

tetrahedra being, respectively, (50, 68), (215, 560) and (1181, 4561). The coarsest tetrahedral mesh has a mesh parameter of $h \approx 1.0$. We set the initial time step to be $\Delta t = 0.04$. Next, we compute the ratio of the error $\|e_{p_h}^N\|_{\varepsilon^{-1}} + \|e_{E_h}^N\|_{\varepsilon} + \|e_{H_h}^N\|_{\mu}$ at the time step corresponding to $T = 1$ for $(h, \Delta t)$ to the error for $(h/2, \Delta t/2)$. For the quadratic basis, this ratio should be approximately 4 computed using that the error is approximately $h^2 + \Delta t^2 + h^2 \Delta t^2$. The two tables then present this ratio for various solution schemes for Examples 3 and 4 using the Crank-Nicolson and implicit leapfrog methods. The computed ratios are in broad agreement with the theoretical expectation given our rather coarse computational estimates. Finite element folklore would require asymptotic estimates to be reliable with around a million degrees of freedom, and our basic computations are quite some away from this requirement. Nevertheless, they add some credibility to our theory and we leave more intensive computational validation to future work.

	$\frac{h}{2}$	$\frac{h}{4}$		$\frac{h}{2}$	$\frac{h}{4}$
$\frac{\Delta t}{2}$	3.50	3.86	$\frac{\Delta t}{2}$	2.17	1.35
$\frac{\Delta t}{4}$	3.20	3.35	$\frac{\Delta t}{4}$	1.97	0.56
$\frac{\Delta t}{8}$	3.08	3.10	$\frac{\Delta t}{8}$	2.28	2.24

TABLE 3.9: The ratio of the error $\|e_{p_h}^N\|_{\varepsilon^{-1}} + \|e_{E_h}^N\|_{\varepsilon} + \|e_{H_h}^N\|_{\mu}$ as in Theorems 3.13 and 3.15 computed at $(h, \Delta t)$ to the error at $(h/2, \Delta t/2)$ for the two dimensional problem in **Example 3.3**. *Left*: The ratios for Crank-Nicolson discretization using quadratic spatial basis. *Right*: The ratios for implicit leapfrog discretization using quadratic spatial basis.

	$\frac{h}{2}$	$\frac{h}{4}$		$\frac{h}{2}$	$\frac{h}{4}$
$\frac{\Delta t}{2}$	3.07	3.19	$\frac{\Delta t}{2}$	2.23	2.06
$\frac{\Delta t}{4}$	3.08	3.19	$\frac{\Delta t}{4}$	2.66	2.22

TABLE 3.10: The ratio of the error $\|e_{p_h}^N\|_{\varepsilon^{-1}} + \|e_{E_h}^N\|_{\varepsilon} + \|e_{H_h}^N\|_{\mu}$ as in Theorems 3.13 and 3.15 computed at $(h, \Delta t)$ to the error at $(h/2, \Delta t/2)$ for the three dimensional problem in **Example 3.4**. *Left*: The ratios for Crank-Nicolson discretization using quadratic spatial basis. *Right*: The ratios for implicit leapfrog discretization using quadratic spatial basis.

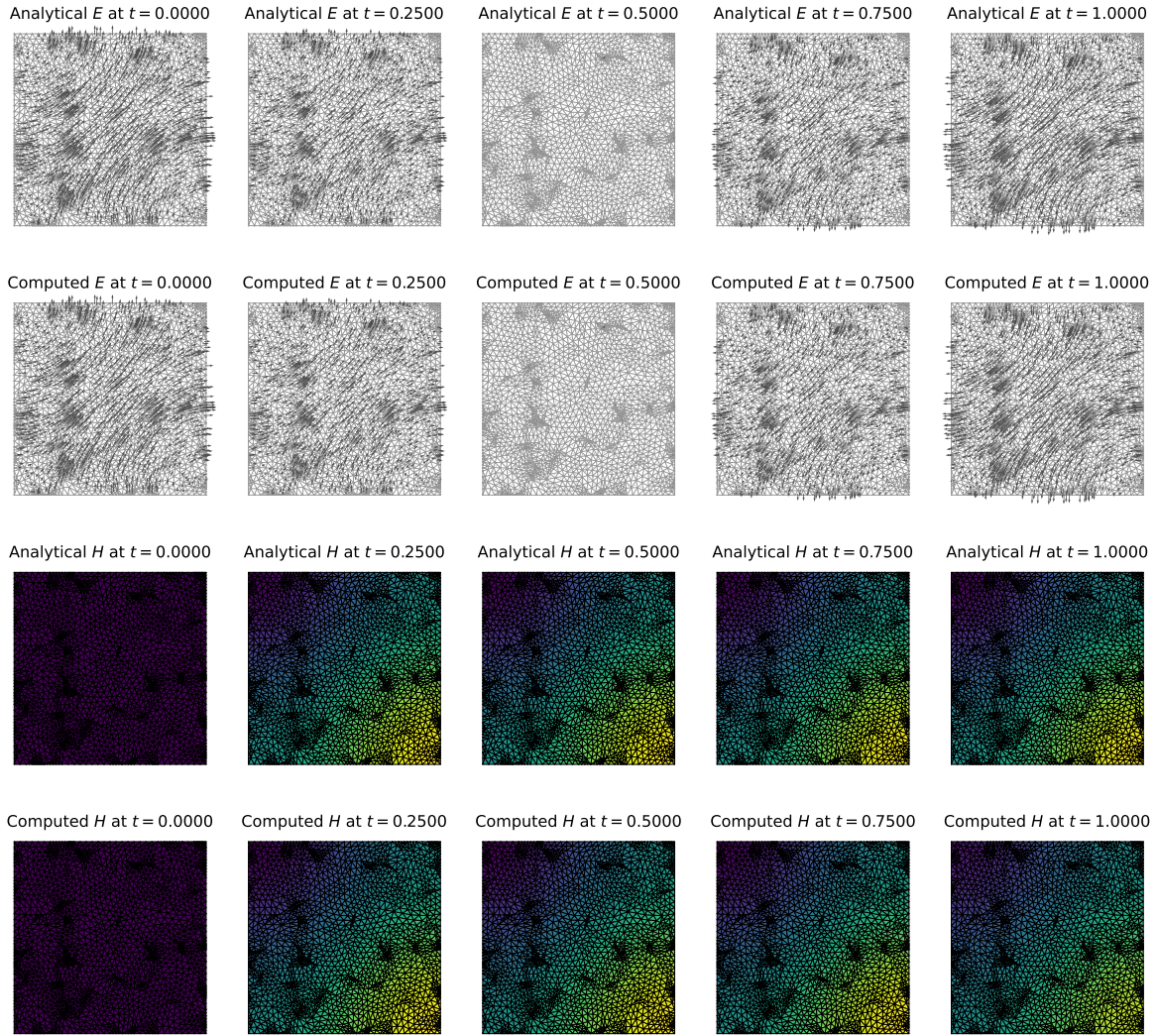


FIGURE 3.1: **Quadratic finite elements, Crank-Nicolson:** Plots of solutions at different time steps for the problem described in **Example 3.1** of Section 3.5 using the Crank-Nicolson time integration and quadratic Whitney forms as basis for the finite dimensional finite element spaces. The solutions for p are not shown due to them being identically equal to 0. The computed solutions for E and H visually match with the analytical solutions near identically.

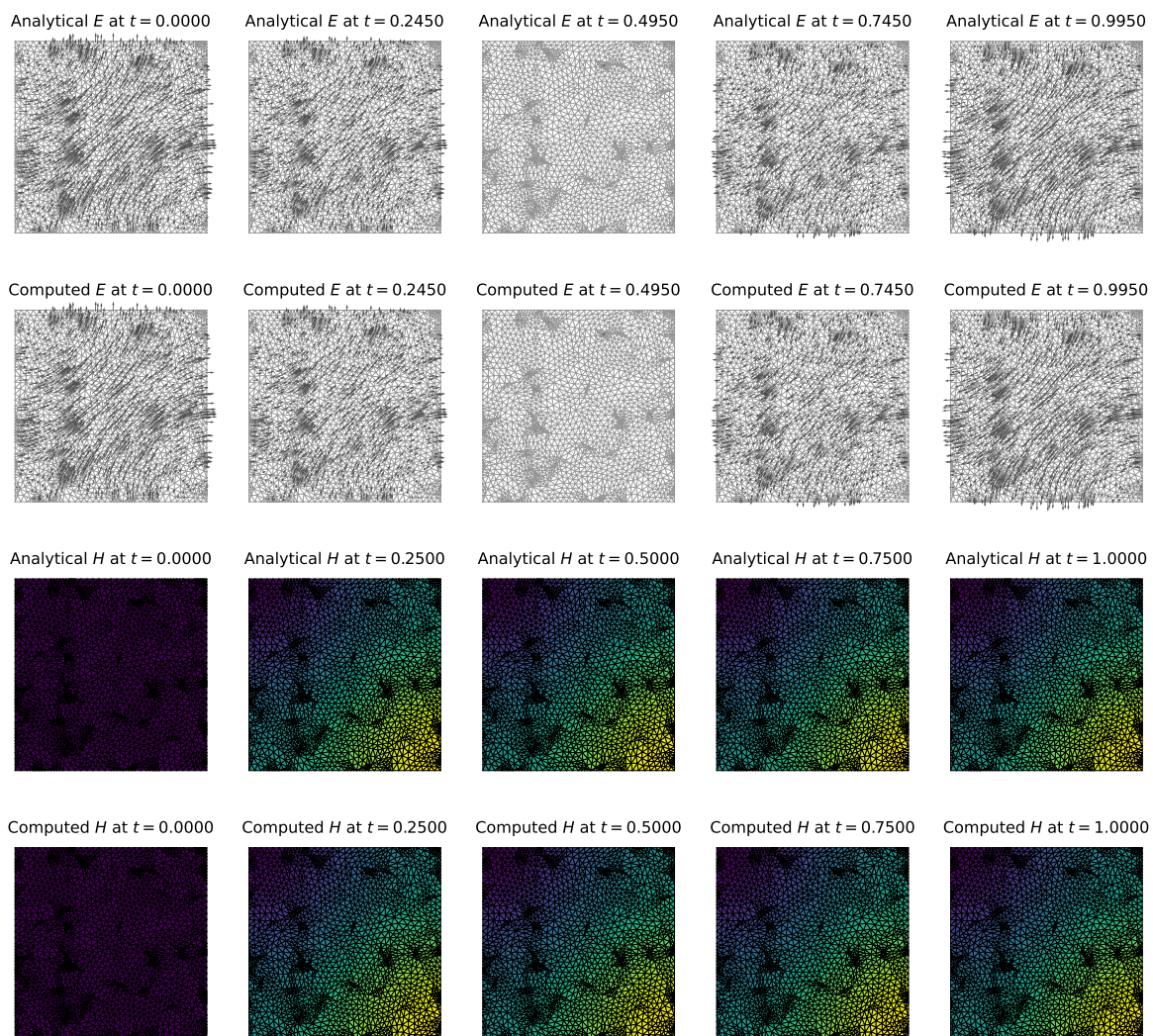


FIGURE 3.2: **Quadratic finite elements, implicit leapfrog:** Plots of solutions at different time steps for the problem described in **Example 3.1** of Section 3.5 using the implicit leapfrog scheme and quadratic Whitney forms as basis for the finite element spaces. The computed solutions for E and H match with the analytical solutions near identically.

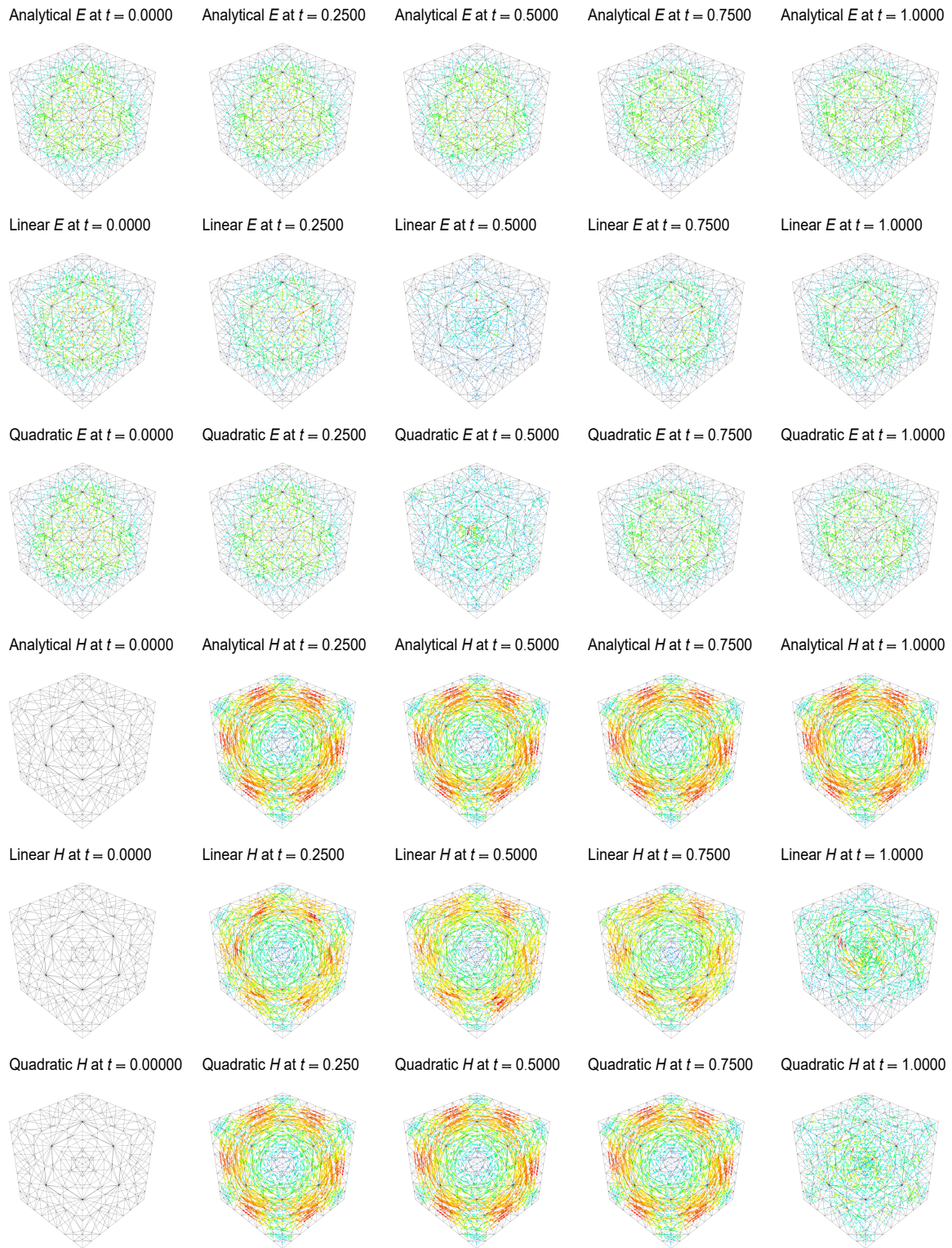


FIGURE 3.3: Linear and Quadratic finite elements, Crank-Nicolson: Plots of solutions at different time steps for the problem described in **Example 3.2** of Section 3.5 using the Crank-Nicolson time integration, and with both linear and quadratic Whitney forms as basis for the finite dimensional finite element spaces. The solutions computed for E and H using the quadratic elements match with the analytical solutions near identically while those computed with the linear elements have a somewhat pronounced approximation error.

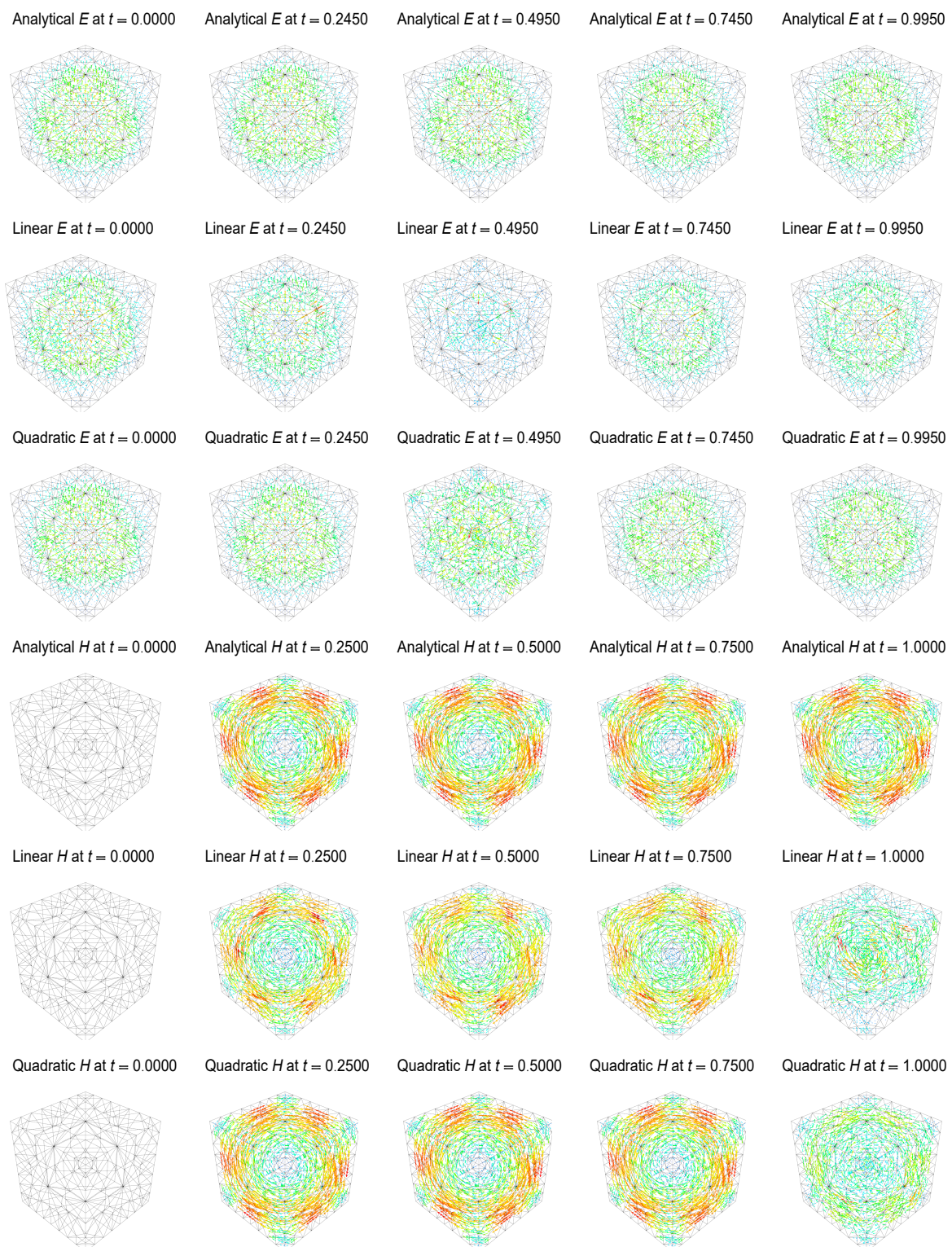


FIGURE 3.4: Linear and Quadratic finite elements, implicit leapfrog: Plots of solutions at different time steps for the problem described in **Example 3.2** of Section 3.5 using the implicit leapfrog time integration, and with both linear and quadratic Whitney forms as basis for the finite element spaces. The solutions computed for E and H using the quadratic elements match with the analytical solutions near identically while those computed with the linear elements have a perceptible approximation error.

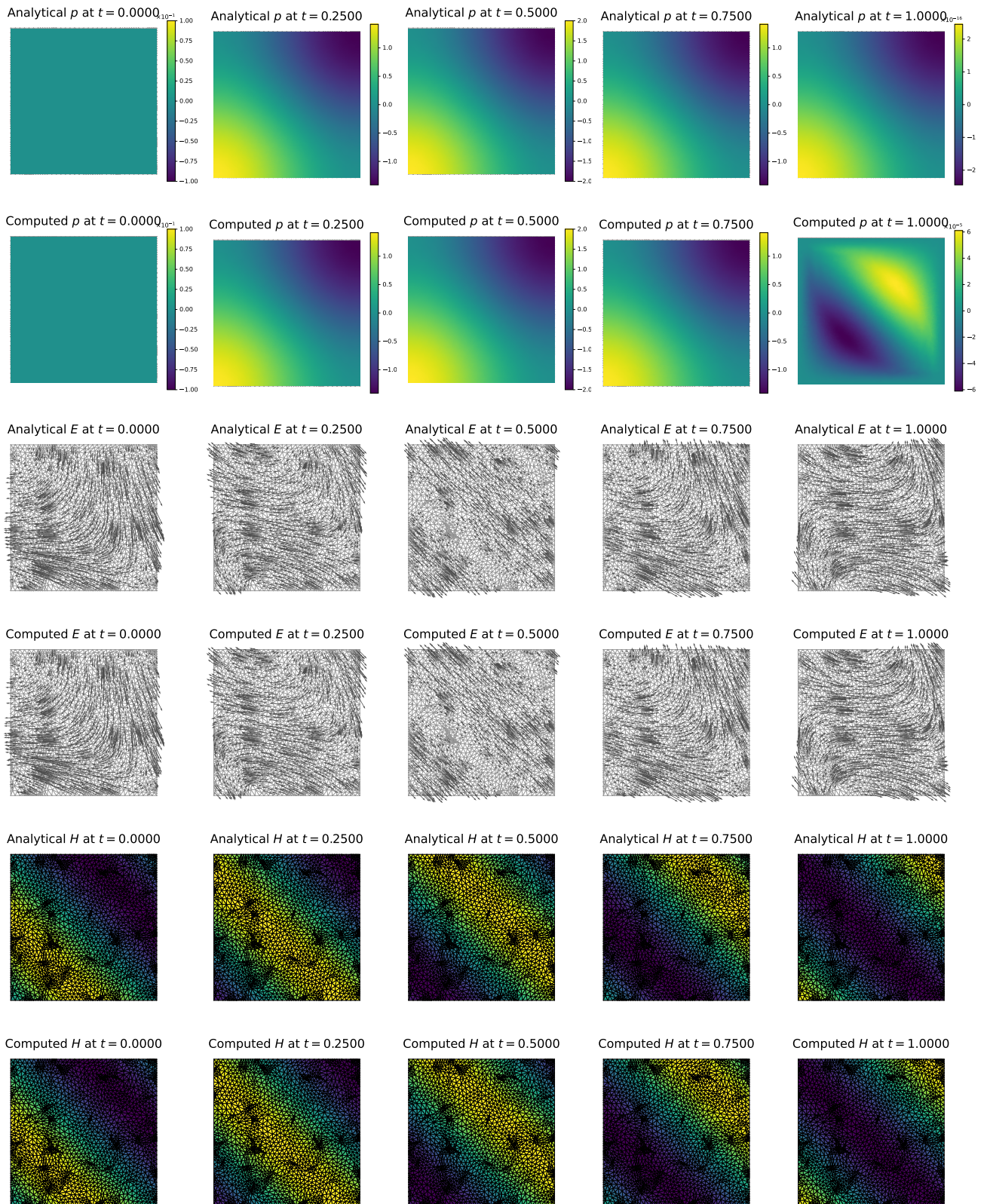


FIGURE 3.5: **Quadratic finite elements, Crank-Nicolson:** Plots of solutions at different time steps for the problem described in **Example 3.3** of Section 3.5 using the Crank-Nicolson time integration and quadratic Whitney forms as basis for the finite element spaces. The computed solutions for p , E and H match with the analytical solutions near identically.

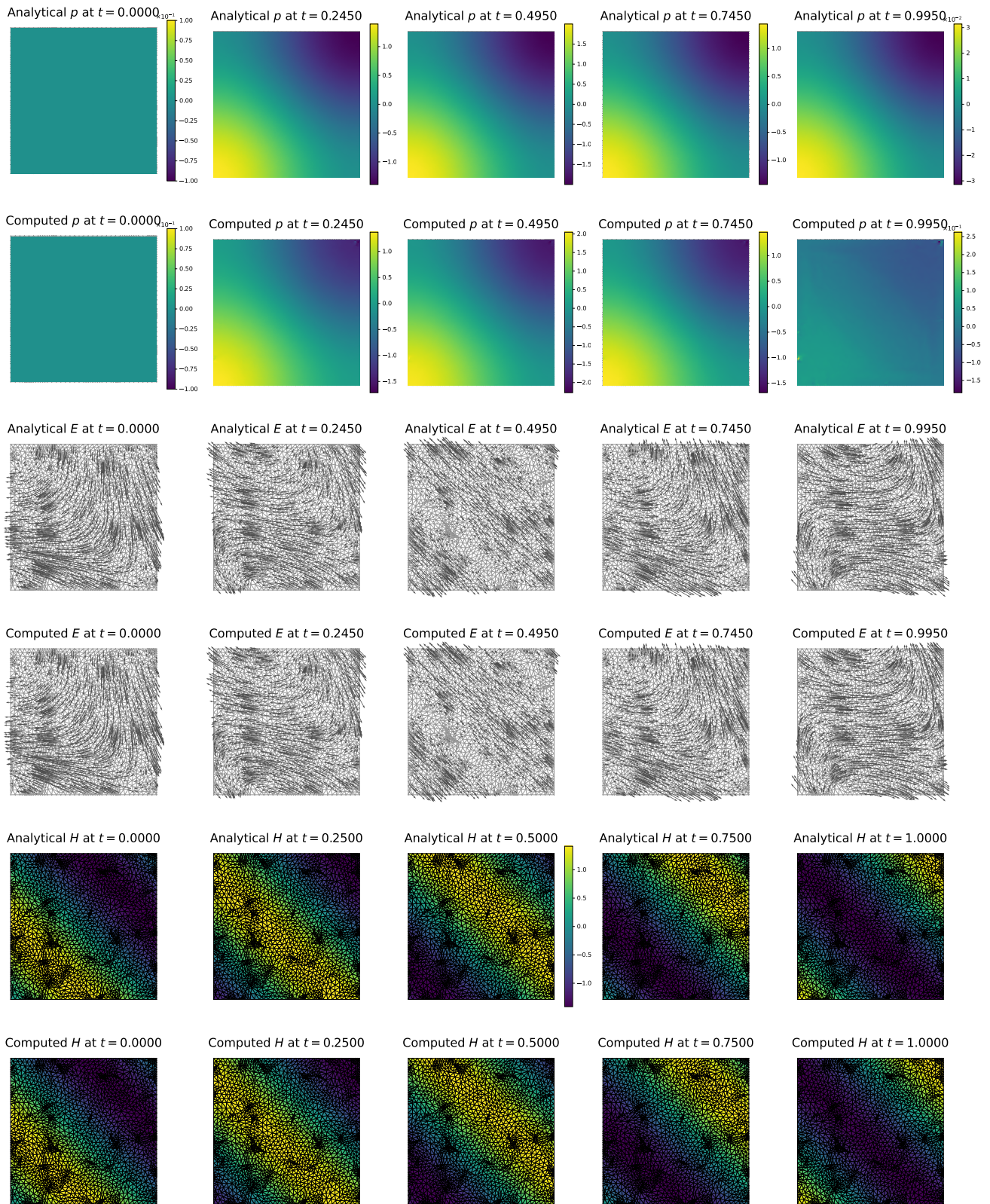


FIGURE 3.6: **Quadratic finite elements, implicit leapfrog:** Plots of solutions at different time steps for the problem described in **Example 3.3** of Section 3.5 using the implicit leapfrog time integration and quadratic Whitney forms. The computed solutions for p , E and H match with the analytical solutions near identically.

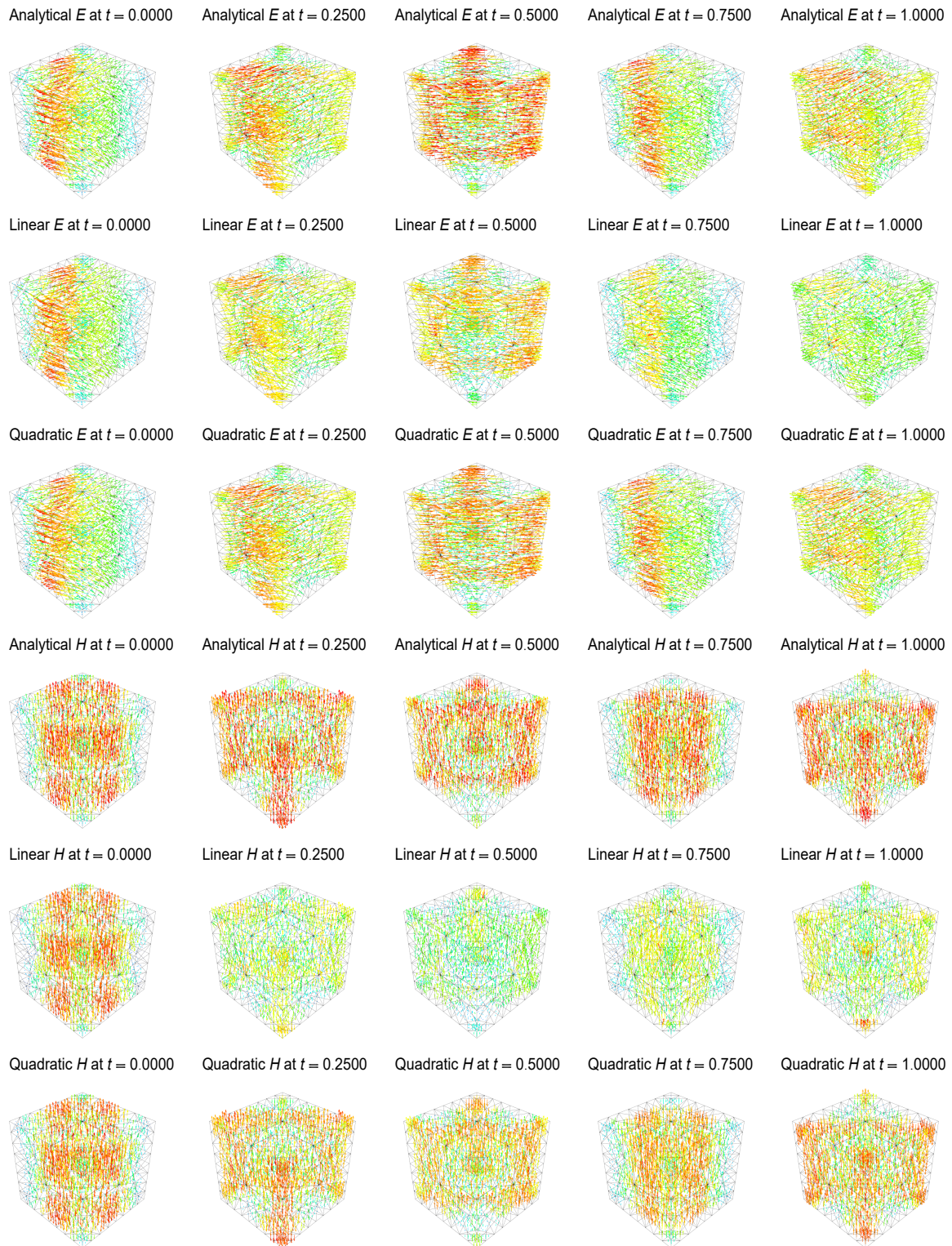


FIGURE 3.7: Linear and Quadratic finite elements, Crank-Nicolson: Plots of solutions at different time steps for the problem described in **Example 3.4** of Section 3.5 using the Crank-Nicolson time integration, and with linear and quadratic Whitney forms. The solutions computed for E and H using the quadratic elements match with the analytical solutions near identically while those computed with the linear elements have a perceptible approximation error.

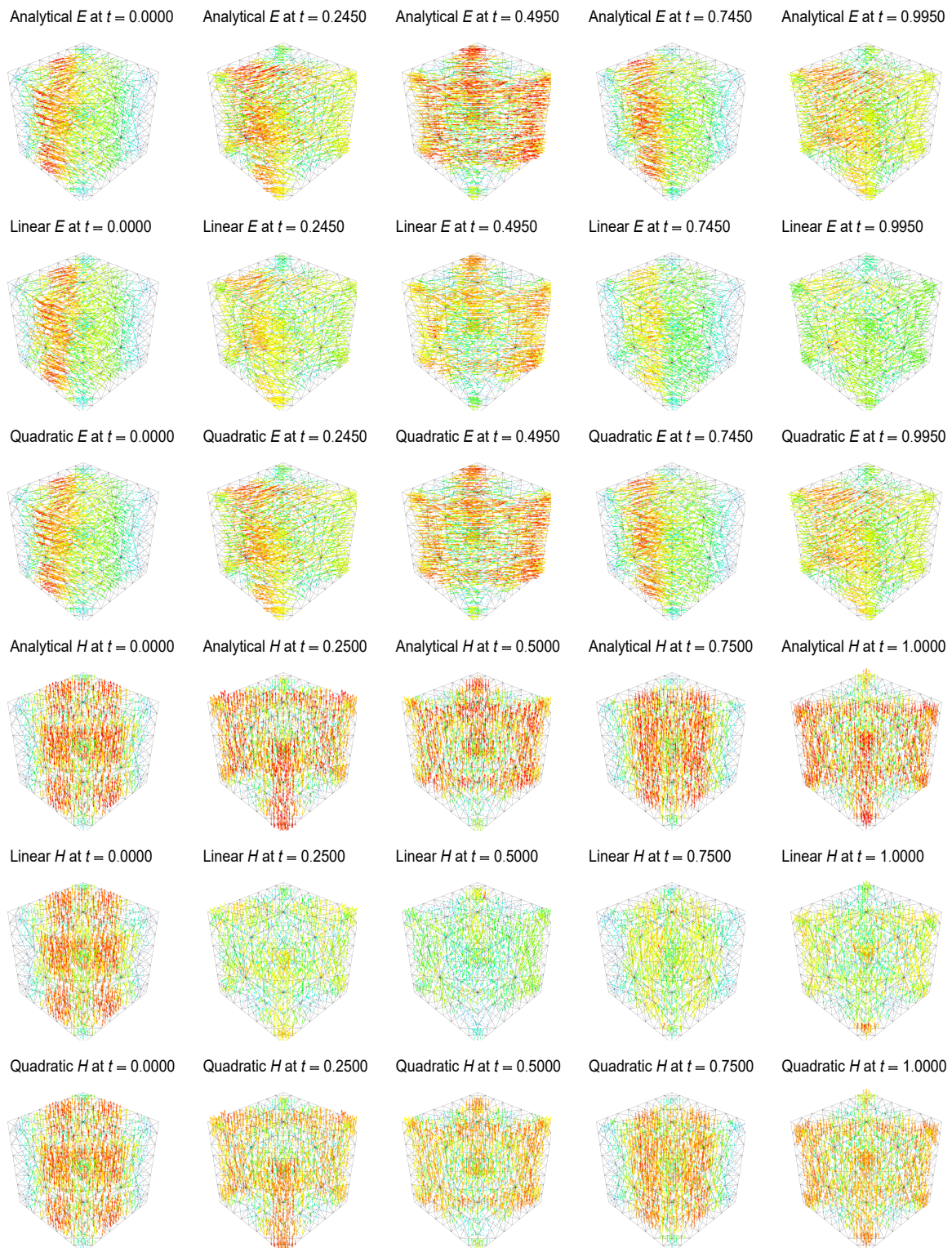


FIGURE 3.8: Linear and Quadratic finite elements, implicit leapfrog: Plots of solutions at different time steps for the problem described in **Example 3.4** of Section 3.5 using the implicit leapfrog time integration with linear and quadratic Whitney basis. The solutions computed for E and H using the quadratic elements match with the analytical solutions near identically while some approximation errors are evident for the computations with the linear basis.

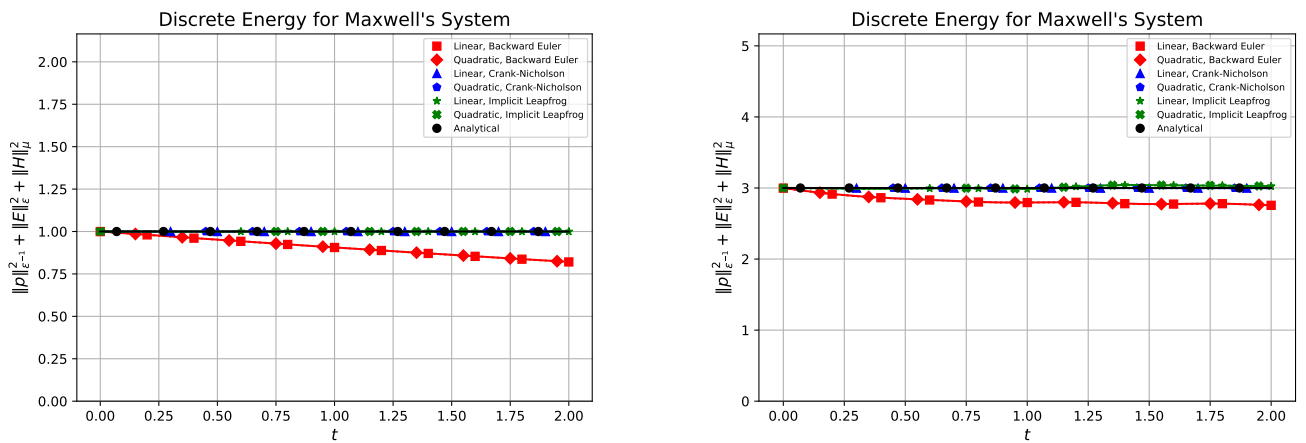


FIGURE 3.9: **Energy Conservation for Examples 1 (left) and 3 (right):** Energy is conserved for both Crank-Nicolson and implicit leapfrog time integration schemes with both linear and quadratic finite elements for the experiments whose solutions are shown in Figures 3.1, 3.2, 3.5 and 3.6. For contrast, we also show the energy decay using the stable but dissipative backward Euler time integration scheme for both these problems.

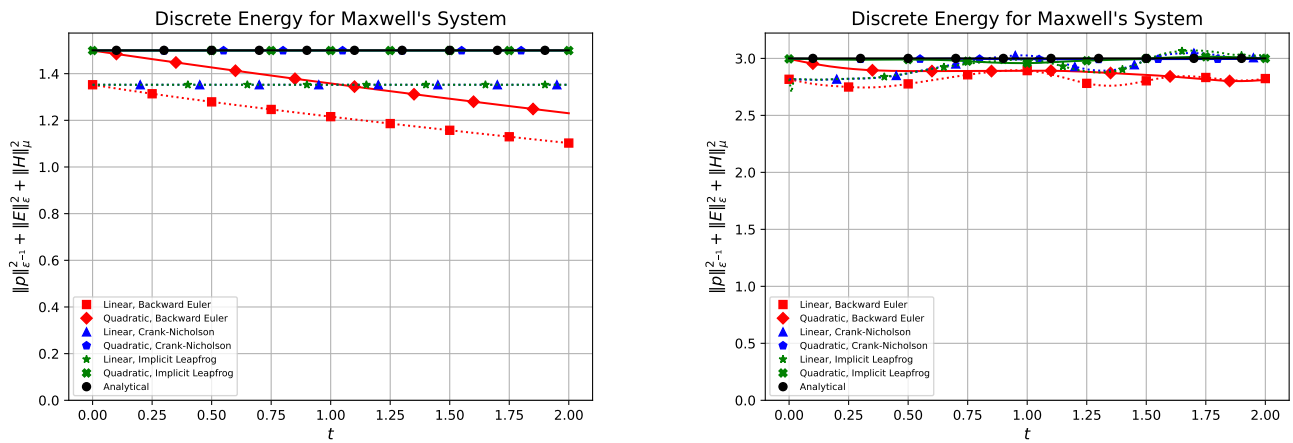


FIGURE 3.10: **Energy Conservation for Examples 2 (left) and 4 (right):** The discrete energy conservation for both Crank-Nicolson and implicit leapfrog time integration schemes in $\mathbb{R}^3$ are numerically more robust with quadratic finite elements for the experiments whose solutions are shown in Figures 3.3, 3.4, 3.7 and 3.8. For contrast, we again show the energy decay for backward Euler time integration scheme with both linear and quadratic finite elements. But, we also empirically validate that computational errors in our second-order accurate in time implicit methods are useful vis-a-vis energy conservation only by also using second-order accurate in space finite elements.

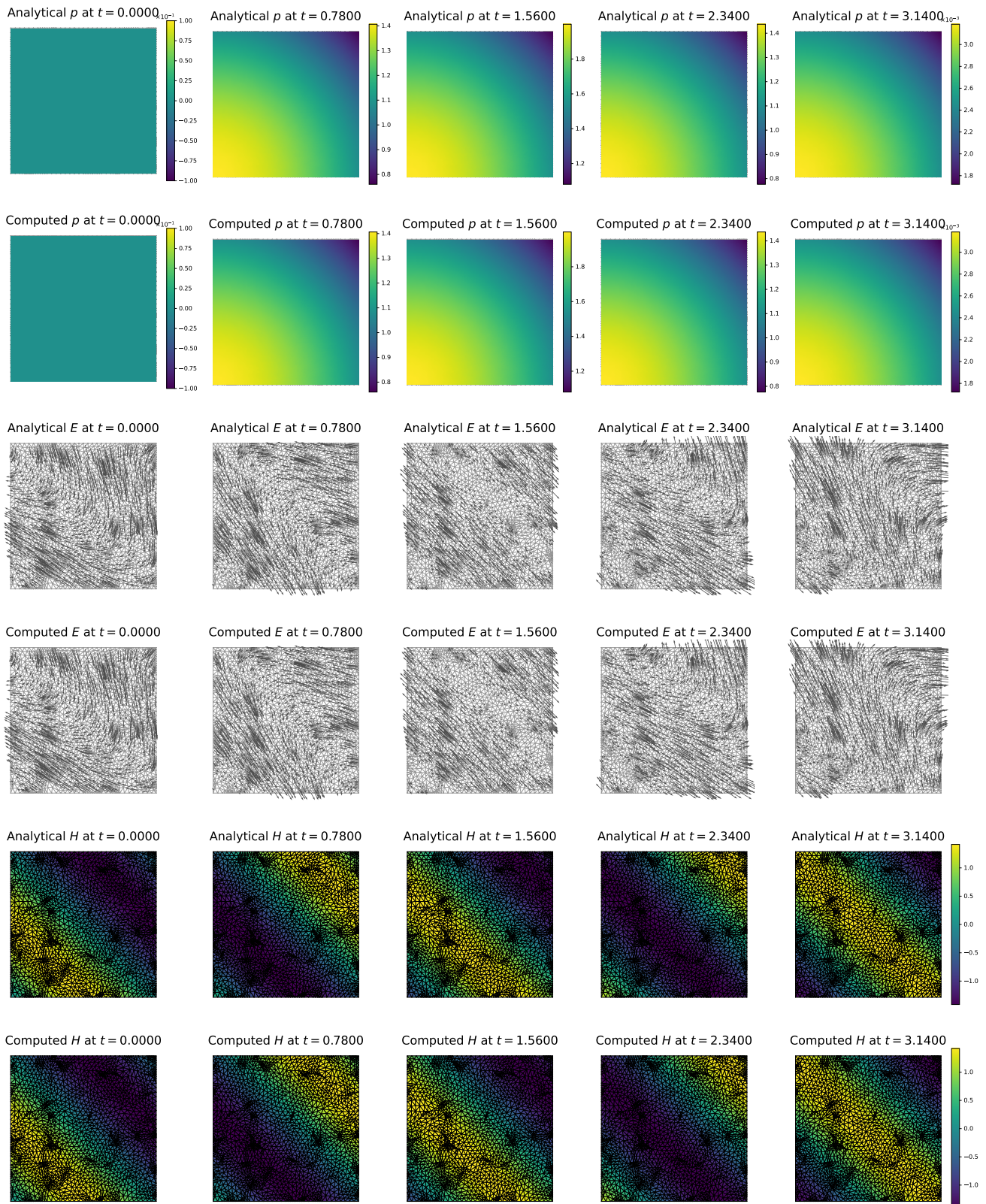


FIGURE 3.11: **Quadratic finite elements, Crank-Nicolson:** Plots of solutions at different time steps for the problem described in **Example 3.5** of Section 3.5 using the Crank-Nicolson time integration and quadratic Whitney forms as basis for the finite element spaces. The computed solutions for p , E and H match with the analytical solutions near identically.

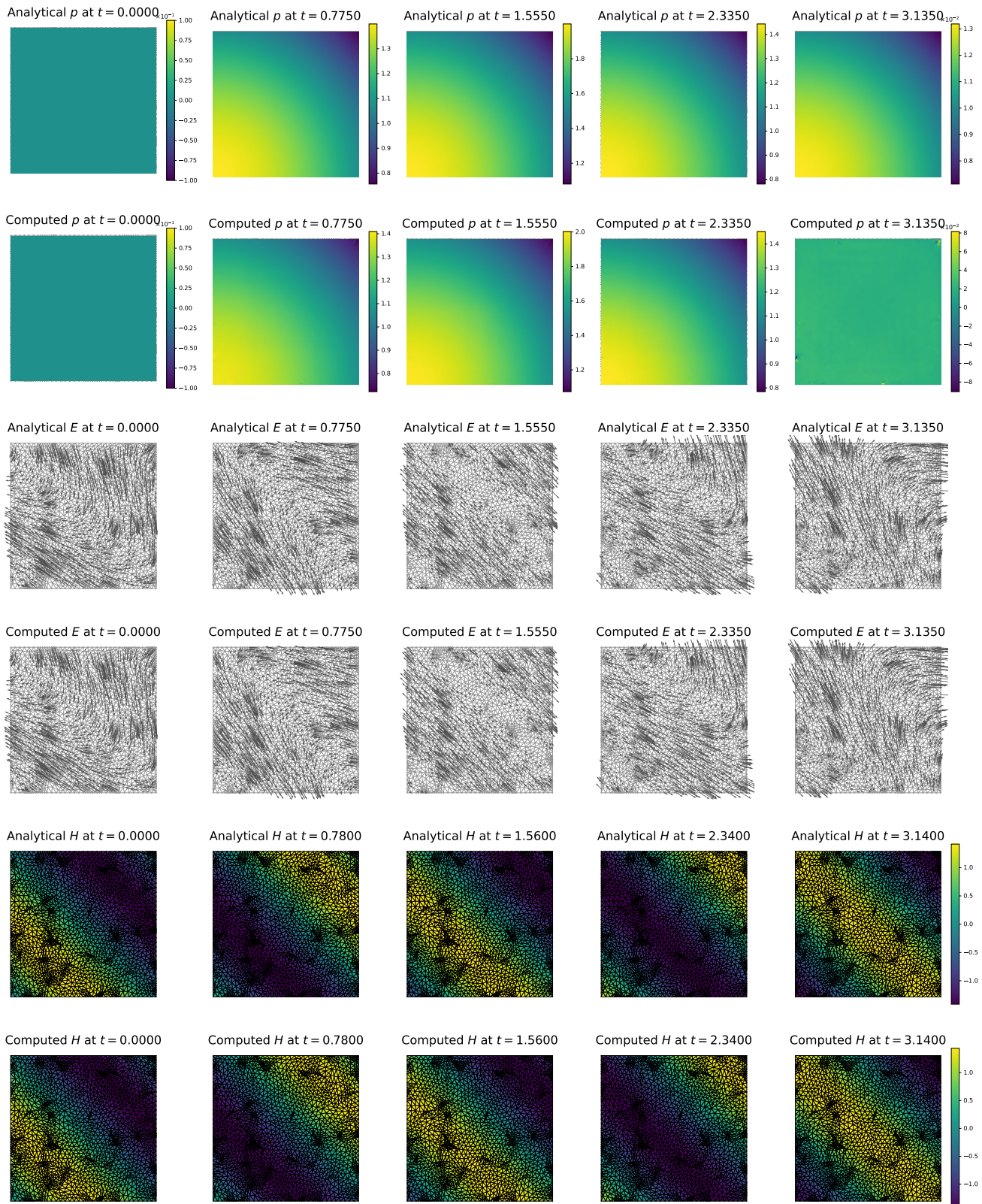


FIGURE 3.12: **Quadratic finite elements, implicit leapfrog:** Plots of solutions at different time steps for the problem described in Example 3.5 of Section 3.5 using the implicit leapfrog time integration and quadratic Whitney forms. The computed solutions for p , E and H match with the analytical solutions near identically.

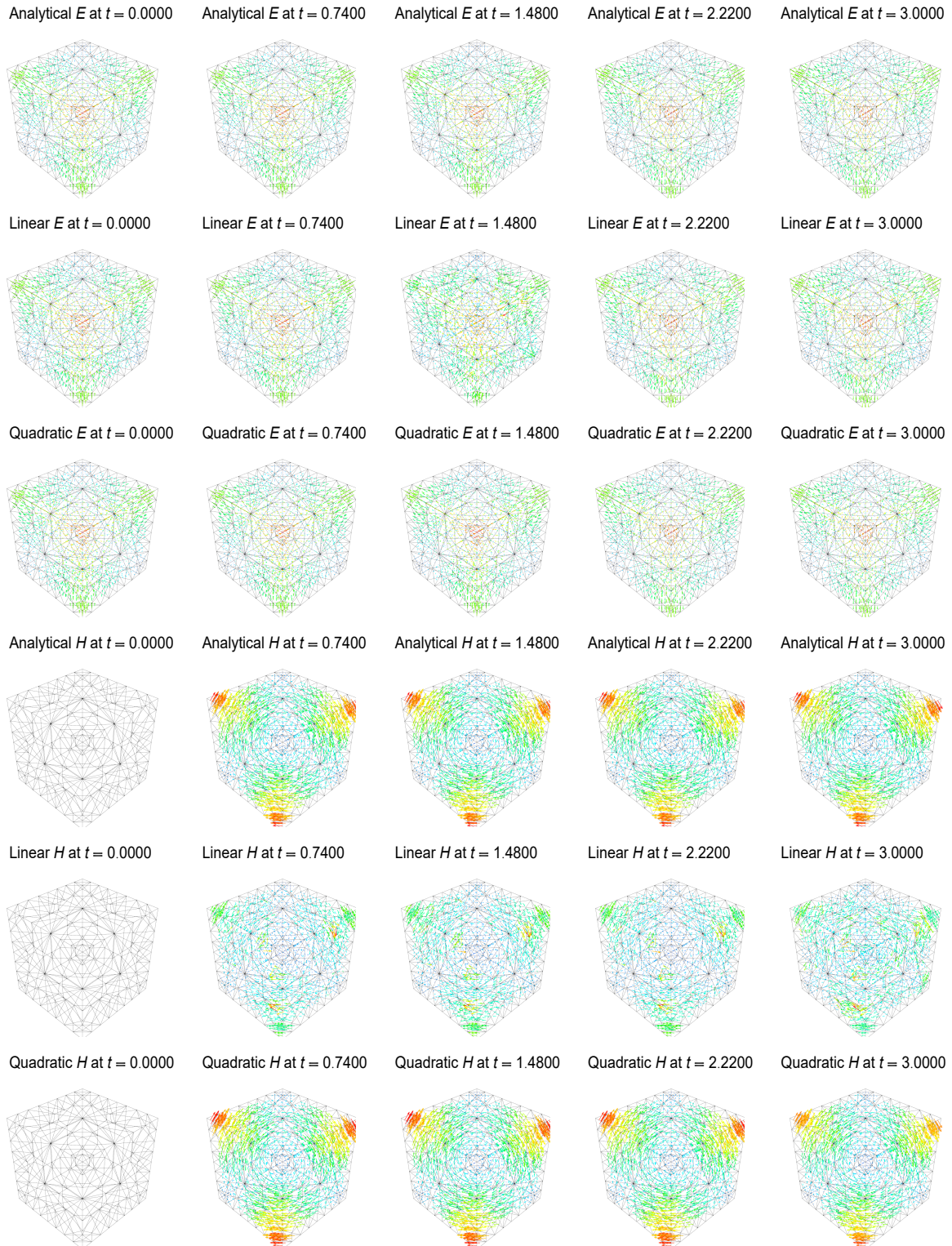


FIGURE 3.13: **Quadratic finite elements, Crank-Nicolson:** Plots of solutions at different time steps for the problem described in **Example 3.6** of Section 3.5 using the Crank-Nicolson time integration and quadratic Whitney forms as basis for the finite element spaces. The computed solutions for p , E and H match with the analytical solutions near identically.

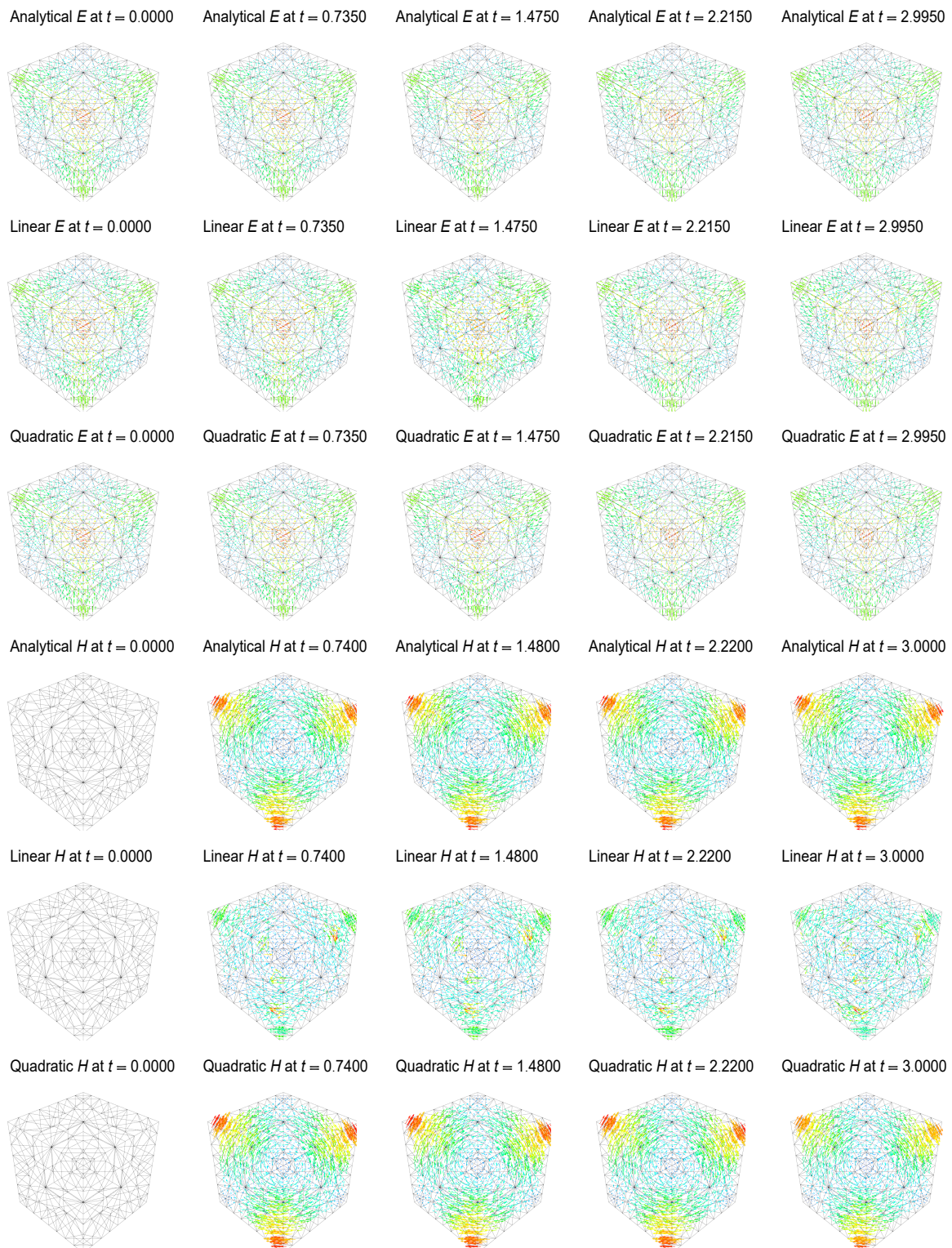


FIGURE 3.14: **Quadratic finite elements, implicit leapfrog:** Plots of solutions at different time steps for the problem described in **Example 3.6** of Section 3.5 using the implicit leapfrog time integration and quadratic Whitney forms. The computed solutions for p , E and H match with the analytical solutions near identically.

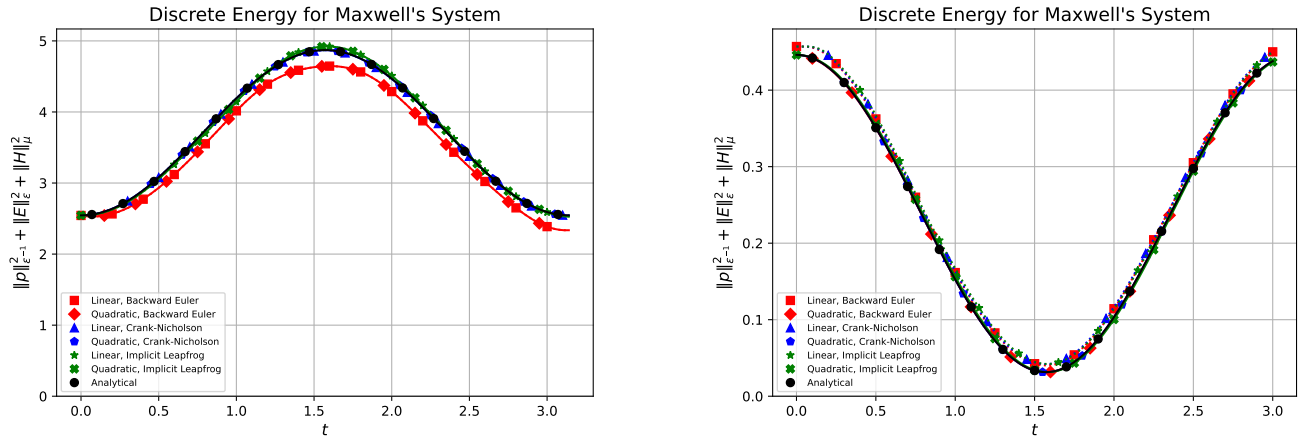


FIGURE 3.15: **Energy Tracking for Examples 5 (left) and 6 (right):** The discrete energy for these problems is time varying but bounded. Both Crank-Nicolson and implicit leapfrog time integration schemes in $\mathbb{R}^2$ (left) and $\mathbb{R}^3$ (right) are able to discretely exactly track this varying energy but the backward Euler scheme for **Example 3.5** is dissipative.

Chapter 4

Two Frameworks and their Fourth-order Implicit Schemes for Time Discretization of Maxwell's Equations

4.1 Introduction

In this chapter, we provide fourth-order implicit time discretizations of Maxwell's equations in a three-field formulation. Our original intent was to develop an arbitrary-order time discretization for this Maxwell's system in conjunction with a compatible spatial discretization of the field variables using finite element exterior calculus (FEEC) [Arn18]. However, our attempt at extending an earlier work [AK25] from its second-order accurate time discretization to merely a fourth-order accurate one led us to through not simply a complicated set of analysis which we believe warrants its own description that we provide hereto, but also that it helped us categorize into two classes a recipe for development of such methods. Our prescriptions therein then helped us methodically concoct these two energy conserving and implicit fourth-order in time discretization of the three-field Maxwell's system. We term these two methodologies as being a *spatial* or a *temporal* strategy. In our chapter here, we describe these two frameworks, provide the corresponding fourth-order time discretization schemes they yield, carry out their respective error analyses for both time semi-discretization and for the full error alongside a FEEC discretization of the spatial solution.

The inspirations for our two frameworks derive from [Fah09] for the spatial strategy and [BTT18] for the temporal strategy. Our main idea in writing out a fourth-order time discretization is rather straightforward: express the time component of the solution for each of the field variables in the Maxwell's system using a Taylor expansion (with the assumption that it is sufficiently regular) around a generic discretized time, and then use the Taylor series for deriving a higher-order and suitably truncated discretization of the first-order time derivatives in the Maxwell's equations. This requires a meaningful way to approximate each of the higher-order time derivatives in the Taylor expansion and our alternatives are to:

1. either use the Maxwell's equations themselves repeatedly to switch all these time derivatives to higher-order spatial derivatives, or
2. directly discretize the higher-order time derivative terms,

in the Taylor approximation of the solution. This can be enjoined with a spatial discretization by way of, as we carry out here, finite element methods to arrive at a numerical solution for the Maxwell's equations. An earlier work [AK25] already provides two flavors of second-order time implicit discretization that is also energy conserving and we seek their corresponding fourth-order generalizations here. [Fah09] set out to provide a strategy for such a solution scheme in the context of discontinuous Galerkin methods by

transforming the higher-order time derivatives to higher-order spatial derivatives. This idea is superficially similar to arbitrary-order non-oscillatory solutions for the advection schemes [MTT01]. Several of Toro's works stem from this viewpoint, however, the similarity ends merely at this idea of recasting time derivative into spatial derivatives. On the other hand, we merely carry forward Fahs's particularly simple ansatz, in itself originally credited to [You01] by Fahs, to its logical conclusion. This paves the way for our spatial strategy discretization that we so name because the temporal higher-order derivatives are replaced by spatial ones. On the other hand, Britt and coauthors in [BTT18] provide us with our idea for the temporal strategy, namely, that we just replace the higher-order time derivatives in a truncated Taylor approximation with suitably appropriate order finite differencing schemes. Again, using this principle, we can massage out a fourth-order time discretization that is implicit and with an energy conservation. Our underlying idea also somewhat superficially resembles or reminds one of the Cauchy-Kowalewskaya procedure which is a higher-order numerical technique in the context of generalized Riemann problems; see, for example, [MB20]. The similarity here is that we also transform a single time derivative in a higher-order time derivative term first by a spatial derivative from Maxwell's equation. We then diverge in our formalism by factoring the spatial derivatives through the time differentiations (given their mutual independence), and in temporal strategy finally approximate these using suitably higher-order finite difference schemes.

Our main contribution in this chapter is outlining and analyzing these two different energy conserving and fourth-order in time discretizations of the three-field formulation of the Maxwell's equations, showing that our proposed schemes are stable and converging in time at the correct rate of four as well as analyzing its convergence in conjunction with a compatible FEEC discretization of this Maxwell's system.

The organization of the rest of this chapter is as follows. In the remainder of this introductory section, we provide a brief overview of the three-field formulation of the Maxwell's equations, the two proposed schemes, a compressed and just necessary background on some of the analysis tools we use, and a formal description of the two frameworks which we have categorized for deriving such higher-order schemes. Sections 4.2 and 4.3 provide the detailed stability and error analysis of our two proposed schemes. Finally, in Section 4.4, we provide some validating numerical computations using our two schemes along with a discussion and future outlook.

4.1.1 Three-field Formulation of Maxwell's Equations

We again consider the three-field (p, E, H) -formulation of the Maxwell's equations which is as follows:

$$\left. \begin{aligned} \frac{\partial p}{\partial t} + \nabla \cdot \varepsilon E &= 0, \\ \nabla p + \varepsilon \frac{\partial E}{\partial t} - \nabla \times H &= 0, \\ \mu \frac{\partial H}{\partial t} + \nabla \times E &= 0, \end{aligned} \right\} \text{ in } \Omega \times (0, T], \quad (4.1a)$$

where $\Omega \subset \mathbb{R}^2/\mathbb{R}^3$ is a domain with Lipschitz boundary $\partial\Omega$, and $T > 0$. We use the following homogeneous boundary conditions:

$$p = 0, E \times n = 0, H \cdot n = 0 \text{ on } \partial\Omega \times (0, T], \quad (4.1b)$$

where n is the unit outward normal to $\partial\Omega$, and with the following initial conditions:

$$p(x, 0) = p_0(x), E(x, 0) = E_0(x), \text{ and } H(x, 0) = H_0(x) \text{ for } x \in \Omega, \quad (4.1c)$$

Furthermore, the initial conditions are required to satisfy $\nabla \cdot (\varepsilon E_0) = p_0$ and $\nabla \cdot (\mu H_0) = 0$ in Ω for consistency with the time evolution. This formulation has already been studied in earlier works such as [APZ13; AHZ17; Adl+21], and in particular we are continuing with the development of, and by suitably generalizing, the

schemes for it and their analysis as in [AK25]. Consequently, all of our background in this chapter is already well stated in [AK25] and we only provide here some essential details. As an example, we wish to restate here again that the homogeneous boundary conditions is merely for simplicity of our analysis and all our results can be extended to the non homogeneous case in a standard manner. We also wish to add that it is possible to extend our method and analysis to nonzero right-hand side functions in Equation (4.1a) but doing so will further complicate the analysis without providing any meaningful additional insight beyond what we state in this chapter and prove for the zero right-hand side functions. A nonzero right-hand side for the Maxwell's system can be discretized using our principles in a straightforward manner.

The variational formulation for Equations (4.1a) to (4.1c) is as following: for $t \in (0, T]$, find $(p(t), E(t), H(t)) \in \dot{H}_{\varepsilon-1}^1(\Omega) \times \dot{H}_{\varepsilon}(\text{curl}; \Omega) \times \dot{H}_{\mu}(\text{div}; \Omega)$ such that:

$$\left\langle \frac{\partial p}{\partial t}, \tilde{p} \right\rangle - \langle \varepsilon E, \nabla \tilde{p} \rangle = 0, \quad \tilde{p} \in \dot{H}_{\varepsilon-1}^1(\Omega), \quad (4.2a)$$

$$\langle \nabla p, \tilde{E} \rangle + \left\langle \varepsilon \frac{\partial E}{\partial t}, \tilde{E} \right\rangle - \langle H, \nabla \times \tilde{E} \rangle = 0, \quad \tilde{E} \in \dot{H}_{\varepsilon}(\text{curl}; \Omega), \quad (4.2b)$$

$$\left\langle \mu \frac{\partial H}{\partial t}, \tilde{H} \right\rangle + \langle \nabla \times E, \tilde{H} \rangle = 0, \quad \tilde{H} \in \dot{H}_{\mu}(\text{div}; \Omega), \quad (4.2c)$$

The stability and uniqueness of solution for this variational formulation can be found in [AK25, Theorem 3.1, Remark 3.1]. As with the description therein, here too we assume that the solution (p, E, H) to the variational formulation has sufficient regularity.

With this brief background, we present below our two fourth-order time implicit discretizations. We denote the first scheme as LF_4 and it is derived using our spatial strategy. LF_4 is a generalization of the second-order time implicit LF_2 scheme of [AK25, Equations 1.4(a)–(c)]. We label our second scheme as TS_4 and it is derived using our temporal strategy. TS_4 should be reminiscent of a fourth-order Crank Nicolson scheme, for example, as in [BT18, Equation 6].

To specify these schemes, we again fix some notation. We consider time $[0, T]$ to be uniformly discretized with a timestep $\Delta t > 0$. Then, $t^n := n\Delta t$ or $t^{n+\frac{1}{2}} := (n + 1/2)\Delta t$ for $n = 0, 1, \dots, N$ where $N\Delta t \approx T$. We also set $u^n := u(x, t^n)$ (and likewise for $u^{n+\frac{1}{2}}$) for $x \in \Omega$ where $u = \{p, E, H\}$ as appropriate.

LF₄ scheme

Find $(p(t^{n+1/2}), E(t^{n+1/2}), H(t^n)) \in \dot{H}_{\varepsilon-1}^1(\Omega) \times \dot{H}_{\varepsilon}(\text{curl}; \Omega) \times \dot{H}_{\mu}(\text{div}; \Omega)$ such that:

$$\left\langle \frac{p^{n+\frac{1}{2}} - p^{n-\frac{1}{2}}}{\Delta t}, \tilde{p} \right\rangle - \left\langle \frac{\varepsilon}{2} (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla \tilde{p} \right\rangle + \frac{\Delta t^2}{12} \left\langle \frac{\varepsilon}{2} \nabla \nabla \cdot (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla \tilde{p} \right\rangle = 0, \quad (4.3a)$$

$$\begin{aligned} & \left\langle \frac{1}{2} \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), \tilde{E} \right\rangle - \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), \nabla \nabla \cdot \tilde{E} \right\rangle + \left\langle \varepsilon \frac{E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}}{\Delta t}, \tilde{E} \right\rangle \\ & - \left\langle \frac{1}{2} (H^{n+1} + H^n), \nabla \times \tilde{E} \right\rangle - \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times (H^{n+1} + H^n), \nabla \times \tilde{E} \right\rangle = 0, \end{aligned} \quad (4.3b)$$

$$\left\langle \mu \frac{H^{n+1} - H^n}{\Delta t}, \tilde{H} \right\rangle + \left\langle \frac{1}{2} \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \tilde{H} \right\rangle$$

$$+ \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \varepsilon^{-1} \mu^{-1} \nabla \times \left(E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \right), \nabla \times \nabla \times \tilde{H} \right\rangle = 0, \quad (4.3c)$$

for all $\tilde{p} \in \mathring{H}_{\varepsilon^{-1}}^1(\Omega)$, $\tilde{E} \in \mathring{H}_{\varepsilon}(\text{curl}; \Omega)$ and $\tilde{H} \in \mathring{H}_{\mu}(\text{div}; \Omega)$. In order to bootstrap the computations from the given initial conditions, we will use the following restricted version of the scheme for the first half time step for p and E , and for the first time step for H :

$$\left\langle \frac{p^{\frac{1}{2}} - p_0}{\Delta t/2}, \tilde{p} \right\rangle - \frac{1}{2} \left\langle \frac{\varepsilon}{2} \left(E^{\frac{1}{2}} + E_0 \right), \nabla \tilde{p} \right\rangle + \frac{1}{8} \cdot \frac{\Delta t^2}{12} \left\langle \frac{\varepsilon}{2} \nabla \nabla \cdot \left(E^{\frac{1}{2}} + E_0 \right), \nabla \tilde{p} \right\rangle = 0, \quad (4.4a)$$

$$\begin{aligned} & \frac{1}{2} \left\langle \frac{1}{2} \nabla \left(p^{\frac{1}{2}} + p_0 \right), \tilde{E} \right\rangle - \frac{1}{8} \cdot \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \nabla \left(p^{\frac{1}{2}} + p_0 \right), \nabla \nabla \cdot \tilde{E} \right\rangle + \left\langle \varepsilon \frac{E^{\frac{1}{2}} - E_0}{\Delta t/2}, \tilde{E} \right\rangle \\ & - \left\langle \frac{1}{2} \left(H^1 + H_0 \right), \nabla \times \tilde{E} \right\rangle - \frac{1}{4} \cdot \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \left(H^1 + H_0 \right), \nabla \times \tilde{E} \right\rangle = 0, \end{aligned} \quad (4.4b)$$

$$\begin{aligned} & \left\langle \mu \frac{H^1 - H_0}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \left\langle \frac{1}{2} \nabla \times \left(E^{\frac{1}{2}} + E_0 \right), \tilde{H} \right\rangle \\ & + \frac{1}{8} \cdot \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \varepsilon^{-1} \mu^{-1} \nabla \times \left(E^{\frac{1}{2}} + E_0 \right), \nabla \times \nabla \times \tilde{H} \right\rangle = 0. \end{aligned} \quad (4.4c)$$

TS₄ scheme

Find $(p(t^{n+1/2}), E(t^{n+1/2}), H(t^n)) \in \mathring{H}_{\varepsilon^{-1}}^1(\Omega) \times \mathring{H}_{\varepsilon}(\text{curl}; \Omega) \times \mathring{H}_{\mu}(\text{div}; \Omega)$ such that:

$$\left\langle \varepsilon^{-1} \frac{p^{n+1} - p^{n-1}}{2\Delta t}, \tilde{p} \right\rangle - \frac{1}{6} \left\langle E^{n+1} + 4E^n + E^{n-1}, \nabla \tilde{p} \right\rangle = 0, \quad (4.5a)$$

$$\begin{aligned} & \frac{1}{6} \left\langle \nabla \left(p^{n+1} + 4p^n + p^{n-1} \right), \tilde{E} \right\rangle + \left\langle \varepsilon \frac{E^{n+1} - E^{n-1}}{2\Delta t}, \tilde{E} \right\rangle \\ & - \frac{1}{6} \left\langle \nabla \times \left(H^{n+1} + 4H^n + H^{n-1} \right), \tilde{E} \right\rangle = 0, \end{aligned} \quad (4.5b)$$

$$\left\langle \mu \frac{H^{n+1} - H^{n-1}}{2\Delta t}, \tilde{H} \right\rangle + \frac{1}{6} \left\langle \nabla \times \left(E^{n+1} + 4E^n + E^{n-1} \right), \tilde{H} \right\rangle = 0. \quad (4.5c)$$

for all $\tilde{p} \in \mathring{H}_{\varepsilon^{-1}}^1(\Omega)$, $\tilde{E} \in \mathring{H}_{\varepsilon}(\text{curl}; \Omega)$ and $\tilde{H} \in \mathring{H}_{\mu}(\text{div}; \Omega)$. Note that this scheme requires no bootstrapping and computations can begin from the provided initial conditions.

4.1.2 Two Strategies for Higher-Order Time Derivatives

We explain our strategies for obtaining a higher-order time discretization by using the following model problem. Let $u : \Omega \times [0, T] \rightarrow \mathbb{R}$ belonging to an appropriate function space (for example, $u \in \mathring{H}_{\varepsilon}(\text{curl}; \Omega; L^2(0, T)) \cap C^\infty(\Omega, (0, T))$) be the solution of the following partial differential equation (PDE):

$$\frac{\partial u}{\partial t} = \nabla \times u \quad \text{on } \Omega \times [0, T]. \quad (4.6)$$

We assume a uniform discretization of $[0, T]$ as specified earlier too and let $t^n := n\Delta t$, $n = 0, 1, \dots, N$ where $N\Delta t \approx T$ for some suitable choice of a fixed time step parameter $\Delta t > 0$. We now Taylor expand the solution around t^n to obtain:

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = \frac{\partial u}{\partial t}(t^n) + \frac{\Delta t^2}{6} \frac{\partial^3 u}{\partial t^3}(t^n) + O(\Delta t^4),$$

where $u^n := u(x, t^n)$ for $x \in \Omega$ and O is the standard Bachmann–Landau notation. Then, our two strategies can be explained as follows.

Spatial Strategy: In the above Taylor series, we transform the higher-order time derivatives into higher-order spatial derivatives using the PDE itself. Thus, in our example, $\frac{\partial^3 u}{\partial t^3} = \nabla \times \nabla \times \nabla \times u$, and so we now set up our spatially derived scheme to be a discretization of the following resultant PDE:

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = \nabla \times u^n + \frac{\Delta t^2}{6} \nabla \times \nabla \times \nabla \times u^n,$$

and we use carefully use FEEC principles for discretizing the spatial solution.

Temporal Strategy: In the temporal strategy, we first replace a single time derivative in all of the higher-order partial time derivatives using the PDE, and then discretize all remaining time derivatives using suitably higher-order finite difference schemes. The finite difference schemes for each of the time derivatives are so chosen as to not worsen the fourth-order accuracy in our time discretization. Thus, in our example, we replace one of the third-order time partial derivatives using Equation (4.6), and the resulting second-order time partial derivative is discretized by a second-order accurate finite difference approximation. In general, we will have more time derivatives and each of them will potentially require a different finite difference scheme. Thus, we obtain $\frac{\partial^3 u}{\partial t^3} = \nabla \times \frac{\partial^2 u}{\partial t^2} \approx \nabla \times \left(\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} \right)$, and set up our discrete scheme to be the following:

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = \nabla \times u^n + \frac{\Delta t^2}{6} \nabla \times \left(\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} \right).$$

In both our above strategies, the spatial component of the solution is discretized using a compatible and a suitably higher-order discretization using the de Rham finite element spaces as in FEEC. Applying these principles to our Maxwell's system in Equation (4.1a) leads us, respectively, to Equations (4.3a) to (4.3c) and Equations (4.5a) to (4.5c) as our proposed fourth-order time discretization schemes.

4.2 Characterization of LF₄ Scheme

4.2.1 Time Discretization Stability

For our implicit LF₄ time discretization scheme as in Equations (4.3a) to (4.3c) with bootstrapping using Equations (4.4a) to (4.4c), we now show that the semidiscretization in time yields a stable method. We also provide the error analysis to show its fourth-order convergence in time in terms of a sufficiently small time step $\Delta t > 0$. For the LF₄ scheme, we define the discrete energy at time t^n to be the following:

$$\mathcal{E}^n := \|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2. \quad (4.7)$$

Theorem 4.1 (Discrete Energy Conservation). *For the semidiscretization using the LF₄ scheme as given in Equations (4.3a) to (4.3c) and (4.4a) to (4.4c), for any fixed time step $\Delta t > 0$ sufficiently small, we have that:*

$$\mathcal{E}^n = \|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 = \|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_{\varepsilon}^2 + \|H_0\|_{\mu}^2 =: \mathcal{E}^0.$$

Proof. Since Equations (4.3a) to (4.3c) are true for all $\tilde{p} \in \dot{H}_{\varepsilon^{-1}}^1(\Omega)$, $\tilde{E} \in \dot{H}_{\varepsilon}(\text{curl}; \Omega)$, $\tilde{H} \in \dot{H}_{\mu}(\text{div}; \Omega)$, using $\tilde{p} = 2\Delta t \varepsilon^{-1} (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}})$, $\tilde{E} = 2\Delta t (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}})$ and $\tilde{H} = 2\Delta t (H^{n+1} + H^n)$ in them, we obtain the following:

$$2\langle p^{n+\frac{1}{2}} - p^{n-\frac{1}{2}}, \varepsilon^{-1} (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}) \rangle - \Delta t \langle E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}, \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}) \rangle + \frac{\Delta t^3}{12} \langle \nabla \nabla \cdot (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}) \rangle = 0,$$

$$\Delta t \langle \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \rangle - \frac{\Delta t^3}{12} \langle \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), \nabla \nabla \cdot (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}) \rangle + 2\langle \varepsilon (E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}), E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \rangle - \Delta t \langle (H^{n+1} + H^n), \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}) \rangle - \frac{\Delta t^3}{12} \langle \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times (H^{n+1} + H^n), \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}) \rangle = 0,$$

$$2\langle \mu (H^{n+1} - H^n), H^{n+1} + H^n \rangle + \Delta t \langle \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), H^{n+1} + H^n \rangle + \frac{\Delta t^3}{12} \langle \varepsilon^{-1} \mu^{-1} \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla \times \nabla \times (H^{n+1} + H^n) \rangle = 0.$$

Adding these equations together and using properties of the inner product, we get that:

$$2\langle \varepsilon^{-1} (p^{n+\frac{1}{2}} - p^{n-\frac{1}{2}}), p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}} \rangle + 2\langle \varepsilon (E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}), E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \rangle + 2\langle \mu (H^{n+1} - H^n), H^{n+1} + H^n \rangle = 0.$$

Summing over $n = 1$ to $N - 1$ leads us to:

$$\|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 - \|E^{\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 - \|H^1\|_{\mu}^2 = 0.$$

Now, $p^{\frac{1}{2}}$, $E^{\frac{1}{2}}$ and H^1 satisfy Equations (4.4a) to (4.4c), and so using $\tilde{p} = 2\Delta t \varepsilon^{-1} (p^{\frac{1}{2}} + p^0)$, $\tilde{E} = 2\Delta t (E^{\frac{1}{2}} + E^0)$ and $\tilde{H} = 4\Delta t (H^1 + H^0)$ and repeating the previous arguments leads us to the following estimate:

$$\|p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|p^0\|_{\varepsilon^{-1}}^2 + \|E^{\frac{1}{2}}\|_{\varepsilon}^2 - \|E^0\|_{\varepsilon}^2 + \|H^1\|_{\mu}^2 - \|H^0\|_{\mu}^2 = 0.$$

Adding these previous two equations, we get that:

$$\|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 = \|p^0\|_{\varepsilon^{-1}}^2 + \|E^0\|_{\varepsilon}^2 + \|H^0\|_{\mu}^2. \quad \square$$

Theorem 4.2 (Discrete Error Estimate). *For the semidiscretization using the LF₄ scheme as in Equations (4.3a) to (4.3c), and (4.4a) to (4.4c), for the solution $(\mathbf{p}, \mathbf{E}, \mathbf{H})$ of the variationally posed Maxwell's system as in Equations (4.2a) to (4.2c) with initial conditions as in Equation (4.1c), assuming sufficient regularity with $\mathbf{p} \in C^5(0, T; \dot{\mathbf{H}}_{\varepsilon^{-1}}^1(\Omega))$, $\mathbf{E} \in C^5(0, T; \dot{\mathbf{H}}_{\varepsilon}(\text{curl}; \Omega))$, and $\mathbf{H} \in C^5(0, T; \dot{\mathbf{H}}_{\mu}(\text{div}; \Omega))$, and for a time step $\Delta t > 0$ sufficiently small, there exists a positive bounded constant C independent of Δt such that:*

$$\|\mathbf{e}_{\mathbf{p}}^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|\mathbf{e}_{\mathbf{E}}^{N-\frac{1}{2}}\|_{\varepsilon} + \|\mathbf{e}_{\mathbf{H}}^N\|_{\mu} \leq C \left[(\Delta t)^4 + \|\mathbf{e}_{\mathbf{p}}^0\|_{\varepsilon^{-1}} + \|\mathbf{e}_{\mathbf{E}}^0\|_{\varepsilon} + \|\mathbf{e}_{\mathbf{H}}^0\|_{\mu} \right],$$

where $\mathbf{e}_{\mathbf{p}}^{n+\frac{1}{2}} := \mathbf{p}(t^{n+\frac{1}{2}}) - \mathbf{p}^{n+\frac{1}{2}}$, $\mathbf{e}_{\mathbf{E}}^{n+\frac{1}{2}} := \mathbf{E}(t^{n+\frac{1}{2}}) - \mathbf{E}^{n+\frac{1}{2}}$ and $\mathbf{e}_{\mathbf{H}}^n := \mathbf{H}(t^n) - \mathbf{H}^n$ are the errors in the time semidiscretization of $\mathbf{p}$, $\mathbf{E}$ and $\mathbf{H}$, respectively.

Proof. Using the Taylor remainder theorem, and expressing $\mathbf{p}(t)$ about $t = t^n$, we have that:

$$\begin{aligned} \mathbf{p}(t) = \mathbf{p}(t^n) &+ \frac{\partial \mathbf{p}}{\partial t}(t^n)(t - t^n) + \frac{\partial^2 \mathbf{p}}{\partial t^2}(t^n) \frac{(t - t^n)^2}{2!} + \frac{\partial^3 \mathbf{p}}{\partial t^3}(t^n) \frac{(t - t^n)^3}{3!} \\ &+ \frac{\partial^4 \mathbf{p}}{\partial t^4}(t^n) \frac{(t - t^n)^4}{4!} + \int_{t^n}^t \frac{(t-s)^4}{4!} \frac{\partial^5 \mathbf{p}}{\partial t^5}(s) ds, \end{aligned}$$

which when evaluated at $t = t^{n+\frac{1}{2}}$ and $t = t^{n-\frac{1}{2}}$ yields:

$$\begin{aligned} \mathbf{p}(t^{n+\frac{1}{2}}) &= \mathbf{p}(t^n) + \frac{\partial \mathbf{p}}{\partial t}(t^n)(t^{n+\frac{1}{2}} - t^n) + \frac{\partial^2 \mathbf{p}}{\partial t^2}(t^n) \frac{(t^{n+\frac{1}{2}} - t^n)^2}{2} + \frac{\partial^3 \mathbf{p}}{\partial t^3}(t^n) \frac{(t^{n+\frac{1}{2}} - t^n)^3}{6} \\ &+ \frac{\partial^4 \mathbf{p}}{\partial t^4}(t^n) \frac{(t^{n+\frac{1}{2}} - t^n)^4}{24} + \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^4}{24} \frac{\partial^5 \mathbf{p}}{\partial t^5}(s) ds, \\ \mathbf{p}(t^{n-\frac{1}{2}}) &= \mathbf{p}(t^n) + \frac{\partial \mathbf{p}}{\partial t}(t^n)(t^{n-\frac{1}{2}} - t^n) + \frac{\partial^2 \mathbf{p}}{\partial t^2}(t^n) \frac{(t^{n-\frac{1}{2}} - t^n)^2}{2} + \frac{\partial^3 \mathbf{p}}{\partial t^3}(t^n) \frac{(t^{n-\frac{1}{2}} - t^n)^3}{6} \\ &+ \frac{\partial^4 \mathbf{p}}{\partial t^4}(t^n) \frac{(t^{n-\frac{1}{2}} - t^n)^4}{24} + \int_{t^n}^{t^{n-\frac{1}{2}}} \frac{(t^{n-\frac{1}{2}} - s)^4}{24} \frac{\partial^5 \mathbf{p}}{\partial t^5}(s) ds. \end{aligned}$$

Subtracting these two equations, and using the result in the inner product term from the semidiscretization of the variational formulation leads to:

$$\left\langle \frac{\mathbf{p}(t^{n+\frac{1}{2}}) - \mathbf{p}(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{\mathbf{p}} \right\rangle = \left\langle \frac{\partial \mathbf{p}}{\partial t}(t^n), \tilde{\mathbf{p}} \right\rangle + \frac{\Delta t^2}{24} \left\langle \frac{\partial^3 \mathbf{p}}{\partial t^3}(t^n), \tilde{\mathbf{p}} \right\rangle + \langle \mathbf{R}_{\mathbf{p}}^n, \tilde{\mathbf{p}} \rangle,$$

in which we have defined that:

$$\mathbf{R}_{\mathbf{p}}^n := \frac{1}{\Delta t} \left[\int_{t^{n-\frac{1}{2}}}^{t^n} \frac{(t^{n-\frac{1}{2}} - s)^4}{24} \frac{\partial^5 \mathbf{p}}{\partial t^5}(s) ds + \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^4}{24} \frac{\partial^5 \mathbf{p}}{\partial t^5}(s) ds \right].$$

Similarly, for E and H, we have the following:

$$\begin{aligned}\langle \varepsilon \frac{E(t^{n+\frac{1}{2}}) - E(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{E} \rangle &= \langle \varepsilon \frac{\partial E}{\partial t}(t^n), \tilde{E} \rangle + \frac{\Delta t^2}{24} \langle \varepsilon \frac{\partial^3 E}{\partial t^3}(t^n), \tilde{E} \rangle + \langle \varepsilon R_E^n, \tilde{E} \rangle, \\ \langle \mu \frac{H(t^{n+1}) - H(t^n)}{\Delta t}, \tilde{H} \rangle &= \langle \mu \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}), \tilde{H} \rangle + \frac{\Delta t^2}{24} \langle \mu \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}), \tilde{H} \rangle + \langle \mu R_H^{n+\frac{1}{2}}, \tilde{H} \rangle,\end{aligned}$$

and in each of which we have defined that:

$$\begin{aligned}R_E^n &:= \frac{1}{\Delta t} \left[\int_{t^{n-\frac{1}{2}}}^{t^n} \frac{(t^{n-\frac{1}{2}} - s)^4}{24} \frac{\partial^5 E}{\partial t^5}(s) ds + \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^4}{24} \frac{\partial^5 E}{\partial t^5}(s) ds \right], \\ R_H^{n+\frac{1}{2}} &:= \frac{1}{\Delta t} \left[\int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^n - s)^4}{24} \frac{\partial^5 H}{\partial t^5}(s) ds + \int_{t^{n+\frac{1}{2}}}^{t^{n+1}} \frac{(t^{n+1} - s)^4}{24} \frac{\partial^5 H}{\partial t^5}(s) ds \right].\end{aligned}$$

Using these terms in the weak formulation as in Equations (4.2a) to (4.2c) at time $t = t^n$ for p and E terms, and at time $t = t^{n+\frac{1}{2}}$ for H and using the Taylor remainder theorem again with:

$$\begin{aligned}u(t^n) &:= \frac{u(t^{n+\frac{1}{2}}) + u(t^{n-\frac{1}{2}})}{2} - r_{1,u}^n, \quad H(t^{n+\frac{1}{2}}) := \frac{H(t^{n+1}) + H(t^n)}{2} - r_{1,H}^{n+\frac{1}{2}}, \\ u(t^n) &:= \frac{u(t^{n+\frac{1}{2}}) + u(t^{n-\frac{1}{2}})}{2} - \frac{\Delta t^2}{8} \frac{\partial^2 u}{\partial t^2}(t^n) - r_{2,u}^n, \\ H(t^{n+\frac{1}{2}}) &:= \frac{H(t^{n+1}) + H(t^n)}{2} - \frac{\Delta t^2}{8} \frac{\partial^2 H}{\partial t^2}(t^{n+\frac{1}{2}}) - r_{2,H}^{n+\frac{1}{2}},\end{aligned}$$

where u is either of p or E, and in which we have defined that:

$$\begin{aligned}r_{1,u}^n &:= \frac{1}{2} \left[\int_{t^n}^{t^{n+\frac{1}{2}}} (t^{n+\frac{1}{2}} - s) \frac{\partial^2 u}{\partial t^2}(s) ds - \int_{t^{n-\frac{1}{2}}}^{t^n} (t^{n-\frac{1}{2}} - s) \frac{\partial^2 u}{\partial t^2}(s) ds \right], \\ r_{1,H}^{n+\frac{1}{2}} &:= \frac{1}{2} \left[\int_{t^{n+\frac{1}{2}}}^{t^{n+1}} (t^{n+1} - s) \frac{\partial^2 H}{\partial t^2}(s) ds - \int_{t^n}^{t^{n+\frac{1}{2}}} (t^n - s) \frac{\partial^2 H}{\partial t^2}(s) ds \right], \\ r_{2,u}^n &:= \frac{1}{2} \left[\int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^3}{6} \frac{\partial^4 u}{\partial t^4}(s) ds - \int_{t^{n-\frac{1}{2}}}^{t^n} \frac{(t^{n-\frac{1}{2}} - s)^3}{6} \frac{\partial^4 u}{\partial t^4}(s) ds \right], \\ r_{2,H}^{n+\frac{1}{2}} &:= \frac{1}{2} \left[\int_{t^{n+\frac{1}{2}}}^{t^{n+1}} \frac{(t^{n+1} - s)^3}{6} \frac{\partial^4 H}{\partial t^4}(s) ds - \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^n - s)^3}{6} \frac{\partial^4 H}{\partial t^4}(s) ds \right].\end{aligned}$$

Using these, we obtain the following:

$$\begin{aligned} & \left\langle \frac{p(t^{n+\frac{1}{2}}) - p(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{p} \right\rangle - \left\langle \frac{\varepsilon}{2} \left(E(t^{n+\frac{1}{2}}) + E(t^{n-\frac{1}{2}}) \right), \nabla \tilde{p} \right\rangle \\ & + \frac{\Delta t^2}{12} \left\langle \frac{\varepsilon}{2} \nabla \nabla \cdot \left(E(t^{n+\frac{1}{2}}) + E(t^{n-\frac{1}{2}}) \right), \nabla \tilde{p} \right\rangle = \left\langle R_p^n + \varepsilon \nabla \cdot r_{2,E}^n - \frac{\Delta t^2}{12} \varepsilon \nabla \cdot \nabla \nabla \cdot r_{1,E}^n, \tilde{p} \right\rangle, \quad (4.8a) \end{aligned}$$

$$\begin{aligned} & \left\langle \frac{1}{2} \nabla \left(p(t^{n+\frac{1}{2}}) + p(t^{n-\frac{1}{2}}) \right), \tilde{E} \right\rangle - \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \nabla \left(p(t^{n+\frac{1}{2}}) + p(t^{n-\frac{1}{2}}) \right), \nabla \nabla \cdot \tilde{E} \right\rangle \\ & + \left\langle \varepsilon \frac{E(t^{n+\frac{1}{2}}) - E(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{E} \right\rangle - \left\langle \frac{1}{2} \left(H(t^{n+1}) + H(t^n) \right), \nabla \times \tilde{E} \right\rangle \\ & - \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \left(H(t^{n+1}) + H(t^n) \right), \nabla \times \tilde{E} \right\rangle \\ & = \left\langle \varepsilon R_E^n + \nabla r_{2,p}^n - \nabla \times r_{2,H}^{n+\frac{1}{2}} - \frac{\Delta t^2}{12} \nabla \nabla \cdot \nabla r_{1,p}^n - \frac{\Delta t^2}{12} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \nabla \times r_{1,H}^{n+\frac{1}{2}}, \tilde{E} \right\rangle, \quad (4.8b) \end{aligned}$$

$$\begin{aligned} & \left\langle \mu \frac{H(t^{n+1}) - H(t^n)}{\Delta t}, \tilde{H} \right\rangle + \left\langle \frac{1}{2} \nabla \times \left(E(t^{n+\frac{1}{2}}) + E(t^{n-\frac{1}{2}}) \right), \tilde{H} \right\rangle \\ & + \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \varepsilon^{-1} \mu^{-1} \nabla \times \left(E(t^{n+\frac{1}{2}}) + E(t^{n-\frac{1}{2}}) \right), \nabla \times \nabla \times \tilde{H} \right\rangle \\ & = \left\langle \mu R_H^{n+\frac{1}{2}} + \nabla \times r_{2,E}^n + \frac{\Delta t^2}{12} \varepsilon^{-1} \mu^{-1} \nabla \times \nabla \times \nabla \times r_{1,E}^n, \tilde{H} \right\rangle. \quad (4.8c) \end{aligned}$$

Then, subtracting these from each of their respective LF₄ scheme equations as in Equations (4.3a) to (4.3c) leads us to the following set of equations:

$$\begin{aligned} & \left\langle \frac{e_p^{n+\frac{1}{2}} - e_p^{n-\frac{1}{2}}}{\Delta t}, \tilde{p} \right\rangle - \frac{1}{2} \left\langle \varepsilon \left(e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}} \right), \nabla \tilde{p} \right\rangle + \frac{\Delta t^2}{24} \left\langle \varepsilon \nabla \nabla \cdot \left(e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}} \right), \nabla \tilde{p} \right\rangle \\ & = \left\langle R_p^n + \varepsilon \nabla \cdot r_{2,E}^n - \frac{\Delta t^2}{12} \varepsilon \nabla \cdot \nabla \nabla \cdot r_{1,E}^n, \tilde{p} \right\rangle, \end{aligned}$$

$$\begin{aligned} & \frac{1}{2} \left\langle \nabla \left(e_p^{n+\frac{1}{2}} + e_p^{n-\frac{1}{2}} \right), \tilde{E} \right\rangle - \frac{\Delta t^2}{24} \left\langle \nabla \left(e_p^{n+\frac{1}{2}} + e_p^{n-\frac{1}{2}} \right), \nabla \nabla \cdot \tilde{E} \right\rangle + \left\langle \varepsilon \frac{e_E^{n+\frac{1}{2}} - e_E^{n-\frac{1}{2}}}{\Delta t}, \tilde{E} \right\rangle \\ & - \frac{1}{2} \left\langle e_H^{n+1} + e_H^n, \nabla \times \tilde{E} \right\rangle - \frac{\Delta t^2}{24} \left\langle \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \left(e_H^{n+1} + e_H^n \right), \nabla \times \tilde{E} \right\rangle \\ & = \left\langle \varepsilon R_E^n + \nabla r_{2,p}^n - \nabla \times r_{2,H}^{n+\frac{1}{2}} - \frac{\Delta t^2}{12} \nabla \nabla \cdot \nabla r_{1,p}^n - \frac{\Delta t^2}{12} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \nabla \times r_{1,H}^{n+\frac{1}{2}}, \tilde{E} \right\rangle, \end{aligned}$$

$$\begin{aligned} & \left\langle \mu \frac{e_H^{n+1} - e_H^n}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \left\langle \nabla \times \left(e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}} \right), \tilde{H} \right\rangle \\ & + \frac{\Delta t^2}{24} \left\langle \varepsilon^{-1} \mu^{-1} \nabla \times \left(e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}} \right), \nabla \times \nabla \times \tilde{H} \right\rangle \\ & = \left\langle \mu R_H^{n+\frac{1}{2}} + \nabla \times r_{2,E}^n + \frac{\Delta t^2}{12} \varepsilon^{-1} \mu^{-1} \nabla \times \nabla \times \nabla \times r_{1,E}^n, \tilde{H} \right\rangle. \end{aligned}$$

Likewise, for the semidiscrete approximation of the initial system as in Equations (4.4a) to (4.4c), we obtain for their errors the following system of equations:

$$\begin{aligned} \left\langle \frac{e_p^{\frac{1}{2}} - e_p^0}{\Delta t/2}, \tilde{p} \right\rangle - \frac{1}{4} \left\langle \varepsilon \left(e_E^{\frac{1}{2}} + e_E^0 \right), \nabla \tilde{p} \right\rangle + \frac{\Delta t^2}{192} \left\langle \varepsilon \nabla \nabla \cdot \left(e_E^{\frac{1}{2}} + e_E^0 \right), \nabla \tilde{p} \right\rangle \\ = \left\langle \frac{1}{2} R_p^0 - \frac{\varepsilon}{2} \nabla \cdot r_{2,E}^0 - \frac{\Delta t^2}{96} \varepsilon \nabla \cdot \nabla \nabla \cdot r_{1,E}^0, \tilde{p} \right\rangle, \end{aligned}$$

$$\begin{aligned} \frac{1}{4} \left\langle \nabla \left(e_p^{\frac{1}{2}} + e_p^0 \right), \tilde{E} \right\rangle - \frac{\Delta t^2}{192} \left\langle \nabla \left(e_p^{\frac{1}{2}} + e_p^0 \right), \nabla \nabla \cdot \tilde{E} \right\rangle + \left\langle \varepsilon \frac{e_E^{\frac{1}{2}} - e_E^0}{\Delta t/2}, \tilde{E} \right\rangle - \frac{1}{2} \left\langle e_H^1 + e_H^0, \nabla \times \tilde{E} \right\rangle \\ - \frac{\Delta t^2}{96} \left\langle \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \left(e_H^1 + e_H^0 \right), \tilde{E} \right\rangle \\ = \left\langle \frac{\varepsilon}{2} R_E^0 + \frac{1}{2} \nabla r_{2,p}^0 - \nabla \times r_{2,H}^{\frac{1}{2}} - \frac{\Delta t^2}{96} \nabla \nabla \cdot \nabla r_{1,p}^0 - \frac{\Delta t^2}{48} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \nabla \times r_{1,H}^{\frac{1}{2}}, \tilde{E} \right\rangle, \end{aligned}$$

$$\begin{aligned} \left\langle \mu \frac{e_H^1 - e_H^0}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{4} \left\langle \nabla \times \left(e_E^{\frac{1}{2}} + e_E^0 \right), \tilde{H} \right\rangle + \frac{\Delta t^2}{192} \left\langle \varepsilon^{-1} \mu^{-1} \nabla \times \left(e_E^{\frac{1}{2}} + e_E^0 \right), \nabla \times \nabla \times \tilde{H} \right\rangle \\ = \left\langle \mu R_H^{\frac{1}{2}} + \frac{1}{2} \nabla \times r_{2,E}^0 + \frac{\Delta t^2}{96} \varepsilon^{-1} \mu^{-1} \nabla \times \nabla \times \nabla \times r_{1,E}^0, \tilde{H} \right\rangle, \end{aligned}$$

and here we define the initial remainder terms to be as follows:

$$\begin{aligned} R_u^0 &:= \frac{1}{\Delta t} \left[\int_{t^0}^{t^{\frac{1}{4}}} \frac{(t^0 - s)^4}{24} \frac{\partial^5 u}{\partial t^5}(s) ds + \int_{t^{\frac{1}{4}}}^{t^{\frac{1}{2}}} \frac{(t^{\frac{1}{2}} - s)^4}{24} \frac{\partial^5 u}{\partial t^5}(s) ds \right], \\ R_H^{\frac{1}{2}} &:= \frac{1}{\Delta t} \left[\int_{t^0}^{t^{\frac{1}{2}}} \frac{(t^0 - s)^4}{24} \frac{\partial^5 H}{\partial t^5}(s) ds + \int_{t^{\frac{1}{2}}}^{t^1} \frac{(t^1 - s)^4}{24} \frac{\partial^5 H}{\partial t^5}(s) ds \right], \\ r_{1,u}^0 &:= \frac{1}{2} \left[\int_{t^{\frac{1}{4}}}^{t^{\frac{1}{2}}} (t^{\frac{1}{2}} - s) \frac{\partial^2 u}{\partial t^2}(s) ds - \int_{t^0}^{t^{\frac{1}{4}}} (t^0 - s) \frac{\partial^2 u}{\partial t^2}(s) ds \right], \\ r_{1,H}^{\frac{1}{2}} &:= \frac{1}{2} \left[\int_{t^{\frac{1}{2}}}^{t^1} (t^1 - s) \frac{\partial^2 H}{\partial t^2}(s) ds - \int_{t^0}^{t^{\frac{1}{2}}} (t^0 - s) \frac{\partial^2 H}{\partial t^2}(s) ds \right], \\ r_{2,u}^0 &:= \frac{1}{2} \left[\int_{t^{\frac{1}{4}}}^{t^{\frac{1}{2}}} \frac{(t^{\frac{1}{2}} - s)^3}{6} \frac{\partial^4 u}{\partial t^4}(s) ds - \int_{t^0}^{t^{\frac{1}{4}}} \frac{(t^0 - s)^3}{6} \frac{\partial^4 u}{\partial t^4}(s) ds \right], \\ r_{2,H}^{\frac{1}{2}} &:= \frac{1}{2} \left[\int_{t^{\frac{1}{2}}}^{t^1} \frac{(t^1 - s)^3}{6} \frac{\partial^4 H}{\partial t^4}(s) ds - \int_{t^0}^{t^{\frac{1}{2}}} \frac{(t^0 - s)^3}{6} \frac{\partial^4 H}{\partial t^4}(s) ds \right]. \end{aligned}$$

Now, in this set of weak formulation equations for the errors, we choose the test functions to be $\tilde{p} = 2\Delta t \varepsilon^{-1} (e_p^{n+\frac{1}{2}} + e_p^{n-\frac{1}{2}})$, $\tilde{E} = 2\Delta t (e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}})$ and $\tilde{H} = 2\Delta t (e_H^n + e_H^{n-1})$. Next, by following essentially the same sequence of steps as in Theorem 4.1, we obtain the estimate for these error terms to be:

$$\begin{aligned} & \|e_p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|e_p^{n-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{n+\frac{1}{2}}\|_{\varepsilon}^2 - \|e_E^{n-\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^{n+1}\|_{\mu}^2 - \|e_H^n\|_{\mu}^2 \\ & \leq \Delta t \left[\|e_p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_p^{n-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|e_E^{n-\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^{n+1}\|_{\mu}^2 + \|e_H^n\|_{\mu}^2 \right] \\ & + \Delta t \left[\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 + \|\nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{2,E}^n\|_{\varepsilon}^2 \right. \\ & \quad \left. + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{2,H}^{n+\frac{1}{2}}\|_{\mu}^2 + \frac{\Delta t^4}{144} \left(\|\nabla \nabla \cdot \nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \nabla \nabla \cdot r_{1,E}^n\|_{\varepsilon}^2 \right. \right. \\ & \quad \left. \left. + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{1,E}^n\|_{\varepsilon}^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{1,H}^{n+\frac{1}{2}}\|_{\mu}^2 \right) \right]. \end{aligned}$$

Summing over $n = 1$ to $N - 1$, we get:

$$\begin{aligned} & \|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|e_p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon}^2 - \|e_E^{\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^N\|_{\mu}^2 - \|e_H^1\|_{\mu}^2 \leq \Delta t \left[\|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 \right. \\ & \quad \left. + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|e_E^{\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^N\|_{\mu}^2 + \|e_H^1\|_{\mu}^2 \right] + 2\Delta t \sum_{n=1}^{N-2} \left[\|e_p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^{n+1}\|_{\mu}^2 \right] \\ & + \Delta t \sum_{n=1}^{N-1} \left[\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 + \|\nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{2,E}^n\|_{\varepsilon}^2 \right. \\ & \quad \left. + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{2,H}^{n+\frac{1}{2}}\|_{\mu}^2 + \frac{\Delta t^4}{144} \left(\|\nabla \nabla \cdot \nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \nabla \nabla \cdot r_{1,E}^n\|_{\varepsilon}^2 \right. \right. \\ & \quad \left. \left. + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{1,E}^n\|_{\varepsilon}^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{1,H}^{n+\frac{1}{2}}\|_{\mu}^2 \right) \right]. \end{aligned}$$

Similarly, for the initial Equations (4.4a) to (4.4c) which bootstrap our computations, by choosing test functions to be $\tilde{p} = 2\Delta t \varepsilon^{-1} (e_p^{\frac{1}{2}} + e_p^0)$, $\tilde{E} = 2\Delta t (e_E^{\frac{1}{2}} + e_E^0)$ and $\tilde{H} = 4\Delta t (e_H^1 + e_H^0)$, we obtain:

$$\begin{aligned} & \|e_p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^{\frac{1}{2}}\|_{\varepsilon}^2 - \|e_E^0\|_{\varepsilon}^2 + \|e_H^1\|_{\mu}^2 - \|e_H^0\|_{\mu}^2 \\ & \leq \Delta t \left[\|e_p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^{\frac{1}{2}}\|_{\varepsilon}^2 + \|e_E^0\|_{\varepsilon}^2 + \|e_H^1\|_{\mu}^2 + \|e_H^0\|_{\mu}^2 \right] \\ & + \Delta t \left[\|R_p^0\|_{\varepsilon^{-1}}^2 + \|R_E^0\|_{\varepsilon}^2 + \|R_H^{\frac{1}{2}}\|_{\mu}^2 + \|\nabla r_{2,p}^0\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{2,E}^0\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{2,E}^0\|_{\varepsilon}^2 \right. \\ & \quad \left. + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{2,H}^{\frac{1}{2}}\|_{\mu}^2 + \frac{\Delta t^4}{144} \left(\|\nabla \nabla \cdot \nabla r_{1,p}^0\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \nabla \nabla \cdot r_{1,E}^0\|_{\varepsilon}^2 \right. \right. \\ & \quad \left. \left. + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{1,E}^0\|_{\varepsilon}^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{1,H}^{\frac{1}{2}}\|_{\mu}^2 \right) \right]. \end{aligned}$$

By next adding the previous two equations, using the initial conditions as in Equation (4.1c), and the positivity of the right hand side terms in the resulting equation, we get that:

$$\begin{aligned} & \|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^N\|_{\mu}^2 \leq \frac{1 + \Delta t}{1 - \Delta t} \left[\|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^0\|_{\varepsilon}^2 + \|e_H^0\|_{\mu}^2 \right] + \\ & \frac{\Delta t}{1 - \Delta t} \sum_{n=0}^{N-1} \left[2 \left(\|e_p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^{n+1}\|_{\mu}^2 \right) + \left(\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \|\nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{2,H}^{n+\frac{1}{2}}\|_{\mu}^2 \\
& + \frac{\Delta t^4}{144} \left(\|\nabla \nabla \cdot \nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \nabla \nabla \cdot r_{1,E}^n\|_{\varepsilon}^2 \right. \\
& \quad \left. + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{1,E}^n\|_{\varepsilon}^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{1,H}^{n+\frac{1}{2}}\|_{\mu}^2 \right).
\end{aligned}$$

Applying the discrete Gronwall inequality 1.10, we obtain the following estimate:

$$\begin{aligned}
\|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^N\|_{\mu}^2 & \leq \left[\frac{6\Delta t}{5} \sum_{n=0}^{N-1} \left(\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 \right. \right. \\
& + \|\nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{2,H}^{n+\frac{1}{2}}\|_{\mu}^2 \\
& + \frac{\Delta t^4}{144} \left(\|\nabla \nabla \cdot \nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \nabla \nabla \cdot r_{1,E}^n\|_{\varepsilon}^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{1,E}^n\|_{\varepsilon}^2 \right. \\
& \quad \left. \left. + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{1,H}^{n+\frac{1}{2}}\|_{\mu}^2 \right) \right] + \frac{7}{5} \left(\|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^0\|_{\varepsilon}^2 + \|e_H^0\|_{\mu}^2 \right) \exp(4T).
\end{aligned}$$

Now, we need to obtain bounding estimates for each of the Taylor remainder terms and to do so, we first consider the first remainder term corresponding to p and argue as follows:

$$\begin{aligned}
\|R_p^n\|_{\varepsilon^{-1}}^2 & = \frac{1}{(24)^2 \Delta t^2} \left\| \int_{t^{n-1}}^{t^{n-\frac{1}{2}}} (t^{n-1} - s)^4 \frac{\partial^5 p}{\partial t^5}(s) ds + \int_{t^{n-\frac{1}{2}}}^{t^n} (t^n - s)^4 \frac{\partial^5 p}{\partial t^5}(s) ds \right\|_{\varepsilon^{-1}}^2, \\
& \leq \frac{1}{(24)^2 \Delta t^2} \left\| \int_{t^{n-1}}^{t^n} (t^n - s)^4 \frac{\partial^5 p}{\partial t^5}(s) ds \right\|_{\varepsilon^{-1}}^2, \quad (\text{using } t^{n-1} < t^n) \\
& \leq \frac{1}{(24)^2 (\Delta t)^2} \int_{t^{n-1}}^{t^n} (s - t^n)^8 ds \int_{t^{n-1}}^{t^n} \left\| \frac{\partial^5 p}{\partial t^5}(s) \right\|_{\varepsilon^{-1}}^2 ds, \quad (\text{by Cauchy-Schwarz}) \\
& = \frac{\Delta t^7}{(24)^2 \cdot 9} \int_{t^{n-1}}^{t^n} \left\| \frac{\partial^5 p}{\partial t^5}(s) \right\|_{\varepsilon^{-1}}^2 ds.
\end{aligned}$$

Summing both sides over $n = 0$ to N , we have that:

$$\sum_{n=0}^N \|R_p^n\|_{\varepsilon^{-1}}^2 \leq \frac{\Delta t^7}{(24)^2 \cdot 9} \int_0^T \left\| \frac{\partial^5 p}{\partial t^5}(s) \right\|_{\varepsilon^{-1}}^2 ds = \frac{\Delta t^7}{(24)^2 \cdot 9} \left\| \frac{\partial^5 p}{\partial t^5} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2.$$

Similarly, for the other Taylor remainder terms, we get that:

$$\begin{aligned}
\|R_E^n\|_{\varepsilon}^2 & \leq \frac{\Delta t^7}{(24)^2 \cdot 9} \left\| \frac{\partial^5 E}{\partial t^5} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 & \leq \frac{\Delta t^7}{(24)^2 \cdot 9} \left\| \frac{\partial^5 H}{\partial t^5} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2,
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1} \|\nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 &\leq \frac{\Delta t^7}{2^7 \cdot 72} \left\| \frac{\partial^4 (\nabla p)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla \cdot r_{2,E}^n\|_{\varepsilon}^2 &\leq \frac{\Delta t^7}{2^7 \cdot 72} \left\| \frac{\partial^4 (\nabla \cdot E)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla \times r_{2,E}^n\|_{\varepsilon}^2 &\leq \frac{\Delta t^7}{2^7 \cdot 72} \left\| \frac{\partial^4 (\nabla \times E)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla \times r_{2,H}^{n+\frac{1}{2}}\|_{\mu}^2 &\leq \frac{\Delta t^7}{2^7 \cdot 72} \left\| \frac{\partial^4 (\nabla \times H)}{\partial t^4} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla \nabla \cdot \nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 &\leq \frac{\Delta t^3}{48} \left\| \frac{\partial^2 (\nabla \nabla \cdot \nabla p)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla \cdot \nabla \nabla \cdot r_{1,E}^n\|_{\varepsilon}^2 &\leq \frac{\Delta t^3}{48} \left\| \frac{\partial^2 (\nabla \cdot \nabla \nabla \cdot E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla \times \nabla \times \nabla \times r_{1,E}^n\|_{\varepsilon}^2 &\leq \frac{\Delta t^3}{48} \left\| \frac{\partial^2 (\nabla \times \nabla \times \nabla \times E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla \times \nabla \times \nabla \times r_{1,H}^{n+\frac{1}{2}}\|_{\mu}^2 &\leq \frac{\Delta t^3}{48} \left\| \frac{\partial^2 (\nabla \times \nabla \times \nabla \times H)}{\partial t^2} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2.
\end{aligned}$$

Finally, using the regularity assumptions for p , E and H , and 1- and 2-norm equivalence, we obtain our required result:

$$\|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon} + \|e_H^N\|_{\mu} \leq C [\Delta t^4 + \|e_p^0\|_{\varepsilon^{-1}} + \|e_E^0\|_{\varepsilon} + \|e_H^0\|_{\mu}]. \quad \square$$

4.2.2 Error Estimate for Full Discretization

We now present the error analysis for the full discretization of the three-field formulation of the Maxwell's equations using a compatible sequence of arbitrary-order de Rham finite elements in conjunction with the implicit LF₄ time integration method. To do so, first we let Π_h^0 , Π_h^1 and Π_h^2 denote the respective smoothed L^2 projection operators. That is, let $\Pi_h^0 : \dot{H}_{\varepsilon^{-1}}^1(\Omega) \rightarrow U_h$, $\Pi_h^1 : \dot{H}_{\varepsilon}(\text{curl}; \Omega) \rightarrow V_h$ and $\Pi_h^2 : \dot{H}_{\mu}(\text{div}; \Omega) \rightarrow W_h$ denote these smoothed L^2 projection operators. Details of these operators is now standard and can be found in many places such as [Sch08; Chr07], [BS08, Lemma 4.3.8], [AFW06, Theorem 5.3], and [EG21a, Lemma 11.9, Corollary 11.11, Theorem 16.10, Theorem 17.5].

We now define the errors for p , E and H at time $(n+1/2)\Delta t$ or $n\Delta t$ under the full discretization to be the following and this is essentially identical to the sequence of analysis for the full error as in [AK25, Section 5] for the method described in that work. So, we have that:

$$e_{p_h}^{n+\frac{1}{2}} := p(t^{n+\frac{1}{2}}) - p_h^{n+\frac{1}{2}} = \eta^{n+\frac{1}{2}} - \eta_h^{n+\frac{1}{2}}, \quad (4.9)$$

$$e_{E_h}^{n+\frac{1}{2}} := E(t^{n+\frac{1}{2}}) - E_h^{n+\frac{1}{2}} = \zeta^{n+\frac{1}{2}} - \zeta_h^{n+\frac{1}{2}}, \quad (4.10)$$

$$e_{H_h}^n := H(t^n) - H_h^n = \xi^n - \xi_h^n, \quad (4.11)$$

and in which we now have the following definitions for the newly introduced terms:

$$\eta^{n+\frac{1}{2}} := p(t^{n+\frac{1}{2}}) - \Pi_h^0 p(t^{n+\frac{1}{2}}), \quad \eta_h^{n+\frac{1}{2}} := p_h^{n+\frac{1}{2}} - \Pi_h^0 p(t^{n+\frac{1}{2}}), \quad (4.12)$$

$$\zeta^{n+\frac{1}{2}} := E(t^{n+\frac{1}{2}}) - \Pi_h^1 E(t^{n+\frac{1}{2}}), \quad \zeta_h^{n+\frac{1}{2}} := E_h^{n+\frac{1}{2}} - \Pi_h^1 E(t^{n+\frac{1}{2}}), \quad (4.13)$$

$$\xi^n := H(t^n) - \Pi_h^2 H(t^n), \quad \xi_h^n := H_h^n - \Pi_h^2 H(t^n). \quad (4.14)$$

For the LF₄ scheme as in Equations (4.3a) to (4.3c), using a de Rham sequence of finite dimensional subspaces of the corresponding function spaces for the spatial discretization of $(p^{n+\frac{1}{2}}, E^{n+\frac{1}{2}}, H^{n+1})$, we obtain the following discrete problem: find $(p_h^{n+\frac{1}{2}}, E_h^{n+\frac{1}{2}}, H_h^{n+1}) \in U_h \times V_h \times W_h \subseteq \mathring{H}_{\varepsilon-1}^1 \times \mathring{H}_\varepsilon(\text{curl}; \Omega) \times \mathring{H}_\mu(\text{div}; \Omega)$ such that:

$$\left\langle \frac{p_h^{n+\frac{1}{2}} - p_h^{n-\frac{1}{2}}}{\Delta t}, \tilde{p} \right\rangle - \left\langle \frac{\varepsilon}{2} (E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}}), \nabla \tilde{p} \right\rangle + \frac{\Delta t^2}{12} \left\langle \frac{\varepsilon}{2} \nabla \nabla \cdot (E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}}), \nabla \tilde{p} \right\rangle = 0, \quad (4.15a)$$

$$\begin{aligned} & \left\langle \frac{1}{2} \nabla (p_h^{n+\frac{1}{2}} + p_h^{n-\frac{1}{2}}), \tilde{E} \right\rangle - \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \nabla (p_h^{n+\frac{1}{2}} + p_h^{n-\frac{1}{2}}), \nabla \nabla \cdot \tilde{E} \right\rangle + \left\langle \varepsilon \frac{E_h^{n+\frac{1}{2}} - E_h^{n-\frac{1}{2}}}{\Delta t}, \tilde{E} \right\rangle \\ & - \left\langle \frac{1}{2} (H_h^{n+1} + H_h^n), \nabla \times \tilde{E} \right\rangle - \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times (H_h^{n+1} + H_h^n), \nabla \times \tilde{E} \right\rangle = 0, \end{aligned} \quad (4.15b)$$

$$\begin{aligned} & \left\langle \mu \frac{H_h^{n+1} - H_h^n}{\Delta t}, \tilde{H} \right\rangle + \left\langle \frac{1}{2} \nabla \times (E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}}), \tilde{H} \right\rangle \\ & + \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \varepsilon^{-1} \mu^{-1} \nabla \times (E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}}), \nabla \times \nabla \times \tilde{H} \right\rangle = 0, \end{aligned} \quad (4.15c)$$

for all $(\tilde{p}, \tilde{E}, \tilde{H}) \in U_h \times V_h \times W_h$, $n = 1, \dots, N-1$. The $n = 0$ bootstrapping as in Equations (4.4a) to (4.4c) leads also to the following discrete problem: find $(p_h^{\frac{1}{2}}, E_h^{\frac{1}{2}}, H_h^1) \in U_h \times V_h \times W_h \subseteq \mathring{H}_{\varepsilon-1}^1 \times \mathring{H}_\varepsilon(\text{curl}; \Omega) \times \mathring{H}_\mu(\text{div}; \Omega)$ such that:

$$\left\langle \frac{p_h^{\frac{1}{2}} - p_h^0}{\Delta t/2}, \tilde{p} \right\rangle - \frac{1}{2} \left\langle \frac{\varepsilon}{2} (E_h^{\frac{1}{2}} + E_h^0), \nabla \tilde{p} \right\rangle + \frac{1}{8} \cdot \frac{\Delta t^2}{12} \left\langle \frac{\varepsilon}{2} \nabla \nabla \cdot (E_h^{\frac{1}{2}} + E_h^0), \nabla \tilde{p} \right\rangle = 0, \quad (4.16a)$$

$$\begin{aligned} & \frac{1}{2} \left\langle \frac{1}{2} \nabla (p_h^{\frac{1}{2}} + p_h^0), \tilde{E} \right\rangle - \frac{1}{8} \cdot \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \nabla (p_h^{\frac{1}{2}} + p_h^0), \nabla \nabla \cdot \tilde{E} \right\rangle + \left\langle \varepsilon \frac{E_h^{\frac{1}{2}} - E_h^0}{\Delta t/2}, \tilde{E} \right\rangle \\ & - \left\langle \frac{1}{2} (H_h^1 + H_h^0), \nabla \times \tilde{E} \right\rangle - \frac{1}{4} \cdot \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \nabla \times \nabla \times (H_h^1 + H_h^0), \nabla \times \tilde{E} \right\rangle = 0, \end{aligned} \quad (4.16b)$$

$$\left\langle \mu \frac{H_h^1 - H_h^0}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \left\langle \frac{1}{2} \nabla \times (E_h^{\frac{1}{2}} + E_h^0), \tilde{H} \right\rangle + \frac{1}{8} \cdot \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \nabla \times (E_h^{\frac{1}{2}} + E_h^0), \nabla \times \nabla \times \tilde{H} \right\rangle = 0, \quad (4.16c)$$

for all $(\tilde{p}, \tilde{E}, \tilde{H}) \in U_h \times V_h \times W_h$ given $(p_h^0, E_h^0, H_h^0) \in U_h \times V_h \times W_h$. For the initial conditions, we set:

$$p_h^0 := \Pi_h^0 p_0, E_h^0 := \Pi_h^1 E_0 \text{ and } H_h^0 := \Pi_h^2 H_0. \quad (4.16d)$$

With this setup, we now state and prove our theorem for the convergence of errors in the full discretization of our system of Maxwell's equations using LF₄ and arbitrary-order de Rham finite elements.

Theorem 4.3 (Full Error Estimate). *Let $p \in C^5(0, T; \dot{H}_{\varepsilon^{-1}}^1(\Omega))$, $E \in C^5(0, T; \dot{H}_{\varepsilon}(\text{curl}; \Omega))$, and $H \in C^5(0, T; \dot{H}_{\mu}(\text{div}; \Omega))$ be the solution to the variational formulation of the Maxwell's equations as in Equations (4.2a) to (4.2c) assuming sufficient regularity, and let $(p_h^{n+\frac{1}{2}}, E_h^{n+\frac{1}{2}}, H_h^{n+1})$ be the solution of the fully discretized Maxwell's equations using the LF₄ scheme as in Equations (4.15a) to (4.15c) and (4.16a) to (4.16d). If the time step $\Delta t > 0$ and the mesh parameter $h > 0$ are sufficiently small, then there exists a positive bounded constant C independent of both Δt and h such that the following error estimate holds:*

$$\|e_{p_h}^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|e_{E_h}^{N-\frac{1}{2}}\|_{\varepsilon} + \|e_{H_h}^N\|_{\mu} \leq C [\Delta t^4 + h^r + h^r \Delta t^4],$$

where the finite element subspaces U_h , V_h and W_h are each spanned by their respective Whitney form basis of polynomial order $r \geq 1$.

Proof. First, we shall subtract the set of equations for the full discretization as in Equations (4.15a) to (4.15c) from Equations (4.8a) to (4.8c), and then use the error terms in Equations (4.9) to (4.11) and thereby obtain:

$$\begin{aligned} \left\langle \frac{e_{p_h}^{n+\frac{1}{2}} - e_{p_h}^{n-\frac{1}{2}}}{\Delta t}, \tilde{p} \right\rangle - \frac{1}{2} \langle \varepsilon (e_{E_h}^{n+\frac{1}{2}} + e_{E_h}^{n-\frac{1}{2}}), \nabla \tilde{p} \rangle + \frac{\Delta t^2}{24} \langle \varepsilon \nabla \nabla \cdot (e_{E_h}^{n+\frac{1}{2}} + e_{E_h}^{n-\frac{1}{2}}), \nabla \tilde{p} \rangle \\ = \langle R_p^n - \varepsilon \nabla \cdot r_{2,E}^n - \frac{\Delta t^2}{12} \varepsilon \nabla \cdot \nabla \nabla \cdot r_{1,E}^n, \tilde{p} \rangle, \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \langle \nabla (e_{p_h}^{n+\frac{1}{2}} + e_{p_h}^{n-\frac{1}{2}}), \tilde{E} \rangle - \frac{\Delta t^2}{24} \langle \nabla (e_{p_h}^{n+\frac{1}{2}} + e_{p_h}^{n-\frac{1}{2}}), \nabla \nabla \cdot \tilde{E} \rangle + \left\langle \varepsilon \frac{e_{E_h}^{n+\frac{1}{2}} - e_{E_h}^{n-\frac{1}{2}}}{\Delta t}, \tilde{E} \right\rangle \\ - \frac{1}{2} \langle e_{H_h}^{n+1} + e_{H_h}^n, \nabla \times \tilde{E} \rangle - \frac{\Delta t^2}{24} \langle \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times (e_{H_h}^{n+1} + e_{H_h}^n), \nabla \times \tilde{E} \rangle \\ = \langle \varepsilon R_E^n + \nabla r_{2,p}^n - \nabla \times r_{2,H}^{n+\frac{1}{2}} - \frac{\Delta t^2}{12} \nabla \nabla \cdot \nabla r_{1,p}^n - \frac{\Delta t^2}{12} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \nabla \times r_{1,H}^{n+\frac{1}{2}}, \tilde{E} \rangle, \end{aligned}$$

$$\begin{aligned} \left\langle \mu \frac{e_{H_h}^{n+1} - e_{H_h}^n}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \langle \nabla \times (e_{E_h}^{n+\frac{1}{2}} + e_{E_h}^{n-\frac{1}{2}}), \tilde{H} \rangle \\ + \frac{\Delta t^2}{24} \langle \varepsilon^{-1} \mu^{-1} \nabla \times (e_{E_h}^{n+\frac{1}{2}} + e_{E_h}^{n-\frac{1}{2}}), \nabla \times \nabla \times \tilde{H} \rangle \\ = \langle \mu R_H^{n+\frac{1}{2}} + \nabla \times r_{2,E}^n + \frac{\Delta t^2}{12} \varepsilon^{-1} \mu^{-1} \nabla \times \nabla \times \nabla \times r_{1,E}^n, \tilde{H} \rangle. \end{aligned}$$

Next, using the values of the error terms $e_{p_h}^n$, $e_{E_h}^n$ and $e_{H_h}^n$ as in Equations (4.9) to (4.11) in the above equations, we get:

$$\left\langle \frac{(\eta^{n+\frac{1}{2}} - \eta^{n-\frac{1}{2}}) - (\eta_h^{n+\frac{1}{2}} - \eta_h^{n-\frac{1}{2}})}{\Delta t}, \tilde{p} \right\rangle - \frac{1}{2} \langle \varepsilon ((\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}}) - (\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}})), \nabla \tilde{p} \rangle$$

$$\begin{aligned}
& + \frac{\Delta t^2}{24} \langle \varepsilon \nabla \nabla \cdot \left(\left(\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}} \right) - \left(\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \right) \right), \nabla \tilde{\mathbf{p}} \rangle \\
& = \langle \mathbf{R}_p^n - \varepsilon \nabla \cdot \mathbf{r}_{2,E}^n - \frac{\Delta t^2}{12} \varepsilon \nabla \cdot \nabla \nabla \cdot \mathbf{r}_{1,E}^n, \tilde{\mathbf{p}} \rangle,
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2} \langle \nabla \left(\left(\eta^{n+\frac{1}{2}} + \eta^{n-\frac{1}{2}} \right) - \left(\eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \right) \right), \tilde{\mathbf{E}} \rangle \\
& \quad - \frac{\Delta t^2}{24} \langle \nabla \left(\left(\eta^{n+\frac{1}{2}} + \eta^{n-\frac{1}{2}} \right) - \left(\eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \right) \right), \nabla \nabla \cdot \tilde{\mathbf{E}} \rangle \\
& \quad + \langle \varepsilon \frac{\left(\zeta^{n+\frac{1}{2}} - \zeta^{n-\frac{1}{2}} \right) - \left(\zeta_h^{n+\frac{1}{2}} - \zeta_h^{n-\frac{1}{2}} \right)}{\Delta t}, \tilde{\mathbf{E}} \rangle - \frac{1}{2} \langle \left(\left(\xi^{n+1} + \xi^n \right) - \left(\xi_h^{n+1} - \xi_h^n \right) \right), \nabla \times \tilde{\mathbf{E}} \rangle \\
& \quad - \frac{\Delta t^2}{24} \langle \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \left(\left(\xi^{n+1} + \xi^n \right) - \left(\xi_h^{n+1} - \xi_h^n \right) \right), \nabla \times \tilde{\mathbf{E}} \rangle \\
& = \langle \varepsilon \mathbf{R}_E^n + \nabla \mathbf{r}_{2,p}^n - \nabla \times \mathbf{r}_{2,H}^{n+\frac{1}{2}} - \frac{\Delta t^2}{12} \nabla \nabla \cdot \nabla \mathbf{r}_{1,p}^n - \frac{\Delta t^2}{12} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \nabla \times \mathbf{r}_{1,H}^{n+\frac{1}{2}}, \tilde{\mathbf{E}} \rangle,
\end{aligned}$$

$$\begin{aligned}
& \langle \mu \frac{\left(\xi^{n+1} - \xi^n \right) - \left(\xi_h^{n+1} - \xi_h^n \right)}{\Delta t}, \tilde{\mathbf{H}} \rangle + \frac{1}{2} \langle \nabla \times \left(\left(\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}} \right) - \left(\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \right) \right), \tilde{\mathbf{H}} \rangle \\
& \quad + \frac{\Delta t^2}{24} \langle \varepsilon^{-1} \mu^{-1} \nabla \times \left(\left(\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}} \right) - \left(\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \right) \right), \nabla \times \nabla \times \tilde{\mathbf{H}} \rangle \\
& = \langle \mu \mathbf{R}_H^{n+\frac{1}{2}} + \nabla \times \mathbf{r}_{2,E}^n + \frac{\Delta t^2}{12} \varepsilon^{-1} \mu^{-1} \nabla \times \nabla \times \nabla \times \mathbf{r}_{1,E}^n, \tilde{\mathbf{H}} \rangle.
\end{aligned}$$

Since these equations are true for all $(\tilde{\mathbf{p}}, \tilde{\mathbf{E}}, \tilde{\mathbf{H}}) \in \mathbf{U}_h \times \mathbf{V}_h \times \mathbf{W}_h$, we choose $\tilde{\mathbf{p}} = -2\Delta t \varepsilon^{-1} \left(\eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \right)$, $\tilde{\mathbf{E}} = -2\Delta t \left(\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \right)$ and $\tilde{\mathbf{H}} = -2\Delta t \left(\xi_h^n + \xi_h^{n-1} \right)$ and using the fact that $\nabla \mathbf{U}_h \subseteq \mathbf{V}_h$ and $\nabla \times \mathbf{V}_h \subseteq \mathbf{W}_h$, we obtain:

$$\begin{aligned}
& 2\langle \varepsilon^{-1} \left(\eta_h^{n+\frac{1}{2}} - \eta_h^{n-\frac{1}{2}} \right), \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle + 2\langle \varepsilon \left(\zeta_h^{n+\frac{1}{2}} - \zeta_h^{n-\frac{1}{2}} \right), \zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \rangle \\
& \quad + 2\langle \mu \left(\xi_h^n - \xi_h^{n-1} \right), \xi_h^n + \xi_h^{n-1} \rangle = 2\langle \varepsilon^{-1} \left(\eta^{n+\frac{1}{2}} - \eta^{n-\frac{1}{2}} \right), \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle \\
& \quad + 2\langle \varepsilon \left(\zeta^{n+\frac{1}{2}} - \zeta^{n-\frac{1}{2}} \right), \zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \rangle + 2\langle \mu \left(\xi^n - \xi^{n-1} \right), \xi_h^n + \xi_h^{n-1} \rangle \\
& \quad + 2\Delta t \langle -\mathbf{R}_p^n + \varepsilon \nabla \cdot \mathbf{r}_{2,E}^n + \frac{\Delta t^2}{12} \varepsilon \nabla \cdot \nabla \nabla \cdot \mathbf{r}_{1,E}^n, \varepsilon^{-1} \left(\eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \right) \rangle \\
& \quad + 2\Delta t \langle -\varepsilon \mathbf{R}_E^n - \nabla \mathbf{r}_{2,p}^n + \nabla \times \mathbf{r}_{2,H}^{n+\frac{1}{2}} + \frac{\Delta t^2}{12} \nabla \nabla \cdot \nabla \mathbf{r}_{1,p}^n + \frac{\Delta t^2}{12} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \nabla \times \mathbf{r}_{1,H}^{n+\frac{1}{2}}, \zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \rangle \\
& \quad + 2\Delta t \langle -\mu \mathbf{R}_H^{n+\frac{1}{2}} - \nabla \times \mathbf{r}_{2,E}^n + \frac{\Delta t^2}{12} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \nabla \times \mathbf{r}_{1,E}^n, \xi_h^n + \xi_h^{n-1} \rangle, \quad (4.17)
\end{aligned}$$

Consider that $\varepsilon^{-1} \left(\eta^{n+\frac{1}{2}} - \eta^{n-\frac{1}{2}} \right) = \varepsilon^{-1} \left(\mathbf{I} - \Pi_h^0 \right) \left(\mathbf{p}(t^{n+\frac{1}{2}}) - \mathbf{p}(t^{n-\frac{1}{2}}) \right)$ by Equation (4.12). Using the Taylor theorem with remainder as in Theorem 4.2, applying the Cauchy-Schwarz, AM-GM, and Triangle inequalities, we have the following resulting inequality:

$$2\langle \varepsilon^{-1} \left(\eta^{n+\frac{1}{2}} - \eta^{n-\frac{1}{2}} \right), \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle = 2\Delta t \langle \varepsilon^{-1} \left(\mathbf{I} - \Pi_h^0 \right) \left(\frac{\partial \mathbf{p}}{\partial t}(t^n) \right), \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle$$

$$\begin{aligned}
& + \frac{\Delta t^2}{24} \frac{\partial^3 p}{\partial t^3(t^n)} \Big), \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \Big\rangle \\
& + 2\Delta t \langle \varepsilon^{-1} (I - \Pi_h^0) R_p^n, \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle \\
& \leq \Delta t \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 \right. \\
& \quad \left. + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 \right] + 6\Delta t \left[\|\eta_h^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\eta_h^{n-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 \right].
\end{aligned}$$

Similarly, using Equations (4.13) and (4.14) for the error terms for E and H, we obtain:

$$\begin{aligned}
2\langle \varepsilon (\zeta_h^{n+\frac{1}{2}} - \zeta_h^{n-\frac{1}{2}}), \zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \rangle & \leq \Delta t \left[\left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 \right. \\
& \quad \left. + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 \right] + 6\Delta t \left[\|\zeta_h^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|\zeta_h^{n-\frac{1}{2}}\|_{\varepsilon}^2 \right], \\
2\langle \mu (\xi_h^{n+1} - \xi_h^n), \xi_h^{n+1} + \xi_h^n \rangle & \leq \Delta t \left[\left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right. \\
& \quad \left. + \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu}^2 \right] + 6\Delta t \left[\|\xi_h^{n+1}\|_{\mu}^2 + \|\xi_h^n\|_{\mu}^2 \right].
\end{aligned}$$

Using these inequalities for η , ζ and ξ in Equation (4.17), we thus obtain the following estimate:

$$\begin{aligned}
2 \left[\|\eta_h^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|\eta_h^{n-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{n+\frac{1}{2}}\|_{\varepsilon}^2 - \|\zeta_h^{n-\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^{n+1}\|_{\mu}^2 - \|\xi_h^n\|_{\mu}^2 \right] & \leq \\
& \Delta t \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 \right. \\
& \quad + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 + \\
& \quad \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 + \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu}^2 \\
& + 8(\|\eta_h^n\|_{\varepsilon^{-1}}^2 + \|\eta_h^{n-1}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^n\|_{\varepsilon}^2 + \|\zeta_h^{n-1}\|_{\varepsilon}^2 + \|\xi_h^n\|_{\mu}^2 + \|\xi_h^{n-1}\|_{\mu}^2) + \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 \\
& + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 + \|\nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{1,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{1,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{1,H}^{n+\frac{1}{2}}\|_{\mu}^2 \\
& + \frac{\Delta t^4}{144} \left(\|\nabla \nabla \cdot \nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \nabla \nabla \cdot r_{2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{2,E}^n\|_{\varepsilon}^2 \right. \\
& \quad \left. + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{2,H}^{n+\frac{1}{2}}\|_{\mu}^2 \right) \Big].
\end{aligned}$$

Summing from $n = 1$ to $N - 1$, we get:

$$\begin{aligned}
\|\eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^N\|_{\mu}^2 & \leq 8\Delta t \sum_{n=1}^{N-1} \left[\|\eta_h^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^n\|_{\mu}^2 \right] \\
& + \Delta t \sum_{n=0}^N \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 \right.
\end{aligned}$$

$$\begin{aligned}
& + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t} (t^{n-\frac{1}{2}}) \right\|_\varepsilon^2 + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3} (t^n) \right\|_\varepsilon^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_\varepsilon^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t} (t^{n+\frac{1}{2}}) \right\|_\mu^2 \\
& + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3} (t^{n+\frac{1}{2}}) \right\|_\mu^2 + \left\| (I - \Pi_h^2) R_H^n \right\|_\mu^2 + \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_\varepsilon^2 + \|R_H^{n+\frac{1}{2}}\|_\mu^2 + \|\nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 \\
& + \|\nabla \cdot r_{1,E}^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{1,E}^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{1,H}^{n+\frac{1}{2}}\|_\mu^2 + \frac{\Delta t^4}{144} \left(\|\nabla \nabla \cdot \nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 \right. \\
& \left. + \|\nabla \cdot \nabla \nabla \cdot r_{2,E}^n\|_\varepsilon^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{2,E}^n\|_\varepsilon^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{2,H}^{n+\frac{1}{2}}\|_\mu^2 \right) \\
& + \left[\|\eta_h^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{\frac{1}{2}}\|_\varepsilon^2 + \|\xi_h^1\|_\mu^2 \right].
\end{aligned}$$

Likewise, for the semidiscrete approximation of the initial system as in Equations (4.4a) to (4.4c), we obtain for their errors the following system of equations:

$$\begin{aligned}
\left\langle \frac{e_{p_h}^{\frac{1}{2}} - e_{p_h}^0}{\Delta t/2}, \tilde{p} \right\rangle - \frac{1}{4} \left\langle \varepsilon \left(e_{E_h}^{\frac{1}{2}} + e_{E_h}^0 \right), \nabla \tilde{p} \right\rangle + \frac{\Delta t^2}{192} \left\langle \varepsilon \nabla \nabla \cdot \left(e_{E_h}^{\frac{1}{2}} + e_{E_h}^0 \right), \nabla \tilde{p} \right\rangle \\
= \left\langle \frac{1}{2} R_p^0 - \frac{\varepsilon}{2} \nabla \cdot r_{2,E}^0 - \frac{\Delta t^2}{96} \varepsilon \nabla \cdot \nabla \nabla \cdot r_{1,E}^0, \tilde{p} \right\rangle,
\end{aligned}$$

$$\begin{aligned}
\frac{1}{4} \left\langle \nabla \left(e_p^{\frac{1}{2}} + e_p^0 \right), \tilde{E} \right\rangle - \frac{\Delta t^2}{192} \left\langle \nabla \left(e_{p_h}^{\frac{1}{2}} + e_{p_h}^0 \right), \nabla \nabla \cdot \tilde{E} \right\rangle + \left\langle \varepsilon \frac{e_{E_h}^{\frac{1}{2}} - e_{E_h}^0}{\Delta t/2}, \tilde{E} \right\rangle \\
- \frac{1}{2} \left\langle e_{H_h}^1 + e_{H_h}^0, \nabla \times \tilde{E} \right\rangle - \frac{\Delta t^2}{96} \left\langle \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \left(e_{H_h}^1 + e_{H_h}^0 \right), \tilde{E} \right\rangle \\
= \left\langle \frac{\varepsilon}{2} R_E^0 + \frac{1}{2} \nabla r_{2,p}^0 - \nabla \times r_{2,H}^{\frac{1}{2}} - \frac{\Delta t^2}{96} \nabla \nabla \cdot \nabla r_{1,p}^0 - \frac{\Delta t^2}{48} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \nabla \times r_{1,H}^{\frac{1}{2}}, \tilde{E} \right\rangle,
\end{aligned}$$

$$\begin{aligned}
\left\langle \mu \frac{e_{H_h}^1 - e_{H_h}^0}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{4} \left\langle \nabla \times \left(e_{E_h}^{\frac{1}{2}} + e_{E_h}^0 \right), \tilde{H} \right\rangle + \frac{\Delta t^2}{192} \left\langle \varepsilon^{-1} \mu^{-1} \nabla \times \left(e_{E_h}^{\frac{1}{2}} + e_{E_h}^0 \right), \nabla \times \nabla \times \tilde{H} \right\rangle \\
= \left\langle \mu R_H^{\frac{1}{2}} + \frac{1}{2} \nabla \times r_{2,E}^0 + \frac{\Delta t^2}{96} \varepsilon^{-1} \mu^{-1} \nabla \times \nabla \times \nabla \times r_{1,E}^0, \tilde{H} \right\rangle,
\end{aligned}$$

Then, following the same sequence of steps for the initial time step and furthermore adding the resulting equation to the inequality for the general case, we arrive at the following:

$$\begin{aligned}
\|\eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{N-\frac{1}{2}}\|_\varepsilon^2 + \|\xi_h^N\|_\mu^2 & \leq \frac{1+4\Delta t}{1-4\Delta t} \left[\|\eta_h^0\|_{\varepsilon^{-1}}^2 + \|\zeta_h^0\|_\varepsilon^2 + \|\xi_h^0\|_\mu^2 \right] \\
& + \frac{8\Delta t}{1-4\Delta t} \sum_{n=0}^{N-1} \left[\|\eta_h^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{n+\frac{1}{2}}\|_\varepsilon^2 + \|\xi_h^{n+1}\|_\mu^2 \right] + \frac{\Delta t}{1-4\Delta t} \sum_{n=0}^{N-1} \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t} (t^n) \right\|_{\varepsilon^{-1}}^2 \right. \\
& + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3} (t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t} (t^n) \right\|_\varepsilon^2 \\
& + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3} (t^n) \right\|_\varepsilon^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_\varepsilon^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t} (t^{n+\frac{1}{2}}) \right\|_\mu^2 \\
& + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3} (t^{n+\frac{1}{2}}) \right\|_\mu^2 + \left\| (I - \Pi_h^2) R_H^n \right\|_\mu^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t} (t^{n+\frac{1}{2}}) \right\|_\mu^2 \Big]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^n) \frac{\partial^3 H}{\partial t^3} (t^{n+\frac{1}{2}}) \right\|_\mu^2 + \left\| (I - \Pi_h^n) R_H^{n+\frac{1}{2}} \right\|_\mu^2 + \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_\varepsilon^2 + \|R_H^{n+\frac{1}{2}}\|_\mu^2 \\
& + \|\nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{1,E}^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{1,E}^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{1,H}^{n+\frac{1}{2}}\|_\mu^2 \\
& + \frac{\Delta t^4}{144} \left(\|\nabla \nabla \cdot \nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \nabla \nabla \cdot r_{2,E}^n\|_\varepsilon^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{2,E}^n\|_\varepsilon^2 \right. \\
& \quad \left. + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{2,H}^{n+\frac{1}{2}}\|_\mu^2 \right).
\end{aligned}$$

Applying the discrete Gronwall inequality 1.10 with $\Delta t < 1/24$, we get that:

$$\begin{aligned}
& \|\eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{N-\frac{1}{2}}\|_\varepsilon^2 + \|\xi_h^N\|_\mu^2 \leq \left[\frac{6\Delta t}{5} \sum_{n=0}^{N-1} \left(\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t} (t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3} (t^n) \right\|_{\varepsilon^{-1}}^2 \right. \right. \\
& \quad + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t} (t^n) \right\|_\varepsilon^2 + \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3} (t^n) \right\|_\varepsilon^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_\varepsilon^2 \\
& \quad + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t} (t^{n+\frac{1}{2}}) \right\|_\mu^2 + \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3} (t^{n+\frac{1}{2}}) \right\|_\mu^2 + \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_\mu^2 + \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_\varepsilon^2 \\
& \quad + \|R_H^{n+\frac{1}{2}}\|_\mu^2 + \|\nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{1,E}^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{1,E}^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{1,H}^{n+\frac{1}{2}}\|_\mu^2 \\
& \quad + \frac{\Delta t^4}{144} \left(\|\nabla \nabla \cdot \nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \nabla \nabla \cdot r_{2,E}^n\|_\varepsilon^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{2,E}^n\|_\varepsilon^2 \right. \\
& \quad \left. + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{2,H}^{n+\frac{1}{2}}\|_\mu^2 \right) + \frac{7}{5} \left(\|\eta_h^0\|_{\varepsilon^{-1}}^2 + \|\zeta_h^0\|_\varepsilon^2 + \|\xi_h^0\|_\mu^2 \right) \exp(16T).
\end{aligned}$$

Using our estimates for the Taylor remainders from Theorem 4.2 for these terms on the right hand side of the above inequality, we further get that:

$$\begin{aligned}
& \Delta t \sum_{n=0}^N \left[\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_\varepsilon^2 + \|R_H^n\|_\mu^2 \right] \\
& \leq \Delta t^8 \left[\left\| \frac{\partial^5 p}{\partial t^5} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^5 E}{\partial t^5} \right\|_{L^2(0,T;L_\varepsilon^2(\Omega))}^2 + \left\| \frac{\partial^5 H}{\partial t^5} \right\|_{L^2(0,T;L_\mu^2(\Omega))}^2 \right],
\end{aligned}$$

and that:

$$\begin{aligned}
& \Delta t \sum_{n=0}^N \left[\|\nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{2,E}^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{2,E}^n\|_\varepsilon^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{2,H}^{n+\frac{1}{2}}\|_\mu^2 \right. \\
& \quad + \frac{\Delta t^4}{144} \left(\|\nabla \nabla \cdot \nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \nabla \nabla \cdot r_{1,E}^n\|_\varepsilon^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{1,E}^n\|_\varepsilon^2 \right. \\
& \quad \left. + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{1,H}^{n+\frac{1}{2}}\|_\mu^2 \right) \left. \right] \leq \Delta t^8 \left[\left\| \frac{\partial^4 (\nabla p)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^4 (\nabla \cdot E)}{\partial t^4} \right\|_{L^2(0,T;L_\varepsilon^2(\Omega))}^2 \right. \\
& \quad + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^4 (\nabla \times E)}{\partial t^4} \right\|_{L^2(0,T;L_\varepsilon^2(\Omega))}^2 + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^4 (\nabla \times H)}{\partial t^4} \right\|_{L^2(0,T;L_\mu^2(\Omega))}^2 \left. \right]
\end{aligned}$$

$$\begin{aligned}
& + \left\| \frac{\partial^2 (\nabla \nabla \cdot \nabla p)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^2 (\nabla \cdot \nabla \nabla \cdot E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 \\
& + \varepsilon^{-3} \mu^{-3} \left\| \frac{\partial^2 (\nabla \times \nabla \times \nabla \times E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \varepsilon^{-3} \mu^{-3} \left\| \frac{\partial^2 (\nabla \times \nabla \times \nabla \times H)}{\partial t^2} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \Bigg].
\end{aligned}$$

Now, for $p \in \dot{H}_{\varepsilon^{-1}}^1(\Omega)$, $E \in \dot{H}_{\varepsilon}(\text{curl}; \Omega)$ and $H \in \dot{H}_{\mu}(\text{div}; \Omega)$, there exists positive bounded constants $C_{1,p}$, $C_{2,p}$, $C_{3,p}$, $C_{1,E}$, $C_{2,E}$, $C_{3,E}$, $C_{1,H}$, $C_{2,H}$, and $C_{3,H}$ such that we have the following error bounds for the L^2 projections:

$$\begin{aligned}
\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}} & \leq C_{1,p} h^r \left\| \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}, \\
\left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}} & \leq C_{2,p} h^r \left\| \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}, \\
\left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon} & \leq C_{1,E} h^r \left\| \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}, \\
\left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon} & \leq C_{2,E} h^r \left\| \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}, \\
\left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}) \right\|_{\mu} & \leq C_{1,H} h^r \left\| \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}) \right\|_{\mu}, \\
\left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}) \right\|_{\mu} & \leq C_{2,H} h^r \left\| \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}) \right\|_{\mu}, \\
\left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}} & \leq C_{3,p} h^r \left\| R_p^n \right\|_{\varepsilon^{-1}}, \\
\left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon} & \leq C_{3,E} h^r \left\| R_E^n \right\|_{\varepsilon}, \\
\left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu} & \leq C_{3,H} h^r \left\| R_H^{n+\frac{1}{2}} \right\|_{\mu}.
\end{aligned}$$

Set $C_0 := \max\{C_{1,p}, C_{2,p}, C_{3,p}, C_{1,E}, C_{2,E}, C_{3,E}, C_{1,H}, C_{2,H}, C_{3,H}\}$. We therefore have that:

$$\begin{aligned}
& \Delta t \sum_{n=0}^{N-1} \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right] \\
& \leq C_0 h^{2r} \sum_{n=0}^{N-1} \Delta t \left[\left\| \frac{\partial p}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial E}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\varepsilon}^2 + \left\| \frac{\partial H}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\mu}^2 \right] \\
& \leq C_0 h^{2r} \int_0^T \left[\left\| \frac{\partial p}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial E}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\varepsilon}^2 + \left\| \frac{\partial H}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\mu}^2 \right] dt \\
& = C_0 h^{2r} \left[\left\| \frac{\partial p}{\partial t} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial E}{\partial t} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial H}{\partial t} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right],
\end{aligned}$$

$$\begin{aligned}
& \Delta t \sum_{n=0}^{N-1} \left[\left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right] \\
& \leq C_0 h^{2r} \sum_{n=0}^{N-1} \Delta t \left[\left\| \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 + \left\| \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right] \\
& \leq C_0 h^{2r} \int_0^T \left[\left\| \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 + \left\| \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right] dt \\
& = C_0 h^{2r} \left[\left\| \frac{\partial^3 p}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^3 E}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial^3 H}{\partial t^3} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right],
\end{aligned}$$

and likewise for the L^2 projection of the remainder terms:

$$\begin{aligned}
& \Delta t \sum_{n=0}^N \left[\left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}} + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon} + \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu} \right], \\
& \leq C_0 h^{2r} \sum_{n=0}^N \Delta t \left[\|R_p^n\|_{L_{\varepsilon^{-1}}^2(\Omega)} + \|R_E^n\|_{L_{\varepsilon}^2(\Omega)} + \|R_H^{n+\frac{1}{2}}\|_{L_{\mu}^2(\Omega)} \right], \\
& \leq C_0 h^{2r} \Delta t^8 \left[\left\| \frac{\partial^5 p}{\partial t^5} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^5 E}{\partial t^5} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial^5 H}{\partial t^5} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right].
\end{aligned}$$

Now, we take $p_h^0 = \Pi_h^0 p_0$, $E_h^0 = \Pi_h^1 E_0$, and $H_h^0 = \Pi_h^2 H_0$, and note that there exists positive bounded constants M_1 , M_2 , M_3 and M_4 such that:

$$\begin{aligned}
& \left\| \frac{\partial p}{\partial t} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial E}{\partial t} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial H}{\partial t} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \leq M_1, \\
& \left\| \frac{\partial^3 p}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^3 E}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial^3 H}{\partial t^3} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \leq M_2, \\
& \left\| \frac{\partial^5 p}{\partial t^5} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^5 E}{\partial t^5} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial^5 H}{\partial t^5} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \leq M_3, \\
& \left\| \frac{\partial^4(\nabla p)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^4(\nabla \cdot E)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^4(\nabla \times E)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 \\
& \quad + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^4(\nabla \times H)}{\partial t^4} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 + \left\| \frac{\partial^2(\nabla \nabla \cdot \nabla p)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 \\
& \quad + \left\| \frac{\partial^2(\nabla \cdot \nabla \nabla \cdot E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^2(\nabla \times \nabla \times \nabla \times E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 \\
& \quad + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^2(\nabla \times \nabla \times \nabla \times H)}{\partial t^2} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \leq M_4.
\end{aligned}$$

Consequently, we have the following estimate:

$$\|\eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^N\|_{\mu}^2 \leq \widetilde{C} [h^{2r} + h^{2r} \Delta t^8 + \Delta t^8],$$

where $\widetilde{C} = 2 \max\{C_0 M_1, C_0 M_2, C_0 M_3, M_3, M_4\} \exp(16T)$ which then using the equivalence between 1- and 2-norms gives us that:

$$\|\eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|\zeta_h^{N-\frac{1}{2}}\|_{\varepsilon} + \|\xi_h^N\|_{\mu} \leq C_1 [h^r + h^r \Delta t^4 + \Delta t^4],$$

where $C_1 = 2\sqrt{\widetilde{C}}$. Also, there are positive bounded constants $\widetilde{C}_2, \widetilde{C}_3$ and $\widetilde{C}_4$ such that:

$$\begin{aligned} \|\eta^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} &= \|(I - \Pi_h^0)p(t^{N-\frac{1}{2}})\|_{\varepsilon^{-1}} \leq \widetilde{C}_2 h^r \|p(t^{N-\frac{1}{2}})\|_{L_{\varepsilon^{-1}}^2(\Omega)} \implies \|\eta^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} \leq C_2 h^r, \\ \|\zeta^{N-\frac{1}{2}}\|_{\varepsilon} &= \|(I - \Pi_h^1)E(t^{N-\frac{1}{2}})\|_{\varepsilon} \leq \widetilde{C}_3 h^r \|E(t^{N-\frac{1}{2}})\|_{L_{\varepsilon}^2(\Omega)} \implies \|\zeta^{N-\frac{1}{2}}\|_{\varepsilon} \leq C_3 h^r, \\ \|\xi^N\|_{\mu} &= \|(I - \Pi_h^2)H(t^N)\|_{\mu} \leq \widetilde{C}_4 h^r \|H(t^N)\|_{L_{\mu}^2(\Omega)} \implies \|\xi^N\|_{\mu} \leq C_4 h^r, \end{aligned}$$

in which $C_2 = \widetilde{C}_2 \|p(t^{N-\frac{1}{2}})\|_{L_{\varepsilon^{-1}}^2(\Omega)}$, $C_3 = \widetilde{C}_3 \|E(t^{N-\frac{1}{2}})\|_{L_{\varepsilon}^2(\Omega)}$ and $C_4 = \widetilde{C}_4 \|H(t^N)\|_{L_{\mu}^2(\Omega)}$ are all bounded positive constants due to Theorem 4.1. Finally, this provides us with our desired result by choosing $C = C_1 + C_2 + C_3 + C_4$:

$$\begin{aligned} \|e_{p_h}^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|e_{E_h}^{N-\frac{1}{2}}\|_{\varepsilon} + \|e_{H_h}^N\|_{\mu} &= \|\eta^{N-\frac{1}{2}} - \eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|\zeta^{N-\frac{1}{2}} - \zeta_h^{N-\frac{1}{2}}\|_{\varepsilon} \\ &\quad + \|\xi^N - \xi_h^N\|_{\mu} \\ &\leq C [(\Delta t)^4 + h^r + h^r \Delta t^4]. \end{aligned} \quad \square$$

4.3 Characterization of TS₄ Scheme

4.3.1 Time Discretization Stability

For our implicit TS₄ scheme as in Equations (4.5a) to (4.5c), just like for LF₄ in Section 4.2, we can show that it is stable and converges with the correct fourth-order of approximation in time. The analysis here is merely an appropriate update of the theorems and their proofs in Section 4.2. However, for the TS₄ scheme, we define the discrete energy at time t^n to be the following and it is motivated by its desired conservation:

$$\begin{aligned} \mathcal{E}^n &:= \|p^{n+1} + p^n\|_{\varepsilon^{-1}}^2 + 2\langle p^{n+1}, p^n \rangle_{\varepsilon^{-1}} + \|E^{n+1} + E^n\|_{\varepsilon}^2 + 2\langle E^{n+1}, E^n \rangle_{\varepsilon} \\ &\quad + \|H^{n+1} + H^n\|_{\mu}^2 + 2\langle H^{n+1}, H^n \rangle_{\mu}. \end{aligned} \quad (4.18)$$

In addition, we require as assumption on the inner products which we state as a hypothesis next and we are motivated by our computational experiments to assert this.

Assumption 4.4 (Positivity of Inner products). *The various $\langle \cdot, \cdot \rangle_{\alpha}$ for $\alpha = \varepsilon^{-1}, \varepsilon$ and μ are all positive for all $t \in [0, T]$ for the Maxwell's solution.*

Theorem 4.5 (Discrete Energy Conservation). *For the semidiscretization using the TS₄ scheme as given in Equations (4.5a) to (4.5c) and with Assumption 4.4, for any fixed time step $\Delta t > 0$ sufficiently small, the energy*

$\mathcal{E}^n$ is conserved, that is, $\mathcal{E}^n = \mathcal{E}^{n-1}$ for $n = 1, \dots, N-1$.

Proof. Since Equations (4.5a) to (4.5c) are true for all $\tilde{\mathbf{p}} \in \dot{\mathbf{H}}_{\varepsilon^{-1}}^1(\Omega)$, $\tilde{\mathbf{E}} \in \dot{\mathbf{H}}_{\varepsilon}(\text{curl}; \Omega)$, $\tilde{\mathbf{H}} \in \dot{\mathbf{H}}_{\mu}(\text{div}; \Omega)$, using $\tilde{\mathbf{p}} = 2\Delta t (\mathbf{p}^{n+1} + 4\mathbf{p}^n + \mathbf{p}^{n-1})$, $\tilde{\mathbf{E}} = 2\Delta t (\mathbf{E}^{n+1} + 4\mathbf{E}^n + \mathbf{E}^{n-1})$ and $\tilde{\mathbf{H}} = 2\Delta t (\mathbf{H}^{n+1} + 4\mathbf{H}^n + \mathbf{H}^{n-1})$ in them and adding the resultant equations using properties of the inner product, we obtain:

$$\begin{aligned} \langle \varepsilon^{-1} (\mathbf{p}^{n+1} - \mathbf{p}^{n-1}), \mathbf{p}^{n+1} + 4\mathbf{p}^n + \mathbf{p}^{n-1} \rangle &+ \langle \varepsilon (\mathbf{E}^{n+1} - \mathbf{E}^{n-1}), \mathbf{E}^{n+1} + 4\mathbf{E}^n + \mathbf{E}^{n-1} \rangle \\ &+ \langle \mu (\mathbf{H}^{n+1} - \mathbf{H}^{n-1}), \mathbf{H}^{n+1} + 4\mathbf{H}^n + \mathbf{H}^{n-1} \rangle = 0. \end{aligned}$$

Consider next that:

$$\begin{aligned} &\langle \varepsilon^{-1} (\mathbf{p}^{n+1} - \mathbf{p}^{n-1}), \mathbf{p}^{n+1} + 4\mathbf{p}^n + \mathbf{p}^{n-1} \rangle \\ &= \langle \varepsilon^{-1} (\mathbf{p}^{n+1} + \mathbf{p}^n - \mathbf{p}^n - \mathbf{p}^{n-1}), \mathbf{p}^{n+1} + \mathbf{p}^n + \mathbf{p}^n + \mathbf{p}^{n-1} \rangle + 2\langle \varepsilon^{-1} (\mathbf{p}^{n+1} - \mathbf{p}^{n-1}), \mathbf{p}^n \rangle \\ &= \langle \varepsilon^{-1} (\mathbf{p}^{n+1} + \mathbf{p}^n), \mathbf{p}^{n+1} + \mathbf{p}^n \rangle - \langle \varepsilon^{-1} (\mathbf{p}^n + \mathbf{p}^{n-1}), \mathbf{p}^n + \mathbf{p}^{n-1} \rangle \\ &\quad + 2\langle \varepsilon^{-1} \mathbf{p}^{n+1}, \mathbf{p}^n \rangle - 2\langle \varepsilon^{-1} \mathbf{p}^n, \mathbf{p}^{n-1} \rangle \\ &= \|\mathbf{p}^{n+1} + \mathbf{p}^n\|_{\varepsilon^{-1}}^2 + 2\langle \mathbf{p}^{n+1}, \mathbf{p}^n \rangle_{\varepsilon^{-1}} - \|\mathbf{p}^n + \mathbf{p}^{n-1}\|_{\varepsilon^{-1}}^2 - 2\langle \mathbf{p}^n, \mathbf{p}^{n-1} \rangle_{\varepsilon^{-1}}. \end{aligned}$$

So, we have the following:

$$\begin{aligned} &\langle \varepsilon^{-1} (\mathbf{p}^{n+1} - \mathbf{p}^{n-1}), \mathbf{p}^{n+1} + 4\mathbf{p}^n + \mathbf{p}^{n-1} \rangle \\ &= \|\mathbf{p}^{n+1} + \mathbf{p}^n\|_{\varepsilon^{-1}}^2 + 2\langle \mathbf{p}^{n+1}, \mathbf{p}^n \rangle_{\varepsilon^{-1}} - \|\mathbf{p}^n + \mathbf{p}^{n-1}\|_{\varepsilon^{-1}}^2 - 2\langle \mathbf{p}^n, \mathbf{p}^{n-1} \rangle_{\varepsilon^{-1}}. \end{aligned}$$

We can similarly have the corresponding expressions for $\mathbf{E}$ and $\mathbf{H}$. Using these various expressions in the first equation in our proof, we get that:

$$\begin{aligned} &\|\mathbf{p}^{n+1} + \mathbf{p}^n\|_{\varepsilon^{-1}}^2 + 2\langle \mathbf{p}^{n+1}, \mathbf{p}^n \rangle_{\varepsilon^{-1}} - \|\mathbf{p}^n + \mathbf{p}^{n-1}\|_{\varepsilon^{-1}}^2 - 2\langle \mathbf{p}^n, \mathbf{p}^{n-1} \rangle_{\varepsilon^{-1}} \\ &\quad + \|\mathbf{E}^{n+1} + \mathbf{E}^n\|_{\varepsilon}^2 + 2\langle \mathbf{E}^{n+1}, \mathbf{E}^n \rangle_{\varepsilon} - \|\mathbf{E}^n + \mathbf{E}^{n-1}\|_{\varepsilon}^2 - 2\langle \mathbf{E}^n, \mathbf{E}^{n-1} \rangle_{\varepsilon} \\ &\quad + \|\mathbf{H}^{n+1} + \mathbf{H}^n\|_{\mu}^2 + 2\langle \mathbf{H}^{n+1}, \mathbf{H}^n \rangle_{\mu} - \|\mathbf{H}^n + \mathbf{H}^{n-1}\|_{\mu}^2 - 2\langle \mathbf{H}^n, \mathbf{H}^{n-1} \rangle_{\mu} = 0, \end{aligned}$$

which yields the desired result:

$$\begin{aligned} &\|\mathbf{p}^{n+1} + \mathbf{p}^n\|_{\varepsilon^{-1}}^2 + 2\langle \mathbf{p}^{n+1}, \mathbf{p}^n \rangle_{\varepsilon^{-1}} + \|\mathbf{E}^{n+1} + \mathbf{E}^n\|_{\varepsilon}^2 + 2\langle \mathbf{E}^{n+1}, \mathbf{E}^n \rangle_{\varepsilon} + \\ &\quad \|\mathbf{H}^{n+1} + \mathbf{H}^n\|_{\mu}^2 + 2\langle \mathbf{H}^{n+1}, \mathbf{H}^n \rangle_{\mu} = \|\mathbf{p}^n + \mathbf{p}^{n-1}\|_{\varepsilon^{-1}}^2 + 2\langle \mathbf{p}^n, \mathbf{p}^{n-1} \rangle_{\varepsilon^{-1}} + \\ &\quad \|\mathbf{E}^n + \mathbf{E}^{n-1}\|_{\varepsilon}^2 + 2\langle \mathbf{E}^n, \mathbf{E}^{n-1} \rangle_{\varepsilon} + \|\mathbf{H}^n + \mathbf{H}^{n-1}\|_{\mu}^2 + 2\langle \mathbf{H}^n, \mathbf{H}^{n-1} \rangle_{\mu}. \quad \square \end{aligned}$$

Theorem 4.6 (Discrete Error Estimate). *For the semidiscretization TS_4 as given in Equations (4.5a) to (4.5c), for the solution $(\mathbf{p}, \mathbf{E}, \mathbf{H})$ of the variationally posed Maxwell's system as in Equations (4.2a) to (4.2c) with initial conditions as in Equation (4.1c), with Assumption 4.4 and sufficient regularity $\mathbf{p} \in C^5(0, T; \dot{\mathbf{H}}_{\varepsilon^{-1}}^1(\Omega))$, $\mathbf{E} \in C^5(0, T; \dot{\mathbf{H}}_{\varepsilon}(\text{curl}; \Omega))$, and $\mathbf{H} \in C^5(0, T; \dot{\mathbf{H}}_{\mu}(\text{div}; \Omega))$, and for the time step $\Delta t > 0$ sufficiently small, there exists a positive bounded constant C independent of Δt such that:*

$$\|e_p^{N-1}\|_{\varepsilon^{-1}} + \|e_E^{N-1}\|_{\varepsilon} + \|e_H^{N-1}\|_{\mu}$$

$$\leq C \left[(\Delta t)^4 + \|e_p^0\|_{\varepsilon^{-1}} + \|e_E^0\|_{\varepsilon} + \|e_H^0\|_{\mu} + \|e_p^1\|_{\varepsilon^{-1}} + \|e_E^1\|_{\varepsilon} + \|e_H^1\|_{\mu} \right],$$

where $e_p^n := p(t^n) - p^n$, $e_E^n := E(t^n) - E^n$ and $e_H^n := H(t^n) - H^n$ are the errors in the time semidiscretization of p , E and H , respectively.

Proof. Using the Taylor remainder theorem, and expressing $u(t)$ about $t = t^n$ for $u = \{p, E, H\}$ and $n = 1, \dots, N-1$, we have that:

$$\begin{aligned} u(t) = u(t^n) &+ \frac{\partial u}{\partial t}(t^n)(t - t^n) + \frac{\partial^2 u}{\partial t^2}(t^n) \frac{(t - t^n)^2}{2!} + \frac{\partial^3 u}{\partial t^3}(t^n) \frac{(t - t^n)^3}{3!} \\ &+ \frac{\partial^4 u}{\partial t^4}(t^n) \frac{(t - t^n)^4}{4!} + \int_{t^n}^t \frac{(t-s)^4}{4!} \frac{\partial^5 u}{\partial t^5}(s) ds, \end{aligned}$$

which when evaluated at $t = t^{n+1}$ and $t = t^{n-1}$ yields:

$$\begin{aligned} u(t^{n+1}) = u(t^n) &+ \frac{\partial u}{\partial t}(t^n)(t^{n+1} - t^n) + \frac{\partial^2 u}{\partial t^2}(t^n) \frac{(t^{n+1} - t^n)^2}{2} + \frac{\partial^3 u}{\partial t^3}(t^n) \frac{(t^{n+1} - t^n)^3}{6} \\ &+ \frac{\partial^4 u}{\partial t^4}(t^n) \frac{(t^{n+1} - t^n)^4}{24} + \int_{t^n}^{t^{n+1}} \frac{(t^{n+1} - s)^4}{24} \frac{\partial^5 u}{\partial t^5}(s) ds, \end{aligned}$$

$$\begin{aligned} u(t^{n-1}) = u(t^n) &+ \frac{\partial u}{\partial t}(t^n)(t^{n-1} - t^n) + \frac{\partial^2 u}{\partial t^2}(t^n) \frac{(t^{n-1} - t^n)^2}{2} + \frac{\partial^3 u}{\partial t^3}(t^n) \frac{(t^{n-1} - t^n)^3}{6} \\ &+ \frac{\partial^4 u}{\partial t^4}(t^n) \frac{(t^{n-1} - t^n)^4}{24} + \int_{t^n}^{t^{n-1}} \frac{(t^{n-1} - s)^4}{24} \frac{\partial^5 u}{\partial t^5}(s) ds. \end{aligned}$$

Subtracting the previous two equations and using the result in the variational formulation as in Equations (4.2a) to (4.2c) leads to the following:

$$\left\langle \frac{u(t^{n+1}) - u(t^{n-1})}{2\Delta t}, \tilde{u} \right\rangle = \left\langle \frac{\partial u}{\partial t}(t^n), \tilde{u} \right\rangle + \frac{\Delta t^2}{6} \left\langle \frac{\partial^3 u}{\partial t^3}(t^n), \tilde{u} \right\rangle + \langle R_u^n, \tilde{u} \rangle,$$

in which we have defined that:

$$R_u^n := \begin{cases} 0, & n = 0, \\ \frac{1}{2\Delta t} \left[\int_{t^{n-1}}^{t^n} \frac{(t^{n-1} - s)^4}{24} \frac{\partial^5 u}{\partial t^5}(s) ds + \int_{t^n}^{t^{n+1}} \frac{(t^{n+1} - s)^4}{24} \frac{\partial^5 u}{\partial t^5}(s) ds \right], & n = 1, \dots, N-1, \end{cases}$$

for $u = \{p, E, H\}$. Again, using the Taylor remainder theorem, and expressing $u(t)$ about $t = t^n$ for $u = \{p, E, H\}$ with $n = 0, 1, \dots, N-1$, and evaluating this at $t = t^{n+1}$ and $t = t^{n-1}$ yields:

$$u(t^{n+1}) = u(t^n) + \frac{\partial u}{\partial t}(t^n)(t^{n+1} - t^n) + \frac{\partial^2 u}{\partial t^2}(t^n) \frac{(t^{n+1} - t^n)^2}{2} + \frac{\partial^3 u}{\partial t^3}(t^n) \frac{(t^{n+1} - t^n)^3}{6}$$

$$+ \int_{t^n}^{t^{n+1}} \frac{(t^{n+1} - s)^3}{6} \frac{\partial^4 u}{\partial t^4}(s) ds,$$

$$u(t^{n-1}) = u(t^n) + \frac{\partial u}{\partial t}(t^n)(t^{n-1} - t^n) + \frac{\partial^2 u}{\partial t^2}(t^n) \frac{(t^{n-1} - t^n)^2}{2} + \frac{\partial^3 u}{\partial t^3}(t^n) \frac{(t^{n-1} - t^n)^3}{6} \\ + \int_{t^n}^{t^{n-1}} \frac{(t^{n-1} - s)^3}{6} \frac{\partial^4 u}{\partial t^4}(s) ds.$$

Adding these two last equations, and using the result again in the variational formulation as in Equations (4.2a) to (4.2c) leads to:

$$\left\langle \frac{u(t^{n+1}) - 2u(t^n) + u(t^{n-1}))}{\Delta t^2}, \tilde{u} \right\rangle = \left\langle \frac{\partial^2 u}{\partial t^2}(t^n), \tilde{u} \right\rangle + \langle r_u^n, \tilde{u} \rangle,$$

in which we have again defined that:

$$r_u^n := \begin{cases} 0, & n = 0, \\ \frac{1}{\Delta t^2} \left[- \int_{t^{n-1}}^{t^n} \frac{(t^{n-1} - s)^3}{6} \frac{\partial^4 u}{\partial t^4}(s) ds + \int_{t^n}^{t^{n+1}} \frac{(t^{n+1} - s)^3}{6} \frac{\partial^4 u}{\partial t^4}(s) ds \right], & n = 1, \dots, N-1, \end{cases}$$

for $u = p, E$ or H . Using these terms in the weak formulation as in Equations (4.2a) to (4.2c) and evaluating at $t = t^n$ for p, E , and H with $n = 1, \dots, N-1$, we are led to:

$$\left\langle \varepsilon^{-1} \frac{p(t^{n+1}) - p(t^{n-1}))}{2\Delta t}, \tilde{p} \right\rangle - \frac{1}{6} \langle E(t^{n+1}) + 4E(t^n) + E(t^{n-1}), \nabla \tilde{p} \rangle = \left\langle \varepsilon^{-1} R_p^n + \frac{\Delta t^2}{6} \nabla \cdot r_E^n, \tilde{p} \right\rangle, \quad (4.19a)$$

$$\frac{1}{6} \langle \nabla \left(p(t^{n+1}) + 4p(t^n) + p(t^{n-1}) \right), \tilde{E} \rangle + \left\langle \varepsilon \frac{E(t^{n+1}) - E(t^{n-1}))}{2\Delta t}, \tilde{E} \right\rangle \\ - \frac{1}{6} \langle \nabla \times \left(H(t^{n+1}) + 4H(t^n) + H(t^{n-1}) \right), \tilde{E} \rangle = \left\langle \varepsilon R_E^n + \frac{\Delta t^2}{6} (\nabla r_p^n - \nabla \times r_H^n), \tilde{E} \right\rangle, \quad (4.19b)$$

$$\left\langle \mu \frac{H(t^{n+1}) - H(t^{n-1}))}{2\Delta t}, \tilde{H} \right\rangle + \frac{1}{6} \langle \nabla \times \left(E(t^{n+1}) + 4E(t^n) + E(t^{n-1}) \right), \tilde{H} \rangle \\ = \left\langle \mu R_H^n + \frac{\Delta t^2}{6} \nabla \times r_E^n, \tilde{H} \right\rangle. \quad (4.19c)$$

Then, subtracting the TS₄ scheme equations as in Equations (4.5a) to (4.5c) leads us to the following set of equations:

$$\left\langle \varepsilon^{-1} \frac{e_p^{n+1} - e_p^{n-1}}{2\Delta t}, \tilde{p} \right\rangle - \frac{1}{6} \langle e_E^{n+1} + 4e_E^n + e_E^{n-1}, \nabla \tilde{p} \rangle = \left\langle \varepsilon^{-1} R_p^n + \frac{\Delta t^2}{6} \nabla \cdot r_E^n, \tilde{p} \right\rangle, \\ \frac{1}{6} \langle \nabla \left(e_p^{n+1} + 4e_p^n + e_p^{n-1} \right), \tilde{E} \rangle + \left\langle \varepsilon \frac{e_E^{n+1} - e_E^{n-1}}{2\Delta t}, \tilde{E} \right\rangle - \frac{1}{6} \langle \nabla \times \left(e_H^{n+1} + 4e_H^n + e_H^{n-1} \right), \tilde{E} \rangle \\ = \left\langle \varepsilon R_E^n + \frac{\Delta t^2}{6} (\nabla r_p^n - \nabla \times r_H^n), \tilde{E} \right\rangle, \\ \left\langle \mu \frac{e_H^{n+1} - e_H^{n-1}}{2\Delta t}, \tilde{H} \right\rangle + \frac{1}{6} \langle \nabla \times \left(e_E^{n+1} + 4e_E^n + e_E^{n-1} \right), \tilde{H} \rangle = \left\langle \mu R_H^n + \frac{\Delta t^2}{6} \nabla \times r_E^n, \tilde{H} \right\rangle.$$

Now, in this set of weak formulation equations for the errors, we choose the test functions to be $\tilde{p} = 2\Delta t (e_p^{n+1} + 4e_p^n + e_p^{n-1})$, $\tilde{E} = 2\Delta t (e_E^{n+1} + 4e_E^n + e_E^{n-1})$ and $\tilde{H} = 2\Delta t (e_H^{n+1} + 4e_H^n + e_H^{n-1})$. Next, by following

essentially the same sequence of steps as in Theorem 4.5, we obtain the estimates for these error terms to be:

$$\begin{aligned}
& \|e_p^{n+1} + e_p^n\|_{\varepsilon^{-1}} + 2\langle e_p^{n+1}, e_p^n \rangle_{\varepsilon^{-1}} - \|e_p^n + e_p^{n-1}\|_{\varepsilon^{-1}} - 2\langle e_p^n, e_p^{n-1} \rangle_{\varepsilon^{-1}} + \|e_E^{n+1} + e_E^n\|_{\varepsilon} \\
& + 2\langle e_E^{n+1}, e_E^n \rangle_{\varepsilon} - \|e_E^n + e_E^{n-1}\|_{\varepsilon} - 2\langle e_E^n, e_E^{n-1} \rangle_{\varepsilon} + \|e_H^{n+1} + e_H^n\|_{\mu} + 2\langle e_H^{n+1}, e_H^n \rangle_{\mu} \\
& - \|e_H^n + e_H^{n-1}\|_{\mu} - 2\langle e_H^n, e_H^{n-1} \rangle_{\mu} = 2\Delta t \left(\langle \varepsilon^{-1} R_p^n + \frac{\Delta t^2}{6} \nabla \cdot r_E^n, e_p^{n+1} + 4e_p^n + e_p^{n-1} \rangle + \right. \\
& \quad \left. \langle \varepsilon R_E^n + \frac{\Delta t^2}{6} (\nabla r_p^n - \nabla \times r_H^n), e_E^{n+1} + 4e_E^n + e_E^{n-1} \rangle \right. \\
& \quad \left. + \langle \mu R_H^n + \frac{\Delta t^2}{6} \nabla \times r_E^n, e_H^{n+1} + 4e_H^n + e_H^{n-1} \rangle \right).
\end{aligned}$$

Applying the Cauchy-Schwarz, triangle and AM-GM inequalities appropriately on right-hand side terms, and using that $\|a + b\|^2 = \|a\|^2 + \|b\|^2 + 2\langle a, b \rangle$ for the left-hand side terms of the above equation, we obtain:

$$\begin{aligned}
& \|e_p^{n+1}\|_{\varepsilon^{-1}}^2 + \|e_p^n\|_{\varepsilon^{-1}}^2 + 4\langle e_p^{n+1}, e_p^n \rangle_{\varepsilon^{-1}} - \|e_p^n\|_{\varepsilon^{-1}}^2 - \|e_p^{n-1}\|_{\varepsilon^{-1}}^2 - 4\langle e_p^n, e_p^{n-1} \rangle_{\varepsilon^{-1}} \\
& + \|e_E^{n+1}\|_{\varepsilon}^2 + \|e_E^n\|_{\varepsilon}^2 + 4\langle e_E^{n+1}, e_E^n \rangle_{\varepsilon} - \|e_E^n\|_{\varepsilon}^2 - \|e_E^{n-1}\|_{\varepsilon}^2 - 4\langle e_E^n, e_E^{n-1} \rangle_{\varepsilon} \\
& + \|e_H^{n+1}\|_{\mu}^2 + \|e_H^n\|_{\mu}^2 + 4\langle e_H^{n+1}, e_H^n \rangle_{\mu} - \|e_H^n\|_{\mu}^2 - \|e_H^{n-1}\|_{\mu}^2 - 4\langle e_H^n, e_H^{n-1} \rangle_{\mu} \\
& \leq 3\Delta t \left[\|e_p^{n+1}\|_{\varepsilon^{-1}}^2 + \|e_p^n\|_{\varepsilon^{-1}}^2 + \|e_p^{n-1}\|_{\varepsilon^{-1}}^2 + \|e_E^{n+1}\|_{\varepsilon}^2 + \|e_E^n\|_{\varepsilon}^2 + \|e_E^{n-1}\|_{\varepsilon}^2 \right. \\
& \quad \left. + \|e_H^{n+1}\|_{\mu}^2 + \|e_H^n\|_{\mu}^2 + \|e_H^{n-1}\|_{\mu}^2 \right] + 18\Delta t \left[\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^n\|_{\mu}^2 \right. \\
& \quad \left. + \frac{\Delta t^4}{36} \left(\|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \left(\|\nabla \times r_E^n\|_{\varepsilon}^2 + \|\nabla \times r_H^n\|_{\mu}^2 \right) \right) \right].
\end{aligned}$$

Summing over $n = 1$ to $N - 2$ on both sides, we get:

$$\begin{aligned}
& \|e_p^{N-1}\|_{\varepsilon^{-1}}^2 + \|e_p^{N-2}\|_{\varepsilon^{-1}}^2 + 4\langle e_p^{N-1}, e_p^{N-2} \rangle_{\varepsilon^{-1}} - \|e_p^1\|_{\varepsilon^{-1}}^2 - \|e_p^0\|_{\varepsilon^{-1}}^2 - 4\langle e_p^1, e_p^0 \rangle_{\varepsilon^{-1}} \\
& + \|e_E^{N-1}\|_{\varepsilon}^2 + \|e_E^{N-2}\|_{\varepsilon}^2 + 4\langle e_E^{N-1}, e_E^{N-2} \rangle_{\varepsilon} - \|e_E^1\|_{\varepsilon}^2 - \|e_E^0\|_{\varepsilon}^2 - 4\langle e_E^1, e_E^0 \rangle_{\varepsilon} + \|e_H^{N-1}\|_{\mu}^2 + \|e_H^{N-2}\|_{\mu}^2 \\
& + 4\langle e_H^{N-1}, e_H^{N-2} \rangle_{\mu} - \|e_H^1\|_{\mu}^2 - \|e_H^0\|_{\mu}^2 - 4\langle e_H^1, e_H^0 \rangle_{\mu} \leq 9\Delta t \sum_{n=0}^{N-1} \left[\|e_p^n\|_{\varepsilon^{-1}}^2 + \|e_E^n\|_{\varepsilon}^2 + \|e_H^n\|_{\mu}^2 \right] \\
& + 18\Delta t \sum_{n=1}^{N-2} \left[\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^n\|_{\mu}^2 \right. \\
& \quad \left. + \frac{\Delta t^4}{36} \left(\|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \left(\|\nabla \times r_E^n\|_{\varepsilon}^2 + \|\nabla \times r_H^n\|_{\mu}^2 \right) \right) \right].
\end{aligned}$$

Since $\langle \cdot, \cdot \rangle_{\alpha}$ are non-negative for $\alpha = \{\varepsilon^{-1}, \varepsilon, \mu\}$, thus using the Cauchy-Schwarz inequality we obtain that:

$$\begin{aligned}
& \|e_p^{N-1}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-1}\|_{\varepsilon}^2 + \|e_H^{N-1}\|_{\mu}^2 \leq 3 \left(\|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^0\|_{\varepsilon}^2 + \|e_H^0\|_{\mu}^2 + \|e_p^1\|_{\varepsilon^{-1}}^2 + \|e_E^1\|_{\varepsilon}^2 \right. \\
& \quad \left. + \|e_H^1\|_{\mu}^2 \right) + 18\Delta t \sum_{n=0}^{N-1} \left[\|e_p^n\|_{\varepsilon^{-1}}^2 + \|e_E^n\|_{\varepsilon}^2 + \|e_H^n\|_{\mu}^2 \right] + 18\Delta t \sum_{n=0}^{N-1} \left[\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 \right. \\
& \quad \left. + \|R_H^n\|_{\mu}^2 + \frac{\Delta t^4}{36} \left(\|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \left(\|\nabla \times r_E^n\|_{\varepsilon}^2 + \|\nabla \times r_H^n\|_{\mu}^2 \right) \right) \right].
\end{aligned}$$

Applying the discrete Gronwall inequality 1.10 similarly as in Theorem 4.2, we obtain the estimate:

$$\begin{aligned}
& \|e_p^{N-1}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-1}\|_{\varepsilon}^2 + \|e_H^{N-1}\|_{\mu}^2 \leq 3 \left(\|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^0\|_{\varepsilon}^2 + \|e_H^0\|_{\mu}^2 + \|e_p^1\|_{\varepsilon^{-1}}^2 + \|e_E^1\|_{\varepsilon}^2 \right. \\
& \quad \left. + \|e_H^1\|_{\mu}^2 \right) + 18\Delta t \sum_{n=0}^{N-1} \left[\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^n\|_{\mu}^2 \right. \\
& \quad \left. + \frac{\Delta t^4}{36} \left(\|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_{\varepsilon}^2 + \varepsilon^{-1}\mu^{-1} \left(\|\nabla \times r_E^n\|_{\varepsilon}^2 + \|\nabla \times r_H^n\|_{\mu}^2 \right) \right) \right] \exp(36T).
\end{aligned}$$

Now, we need to obtain bounding estimates for each of the Taylor remainder terms. So with $u = \{p, E, H\}$ and $n = 0, 1, \dots, N-1$, we argue as following:

$$\begin{aligned}
\|R_u^n\|_{\alpha}^2 &= \frac{1}{4(24)^2\Delta t^2} \left\| \int_{t^{n-1}}^{t^n} (t^{n-1} - s)^4 \frac{\partial^5 u}{\partial t^5}(s) ds + \int_{t^n}^{t^{n+1}} (t^{n+1} - s)^4 \frac{\partial^5 u}{\partial t^5}(s) ds \right\|_{\alpha}^2, \\
&\leq \frac{1}{(48)^2(\Delta t)^2} \int_{t^{n-1}}^{t^{n+1}} (s - t^{n+1})^8 ds \int_{t^{n-1}}^{t^{n+1}} \left\| \frac{\partial^5 u}{\partial t^5}(s) \right\|_{\alpha}^2 ds, \\
&\quad \text{(using } t^{n-1} < t^{n+1} \text{ and Cauchy-Schwarz)} \\
&= \frac{2}{81} \Delta t^7 \int_{t^{n-1}}^{t^{n+1}} \left\| \frac{\partial^5 u}{\partial t^5}(s) \right\|_{\alpha}^2 ds,
\end{aligned}$$

and now summing both sides over $n = 0$ to $N-1$, we have that:

$$\sum_{n=0}^{N-1} \|R_u^n\|_{\alpha}^2 \leq \frac{2}{81} \Delta t^7 \int_0^T \left\| \frac{\partial^5 u}{\partial t^5}(s) \right\|_{\alpha}^2 ds = \frac{2}{81} \Delta t^7 \left\| \frac{\partial^5 u}{\partial t^5} \right\|_{L^2(0,T;L_{\alpha}^2(\Omega))}^2.$$

Similarly, for any operator Φ acting on u belonging to an appropriate function space on Ω , using the Cauchy-Schwarz and Chebyshev inequalities, we have that:

$$\sum_{n=0}^{N-1} \|\Phi(r_u^n)\|_{\alpha}^2 \leq \frac{\Delta t^3}{252} \int_0^T \left\| \frac{\partial^4(\Phi(u))}{\partial t^4}(s) \right\|_{\alpha}^2 ds = \frac{\Delta t^3}{252} \left\| \frac{\partial^4(\Phi(u))}{\partial t^4} \right\|_{L^2(0,T;L_{\alpha}^2(\Omega))}^2,$$

for $u = \{p, E, H\}$ and $n = 0, 1, \dots, N-1$. Finally, using the regularity assumptions for p, E and H , and 1- and 2-norm equivalences, we obtain the required result:

$$\begin{aligned}
\|e_p^{N-1}\|_{\varepsilon^{-1}} + \|e_E^{N-1}\|_{\varepsilon} + \|e_H^{N-1}\|_{\mu} &\leq C \left[(\Delta t)^4 + \|e_p^0\|_{\varepsilon^{-1}} + \|e_E^0\|_{\varepsilon} + \|e_H^0\|_{\mu} \right. \\
&\quad \left. + \|e_p^1\|_{\varepsilon^{-1}} + \|e_E^1\|_{\varepsilon} + \|e_H^1\|_{\mu} \right]. \quad \square
\end{aligned}$$

4.3.2 Error Estimate for Full Discretization

The full error analysis of the three-field Maxwell's systems with arbitrary-order de Rham finite elements in conjunction with TS_4 is nearly identical to that for the LF_4 case in Section 4.2. Like earlier, we again define the errors for p, E and H at time $n\Delta t$ under the full discretization as the following:

$$e_{p_h}^n := p(t^n) - p_h^n = \eta^n - \eta_h^n, \quad (4.20)$$

$$e_{E_h}^n := E(t^n) - E_h^n = \zeta^n - \zeta_h^n, \quad (4.21)$$

$$e_{H_h}^n := H(t^n) - H_h^n = \xi^n - \xi_h^n, \quad (4.22)$$

and in which we again have the following definitions for the above newly introduced terms:

$$\eta^n := p(t^n) - \Pi_h^0 p(t^n), \quad \eta_h^n := p_h^n - \Pi_h^0 p(t^n), \quad (4.23)$$

$$\zeta^n := E(t^n) - \Pi_h^1 E(t^n), \quad \zeta_h^n := E_h^n - \Pi_h^1 E(t^n), \quad (4.24)$$

$$\xi^n := H(t^n) - \Pi_h^2 H(t^n), \quad \xi_h^n := H_h^n - \Pi_h^2 H(t^n). \quad (4.25)$$

For the TS₄ scheme as in Equations (4.5a) to (4.5c), using a de Rham sequence of finite dimensional subspaces of the corresponding function spaces for the spatial discretization of (p^n, E^n, H^n) , we obtain the following discrete problem: find $(p_h^n, E_h^n, H_h^n) \in U_h \times V_h \times W_h \subseteq \dot{H}_{\varepsilon^{-1}}^1 \times \dot{H}_\varepsilon(\text{curl}; \Omega) \times \dot{H}_\mu(\text{div}; \Omega)$ such that:

$$\left\langle \varepsilon^{-1} \frac{p_h^{n+1} - p_h^{n-1}}{2\Delta t}, \tilde{p} \right\rangle - \frac{1}{6} \langle E_h^{n+1} + 4E_h^n + E_h^{n-1}, \nabla \tilde{p} \rangle = 0, \quad (4.26a)$$

$$\begin{aligned} \frac{1}{6} \langle \nabla (p_h^{n+1} + 4p_h^n + p_h^{n-1}), \tilde{E} \rangle + \left\langle \varepsilon \frac{E_h^{n+1} - E_h^{n-1}}{2\Delta t}, \tilde{E} \right\rangle \\ - \frac{1}{6} \langle \nabla \times (H_h^{n+1} + 4H_h^n + H_h^{n-1}), \tilde{E} \rangle = 0, \end{aligned} \quad (4.26b)$$

$$\left\langle \mu \frac{H_h^{n+1} - H_h^{n-1}}{2\Delta t}, \tilde{H} \right\rangle + \frac{1}{6} \langle \nabla \times (E_h^{n+1} + 4E_h^n + E_h^{n-1}), \tilde{H} \rangle = 0. \quad (4.26c)$$

for all $(\tilde{p}, \tilde{E}, \tilde{H}) \in U_h \times V_h \times W_h$, $n = 1, \dots, N-1$. As earlier, we again have the same L^2 smoothed projection operators, namely, $\Pi_h^0 : \dot{H}_{\varepsilon^{-1}}^1(\Omega) \rightarrow U_h$, $\Pi_h^1 : \dot{H}_\varepsilon(\text{curl}; \Omega) \rightarrow V_h$ and $\Pi_h^2 : \dot{H}_\mu(\text{div}; \Omega) \rightarrow W_h$, and for the initial conditions, we simply again set:

$$u_h^0 := \Pi_h^k u_0 \text{ and } u_h^1 := \Pi_h^k u_1, \text{ for } u = \{p, E, H\} \text{ and } k = 0, 1 \text{ or } 2 \text{ respectively.} \quad (4.26d)$$

With this setup, we now state and prove our theorem for the convergence of the errors in the full discretization of our system of Maxwell's equations using TS₄ and arbitrary-order de Rham finite elements.

Theorem 4.7 (Full Error Estimate). *Let $p \in C^5(0, T; \dot{H}_{\varepsilon^{-1}}^1(\Omega))$, $E \in C^5(0, T; \dot{H}_\varepsilon(\text{curl}; \Omega))$, and $H \in C^5(0, T; \dot{H}_\mu(\text{div}; \Omega))$ be the solution to the variational formulation of the Maxwell's equations as in Equations (4.2a) to (4.2c) with sufficient regularity, and let (p_h^n, E_h^n, H_h^n) be the solution of the fully discretized Maxwell's equations using TS₄ scheme as in Equations (4.16d) and (4.26a) to (4.26c). With Assumption 4.4, if the time step $\Delta t > 0$ and the mesh parameter $h > 0$ are sufficiently small, then there exists a positive bounded constant C independent of both Δt and h such that the following error estimate holds:*

$$\|e_{p_h}^{N-1}\|_{\varepsilon^{-1}} + \|e_{E_h}^{N-1}\|_\varepsilon + \|e_{H_h}^{N-1}\|_\mu \leq C [\Delta t^4 + h^r + h^r \Delta t^4],$$

where the finite element subspaces U_h , V_h and W_h are each spanned by their respective Whitney form basis of polynomial order $r \geq 1$.

Proof. First, we shall subtract the set of equations for the full discretization as in Equations (4.26a) to (4.26c) from Equations (4.19a) to (4.19c), and then use the error terms in Equations (4.20) to (4.22) and thereby obtain:

$$\left\langle \varepsilon^{-1} \frac{e_{p_h}^{n+1} - e_{p_h}^{n-1}}{2\Delta t}, \tilde{p} \right\rangle - \frac{1}{6} \langle e_{E_h}^{n+1} + 4e_{E_h}^n + e_{E_h}^{n-1}, \nabla \tilde{p} \rangle = \left\langle \varepsilon^{-1} R_p^n + \frac{\Delta t^2}{6} \nabla \cdot r_E^n, \tilde{p} \right\rangle,$$

$$\begin{aligned}
& \frac{1}{6} \langle \nabla (e_{p_h}^{n+1} + 4e_{p_h}^n + e_{p_h}^{n-1}), \tilde{E} \rangle + \langle \varepsilon \frac{e_{E_h}^{n+1} - e_{E_h}^{n-1}}{2\Delta t}, \tilde{E} \rangle \\
& - \frac{1}{6} \langle \nabla \times (e_{H_h}^{n+1} + 4e_{H_h}^n + e_{H_h}^{n-1}), \tilde{E} \rangle = \langle \varepsilon R_E^n + \frac{\Delta t^2}{6} (\nabla r_p^n - \nabla \times r_H^n), \tilde{E} \rangle, \\
& \langle \mu \frac{e_{H_h}^{n+1} - e_{H_h}^{n-1}}{2\Delta t}, \tilde{H} \rangle + \frac{1}{6} \langle \nabla \times (e_{E_h}^{n+1} + 4e_{E_h}^n + e_{E_h}^{n-1}), \tilde{H} \rangle = \langle \mu R_H^n + \frac{\Delta t^2}{6} \nabla \times r_E^n, \tilde{H} \rangle.
\end{aligned}$$

Next, using the values of the error terms $e_{p_h}^n$, $e_{E_h}^n$ and $e_{H_h}^n$ as in Equations (4.20) to (4.22) in the above equations, we get:

$$\begin{aligned}
& \langle \varepsilon^{-1} \frac{(\eta^{n+1} - \eta^{n-1}) - (\eta_h^{n+1} - \eta_h^{n-1})}{2\Delta t}, \tilde{p} \rangle \\
& - \frac{1}{6} \langle (\zeta^{n+1} + 4\zeta^n + \zeta^{n-1}) - (\zeta_h^{n+1} + 4\zeta_h^n + \zeta_h^{n-1}), \nabla \tilde{p} \rangle = \langle \varepsilon^{-1} R_p^n + \frac{\Delta t^2}{6} \nabla \cdot r_E^n, \tilde{p} \rangle,
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{6} \langle \nabla (\eta^{n+1} + 4\eta^n + \eta^{n-1}) - \nabla (\eta_h^{n+1} + 4\eta_h^n + \eta_h^{n-1}), \tilde{E} \rangle \\
& + \langle \varepsilon \frac{(\zeta^{n+1} - \zeta^{n-1}) - (\zeta_h^{n+1} - \zeta_h^{n-1})}{2\Delta t}, \tilde{E} \rangle \\
& - \frac{1}{6} \langle (\xi^{n+1} + 4\xi^n + \xi^{n-1}) - (\xi_h^{n+1} + 4\xi_h^n + \xi_h^{n-1}), \nabla \times \tilde{E} \rangle \\
& = \langle \varepsilon R_E^n + \frac{\Delta t^2}{6} (\nabla r_p^n - \nabla \times r_H^n), \tilde{E} \rangle,
\end{aligned}$$

$$\begin{aligned}
& \langle \mu \frac{(\xi^{n+1} - \xi^{n-1}) - (\xi_h^{n+1} - \xi_h^{n-1})}{2\Delta t}, \tilde{H} \rangle \\
& + \frac{1}{6} \langle \nabla \times (\zeta^{n+1} + 4\zeta^n + \zeta^{n-1}) - \nabla \times (\zeta_h^{n+1} + 4\zeta_h^n + \zeta_h^{n-1}), \tilde{H} \rangle = \langle \mu R_H^n + \frac{\Delta t^2}{6} \nabla \times r_E^n, \tilde{H} \rangle.
\end{aligned}$$

Since these equations are true for all $(\tilde{p}, \tilde{E}, \tilde{H}) \in U_h \times V_h \times W_h$, we choose $\tilde{p} = -2\Delta t (\eta_h^{n+1} + 4\eta_h^n + \eta_h^{n-1})$, $\tilde{E} = -2\Delta t (\zeta_h^{n+1} + 4\zeta_h^n + \zeta_h^{n-1})$ and $\tilde{H} = -2\Delta t (\xi_h^{n+1} + 4\xi_h^n + \xi_h^{n-1})$ and using the fact that $\nabla U_h \subseteq V_h$ and $\nabla \times V_h \subseteq W_h$, we obtain that:

$$\begin{aligned}
& \langle \varepsilon^{-1} (\eta^{n+1} - \eta^{n-1}), \eta_h^{n+1} + 4\eta_h^n + \eta_h^{n-1} \rangle + \langle \varepsilon (\zeta^{n+1} - \zeta^{n-1}), \zeta_h^{n+1} + 4\zeta_h^n + \zeta_h^{n-1} \rangle \\
& + \langle \mu (\xi^{n+1} - \xi^{n-1}), \xi_h^{n+1} + 4\xi_h^n + \xi_h^{n-1} \rangle = \langle \varepsilon^{-1} (\eta^{n+1} - \eta^{n-1}), \eta_h^{n+1} + 4\eta_h^n + \eta_h^{n-1} \rangle \\
& + \langle \varepsilon (\zeta^{n+1} - \zeta^{n-1}), \zeta_h^{n+1} + 4\zeta_h^n + \zeta_h^{n-1} \rangle + \langle \mu (\xi^{n+1} - \xi^{n-1}), \xi_h^{n+1} + 4\xi_h^n + \xi_h^{n-1} \rangle \\
& + 2\Delta t \left[\langle -\varepsilon^{-1} R_p^n + \frac{\Delta t^2}{6} \nabla \cdot r_E^n, \eta_h^{n+1} + 4\eta_h^n + \eta_h^{n-1} \rangle \right. \\
& + \langle -\varepsilon R_E^n + \frac{\Delta t^2}{6} (\nabla r_p^n - \nabla \times r_H^n), \zeta_h^{n+1} + 4\zeta_h^n + \zeta_h^{n-1} \rangle \\
& \left. + \langle -\mu R_H^n + \frac{\Delta t^2}{6} \nabla \times r_E^n, \xi_h^{n+1} + 4\xi_h^n + \xi_h^{n-1} \rangle \right]. \quad (4.27)
\end{aligned}$$

Consider now that $\varepsilon^{-1} (\eta^{n+1} - \eta^{n-1}) = \varepsilon^{-1} (I - \Pi_h^0) (p(t^{n+1}) - p(t^{n-1}))$ by Equation (4.23). Using the Taylor theorem with remainder as in Theorem 4.6, applying the Cauchy-Schwarz, AM-GM, and triangle inequalities,

we have the following inequality:

$$\begin{aligned}
\langle \varepsilon^{-1} (\eta^{n+1} - \eta^{n-1}), \eta_h^{n+1} + 4\eta_h^n + \eta_h^{n-1} \rangle &= 2\Delta t \langle \varepsilon^{-1} (I - \Pi_h^0) \left(\frac{\partial p}{\partial t}(t^n) \right. \\
&\quad \left. + \frac{\Delta t^2}{6} \frac{\partial^3 p}{\partial t^3}(t^n) \right), \eta_h^{n+1} + 4\eta_h^n + \eta_h^{n-1} \rangle \\
&\quad + 2\Delta t \langle \varepsilon^{-1} (I - \Pi_h^0) R_p^n, \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle \\
&\leq 18\Delta t \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 \right. \\
&\quad \left. + \frac{\Delta t^4}{36} \left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 \right] \\
&\quad + 3\Delta t \left[\|\eta_h^{n+1}\|_{\varepsilon^{-1}}^2 + \|\eta_h^n\|_{\varepsilon^{-1}}^2 + \|\eta_h^{n-1}\|_{\varepsilon^{-1}}^2 \right].
\end{aligned}$$

Similarly, using Equations (4.24) and (4.25) for the full error terms for E and H, we obtain the following:

$$\begin{aligned}
\langle \varepsilon (\zeta^{n+1} - \zeta^{n-1}), \zeta_h^{n+1} + 4\zeta_h^n + \zeta_h^{n-1} \rangle &\leq 18\Delta t \left[\left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \frac{\Delta t^4}{36} \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 \right. \\
&\quad \left. + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 \right] + 3\Delta t \left[\|\zeta_h^{n+1}\|_{\varepsilon}^2 + \|\zeta_h^n\|_{\varepsilon}^2 + \|\zeta_h^{n-1}\|_{\varepsilon}^2 \right], \\
\langle \mu (\xi^{n+1} - \xi^{n-1}), \xi_h^{n+1} + 4\xi_h^n + \xi_h^{n-1} \rangle &\leq 18\Delta t \left[\left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^n) \right\|_{\mu}^2 + \frac{\Delta t^4}{36} \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3}(t^n) \right\|_{\mu}^2 \right. \\
&\quad \left. + \left\| (I - \Pi_h^2) R_H^n \right\|_{\mu}^2 \right] + 3\Delta t \left[\|\xi_h^{n+1}\|_{\mu}^2 + \|\xi_h^n\|_{\mu}^2 + \|\xi_h^{n-1}\|_{\mu}^2 \right].
\end{aligned}$$

Using these inequalities for η , ζ and ξ in Equation (4.27), we thus obtain the following estimate:

$$\begin{aligned}
&\|\eta_h^{n+1}\|_{\varepsilon^{-1}}^2 + \|\eta_h^n\|_{\varepsilon^{-1}}^2 + 4\langle \eta_h^{n+1}, \eta_h^n \rangle_{\varepsilon^{-1}} - \|\eta_h^n\|_{\varepsilon^{-1}}^2 - \|\eta_h^{n-1}\|_{\varepsilon^{-1}}^2 - 4\langle \eta_h^n, \eta_h^{n-1} \rangle_{\varepsilon^{-1}} + \|\zeta_h^{n+1}\|_{\varepsilon}^2 + \|\zeta_h^n\|_{\varepsilon}^2 \\
&+ 4\langle \zeta_h^{n+1}, \zeta_h^n \rangle_{\varepsilon} - \|\zeta_h^n\|_{\varepsilon}^2 - \|\zeta_h^{n-1}\|_{\varepsilon}^2 - 4\langle \zeta_h^n, \zeta_h^{n-1} \rangle_{\varepsilon} + \|\xi_h^{n+1}\|_{\mu}^2 + \|\xi_h^n\|_{\mu}^2 + 4\langle \xi_h^{n+1}, \xi_h^n \rangle_{\mu} - \|\xi_h^n\|_{\mu}^2 - \|\xi_h^{n-1}\|_{\mu}^2 \\
&- 4\langle \xi_h^n, \xi_h^{n-1} \rangle_{\mu} \leq 18\Delta t \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^n) \right\|_{\mu}^2 \right. \\
&\quad \left. + \frac{\Delta t^4}{36} \left(\left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3}(t^n) \right\|_{\mu}^2 \right) \right. \\
&\quad \left. + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) R_H^n \right\|_{\mu}^2 + \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^n\|_{\mu}^2 \right. \\
&\quad \left. + \frac{\Delta t^4}{36} \left(\|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \left(\|\nabla \times r_E^n\|_{\varepsilon}^2 + \|\nabla \times r_H^n\|_{\mu}^2 \right) \right) \right] \\
&\quad + 6\Delta t \left[\|\eta_h^{n+1}\|_{\varepsilon^{-1}}^2 + \|\eta_h^n\|_{\varepsilon^{-1}}^2 + \|\eta_h^{n-1}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{n+1}\|_{\varepsilon}^2 + \|\zeta_h^n\|_{\varepsilon}^2 + \|\zeta_h^{n-1}\|_{\varepsilon}^2 \right. \\
&\quad \left. + \|\xi_h^{n+1}\|_{\mu}^2 + \|\xi_h^n\|_{\mu}^2 + \|\xi_h^{n-1}\|_{\mu}^2 \right].
\end{aligned}$$

Summing from $n = 1$ to $N - 2$ and using the Cauchy-Schwarz inequality along with non-negativity of inner products $\langle \cdot, \cdot \rangle_{\alpha}$ for $\alpha = \{\varepsilon^{-1}, \varepsilon, \mu\}$, we have that:

$$\begin{aligned}
& \|\eta_h^{N-1}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{N-1}\|_{\varepsilon}^2 + \|\xi_h^{N-1}\|_{\mu}^2 \leq 3 \left(\|\eta_h^0\|_{\varepsilon^{-1}}^2 + \|\zeta_h^0\|_{\varepsilon}^2 + \|\xi_h^0\|_{\mu}^2 + \|\eta_h^1\|_{\varepsilon^{-1}}^2 + \|\zeta_h^1\|_{\varepsilon}^2 + \|\xi_h^1\|_{\mu}^2 \right) \\
& + 18\Delta t \sum_{n=0}^{N-1} \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^n) \right\|_{\mu}^2 \right. \\
& + \frac{\Delta t^4}{36} \left(\left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3}(t^n) \right\|_{\mu}^2 \right) + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 \\
& + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) R_H^n \right\|_{\mu}^2 + \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^n\|_{\mu}^2 + \frac{\Delta t^4}{36} \left(\|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_{\varepsilon}^2 \right. \\
& \left. \left. + \varepsilon^{-1} \mu^{-1} \left(\|\nabla \times r_E^n\|_{\varepsilon}^2 + \|\nabla \times r_H^n\|_{\mu}^2 \right) \right) \right] + 18\Delta t \sum_{n=0}^{N-1} \left[\|\eta_h^n\|_{\varepsilon^{-1}}^2 + \|\zeta_h^n\|_{\varepsilon}^2 + \|\xi_h^n\|_{\mu}^2 \right].
\end{aligned}$$

Now, applying the discrete Gronwall inequality 1.10 with $\Delta t < 1/24$, we get that:

$$\begin{aligned}
& \|\eta_h^{N-1}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{N-1}\|_{\varepsilon}^2 + \|\xi_h^{N-1}\|_{\mu}^2 \leq \left[3 \left(\|\eta_h^0\|_{\varepsilon^{-1}}^2 + \|\zeta_h^0\|_{\varepsilon}^2 + \|\xi_h^0\|_{\mu}^2 + \|\eta_h^1\|_{\varepsilon^{-1}}^2 + \|\zeta_h^1\|_{\varepsilon}^2 + \|\xi_h^1\|_{\mu}^2 \right) \right. \\
& + 18\Delta t \sum_{n=0}^{N-1} \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^n) \right\|_{\mu}^2 \right. \\
& + \frac{\Delta t^4}{36} \left(\left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3}(t^n) \right\|_{\mu}^2 \right) + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 \\
& + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) R_H^n \right\|_{\mu}^2 + \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^n\|_{\mu}^2 + \frac{\Delta t^4}{36} \left(\|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_{\varepsilon}^2 \right. \\
& \left. \left. + \varepsilon^{-1} \mu^{-1} \left(\|\nabla \times r_E^n\|_{\varepsilon}^2 + \|\nabla \times r_H^n\|_{\mu}^2 \right) \right) \right] \exp(36T).
\end{aligned}$$

Using our estimates for the Taylor remainders from Theorem 4.2 for the terms on the right-hand side of the above inequality, we further get that:

$$\begin{aligned}
18\Delta t \sum_{n=0}^{N-1} \left[\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^n\|_{\mu}^2 \right] & \leq \\
& \frac{4}{9} \Delta t^8 \left[\left\| \frac{\partial^5 p}{\partial t^5} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^5 E}{\partial t^5} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial^5 H}{\partial t^5} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right],
\end{aligned}$$

and that:

$$\begin{aligned}
& \frac{\Delta t^5}{2} \sum_{n=0}^N \left[\|\nabla r_p^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_E^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \left(\|\nabla \times r_E^n\|_{\varepsilon}^2 + \|\nabla \times r_H^n\|_{\mu}^2 \right) \right] \\
& \leq \frac{\Delta t^8}{504} \left[\left\| \frac{\partial^4 (\nabla p)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^4 (\nabla \cdot E)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 \right. \\
& \left. + \varepsilon^{-1} \mu^{-1} \left(\left\| \frac{\partial^4 (\nabla \times E)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial^4 (\nabla \times H)}{\partial t^4} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right) \right].
\end{aligned}$$

Now, for $p \in \dot{H}_{\varepsilon^{-1}}^1(\Omega)$, $E \in \dot{H}_{\varepsilon}(\text{curl}; \Omega)$ and $H \in \dot{H}_{\mu}(\text{div}; \Omega)$, there exists positive bounded constants $C_{1,u}$, $C_{2,u}$, $C_{3,u}$ for $u = \{p, E, H\}$ such that we have the following error bounds for the L^2 projections:

$$\begin{aligned} \left\| (I - \Pi_h^k) \frac{\partial u}{\partial t}(t^n) \right\|_{\alpha} &\leq C_{1,u} h^r \left\| \frac{\partial u}{\partial t}(t^n) \right\|_{\alpha}, \left\| (I - \Pi_h^k) \frac{\partial^3 u}{\partial t^3}(t^n) \right\|_{\alpha} \leq C_{2,u} h^r \left\| \frac{\partial^3 u}{\partial t^3}(t^n) \right\|_{\alpha}, \\ \left\| (I - \Pi_h^0) R_p^n \right\|_{\alpha} &\leq C_{3,p} h^r \left\| R_p^n \right\|_{\alpha}, \end{aligned}$$

where $k = 0, 1, 2$ and $\alpha = \{\varepsilon^{-1}, \varepsilon, \mu\}$, respectively. Set $C_0 := \max_{u \in \{p, E, H\}} \{C_{1,u}, C_{2,u}, C_{3,u}\}$. We therefore have that:

$$\begin{aligned} 18\Delta t \sum_{n=0}^{N-1} &\left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^n) \right\|_{\mu}^2 \right] \\ &\leq 18C_0 h^{2r} \sum_{n=0}^{N-1} \Delta t \left[\left\| \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \left\| \frac{\partial H}{\partial t}(t^n) \right\|_{\mu}^2 \right] \\ &\leq 18C_0 h^{2r} \int_0^T \left[\left\| \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \left\| \frac{\partial H}{\partial t}(t^n) \right\|_{\mu}^2 \right] dt \\ &= 18C_0 h^{2r} \left[\left\| \frac{\partial p}{\partial t} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial E}{\partial t} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial H}{\partial t} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right], \end{aligned}$$

$$\begin{aligned} \frac{\Delta t^5}{2} \sum_{n=0}^{N-1} &\left[\left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3}(t^n) \right\|_{\mu}^2 \right] \\ &\leq C_0 h^{2r} \sum_{n=0}^{N-1} \Delta t \left[\left\| \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 + \left\| \frac{\partial^3 H}{\partial t^3}(t^n) \right\|_{\mu}^2 \right] \quad (\because \Delta t < 1) \\ &\leq C_0 h^{2r} \int_0^T \left[\left\| \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 + \left\| \frac{\partial^3 H}{\partial t^3}(t^n) \right\|_{\mu}^2 \right] dt \\ &= C_0 h^{2r} \left[\left\| \frac{\partial^3 p}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^3 E}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial^3 H}{\partial t^3} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right], \end{aligned}$$

and likewise for the L^2 projection of the remainder terms:

$$\begin{aligned} 18\Delta t \sum_{n=0}^{N-1} &\left[\left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}} + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon} + \left\| (I - \Pi_h^2) R_H^n \right\|_{\mu} \right], \\ &\leq C_0 h^{2r} \sum_{n=0}^{N-1} \Delta t \left[\left\| R_p^n \right\|_{L_{\varepsilon^{-1}}^2(\Omega)} + \left\| R_E^n \right\|_{L_{\varepsilon}^2(\Omega)} + \left\| R_H^n \right\|_{L_{\mu}^2(\Omega)} \right], \\ &\leq \frac{4}{9} C_0 h^{2r} \Delta t^8 \left[\left\| \frac{\partial^5 p}{\partial t^5} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^5 E}{\partial t^5} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial^5 H}{\partial t^5} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right]. \end{aligned}$$

Now, we take $u_h^0 = \Pi_h^k u_0$, and $u_h^1 = \Pi_h^k u_1$, for $k = 0, 1, 2$ respectively, for $u = \{p, E, H\}$, and we note that there exists a positive bounded constant M such that:

$$\begin{aligned}
& 18 \left(\left\| \frac{\partial p}{\partial t} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial E}{\partial t} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial H}{\partial t} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right) + \left\| \frac{\partial^3 p}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 \\
& + \left\| \frac{\partial^3 E}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial^3 H}{\partial t^3} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 + \frac{8}{9} \left(\left\| \frac{\partial^5 p}{\partial t^5} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^5 E}{\partial t^5} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 \right. \\
& \left. + \left\| \frac{\partial^5 H}{\partial t^5} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right) + \frac{1}{504} \left(\left\| \frac{\partial^4 (\nabla p)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^4 (\nabla \cdot E)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 \right. \\
& \left. + \varepsilon^{-1} \mu^{-1} \left(\left\| \frac{\partial^4 (\nabla \times E)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial^4 (\nabla \times H)}{\partial t^4} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right) \right) \leq M.
\end{aligned}$$

Consequently, we have the following estimate:

$$\|\eta_h^{N-1}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{N-1}\|_{\varepsilon}^2 + \|\xi_h^{N-1}\|_{\mu}^2 \leq \tilde{C} [h^{2r} + h^{2r} \Delta t^8 + \Delta t^8],$$

where $\tilde{C} = M \exp(36T)$ which then using the equivalence between 1- and 2-norms gives us that:

$$\|\eta_h^{N-1}\|_{\varepsilon^{-1}} + \|\zeta_h^{N-1}\|_{\varepsilon} + \|\xi_h^{N-1}\|_{\mu} \leq C_1 [h^r + h^r \Delta t^4 + \Delta t^4],$$

where $C_1 = \sqrt{3\tilde{C}}$. Also, there are positive bounded constants $\tilde{C}_2$, $\tilde{C}_3$ and $\tilde{C}_4$ such that:

$$\begin{aligned}
\|\eta^{N-1}\|_{\varepsilon^{-1}} &= \|(I - \Pi_h^0)p(t^{N-1})\|_{\varepsilon^{-1}} \leq \tilde{C}_2 h^r \|p(t^{N-1})\|_{L_{\varepsilon^{-1}}^2(\Omega)} \implies \|\eta^{N-1}\|_{\varepsilon^{-1}} \leq C_2 h^r, \\
\|\zeta^{N-1}\|_{\varepsilon} &= \|(I - \Pi_h^1)E(t^{N-1})\|_{\varepsilon} \leq \tilde{C}_3 h^r \|E(t^{N-1})\|_{L_{\varepsilon}^2(\Omega)} \implies \|\zeta^{N-1}\|_{\varepsilon} \leq C_3 h^r, \\
\|\xi^{N-1}\|_{\mu} &= \|(I - \Pi_h^2)H(t^{N-1})\|_{\mu} \leq \tilde{C}_4 h^r \|H(t^{N-1})\|_{L_{\mu}^2(\Omega)} \implies \|\xi^{N-1}\|_{\mu} \leq C_4 h^r,
\end{aligned}$$

in which $C_2 = \tilde{C}_2 \|p(t^{N-1})\|_{L_{\varepsilon^{-1}}^2(\Omega)}$, $C_3 = \tilde{C}_3 \|E(t^{N-1})\|_{L_{\varepsilon}^2(\Omega)}$ and $C_4 = \tilde{C}_4 \|H(t^{N-1})\|_{L_{\mu}^2(\Omega)}$ are all bounded positive constants due to Theorem 4.5. Finally, this provides us with our desired result by choosing $C = C_1 + C_2 + C_3 + C_4$:

$$\begin{aligned}
\|e_{p_h}^{N-1}\|_{\varepsilon^{-1}} + \|e_{E_h}^{N-1}\|_{\varepsilon} + \|e_{H_h}^{N-1}\|_{\mu} &= \|\eta^{N-1} - \eta_h^{N-1}\|_{\varepsilon^{-1}} + \|\zeta^{N-1} - \zeta_h^{N-1}\|_{\varepsilon} + \|\xi^{N-1} - \xi_h^{N-1}\|_{\mu}, \\
&\leq C [(\Delta t)^4 + h^r + h^r \Delta t^4].
\end{aligned}$$

□

4.4 Numerical Results

We now present two validating computational examples in $\mathbb{R}^2$.

Example 4.1: This problem consists of the Maxwell's system being posed on a unit square in $\mathbb{R}^2$ with the analytical solution, material parameters, initial and final times as below:

$$p = 0, \quad E = \begin{bmatrix} \sin \pi y \cos \pi t \\ \sin \pi x \cos \pi t \end{bmatrix}, \quad H = (\cos \pi y - \cos \pi x) \sin \pi t,$$

$$\varepsilon = 1, \quad \mu = 1, \quad T_{\min} = 0, \quad T_{\max} = 1.$$

The computational results of solving this problem with the choices LF_4 and TS_4 in conjunction with linear or quadratic FEEC basis are all shown in Figures 4.1 to 4.4.

Example 4.2: This problem is also on the unit square in $\mathbb{R}^2$ with $p \neq 0$, and with non homogeneous boundary conditions for both p and E and homogeneous boundary conditions for H .

$$p = (\cos \pi x + \cos \pi y) \sin \pi t,$$

$$E = \begin{bmatrix} \sin \pi(\sqrt{2}t - x - y) - \sin \pi x \cos \pi t \\ -\sin \pi(\sqrt{2}t - x - y) - \sin \pi y \cos \pi t \end{bmatrix}, \quad H = -\sqrt{2} \sin \pi(\sqrt{2}t - x - y),$$

$$\epsilon = 1, \quad \mu = 1, \quad T_{\min} = 0, \quad T_{\max} = 1.$$

The computational results of solving this problem with the choices LF_4 and TS_4 in conjunction with linear or quadratic FEEC basis are all shown in Figures 4.5 to 4.8.

In the end, we wish to conclude by summarizing that we have outlined two different strategies for the fourth-order time discretization of the three-field formulation of Maxwell's equations that have resulted in two implicit time integration schemes which we have demonstrated to be energy conserving and convergent. We further generalize this to arbitrary-order schemes for the spatial strategy in the Chapter 5 and leave the precise characterization of a plethora of such higher-order schemes arising from the temporal strategy to a future work.

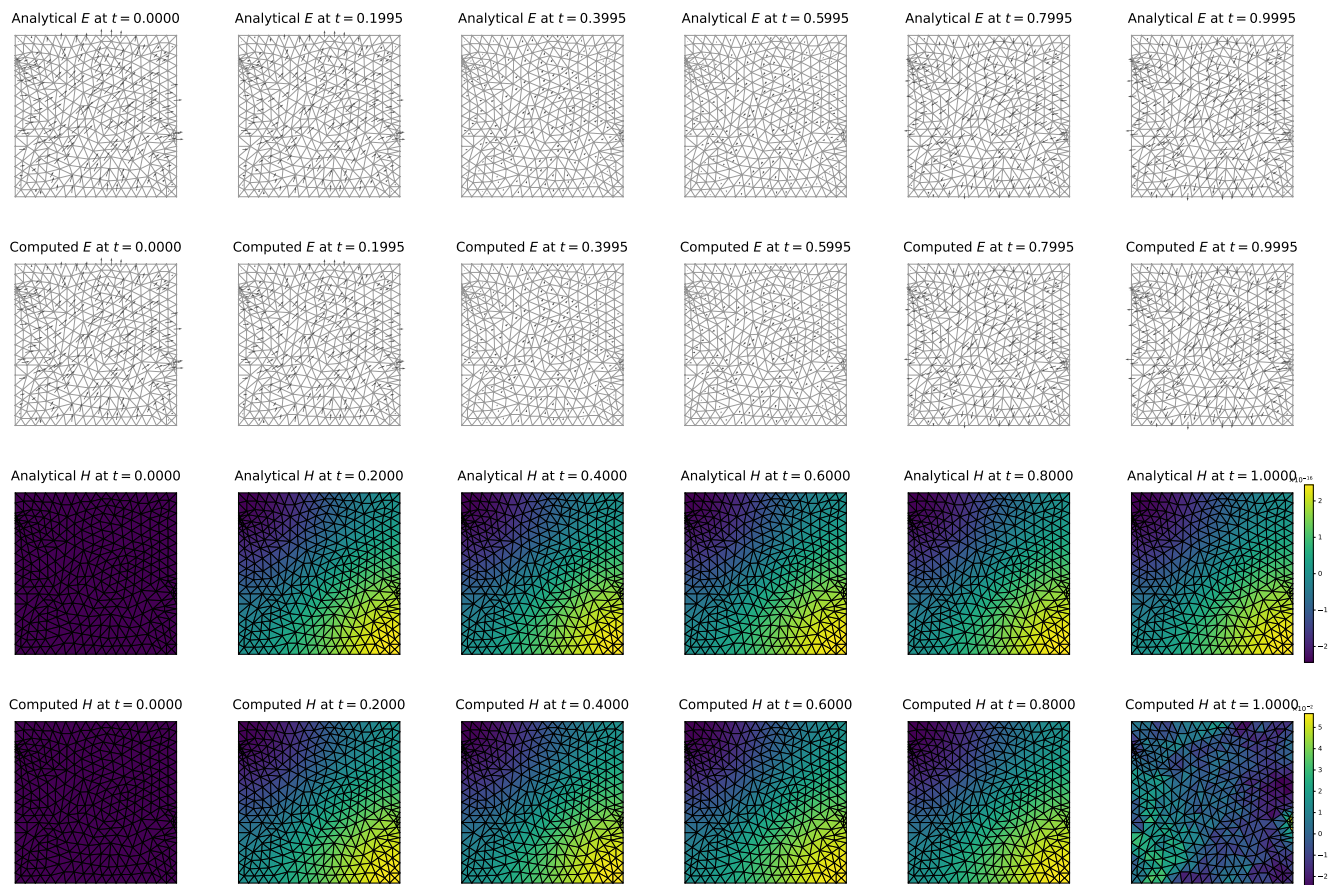


FIGURE 4.1: **Linear finite elements with LF_4** : Plots of solutions at different time steps for the problem described in **Example 4.1** of Section 4.4 using the LF_4 and linear Whitney forms as basis for the FEEC spaces. The solutions for p are not shown due to them being identically equal to 0. The computed solutions for E and H visually match with the analytical solutions nearly identically.

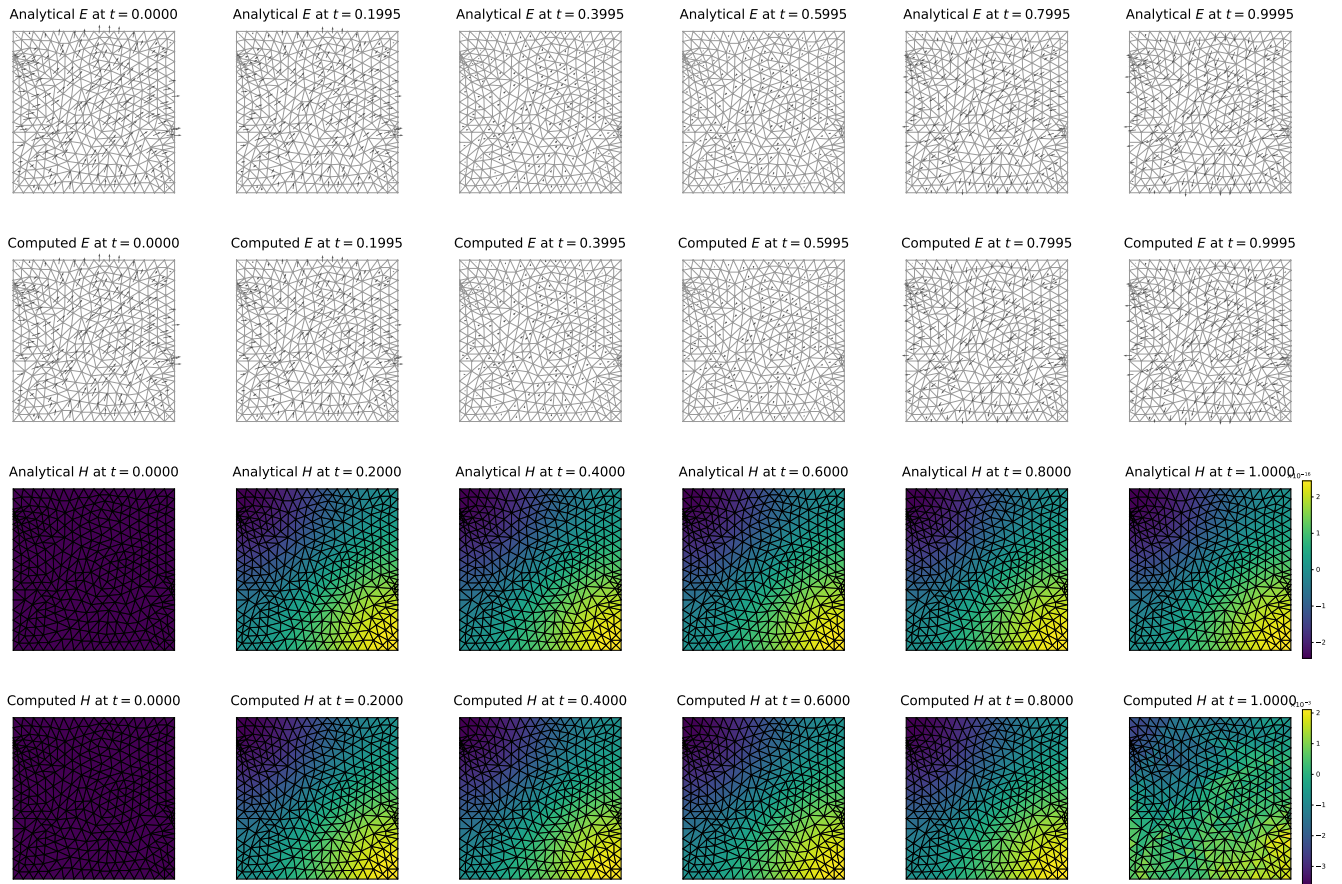


FIGURE 4.2: **Quadratic finite elements with LF₄**: Plots of solutions at different time steps for **Example 4.1** of Section 4.4 using LF₄ and quadratic Whitney forms as basis for the FEEC spaces. The solutions for $\mathbf{p}$ are not shown due to them being identically equal to 0. The computed solutions for $\mathbf{E}$ and $\mathbf{H}$ visually match with the analytical solutions nearly identically.

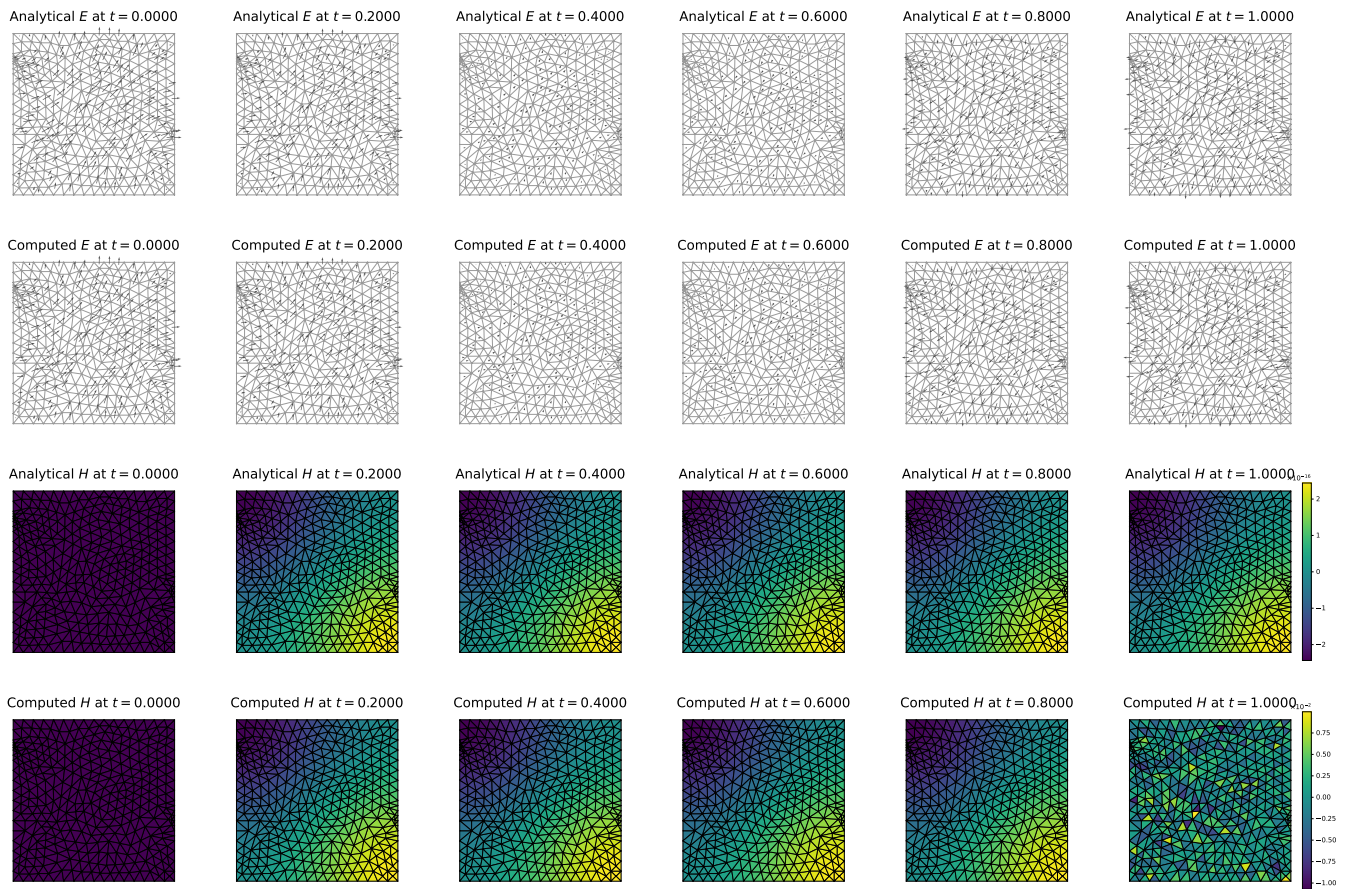


FIGURE 4.3: **Linear finite elements with TS_4** : Plots of solutions at different time steps for **Example 4.1** of Section 4.4 using TS_4 and linear Whitney forms for the FEEC spaces. The solutions for p are again not shown due to them being identically 0.

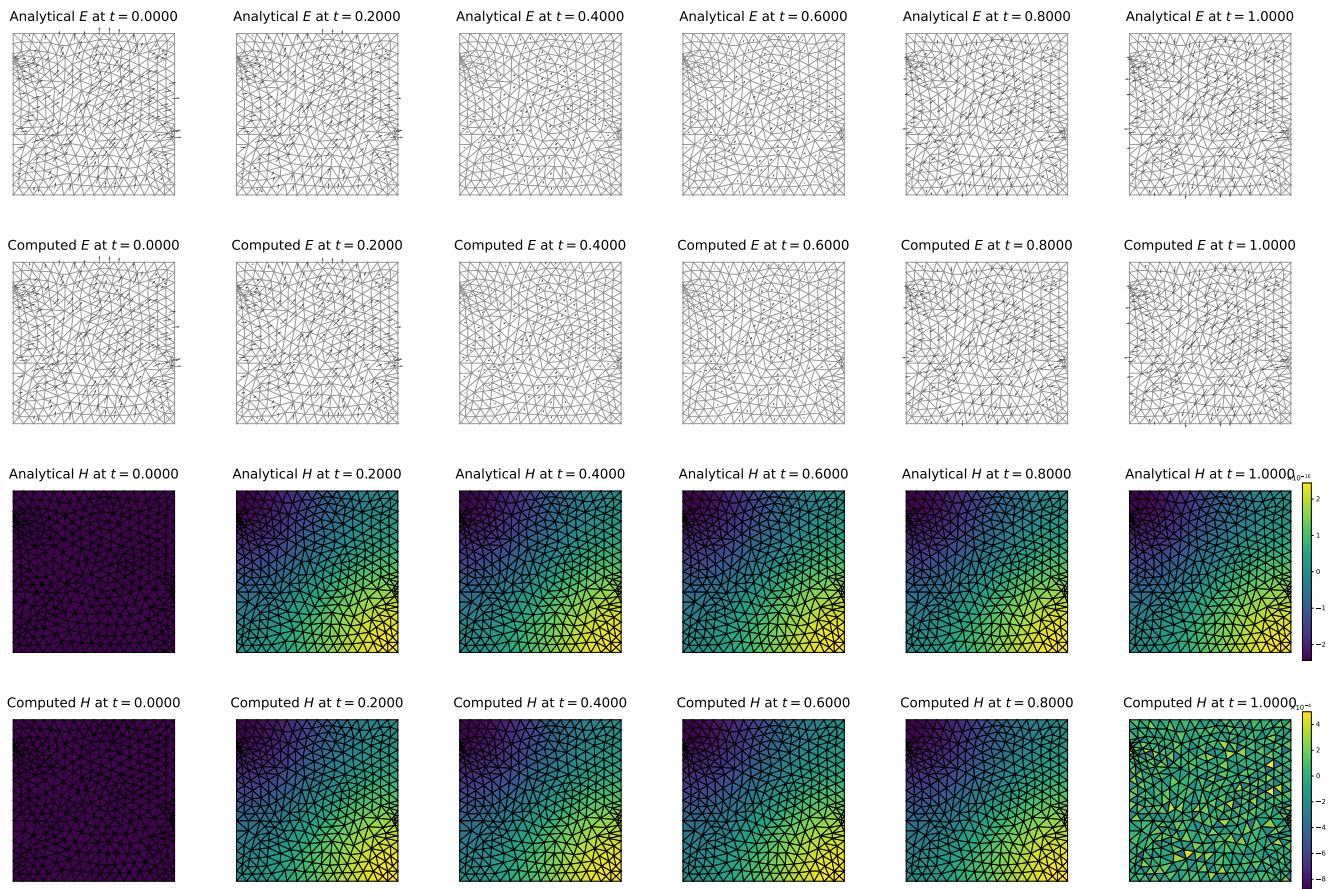


FIGURE 4.4: **Quadratic finite elements with TS_4** : Plots of solutions at different time steps for **Example 4.1** of Section 4.4 using TS_4 and quadratic Whitney forms for the FEEC spaces.

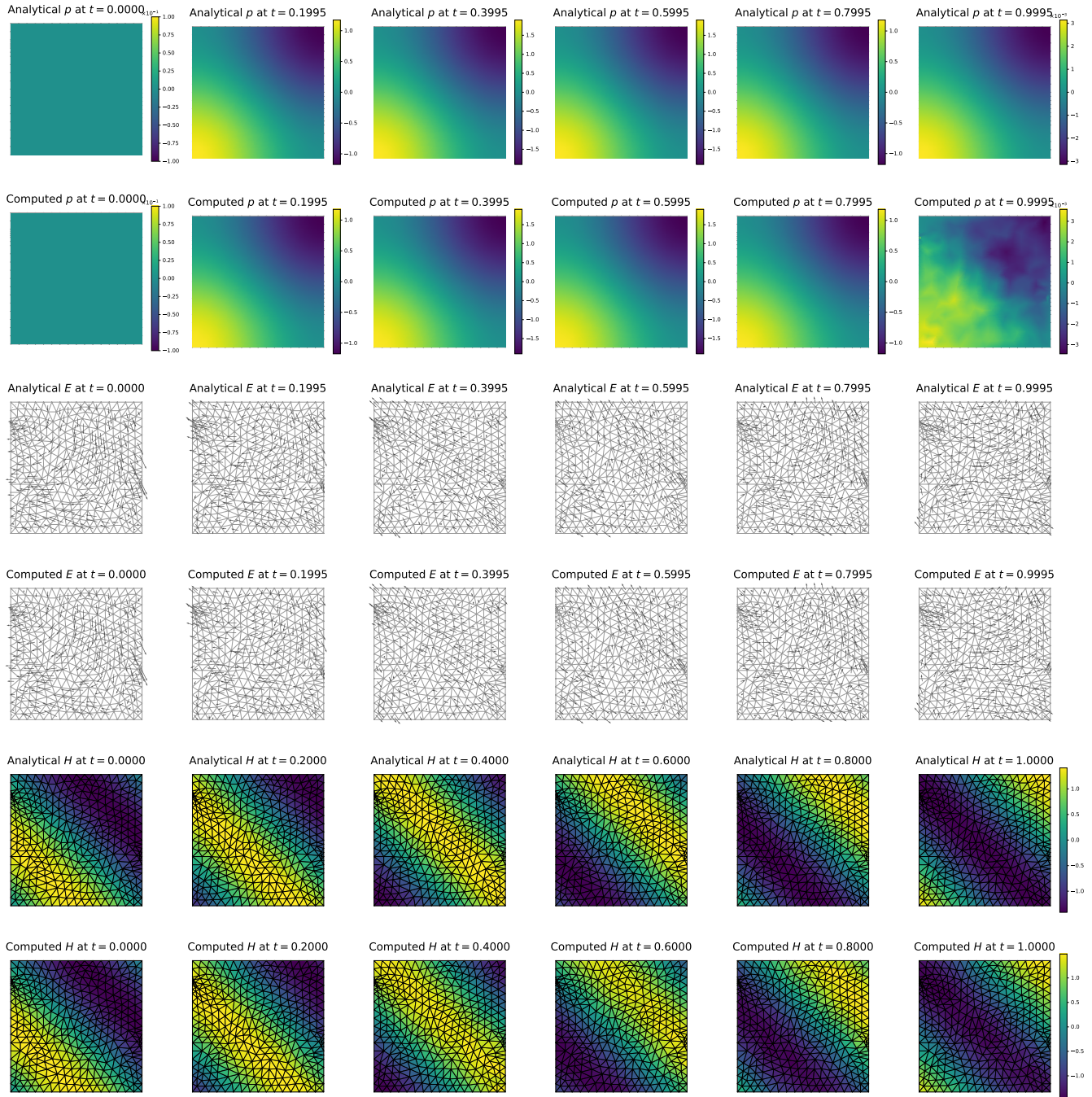


FIGURE 4.5: **Linear finite elements with LF_4** : Plots of solutions at different time steps for **Example 4.2** of Section 4.4 using the LF_4 and linear Whitney forms in FEEC. The computed solutions for p , E and H visually match with the analytical solutions nearly identically.

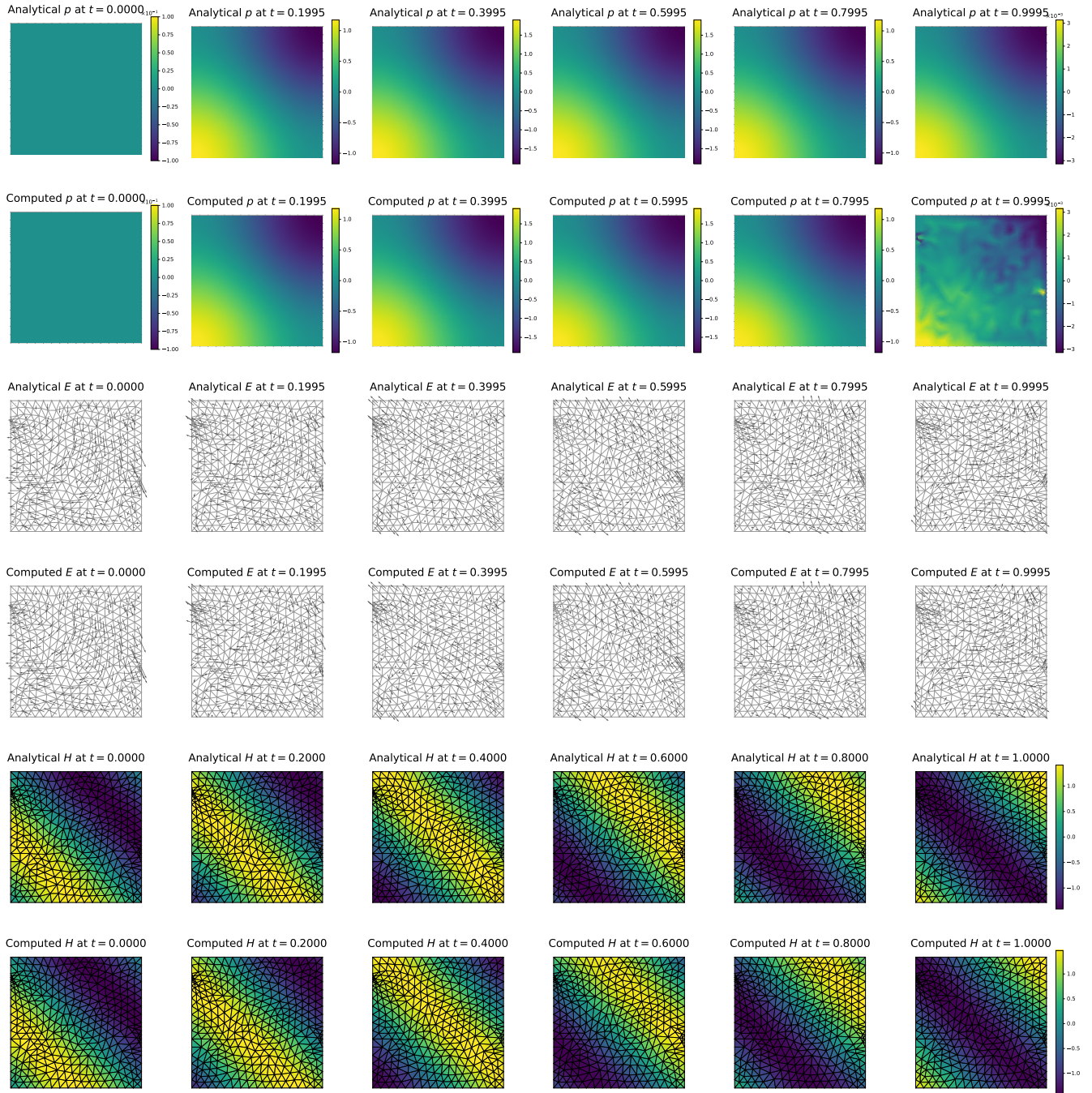


FIGURE 4.6: **Quadratic finite elements with LF_4** : Plots of solutions at different time steps for **Example 4.2** of Section 4.4 using the LF_4 and quadratic Whitney forms in FEEC. The computed solutions for p , E and H visually match with the analytical solutions nearly identically.

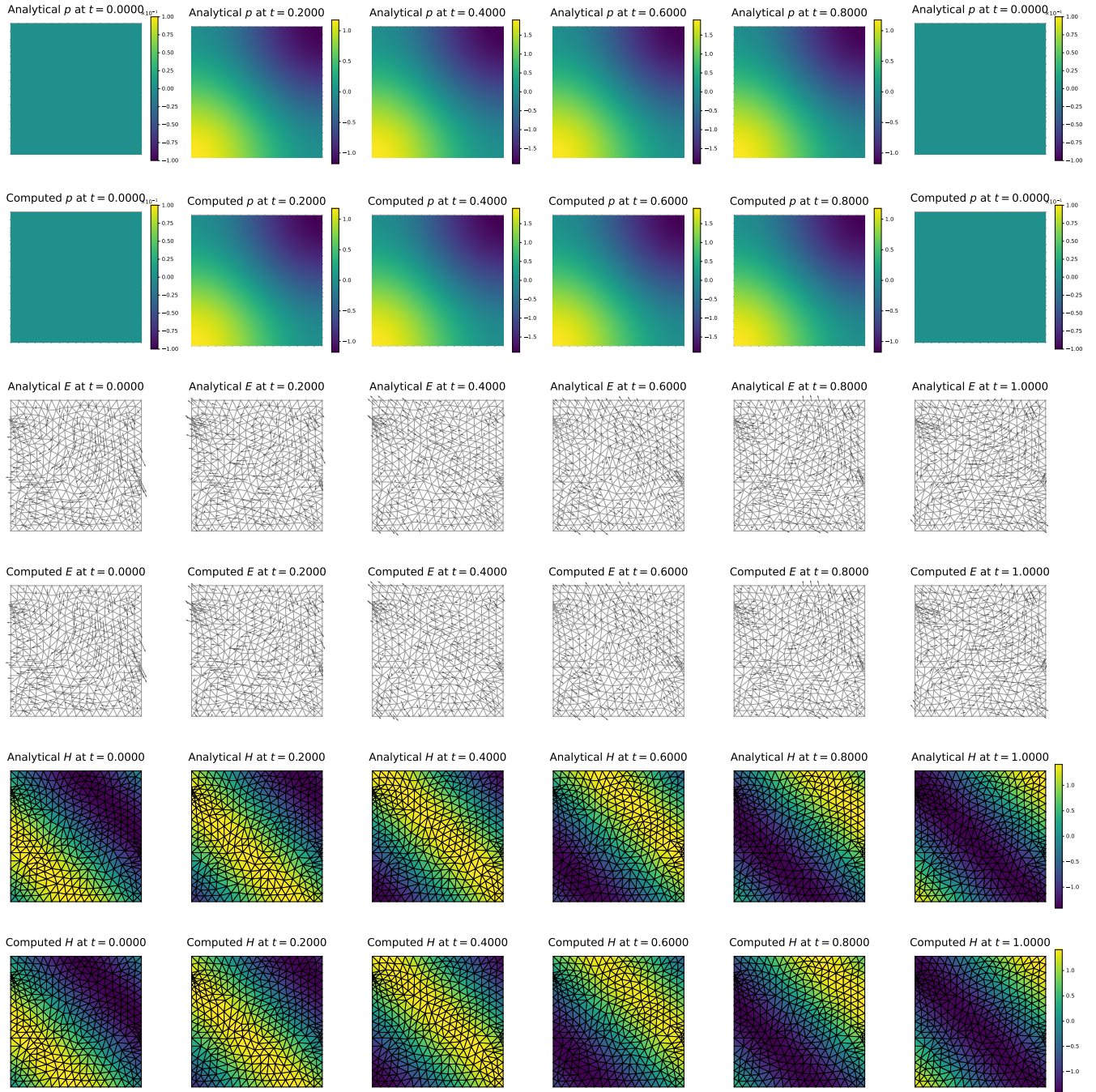


FIGURE 4.7: **Linear finite elements with TS₄**: Plots of solutions at different time steps for **Example 4.2** of Section 4.4 using the TS₄ and linear Whitney forms in FEEC. The computed solutions for p , E and H again visually match with the analytical solutions nearly identically.

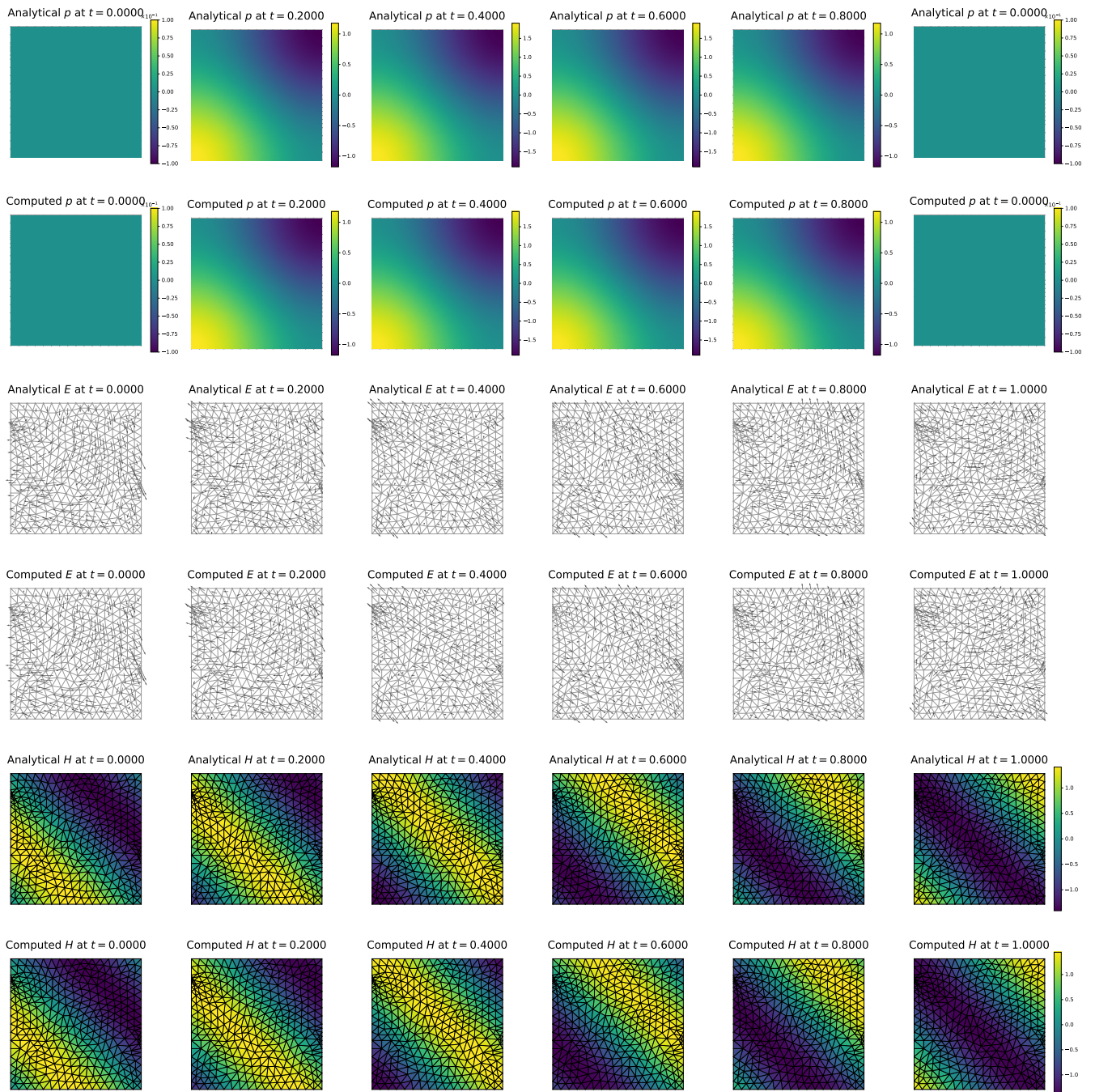


FIGURE 4.8: **Quadratic finite elements with TS_4** : Plots of solutions at different time steps for **Example 4.2** of Section 4.4 using the TS_4 and quadratic Whitney forms in FEEC. The computed solutions for p , E and H yet again visually match with the analytical solutions nearly identically.

Chapter 5

An Arbitrarily Higher-Order Time Implicit Scheme for Maxwell's Equations

5.1 Introduction

5.1.1 Three-field Formulation of Maxwell's Equations

We again begin by providing the three field (p, E, H) formulation of the Maxwell's equations as below for quick reference:

$$\left. \begin{aligned} \frac{\partial p}{\partial t} + \nabla \cdot \varepsilon E &= 0, \\ \nabla p + \varepsilon \frac{\partial E}{\partial t} - \nabla \times H &= 0, \\ \mu \frac{\partial H}{\partial t} + \nabla \times E &= 0, \end{aligned} \right\} \text{ in } \Omega \times (0, T], \quad (5.1a)$$

where $\Omega \subset \mathbb{R}^2/\mathbb{R}^3$ is a domain with Lipschitz boundary $\partial\Omega$, and $T > 0$. We use the following homogeneous boundary conditions:

$$p = 0, E \times n = 0, H \cdot n = 0 \text{ on } \partial\Omega \times (0, T], \quad (5.1b)$$

where n is the unit outward normal to $\partial\Omega$. We wish to state that the homogeneous boundary conditions are merely for convenience and all of our work can be extended to non homogeneous boundary conditions in a very standard way. Finally, we use the following initial conditions:

$$p(x, 0) = p_0(x), E(x, 0) = E_0(x), \text{ and } H(x, 0) = H_0(x) \text{ for } x \in \Omega, \quad (5.1c)$$

and in which for the sake of consistency as to why p was introduced, the initial conditions are required to also satisfy $\nabla \cdot (\varepsilon E_0) = p_0$ while physics provides for $\nabla \cdot (\mu H_0) = 0$ in Ω . Given this setup, we also wish to add that it is possible to extend our method and analysis to nonzero right hand sides in Equation (5.1a) but it makes the already complicated analysis quite rather unwieldy and also requires more hypotheses on these forcing functions. As a tradeoff, we realized that this sort of absolute generalization does not lend itself to gaining any meaningful insights beyond what we can already derive from our work. Therefore, while it is possible to work with a specific instance of a higher-order scheme with forcing functions, we leave this altogether out of our discussion.

We next cast Equations (5.1a) to (5.1c) into the following variational formulation: for $t \in (0, T]$, find $(p(t), E(t), H(t)) \in \mathring{H}_{\varepsilon^{-1}}^1(\Omega) \times \mathring{H}_{\varepsilon}(\text{curl}; \Omega) \times \mathring{H}_{\mu}(\text{div}; \Omega)$ such that:

$$\left\langle \frac{\partial p}{\partial t}, \tilde{p} \right\rangle - \langle \varepsilon E, \nabla \tilde{p} \rangle = 0, \quad \tilde{p} \in \mathring{H}_{\varepsilon^{-1}}^1(\Omega), \quad (5.2a)$$

$$\langle \nabla p, \tilde{E} \rangle + \langle \varepsilon \frac{\partial E}{\partial t}, \tilde{E} \rangle - \langle H, \nabla \times \tilde{E} \rangle = 0, \quad \tilde{E} \in \dot{H}_\varepsilon(\text{curl}; \Omega), \quad (5.2b)$$

$$\langle \mu \frac{\partial H}{\partial t}, \tilde{H} \rangle + \langle \nabla \times E, \tilde{H} \rangle = 0, \quad \tilde{H} \in \dot{H}_\mu(\text{div}; \Omega), \quad (5.2c)$$

For the sake of analysis, we require that the solutions (p, E, H) have sufficient regularity. Given this background about our system on Maxwell's equations, we next state our higher-order discretization, which we are generalizing using the spatial strategy as discussed in Section 4.1.2 of Chapter 4. We first provide an illustrative example for the sixth-order scheme, and then we provide the full generality for the R^{th} -order scheme.

5.1.2 Warmup Example: LF_6 Scheme

Find $(p(t^{n+1/2}), E(t^{n+1/2}), H(t^n)) \in \dot{H}_{\varepsilon^{-1}}^1(\Omega) \times \dot{H}_\varepsilon(\text{curl}; \Omega) \times \dot{H}_\mu(\text{div}; \Omega)$ such that:

$$\begin{aligned} \langle \frac{p^{n+\frac{1}{2}} - p^{n-\frac{1}{2}}}{\Delta t}, \tilde{p} \rangle - \langle \frac{\varepsilon}{2} (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla \tilde{p} \rangle + \frac{\Delta t^2}{12} \langle \frac{\varepsilon}{2} \nabla \nabla \cdot (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla \tilde{p} \rangle \\ - \frac{\Delta t^4}{120} \langle \frac{\varepsilon}{2} (\nabla \nabla \cdot)^2 (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla \tilde{p} \rangle = 0, \end{aligned} \quad (5.3a)$$

$$\begin{aligned} \langle \frac{1}{2} \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), \tilde{E} \rangle - \frac{\Delta t^2}{12} \langle \frac{1}{2} \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), \nabla \nabla \cdot \tilde{E} \rangle + \frac{\Delta t^4}{120} \langle \frac{1}{2} \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), (\nabla \nabla \cdot)^2 \tilde{E} \rangle \\ + \langle \varepsilon \frac{E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}}{\Delta t}, \tilde{E} \rangle - \langle \frac{1}{2} (H^{n+1} + H^n), \nabla \times \tilde{E} \rangle - \frac{\Delta t^2}{12} \langle \frac{1}{2} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times (H^{n+1} + H^n), \nabla \times \tilde{E} \rangle \\ - \frac{\Delta t^4}{120} \langle \frac{1}{2} \mu^{-2} \varepsilon^{-2} (\nabla \times \nabla \times)^2 (H^{n+1} + H^n), \nabla \times \tilde{E} \rangle = 0, \end{aligned} \quad (5.3b)$$

$$\begin{aligned} \langle \mu \frac{H^{n+1} - H^n}{\Delta t}, \tilde{H} \rangle + \langle \frac{1}{2} \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \tilde{H} \rangle + \frac{\Delta t^2}{12} \langle \frac{1}{2} \varepsilon^{-1} \mu^{-1} \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla \times \nabla \times \tilde{H} \rangle \\ + \frac{\Delta t^4}{120} \langle \frac{1}{2} \varepsilon^{-2} \mu^{-2} \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), (\nabla \times \nabla \times)^2 \tilde{H} \rangle = 0, \end{aligned} \quad (5.3c)$$

for all $\tilde{p} \in \dot{H}_{\varepsilon^{-1}}^1(\Omega)$, $\tilde{E} \in \dot{H}_\varepsilon(\text{curl}; \Omega)$ and $\tilde{H} \in \dot{H}_\mu(\text{div}; \Omega)$. In this description, $(\nabla \nabla \cdot)^2$ stands for the operator $(\nabla \nabla \cdot \nabla \nabla \cdot)$ and likewise $(\nabla \times \nabla \times)^2$ denotes $(\nabla \times \nabla \times \nabla \times \nabla \times)$. In order to bootstrap the computations, we will use the following scheme for the first half time step for p and E , and for the first time step for H :

$$\begin{aligned} \langle \frac{p^{\frac{1}{2}} - p_0}{\Delta t/2}, \tilde{p} \rangle - \frac{1}{2} \langle \frac{\varepsilon}{2} (E^{\frac{1}{2}} + E_0), \nabla \tilde{p} \rangle + \frac{1}{8} \cdot \frac{\Delta t^2}{12} \langle \frac{\varepsilon}{2} \nabla \nabla \cdot (E^{\frac{1}{2}} + E_0), \nabla \tilde{p} \rangle \\ - \frac{1}{32} \cdot \frac{\Delta t^4}{120} \langle \frac{\varepsilon}{2} (\nabla \nabla \cdot)^2 (E^{\frac{1}{2}} + E_0), \nabla \tilde{p} \rangle = 0, \end{aligned} \quad (5.4a)$$

$$\begin{aligned}
& \frac{1}{2} \langle \frac{1}{2} \nabla (p^{\frac{1}{2}} + p_0), \tilde{E} \rangle - \frac{1}{8} \cdot \frac{\Delta t^2}{12} \langle \frac{1}{2} \nabla (p^{\frac{1}{2}} + p_0), \nabla \nabla \cdot \tilde{E} \rangle + \frac{1}{32} \cdot \frac{\Delta t^4}{120} \langle \frac{1}{2} \nabla (p^{\frac{1}{2}} + p_0), (\nabla \nabla \cdot)^2 \tilde{E} \rangle \\
& + \langle \varepsilon \frac{E^{\frac{1}{2}} - E_0}{\Delta t/2}, \tilde{E} \rangle - \langle \frac{1}{2} (H^1 + H_0), \nabla \times \tilde{E} \rangle - \frac{1}{4} \cdot \frac{\Delta t^2}{12} \langle \frac{1}{2} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times (H^1 + H_0), \nabla \times \tilde{E} \rangle \\
& + \frac{1}{16} \cdot \frac{\Delta t^4}{120} \langle \frac{1}{2} \mu^{-2} \varepsilon^{-2} (\nabla \times \nabla \times)^2 (H^1 + H_0), \nabla \times \tilde{E} \rangle = 0, \quad (5.4b)
\end{aligned}$$

$$\begin{aligned}
& \langle \mu \frac{H^1 - H_0}{\Delta t}, \tilde{H} \rangle + \frac{1}{2} \langle \frac{1}{2} \nabla \times (E^{\frac{1}{2}} + E_0), \tilde{H} \rangle + \frac{1}{8} \cdot \frac{\Delta t^2}{12} \langle \frac{1}{2} \varepsilon^{-1} \mu^{-1} \nabla \times (E^{\frac{1}{2}} + E_0), \nabla \times \nabla \times \tilde{H} \rangle \\
& + \frac{1}{32} \cdot \frac{\Delta t^4}{120} \langle \frac{1}{2} \varepsilon^{-2} \mu^{-2} \nabla \times (E^{\frac{1}{2}} + E_0), (\nabla \times \nabla \times)^2 \tilde{H} \rangle = 0. \quad (5.4c)
\end{aligned}$$

We now rewrite LF_6 using some additional notation with summations as next. This is so as to make it easier to then state the arbitrarily higher-order scheme LF_R .

5.1.3 LF_6 Scheme in a Sum Notation

Find $(p(t^{n+1/2}), E(t^{n+1/2}), H(t^n)) \in \dot{H}_{\varepsilon^{-1}}^1(\Omega) \times \dot{H}_{\varepsilon}(\text{curl}; \Omega) \times \dot{H}_{\mu}(\text{div}; \Omega)$ such that:

$$\begin{aligned}
& \langle \frac{p^{n+\frac{1}{2}} - p^{n-\frac{1}{2}}}{\Delta t}, \tilde{p} \rangle - \langle \frac{\varepsilon}{2} (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla \tilde{p} \rangle \\
& + \sum_{a=1}^{\frac{6}{2}-1} \sum_{k_a=1}^{\frac{6}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \langle \frac{\varepsilon}{2} (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla \tilde{p} \rangle = 0, \quad (5.5a)
\end{aligned}$$

$$\begin{aligned}
& \langle \frac{1}{2} \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), \tilde{E} \rangle - \sum_{a=1}^{\frac{6}{2}-1} \sum_{k_a=1}^{\frac{6}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \langle \frac{1}{2} \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \tilde{E} \rangle \\
& + \langle \varepsilon \frac{E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}}{\Delta t}, \tilde{E} \rangle - \langle \frac{1}{2} (H^{n+1} + H^n), \nabla \times \tilde{E} \rangle \\
& + \sum_{a=1}^{\frac{6}{2}-1} \sum_{k_a=1}^{\frac{6}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} \langle \frac{1}{2} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} (H^{n+1} + H^n), \nabla \times \tilde{E} \rangle = 0, \quad (5.5b)
\end{aligned}$$

$$\begin{aligned}
& \langle \mu \frac{H^{n+1} - H^n}{\Delta t}, \tilde{H} \rangle + \langle \frac{1}{2} \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \tilde{H} \rangle \\
& - \sum_{a=1}^{\frac{6}{2}-1} \sum_{k_a=1}^{\frac{6}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} \langle \frac{1}{2} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \tilde{H} \rangle = 0, \quad (5.5c)
\end{aligned}$$

for all $\tilde{p} \in \dot{H}_{\varepsilon^{-1}}^1(\Omega)$, $\tilde{E} \in \dot{H}_{\varepsilon}(\text{curl}; \Omega)$ and $\tilde{H} \in \dot{H}_{\mu}(\text{div}; \Omega)$. The bootstrapping equations using the initial values at $t = 0$ are:

$$\begin{aligned} \left\langle \frac{p^{\frac{1}{2}} - p_0}{\Delta t/2}, \tilde{p} \right\rangle - \frac{1}{2} \left\langle \frac{\varepsilon}{2} (E^{\frac{1}{2}} + E_0), \nabla \tilde{p} \right\rangle \\ + \sum_{a=1}^{\frac{6}{2}-1} \sum_{k_a=1}^{\frac{6}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m+1}} f(a) \left\langle \frac{\varepsilon}{2} (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} (E^{\frac{1}{2}} + E_0), \nabla \tilde{p} \right\rangle = 0, \quad (5.6a) \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \left\langle \frac{1}{2} \nabla (p^{\frac{1}{2}} + p_0), \tilde{E} \right\rangle - \sum_{a=1}^{\frac{6}{2}-1} \sum_{k_a=1}^{\frac{6}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m+1}} f(a) \left\langle \frac{1}{2} \nabla (p^{\frac{1}{2}} + p_0), (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \tilde{E} \right\rangle \\ + \left\langle \varepsilon \frac{E^{\frac{1}{2}} - E_0}{\Delta t/2}, \tilde{E} \right\rangle - \left\langle \frac{1}{2} (H^1 + H_0), \nabla \times \tilde{E} \right\rangle \\ + \sum_{a=1}^{\frac{6}{2}-1} \sum_{k_a=1}^{\frac{6}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m}} f(a) (-1)^{\sum_{m=1}^a k_m} \left\langle \frac{1}{2} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} (H^1 + H_0), \nabla \times \tilde{E} \right\rangle = 0, \quad (5.6b) \end{aligned}$$

$$\begin{aligned} \left\langle \mu \frac{H^1 - H_0}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \left\langle \frac{1}{2} \nabla \times (E^{\frac{1}{2}} + E_0), \tilde{H} \right\rangle \\ - \sum_{a=1}^{\frac{6}{2}-1} \sum_{k_a=1}^{\frac{6}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m+1}} f(a) (-1)^{\sum_{m=1}^a k_m} \left\langle \frac{1}{2} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} \nabla \times (E^{\frac{1}{2}} + E_0), (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \tilde{H} \right\rangle = 0, \quad (5.6c) \end{aligned}$$

where

$$f(a) := \left(\prod_{c=2}^a \frac{1}{(2k_c)!} \right) \left(\frac{\Delta t}{2} \right)^{2 \sum_{d=1}^a k_d}, \quad C_{k_1} := \left(\frac{1}{(2k_1)!} - \frac{1}{(2k_1+1)!} \right). \quad (5.6d)$$

We can now state our LF_R scheme and we do so by appropriately generalizing from the sixth-order in the LF_6 expressions as in Equations (5.5a) to (5.5c) and (5.6a) to (5.6c) to R^{th} -order.

5.1.4 The LF_R Scheme

Find $(p(t^{n+1/2}), E(t^{n+1/2}), H(t^n)) \in \dot{H}_{\varepsilon^{-1}}^1(\Omega) \times \dot{H}_{\varepsilon}(\text{curl}; \Omega) \times \dot{H}_{\mu}(\text{div}; \Omega)$ such that:

$$\left\langle \frac{p^{n+\frac{1}{2}} - p^{n-\frac{1}{2}}}{\Delta t}, \tilde{p} \right\rangle - \left\langle \frac{\varepsilon}{2} (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla \tilde{p} \right\rangle$$

$$+ \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \left\langle \frac{\varepsilon}{2} (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \left(E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \right), \nabla \tilde{p} \right\rangle = 0, \quad (5.7a)$$

$$\begin{aligned} & \left\langle \frac{1}{2} \nabla \left(p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}} \right), \tilde{E} \right\rangle - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \left\langle \frac{1}{2} \nabla \left(p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}} \right), (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \tilde{E} \right\rangle \\ & + \left\langle \varepsilon \frac{E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}}{\Delta t}, \tilde{E} \right\rangle - \left\langle \frac{1}{2} \left(H^{n+1} + H^n \right), \nabla \times \tilde{E} \right\rangle \\ & + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} \left\langle \frac{1}{2} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \left(H^{n+1} + H^n \right), \nabla \times \tilde{E} \right\rangle = 0, \quad (5.7b) \end{aligned}$$

$$\begin{aligned} & \left\langle \mu \frac{H^{n+1} - H^n}{\Delta t}, \tilde{H} \right\rangle + \left\langle \frac{1}{2} \nabla \times \left(E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \right), \tilde{H} \right\rangle \\ & - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} \left\langle \frac{1}{2} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} \nabla \times \left(E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \right), (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \tilde{H} \right\rangle = 0, \quad (5.7c) \end{aligned}$$

for all $\tilde{p} \in \dot{H}_{\varepsilon^{-1}}^1(\Omega)$, $\tilde{E} \in \dot{H}_{\varepsilon}(\text{curl}; \Omega)$ and $\tilde{H} \in \dot{H}_{\mu}(\text{div}; \Omega)$ with bootstrapping using the given initial values at $t = 0$ for the first half time step for p and E , and for the first time step for H :

$$\begin{aligned} & \left\langle \frac{p^{\frac{1}{2}} - p_0}{\Delta t/2}, \tilde{p} \right\rangle - \frac{1}{2} \left\langle \frac{\varepsilon}{2} \left(E^{\frac{1}{2}} + E_0 \right), \nabla \tilde{p} \right\rangle \\ & + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m+1}} f(a) \left\langle \frac{\varepsilon}{2} (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \left(E^{\frac{1}{2}} + E_0 \right), \nabla \tilde{p} \right\rangle = 0, \quad (5.8a) \end{aligned}$$

$$\begin{aligned} & \frac{1}{2} \left\langle \frac{1}{2} \nabla \left(p^{\frac{1}{2}} + p_0 \right), \tilde{E} \right\rangle - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m+1}} f(a) \left\langle \frac{1}{2} \nabla \left(p^{\frac{1}{2}} + p_0 \right), (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \tilde{E} \right\rangle \\ & + \left\langle \varepsilon \frac{E^{\frac{1}{2}} - E_0}{\Delta t/2}, \tilde{E} \right\rangle - \left\langle \frac{1}{2} \left(H^1 + H_0 \right), \nabla \times \tilde{E} \right\rangle \\ & + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m}} f(a) (-1)^{\sum_{m=1}^a k_m} \left\langle \frac{1}{2} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \left(H^1 + H_0 \right), \nabla \times \tilde{E} \right\rangle = 0, \quad (5.8b) \end{aligned}$$

$$\begin{aligned}
& \left\langle \mu \frac{H^1 - H_0}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \left\langle \frac{1}{2} \nabla \times (E^{\frac{1}{2}} + E_0), \tilde{H} \right\rangle \\
& - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m+1}} f(a) (-1)^{\sum_{m=1}^a k_m} \left\langle \frac{1}{2} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} \nabla \times (E^{\frac{1}{2}} + E_0), (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \tilde{H} \right\rangle = 0, \quad (5.8c)
\end{aligned}$$

where $f(a)$ and C_{k_1} are defined to be same as in (5.6d)

5.1.5 Organization

In Section 5.3, we prove the energy conservation of LF_R , and analyze and characterize the asymptotic convergence of the semidiscretization error for the Maxwell's system as well as the error for the full discretization of this system in conjunction with compatible finite elements as drawn from finite element exterior calculus [Arn18]. In Section 5.4, we provide some computational examples and end with a brief summary of this chapter.

5.2 Characterization of Implicit LF_6 Scheme

For our implicit LF_6 time discretization scheme, we next show that this semidiscretization in time yields stable methods and we provide error estimates which show that this scheme is correspondingly convergent in each of its chosen sufficiently small but fixed time step.

5.2.1 Time Discretization Stability

Theorem 5.1 (Discrete Energy Estimate). *For the semidiscretization using the LF_6 scheme as given in Equations (5.3a) to (5.3c) and (5.4a) to (5.4c), for any time step Δt sufficiently small, there exists a positive bounded constant C independent of Δt such that:*

$$\mathcal{E}^N := \|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 = \|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_{\varepsilon}^2 + \|H_0\|_{\mu}^2 =: \mathcal{E}^0.$$

Proof. Since Equations (5.3a) to (5.3c) are true for all $\tilde{p} \in \mathring{H}_{\varepsilon^{-1}}^1(\Omega)$, $\tilde{E} \in \mathring{H}_{\varepsilon}(\text{curl}; \Omega)$, $\tilde{H} \in \mathring{H}_{\mu}(\text{div}; \Omega)$, using $\tilde{p} = 2\Delta t \varepsilon^{-1} (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}})$, $\tilde{E} = 2\Delta t (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}})$ and $\tilde{H} = 2\Delta t (H^{n+1} + H^n)$ in them, we obtain the following:

$$\begin{aligned}
& 2 \left\langle p^{n+\frac{1}{2}} - p^{n-\frac{1}{2}}, \varepsilon^{-1} (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}) \right\rangle - \Delta t \left\langle E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}, \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}) \right\rangle \\
& + \frac{\Delta t^3}{12} \left\langle \nabla \nabla \cdot (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}) \right\rangle - \frac{\Delta t^5}{120} \left\langle (\nabla \nabla \cdot)^2 (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}) \right\rangle = 0,
\end{aligned}$$

$$\begin{aligned} & \Delta t \langle \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \rangle - \frac{\Delta t^3}{12} \langle \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), \nabla \nabla \cdot (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}) \rangle \\ & + \frac{\Delta t^5}{120} \langle \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), (\nabla \nabla \cdot)^2 (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}) \rangle + 2 \langle \varepsilon (E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}), E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \rangle \\ & - \Delta t \langle (H^{n+1} + H^n), \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}) \rangle - \frac{\Delta t^3}{12} \langle \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times (H^{n+1} + H^n), \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}) \rangle \\ & - \frac{\Delta t^5}{120} \langle \mu^{-2} \varepsilon^{-2} (\nabla \times \nabla \times)^2 (H^{n+1} + H^n), \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}) \rangle = 0, \end{aligned}$$

$$\begin{aligned} & 2 \langle \mu (H^{n+1} - H^n), H^{n+1} + H^n \rangle + \Delta t \langle \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), H^{n+1} + H^n \rangle \\ & + \frac{\Delta t^3}{12} \langle \varepsilon^{-1} \mu^{-1} \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla \times \nabla \times (H^{n+1} + H^n) \rangle \\ & + \frac{\Delta t^5}{120} \langle \varepsilon^{-2} \mu^{-2} \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), (\nabla \times \nabla \times)^2 (H^{n+1} + H^n) \rangle = 0. \end{aligned}$$

Using the symmetry of the inner product and adding the above equations, we next obtain:

$$\begin{aligned} & 2 \langle \varepsilon^{-1} (p^{n+\frac{1}{2}} - p^{n-\frac{1}{2}}), p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}} \rangle + 2 \langle \varepsilon (E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}), E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \rangle \\ & + 2 \langle \mu (H^{n+1} - H^n), H^{n+1} + H^n \rangle = 0. \end{aligned}$$

Summing over $n = 1$ to $N - 1$ leads us to:

$$\|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 - \|E^{\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 - \|H^1\|_{\mu}^2 = 0$$

Now, $p^{\frac{1}{2}}$, $E^{\frac{1}{2}}$ and H^1 satisfy Equations (5.4a) to (5.4c), and so using $\tilde{p} = 2\Delta t \varepsilon^{-1} (p^{\frac{1}{2}} + p^0)$, $\tilde{E} = 2\Delta t (E^{\frac{1}{2}} + E^0)$ and $\tilde{H} = 4\Delta t (H^1 + H^0)$ there and repeating the arguments presented above leads us to the following estimate:

$$\|p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|p^0\|_{\varepsilon^{-1}}^2 + \|E^{\frac{1}{2}}\|_{\varepsilon}^2 - \|E^0\|_{\varepsilon}^2 + \|H^1\|_{\mu}^2 - \|H^0\|_{\mu}^2 = 0$$

Adding these last two equalities, we get:

$$\|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 = \|p^0\|_{\varepsilon^{-1}}^2 + \|E^0\|_{\varepsilon}^2 + \|H^0\|_{\mu}^2$$

□

Theorem 5.2 (Discrete Error Estimate). *For the semidiscretization using the LF₆ scheme as given in Equations (5.3a) to (5.3c), and (5.4a) to (5.4c), for the solution (p, E, H) of Equations (5.2a) to (5.2c) with initial conditions as in Equation (5.1c) and assuming sufficient regularity with $p \in C^7(0, T; \dot{H}_{\varepsilon^{-1}}^1(\Omega))$, $E \in C^7(0, T; \dot{H}_{\varepsilon}(\text{curl}; \Omega))$, and $H \in C^7(0, T; \dot{H}_{\mu}(\text{div}; \Omega))$, and for the time step Δt sufficiently small, there exists a positive bounded constant C independent of Δt such that:*

$$\|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon} + \|e_H^N\|_{\mu} \leq C [(\Delta t)^6 + \|e_p^0\|_{\varepsilon^{-1}} + \|e_E^0\|_{\varepsilon} + \|e_H^0\|_{\mu}],$$

where $e_p^{n+\frac{1}{2}} := p(t^{n+\frac{1}{2}}) - p^{n+\frac{1}{2}}$, $e_E^{n+\frac{1}{2}} := E(t^{n+\frac{1}{2}}) - E^{n+\frac{1}{2}}$ and $e_H^n := H(t^n) - H^n$ are the errors in the time semidiscretization of p , E and H , respectively and at the indicated time indices.

Proof. Using the Taylor remainder theorem up to 7th, and expressing $p(t)$ about $t = t^n$, we have that:

$$p(t) = p(t^n) + \frac{\partial p}{\partial t}(t^n)(t - t^n) + \frac{\partial^2 p}{\partial t^2}(t^n)\frac{(t - t^n)^2}{2!} + \frac{\partial^3 p}{\partial t^3}(t^n)\frac{(t - t^n)^3}{3!} + \frac{\partial^4 p}{\partial t^4}(t^n)\frac{(t - t^n)^4}{4!} \\ + \frac{\partial^5 p}{\partial t^5}(t^n)\frac{(t - t^n)^5}{5!} + \frac{\partial^6 p}{\partial t^6}(t^n)\frac{(t - t^n)^6}{6!} + \int_{t^n}^t \frac{(t - s)^6}{6!} \frac{\partial^7 p}{\partial t^7}(s) ds,$$

which when evaluated at $t = t^{n+\frac{1}{2}}$ and $t = t^{n-\frac{1}{2}}$ yields:

$$p(t^{n+\frac{1}{2}}) = p(t^n) + \frac{\partial p}{\partial t}(t^n)(t^{n+\frac{1}{2}} - t^n) + \frac{\partial^2 p}{\partial t^2}(t^n)\frac{(t^{n+\frac{1}{2}} - t^n)^2}{2} + \frac{\partial^3 p}{\partial t^3}(t^n)\frac{(t^{n+\frac{1}{2}} - t^n)^3}{6} \\ + \frac{\partial^4 p}{\partial t^4}(t^n)\frac{(t^{n+\frac{1}{2}} - t^n)^4}{24} + \frac{\partial^5 p}{\partial t^5}(t^n)\frac{(t^{n+\frac{1}{2}} - t^n)^5}{120} + \frac{\partial^6 p}{\partial t^6}(t^n)\frac{(t^{n+\frac{1}{2}} - t^n)^6}{720} + \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^6}{720} \frac{\partial^7 p}{\partial t^7}(s) ds, \\ p(t^{n-\frac{1}{2}}) = p(t^n) + \frac{\partial p}{\partial t}(t^n)(t^{n-\frac{1}{2}} - t^n) + \frac{\partial^2 p}{\partial t^2}(t^n)\frac{(t^{n-\frac{1}{2}} - t^n)^2}{2} + \frac{\partial^3 p}{\partial t^3}(t^n)\frac{(t^{n-\frac{1}{2}} - t^n)^3}{6} \\ + \frac{\partial^4 p}{\partial t^4}(t^n)\frac{(t^{n-\frac{1}{2}} - t^n)^4}{24} + \frac{\partial^5 p}{\partial t^5}(t^n)\frac{(t^{n-\frac{1}{2}} - t^n)^5}{120} + \frac{\partial^6 p}{\partial t^6}(t^n)\frac{(t^{n-\frac{1}{2}} - t^n)^6}{720} + \int_{t^n}^{t^{n-\frac{1}{2}}} \frac{(t^{n-\frac{1}{2}} - s)^6}{720} \frac{\partial^7 p}{\partial t^7}(s) ds.$$

Subtracting these two equations, and using the result in the inner product term from the semidiscretization of the variational formulation leads to:

$$\left\langle \frac{p(t^{n+\frac{1}{2}}) - p(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{p} \right\rangle = \left\langle \frac{\partial p}{\partial t}(t^n), \tilde{p} \right\rangle + \frac{\Delta t^2}{24} \left\langle \frac{\partial^3 p}{\partial t^3}(t^n), \tilde{p} \right\rangle + \frac{\Delta t^4}{1920} \left\langle \frac{\partial^5 p}{\partial t^5}(t^n), \tilde{p} \right\rangle + \langle R_p^n, \tilde{p} \rangle,$$

in which we have defined that:

$$R_p^n := \frac{1}{\Delta t} \left[\int_{t^{n-\frac{1}{2}}}^{t^n} \frac{(t^{n-\frac{1}{2}} - s)^6}{720} \frac{\partial^7 p}{\partial t^7}(s) ds + \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^6}{720} \frac{\partial^7 p}{\partial t^7}(s) ds \right].$$

Similarly, for E and H , we have the following:

$$\left\langle \varepsilon \frac{E(t^{n+\frac{1}{2}}) - E(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{E} \right\rangle = \left\langle \varepsilon \frac{\partial E}{\partial t}(t^n), \tilde{E} \right\rangle + \frac{\Delta t^3}{24} \left\langle \varepsilon \frac{\partial^3 E}{\partial t^3}(t^n), \tilde{E} \right\rangle + \frac{\Delta t^5}{1920} \left\langle \varepsilon \frac{\partial^5 E}{\partial t^5}(t^n), \tilde{E} \right\rangle + \langle \varepsilon R_E^n, \tilde{E} \rangle, \\ \left\langle \mu \frac{H(t^{n+\frac{1}{2}}) - H(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{H} \right\rangle = \left\langle \mu \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}), \tilde{H} \right\rangle + \frac{\Delta t^3}{24} \left\langle \mu \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}), \tilde{H} \right\rangle + \frac{\Delta t^5}{1920} \left\langle \mu \frac{\partial^5 H}{\partial t^5}(t^{n+\frac{1}{2}}), \tilde{H} \right\rangle \\ + \langle \mu R_H^{n+\frac{1}{2}}, \tilde{H} \rangle,$$

and in each of which we have defined that:

$$R_E^n := \frac{1}{\Delta t} \left[\int_{t^{n-\frac{1}{2}}}^{t^n} \frac{(t^{n-\frac{1}{2}} - s)^6}{720} \frac{\partial^7 E}{\partial t^7}(s) ds + \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^6}{720} \frac{\partial^7 E}{\partial t^7}(s) ds \right],$$

$$R_H^{n+\frac{1}{2}} := \frac{1}{\Delta t} \left[\int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^n - s)^6}{720} \frac{\partial^7 H}{\partial t^7}(s) ds + \int_{t^{n+\frac{1}{2}}}^{t^{n+1}} \frac{(t^{n+1} - s)^6}{720} \frac{\partial^7 H}{\partial t^7}(s) ds \right].$$

Using these terms in the weak formulation as in Equations (5.2a) to (5.2c) at time $t = t^n$ for p and E terms, and at time $t = t^{n+\frac{1}{2}}$ for H and using the Taylor remainder theorem again with:

$$u(t^n) := \frac{u(t^{n+\frac{1}{2}}) + u(t^{n-\frac{1}{2}})}{2} - r_{1,u}^n, \quad H(t^{n+\frac{1}{2}}) := \frac{H(t^{n+1}) + H(t^n)}{2} - r_{1,H}^{n+\frac{1}{2}},$$

$$u(t^n) := \frac{u(t^{n+\frac{1}{2}}) + u(t^{n-\frac{1}{2}})}{2} - \frac{\Delta t^2}{8} \frac{\partial^2 u}{\partial t^2}(t^n) - r_{2,u}^n, \quad H(t^{n+\frac{1}{2}}) := \frac{H(t^{n+1}) + H(t^n)}{2} - \frac{\Delta t^2}{8} \frac{\partial^2 H}{\partial t^2}(t^{n+\frac{1}{2}}) - r_{2,H}^{n+\frac{1}{2}},$$

$$u(t^n) := \frac{u(t^{n+\frac{1}{2}}) + u(t^{n-\frac{1}{2}})}{2} - \frac{\Delta t^2}{8} \frac{\partial^2 u}{\partial t^2}(t^n) - \frac{\Delta t^4}{384} \frac{\partial^4 u}{\partial t^4}(t^n) - r_{3,u}^n,$$

$$H(t^{n+\frac{1}{2}}) := \frac{H(t^{n+1}) + H(t^n)}{2} - \frac{\Delta t^2}{8} \frac{\partial^2 H}{\partial t^2}(t^{n+\frac{1}{2}}) - \frac{\Delta t^4}{384} \frac{\partial^4 H}{\partial t^4}(t^{n+\frac{1}{2}}) - r_{3,H}^{n+\frac{1}{2}},$$

where u is either of p or E , and in which we have defined that:

$$\begin{aligned} r_{1,u}^n &:= \frac{1}{2} \left[\int_{t^n}^{t^{n+\frac{1}{2}}} (t^{n+\frac{1}{2}} - s) \frac{\partial^2 u}{\partial t^2}(s) ds - \int_{t^{n-\frac{1}{2}}}^{t^n} (t^{n-\frac{1}{2}} - s) \frac{\partial^2 u}{\partial t^2}(s) ds \right], \\ r_{1,H}^{n+\frac{1}{2}} &:= \frac{1}{2} \left[\int_{t^{n+\frac{1}{2}}}^{t^{n+1}} (t^{n+1} - s) \frac{\partial^2 H}{\partial t^2}(s) ds - \int_{t^n}^{t^{n+\frac{1}{2}}} (t^n - s) \frac{\partial^2 H}{\partial t^2}(s) ds \right], \\ r_{2,u}^n &:= \frac{1}{2} \left[\int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^3}{6} \frac{\partial^4 u}{\partial t^4}(s) ds - \int_{t^{n-\frac{1}{2}}}^{t^n} \frac{(t^{n-\frac{1}{2}} - s)^3}{6} \frac{\partial^4 u}{\partial t^4}(s) ds \right], \\ r_{2,H}^{n+\frac{1}{2}} &:= \frac{1}{2} \left[\int_{t^{n+\frac{1}{2}}}^{t^{n+1}} \frac{(t^{n+1} - s)^3}{6} \frac{\partial^4 H}{\partial t^4}(s) ds - \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^n - s)^3}{6} \frac{\partial^4 H}{\partial t^4}(s) ds \right], \\ r_{3,u}^n &:= \frac{1}{2} \left[\int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^5}{120} \frac{\partial^6 u}{\partial t^6}(s) ds - \int_{t^{n-\frac{1}{2}}}^{t^n} \frac{(t^{n-\frac{1}{2}} - s)^5}{120} \frac{\partial^6 u}{\partial t^6}(s) ds \right], \\ r_{3,H}^{n+\frac{1}{2}} &:= \frac{1}{2} \left[\int_{t^{n+\frac{1}{2}}}^{t^{n+1}} \frac{(t^{n+1} - s)^5}{120} \frac{\partial^6 H}{\partial t^6}(s) ds - \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^n - s)^5}{120} \frac{\partial^6 H}{\partial t^6}(s) ds \right], \end{aligned}$$

we obtain:

$$\begin{aligned}
& \left\langle \frac{p(t^{n+\frac{1}{2}}) - p(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{p} \right\rangle - \left\langle \frac{\varepsilon}{2} \left(E(t^{n+\frac{1}{2}}) + E(t^{n-\frac{1}{2}}) \right), \nabla \tilde{p} \right\rangle + \frac{\Delta t^2}{12} \left\langle \frac{\varepsilon}{2} \nabla \nabla \cdot \left(E(t^{n+\frac{1}{2}}) + E(t^{n-\frac{1}{2}}) \right), \nabla \tilde{p} \right\rangle \\
& - \frac{\Delta t^4}{120} \left\langle \frac{\varepsilon}{2} (\nabla \nabla \cdot)^2 \left(E(t^{n+\frac{1}{2}}) + E(t^{n-\frac{1}{2}}) \right), \nabla \tilde{p} \right\rangle = \langle R_p^n + \varepsilon \nabla \cdot r_{3,E}^n - \frac{\Delta t^2}{12} \varepsilon \nabla \cdot \nabla \nabla \cdot r_{2,E}^n + \frac{\Delta t^4}{120} \varepsilon \nabla \cdot (\nabla \nabla \cdot)^2 r_{1,E}^n, \tilde{p} \rangle, \quad (5.9a)
\end{aligned}$$

$$\begin{aligned}
& \left\langle \frac{1}{2} \nabla \left(p(t^{n+\frac{1}{2}}) + p(t^{n-\frac{1}{2}}) \right), \tilde{E} \right\rangle - \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \nabla \left(p(t^{n+\frac{1}{2}}) + p(t^{n-\frac{1}{2}}) \right), \nabla \nabla \cdot \tilde{E} \right\rangle \\
& + \frac{\Delta t^4}{120} \left\langle \frac{1}{2} \nabla \left(p(t^{n+\frac{1}{2}}) + p(t^{n-\frac{1}{2}}) \right), (\nabla \nabla \cdot)^2 \tilde{E} \right\rangle + \left\langle \varepsilon \frac{E(t^{n+\frac{1}{2}}) - E(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{E} \right\rangle - \left\langle \frac{1}{2} \left(H(t^{n+1}) + H(t^n) \right), \nabla \times \tilde{E} \right\rangle \\
& - \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \left(H(t^{n+1}) + H(t^n) \right), \nabla \times \tilde{E} \right\rangle - \frac{\Delta t^4}{120} \left\langle \frac{1}{2} \mu^{-2} \varepsilon^{-2} (\nabla \times \nabla \times)^2 \left(H(t^{n+1}) + H(t^n) \right), \nabla \times \tilde{E} \right\rangle \\
& = \langle \varepsilon R_E^n + \nabla r_{3,p}^n - \nabla \times r_{3,H}^{n+\frac{1}{2}} - \frac{\Delta t^2}{12} \nabla \nabla \cdot \nabla r_{2,p}^n - \frac{\Delta t^2}{12} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \nabla \times r_{2,H}^{n+\frac{1}{2}} \\
& + \frac{\Delta t^4}{120} (\nabla \nabla \cdot)^2 \nabla r_{1,p}^n - \frac{\Delta t^4}{120} \mu^{-2} \varepsilon^{-2} \nabla \times (\nabla \times \nabla \times)^2 r_{2,H}^{n+\frac{1}{2}}, \tilde{E} \rangle, \quad (5.9b)
\end{aligned}$$

$$\begin{aligned}
& \left\langle \mu \frac{H(t^{n+1}) - H(t^n)}{\Delta t}, \tilde{H} \right\rangle + \left\langle \frac{1}{2} \nabla \times \left(E(t^{n+\frac{1}{2}}) + E(t^{n-\frac{1}{2}}) \right), \tilde{H} \right\rangle \\
& + \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \varepsilon^{-1} \mu^{-1} \nabla \times \left(E(t^{n+\frac{1}{2}}) + E(t^{n-\frac{1}{2}}) \right), \nabla \times \nabla \times \tilde{H} \right\rangle \\
& + \frac{\Delta t^4}{120} \left\langle \frac{1}{2} \varepsilon^{-2} \mu^{-2} \nabla \times \left(E(t^{n+\frac{1}{2}}) + E(t^{n-\frac{1}{2}}) \right), (\nabla \times \nabla \times)^2 \tilde{H} \right\rangle \\
& = \langle \mu R_H^{n+\frac{1}{2}} + \nabla \times r_{3,E}^n + \frac{\Delta t^2}{12} \varepsilon^{-1} \mu^{-1} \nabla \times \nabla \times \nabla \times r_{2,E}^n + \frac{\Delta t^4}{120} \varepsilon^{-2} \mu^{-2} (\nabla \times \nabla \times)^2 \nabla \times r_{1,E}^n, \tilde{H} \rangle. \quad (5.9c)
\end{aligned}$$

Then subtracting the LF₆ scheme expressions as in Equations (5.3a) to (5.3c) leads us to the following set of equations:

$$\begin{aligned}
& \left\langle \frac{e_p^{n+\frac{1}{2}} - e_p^{n-\frac{1}{2}}}{\Delta t}, \tilde{p} \right\rangle - \frac{1}{2} \left\langle \varepsilon \left(e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}} \right), \nabla \tilde{p} \right\rangle + \frac{\Delta t^2}{24} \left\langle \varepsilon \nabla \nabla \cdot \left(e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}} \right), \nabla \tilde{p} \right\rangle \\
& - \frac{\Delta t^4}{120} \left\langle \varepsilon (\nabla \nabla \cdot)^2 \left(e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}} \right), \nabla \tilde{p} \right\rangle = \langle R_p^n + \varepsilon \nabla \cdot r_{3,E}^n - \frac{\Delta t^2}{12} \varepsilon \nabla \cdot \nabla \nabla \cdot r_{2,E}^n + \frac{\Delta t^4}{120} \varepsilon \nabla \cdot (\nabla \nabla \cdot)^2 r_{1,E}^n, \tilde{p} \rangle,
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2} \left\langle \nabla \left(e_p^{n+\frac{1}{2}} + e_p^{n-\frac{1}{2}} \right), \tilde{E} \right\rangle - \frac{\Delta t^2}{24} \left\langle \nabla \left(e_p^{n+\frac{1}{2}} + e_p^{n-\frac{1}{2}} \right), \nabla \nabla \cdot \tilde{E} \right\rangle + \frac{\Delta t^4}{120} \left\langle \nabla \left(e_p^{n+\frac{1}{2}} + e_p^{n-\frac{1}{2}} \right), (\nabla \nabla \cdot)^2 \tilde{E} \right\rangle \\
& + \left\langle \varepsilon \frac{e_E^{n+\frac{1}{2}} - e_E^{n-\frac{1}{2}}}{\Delta t}, \tilde{E} \right\rangle - \frac{1}{2} \left\langle e_H^{n+1} + e_H^n, \nabla \times \tilde{E} \right\rangle - \frac{\Delta t^2}{24} \left\langle \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \left(e_H^{n+1} + e_H^n \right), \nabla \times \tilde{E} \right\rangle \\
& - \frac{\Delta t^4}{120} \left\langle \mu^{-2} \varepsilon^{-2} (\nabla \times \nabla \times)^2 \left(e_H^{n+1} + e_H^n \right), \nabla \times \tilde{E} \right\rangle = \langle \varepsilon R_E^n + \nabla r_{3,p}^n - \nabla \times r_{3,H}^{n+\frac{1}{2}} - \frac{\Delta t^2}{12} \nabla \nabla \cdot \nabla r_{2,p}^n \\
& - \frac{\Delta t^2}{12} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \nabla \times r_{2,H}^{n+\frac{1}{2}} + \frac{\Delta t^4}{120} (\nabla \nabla \cdot)^2 \nabla r_{1,p}^n - \frac{\Delta t^4}{120} \mu^{-2} \varepsilon^{-2} \nabla \times (\nabla \times \nabla \times)^2 r_{1,H}^{n+\frac{1}{2}}, \tilde{E} \rangle,
\end{aligned}$$

$$\begin{aligned}
& \left\langle \mu \frac{e_H^{n+1} - e_H^n}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \left\langle \nabla \times \left(e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}} \right), \tilde{H} \right\rangle + \frac{\Delta t^2}{24} \left\langle \varepsilon^{-1} \mu^{-1} \nabla \times \left(e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}} \right), \nabla \times \nabla \times \tilde{H} \right\rangle \\
& + \frac{\Delta t^4}{120} \left\langle \varepsilon^{-2} \mu^{-2} \nabla \times \left(e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}} \right), (\nabla \times \nabla \times)^2 \tilde{H} \right\rangle = \left\langle \mu R_H^{n+\frac{1}{2}} + \nabla \times r_{3,E}^n + \frac{\Delta t^2}{12} \varepsilon^{-1} \mu^{-1} \nabla \times \nabla \times \nabla \times r_{2,E}^n \right. \\
& \quad \left. + \frac{\Delta t^4}{120} \varepsilon^{-2} \mu^{-2} (\nabla \times \nabla \times)^2 \nabla \times r_{1,E}^n, \tilde{H} \right\rangle.
\end{aligned}$$

Likewise, for the semidiscrete approximation of the initial system as in Equations (5.4a) to (5.4c), we obtain for their errors the following system of equations:

$$\begin{aligned}
& \left\langle \frac{e_p^{\frac{1}{2}} - e_p^0}{\Delta t/2}, \tilde{p} \right\rangle - \frac{1}{4} \left\langle \varepsilon \left(e_E^{\frac{1}{2}} + e_E^0 \right), \nabla \tilde{p} \right\rangle + \frac{\Delta t^2}{192} \left\langle \varepsilon \nabla \nabla \cdot \left(e_E^{\frac{1}{2}} + e_E^0 \right), \nabla \tilde{p} \right\rangle - \frac{\Delta t^4}{7680} \left\langle \varepsilon (\nabla \nabla \cdot)^2 \left(e_E^{\frac{1}{2}} + e_E^0 \right), \nabla \tilde{p} \right\rangle \\
& = \left\langle \frac{1}{2} R_p^0 + \frac{\varepsilon}{2} \nabla \cdot r_{3,E}^0 - \frac{\Delta t^2}{96} \varepsilon \nabla \cdot \nabla \nabla \cdot r_{2,E}^0 + \frac{\Delta t^4}{3840} \varepsilon \nabla \cdot (\nabla \nabla \cdot)^2 r_{1,E}^0, \tilde{p} \right\rangle,
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{4} \left\langle \nabla \left(e_p^{\frac{1}{2}} + e_p^0 \right), \tilde{E} \right\rangle - \frac{\Delta t^2}{192} \left\langle \nabla \left(e_p^{\frac{1}{2}} + e_p^0 \right), \nabla \nabla \cdot \tilde{E} \right\rangle + \frac{\Delta t^4}{7680} \left\langle \nabla \left(e_p^{\frac{1}{2}} + e_p^0 \right), (\nabla \nabla \cdot)^2 \tilde{E} \right\rangle + \left\langle \varepsilon \frac{e_E^{\frac{1}{2}} - e_E^0}{\Delta t/2}, \tilde{E} \right\rangle \\
& - \frac{1}{2} \left\langle e_H^1 + e_H^0, \nabla \times \tilde{E} \right\rangle - \frac{\Delta t^2}{96} \left\langle \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \left(e_H^1 + e_H^0 \right), \tilde{E} \right\rangle - \frac{\Delta t^4}{3840} \left\langle \mu^{-2} \varepsilon^{-2} (\nabla \times \nabla \times)^2 \left(e_H^1 + e_H^0 \right), \tilde{E} \right\rangle \\
& = \left\langle \frac{\varepsilon}{2} R_E^0 + \frac{1}{2} \nabla r_{3,p}^0 - \nabla \times r_{3,H}^{\frac{1}{2}} - \frac{\Delta t^2}{96} \nabla \nabla \cdot \nabla r_{2,p}^0 - \frac{\Delta t^2}{48} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \nabla \times r_{2,H}^{\frac{1}{2}} + \frac{\Delta t^4}{3840} (\nabla \nabla \cdot)^2 \nabla r_{1,p}^0 \right. \\
& \quad \left. - \frac{\Delta t^4}{1920} \mu^{-2} \varepsilon^{-2} \nabla \times (\nabla \times \nabla \times)^2 r_{1,H}^{\frac{1}{2}}, \tilde{E} \right\rangle,
\end{aligned}$$

$$\begin{aligned}
& \left\langle \mu \frac{e_H^1 - e_H^0}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{4} \left\langle \nabla \times \left(e_E^{\frac{1}{2}} + e_E^0 \right), \tilde{H} \right\rangle + \frac{\Delta t^2}{192} \left\langle \varepsilon^{-1} \mu^{-1} \nabla \times \left(e_E^{\frac{1}{2}} + e_E^0 \right), \nabla \times \nabla \times \tilde{H} \right\rangle \\
& + \frac{\Delta t^2}{7680} \left\langle \varepsilon^{-2} \mu^{-2} \nabla \times \left(e_E^{\frac{1}{2}} + e_E^0 \right), (\nabla \times \nabla \times)^2 \tilde{H} \right\rangle = \left\langle \mu R_H^{\frac{1}{2}} + \frac{1}{2} \nabla \times r_{3,E}^0 + \frac{\Delta t^2}{96} \varepsilon^{-1} \mu^{-1} \nabla \times \nabla \times \nabla \times r_{2,E}^0 \right. \\
& \quad \left. + \frac{\Delta t^4}{3840} \varepsilon^{-2} \mu^{-2} (\nabla \times \nabla \times)^2 \nabla \times r_{1,E}^0, \tilde{H} \right\rangle,
\end{aligned}$$

and for which we define the initial remainder terms as follows:

$$\begin{aligned}
R_u^0 &:= \frac{1}{\Delta t} \left[\int_{t^0}^{t^{\frac{1}{4}}} \frac{(t^0 - s)^4}{24} \frac{\partial^5 u}{\partial t^5}(s) ds + \int_{t^{\frac{1}{4}}}^{t^{\frac{1}{2}}} \frac{(t^{\frac{1}{2}} - s)^4}{24} \frac{\partial^5 u}{\partial t^5}(s) ds \right], \\
R_H^{\frac{1}{2}} &:= \frac{1}{\Delta t} \left[\int_{t^0}^{t^{\frac{1}{2}}} \frac{(t^0 - s)^4}{24} \frac{\partial^5 H}{\partial t^5}(s) ds + \int_{t^{\frac{1}{2}}}^{t^1} \frac{(t^1 - s)^4}{24} \frac{\partial^5 H}{\partial t^5}(s) ds \right], \\
r_{1,u}^0 &:= \frac{1}{2} \left[\int_{t^{\frac{1}{4}}}^{t^{\frac{1}{2}}} (t^{\frac{1}{2}} - s) \frac{\partial^2 u}{\partial t^2}(s) ds - \int_{t^0}^{t^{\frac{1}{4}}} (t^0 - s) \frac{\partial^2 u}{\partial t^2}(s) ds \right],
\end{aligned}$$

$$\begin{aligned}
r_{1,H}^{\frac{1}{2}} &:= \frac{1}{2} \left[\int_{t^{\frac{1}{2}}}^{t^1} (t^1 - s) \frac{\partial^2 H}{\partial t^2}(s) ds - \int_{t^0}^{t^{\frac{1}{2}}} (t^0 - s) \frac{\partial^2 H}{\partial t^2}(s) ds \right], \\
r_{2,u}^0 &:= \frac{1}{2} \left[\int_{t^{\frac{1}{4}}}^{t^{\frac{1}{2}}} \frac{(t^{\frac{1}{2}} - s)^3}{6} \frac{\partial^4 u}{\partial t^4}(s) ds - \int_{t^0}^{t^{\frac{1}{4}}} \frac{(t^0 - s)^3}{6} \frac{\partial^4 u}{\partial t^4}(s) ds \right], \\
r_{2,H}^{\frac{1}{2}} &:= \frac{1}{2} \left[\int_{t^{\frac{1}{2}}}^{t^1} \frac{(t^1 - s)^3}{6} \frac{\partial^4 H}{\partial t^4}(s) ds - \int_{t^0}^{t^{\frac{1}{2}}} \frac{(t^0 - s)^3}{6} \frac{\partial^4 H}{\partial t^4}(s) ds \right], \\
r_{3,u}^0 &:= \frac{1}{2} \left[\int_{t^{\frac{1}{4}}}^{t^{\frac{1}{2}}} \frac{(t^{\frac{1}{2}} - s)^5}{120} \frac{\partial^6 u}{\partial t^6}(s) ds - \int_{t^0}^{t^{\frac{1}{4}}} \frac{(t^0 - s)^5}{120} \frac{\partial^6 u}{\partial t^6}(s) ds \right], \\
r_{3,H}^{\frac{1}{2}} &:= \frac{1}{2} \left[\int_{t^{\frac{1}{2}}}^{t^1} \frac{(t^1 - s)^5}{120} \frac{\partial^6 H}{\partial t^6}(s) ds - \int_{t^0}^{t^{\frac{1}{2}}} \frac{(t^0 - s)^5}{120} \frac{\partial^6 H}{\partial t^6}(s) ds \right].
\end{aligned}$$

Now, in this set of weak formulation equations for the errors, we choose the test functions to be $\tilde{p} = 2\Delta t \varepsilon^{-1} (e_p^{n+\frac{1}{2}} + e_p^{n-\frac{1}{2}})$, $\tilde{E} = 2\Delta t (e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}})$ and $\tilde{H} = 2\Delta t (e_H^n + e_H^{n-1})$. Next, by following essentially the same sequence of steps as in Theorem 5.1, we obtain the estimate for these error terms to be:

$$\begin{aligned}
&\|e_p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|e_p^{n-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{n+\frac{1}{2}}\|_{\varepsilon}^2 - \|e_E^{n-\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^{n+1}\|_{\mu}^2 - \|e_H^n\|_{\mu}^2 \leq \Delta t \left[\|e_p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|e_p^{n-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 \right. \\
&\quad \left. + \|e_E^{n+\frac{1}{2}}\|_{\varepsilon}^2 - \|e_E^{n-\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^{n+1}\|_{\mu}^2 - \|e_H^n\|_{\mu}^2 \right] + \Delta t \left[\|\mathbf{R}_p^n\|_{\varepsilon^{-1}}^2 + \|\mathbf{R}_E^n\|_{\varepsilon}^2 + \|\mathbf{R}_H^{n+\frac{1}{2}}\|_{\mu}^2 + \|\nabla \mathbf{r}_{3,p}^n\|_{\varepsilon^{-1}}^2 \right. \\
&\quad \left. + \|\nabla \cdot \mathbf{r}_{3,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times \mathbf{r}_{3,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times \mathbf{r}_{3,H}^{n+\frac{1}{2}}\|_{\mu}^2 + \frac{\Delta t^4}{144} \left(\|\nabla \nabla \cdot \nabla \mathbf{r}_{2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \nabla \nabla \cdot \mathbf{r}_{2,E}^n\|_{\varepsilon}^2 \right. \right. \\
&\quad \left. \left. + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times \mathbf{r}_{2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times \mathbf{r}_{2,H}^{n+\frac{1}{2}}\|_{\mu}^2 \right) + \frac{\Delta t^8}{14400} \left(\|(\nabla \nabla \cdot)^2 \nabla \mathbf{r}_{1,p}^n\|_{\varepsilon^{-1}}^2 \right. \\
&\quad \left. + \|\nabla \cdot (\nabla \nabla \cdot)^2 \mathbf{r}_{1,E}^n\|_{\varepsilon}^2 + \varepsilon^{-5} \mu^{-5} \|(\nabla \times \nabla \times)^2 \nabla \times \mathbf{r}_{1,E}^n\|_{\varepsilon}^2 + \varepsilon^{-5} \mu^{-5} \|\nabla \times (\nabla \times \nabla \times)^2 \mathbf{r}_{1,H}^{n+\frac{1}{2}}\|_{\mu}^2 \right) \Big].
\end{aligned}$$

Now, summing over $n = 1$ to $N - 1$, we get

$$\begin{aligned}
&\|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|e_p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon}^2 - \|e_E^{\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^N\|_{\mu}^2 - \|e_H^1\|_{\mu}^2 \leq \Delta t \left[\|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|e_p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon}^2 \right. \\
&\quad \left. - \|e_E^{\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^N\|_{\mu}^2 - \|e_H^1\|_{\mu}^2 \right] + 2\Delta t \sum_{n=1}^{N-1} \left[\|e_p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^{n+1}\|_{\mu}^2 \right] + \Delta t \sum_{n=1}^{N-1} \left[\|\mathbf{R}_p^n\|_{\varepsilon^{-1}}^2 + \|\mathbf{R}_E^n\|_{\varepsilon}^2 \right. \\
&\quad \left. + \|\mathbf{R}_H^{n+\frac{1}{2}}\|_{\mu}^2 + \|\nabla \mathbf{r}_{3,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \mathbf{r}_{3,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times \mathbf{r}_{3,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times \mathbf{r}_{3,H}^{n+\frac{1}{2}}\|_{\mu}^2 \right. \\
&\quad \left. + \frac{\Delta t^4}{144} \left(\|\nabla \nabla \cdot \nabla \mathbf{r}_{2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \nabla \nabla \cdot \mathbf{r}_{2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times \mathbf{r}_{2,E}^n\|_{\varepsilon}^2 \right. \right. \\
&\quad \left. \left. + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times \mathbf{r}_{2,H}^{n+\frac{1}{2}}\|_{\mu}^2 \right) + \frac{\Delta t^8}{14400} \left(\|(\nabla \nabla \cdot)^2 \nabla \mathbf{r}_{1,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot (\nabla \nabla \cdot)^2 \mathbf{r}_{1,E}^n\|_{\varepsilon}^2 \right. \right.
\end{aligned}$$

$$+\varepsilon^{-5}\mu^{-5}\|(\nabla \times \nabla \times)^2 \nabla \times r_{1,E}^n\|_\varepsilon^2 + \varepsilon^{-5}\mu^{-5}\|\nabla \times (\nabla \times \nabla \times)^2 r_{1,H}^{n+\frac{1}{2}}\|_\mu^2 \Big) \Big].$$

Similarly, for the case of initial equations, choosing test functions to be $\tilde{p} = 2\Delta t \varepsilon^{-1} (e_p^{\frac{1}{2}} + e_p^0)$, $\tilde{E} = 2\Delta t (e_E^{\frac{1}{2}} + e_E^0)$ and $\tilde{H} = 4\Delta t (e_H^1 + e_H^0)$, we get

$$\begin{aligned} & \|e_p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^{\frac{1}{2}}\|_\varepsilon^2 - \|e_E^0\|_\varepsilon^2 + \|e_H^1\|_\mu^2 - \|e_H^0\|_\mu^2 \leq \Delta t \Big[\|e_p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^{\frac{1}{2}}\|_\varepsilon^2 - \|e_E^0\|_\varepsilon^2 \\ & + \|e_H^1\|_\mu^2 - \|e_H^0\|_\mu^2 \Big] + \Delta t \Big[\|R_p^0\|_{\varepsilon^{-1}}^2 + \|R_E^0\|_\varepsilon^2 + \|R_H^{\frac{1}{2}}\|_\mu^2 + \|\nabla r_{3,p}^0\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{3,E}^0\|_\varepsilon^2 + \varepsilon^{-1}\mu^{-1}\|\nabla \times r_{3,E}^0\|_\varepsilon^2 \\ & + \varepsilon^{-1}\mu^{-1}\|\nabla \times r_{3,H}^{\frac{1}{2}}\|_\mu^2 + \frac{\Delta t^4}{144} \Big(\|\nabla \nabla \cdot \nabla r_{2,p}^0\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \nabla \nabla \cdot r_{2,E}^0\|_\varepsilon^2 + \varepsilon^{-3}\mu^{-3}\|\nabla \times \nabla \times \nabla \times r_{2,E}^0\|_\varepsilon^2 \\ & + \varepsilon^{-3}\mu^{-3}\|\nabla \times \nabla \times \nabla \times r_{2,H}^{\frac{1}{2}}\|_\mu^2 \Big) + \frac{\Delta t^8}{14400} \Big(\|(\nabla \nabla \cdot)^2 \nabla r_{1,p}^0\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot (\nabla \nabla \cdot)^2 r_{1,E}^0\|_\varepsilon^2 \\ & + \varepsilon^{-5}\mu^{-5}\|(\nabla \times \nabla \times)^2 \nabla \times r_{1,E}^0\|_\varepsilon^2 + \varepsilon^{-5}\mu^{-5}\|\nabla \times (\nabla \times \nabla \times)^2 r_{1,H}^{\frac{1}{2}}\|_\mu^2 \Big) \Big]. \end{aligned}$$

Adding the previous two equations and using the initial conditions as in Equation (5.1c), and the positivity of all the right-hand side terms, we have that:

$$\begin{aligned} & \|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-\frac{1}{2}}\|_\varepsilon^2 + \|e_H^N\|_\mu^2 \leq \frac{1+\Delta t}{1-\Delta t} \Big[\|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^0\|_\varepsilon^2 + \|e_H^0\|_\mu^2 \Big] + \frac{\Delta t}{1-\Delta t} \sum_{n=0}^{N-1} \Big[2 \Big(\|e_p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 \\ & + \|e_E^{n+\frac{1}{2}}\|_\varepsilon^2 + \|e_H^{n+1}\|_\mu^2 \Big) + \Big(\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_\varepsilon^2 + \|R_H^{n+\frac{1}{2}}\|_\mu^2 + \|\nabla r_{3,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{3,E}^n\|_\varepsilon^2 + \varepsilon^{-1}\mu^{-1}\|\nabla \times r_{3,E}^n\|_\varepsilon^2 \\ & + \varepsilon^{-1}\mu^{-1}\|\nabla \times r_{3,H}^{n+\frac{1}{2}}\|_\mu^2 \Big) + \frac{\Delta t^4}{144} \Big(\|\nabla \nabla \cdot \nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \nabla \nabla \cdot r_{2,E}^n\|_\varepsilon^2 + \varepsilon^{-3}\mu^{-3}\|\nabla \times \nabla \times \nabla \times r_{2,E}^n\|_\varepsilon^2 \\ & + \varepsilon^{-3}\mu^{-3}\|\nabla \times \nabla \times \nabla \times r_{2,H}^{n+\frac{1}{2}}\|_\mu^2 \Big) + \frac{\Delta t^8}{14400} \Big(\|(\nabla \nabla \cdot)^2 \nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot (\nabla \nabla \cdot)^2 r_{1,E}^n\|_\varepsilon^2 \\ & + \varepsilon^{-5}\mu^{-5}\|(\nabla \times \nabla \times)^2 \nabla \times r_{1,E}^n\|_\varepsilon^2 + \varepsilon^{-5}\mu^{-5}\|\nabla \times (\nabla \times \nabla \times)^2 r_{1,H}^{n+\frac{1}{2}}\|_\mu^2 \Big) \Big]. \end{aligned}$$

Applying the discrete Gronwall inequality, we obtain the estimate:

$$\begin{aligned} & \|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-\frac{1}{2}}\|_\varepsilon^2 + \|e_H^N\|_\mu^2 \leq \left[\frac{6\Delta t}{5} \sum_{n=0}^{N-1} \Big(\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_\varepsilon^2 + \|R_H^{n+\frac{1}{2}}\|_\mu^2 \right. \\ & + \|\nabla r_{3,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{3,E}^n\|_\varepsilon^2 + \varepsilon^{-1}\mu^{-1}\|\nabla \times r_{3,E}^n\|_\varepsilon^2 + \varepsilon^{-1}\mu^{-1}\|\nabla \times r_{3,H}^{n+\frac{1}{2}}\|_\mu^2 \\ & + \frac{\Delta t^4}{144} \Big(\|\nabla \nabla \cdot \nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \nabla \nabla \cdot r_{2,E}^n\|_\varepsilon^2 + \varepsilon^{-3}\mu^{-3}\|\nabla \times \nabla \times \nabla \times r_{2,E}^n\|_\varepsilon^2 \\ & + \varepsilon^{-3}\mu^{-3}\|\nabla \times \nabla \times \nabla \times r_{2,H}^{n+\frac{1}{2}}\|_\mu^2 \Big) + \frac{\Delta t^8}{14400} \Big(\|(\nabla \nabla \cdot)^2 \nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot (\nabla \nabla \cdot)^2 r_{1,E}^n\|_\varepsilon^2 \\ & + \varepsilon^{-5}\mu^{-5}\|(\nabla \times \nabla \times)^2 \nabla \times r_{1,E}^n\|_\varepsilon^2 + \varepsilon^{-5}\mu^{-5}\|\nabla \times (\nabla \times \nabla \times)^2 r_{1,H}^{n+\frac{1}{2}}\|_\mu^2 \Big) \\ & \left. + \frac{7}{5} \Big(\|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^0\|_\varepsilon^2 + \|e_H^0\|_\mu^2 \Big) \right] \exp(4T). \end{aligned}$$

Now, we need to obtain bounding estimates for each of the Taylor remainder terms and to do so, we first consider the first remainder term corresponding to p and argue as follows:

$$\begin{aligned}
\|R_p^n\|_{\varepsilon^{-1}}^2 &= \frac{1}{(720)^2 (\Delta t)^2} \left\| \int_{t^{n-1}}^{t^{n-\frac{1}{2}}} (t^{n-1} - s)^6 \frac{\partial^7 p}{\partial t^7}(s) ds + \int_{t^{n-\frac{1}{2}}}^{t^n} (t^n - s)^6 \frac{\partial^7 p}{\partial t^7}(s) ds \right\|_{\varepsilon^{-1}}^2, \\
&\leq \frac{1}{(720)^2 (\Delta t)^2} \left\| \int_{t^{n-1}}^{t^n} (t^n - s)^6 \frac{\partial^7 p}{\partial t^7}(s) ds \right\|_{\varepsilon^{-1}}^2, \quad (\text{using } t^{n-1} < t^n) \\
&\leq \frac{1}{(720)^2 (\Delta t)^2} \int_{t^{n-1}}^{t^n} (s - t^n)^{12} ds \int_{t^{n-1}}^{t^n} \left\| \frac{\partial^7 p}{\partial t^7}(s) \right\|_{\varepsilon^{-1}}^2 ds, \quad (\text{by Cauchy-Schwarz}) \\
&= \frac{(\Delta t)^{11}}{(720)^2 \cdot 13} \int_{t^{n-1}}^{t^n} \left\| \frac{\partial^7 p}{\partial t^7}(s) \right\|_{\varepsilon^{-1}}^2 ds,
\end{aligned}$$

and now summing both sides over $n = 0$ to N , we have that:

$$\sum_{n=0}^N \|R_p^n\|_{\varepsilon^{-1}}^2 \leq \frac{(\Delta t)^{11}}{(720)^2 \cdot 13} \int_0^T \left\| \frac{\partial^7 p}{\partial t^7}(s) \right\|_{\varepsilon^{-1}}^2 ds = \frac{(\Delta t)^{11}}{(720)^2 \cdot 13} \left\| \frac{\partial^7 p}{\partial t^7} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2.$$

Similarly, for the remaining Taylor remainder terms, we have that:

$$\begin{aligned}
\|R_E^n\|_{\varepsilon}^2 &\leq \frac{(\Delta t)^{11}}{(720)^2 \cdot 13} \left\| \frac{\partial^7 E}{\partial t^7} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 &\leq \frac{(\Delta t)^{11}}{(720)^2 \cdot 13} \left\| \frac{\partial^7 H}{\partial t^7} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla r_{3,p}^n\|_{\varepsilon^{-1}}^2 &\leq \frac{(\Delta t)^{11}}{2^{11} \cdot 28800} \left\| \frac{\partial^6 (\nabla p)}{\partial t^6} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla \cdot r_{3,E}^n\|_{\varepsilon}^2 &\leq \frac{(\Delta t)^{11}}{2^{11} \cdot 28800} \left\| \frac{\partial^6 (\nabla \cdot E)}{\partial t^6} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla \times r_{3,E}^n\|_{\varepsilon}^2 &\leq \frac{(\Delta t)^{11}}{2^{11} \cdot 28800} \left\| \frac{\partial^6 (\nabla \times E)}{\partial t^6} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla \times r_{3,H}^{n+\frac{1}{2}}\|_{\mu}^2 &\leq \frac{(\Delta t)^{11}}{2^{11} \cdot 28800} \left\| \frac{\partial^6 (\nabla \times H)}{\partial t^6} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla \nabla \cdot \nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 &\leq \frac{(\Delta t)^7}{2^7 \cdot 72} \left\| \frac{\partial^4 (\nabla \nabla \cdot \nabla p)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla \cdot \nabla \nabla \cdot r_{2,E}^n\|_{\varepsilon}^2 &\leq \frac{(\Delta t)^7}{2^7 \cdot 72} \left\| \frac{\partial^4 (\nabla \cdot \nabla \nabla \cdot E)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2,
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1} \|\nabla \times \nabla \times \nabla \times r_{2,E}^n\|_{\varepsilon}^2 &\leq \frac{(\Delta t)^7}{2^7 \cdot 72} \left\| \frac{\partial^4 (\nabla \times \nabla \times \nabla \times E)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla \times \nabla \times \nabla \times r_{2,H}^{n+\frac{1}{2}}\|_{\mu}^2 &\leq \frac{(\Delta t)^7}{2^7 \cdot 72} \left\| \frac{\partial^4 (\nabla \times \nabla \times \nabla \times H)}{\partial t^4} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|(\nabla \nabla \cdot)^2 \nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 &\leq \frac{(\Delta t)^3}{48} \left\| \frac{\partial^2 ((\nabla \nabla \cdot)^2 \nabla p)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla \cdot (\nabla \nabla \cdot)^2 r_{1,E}^n\|_{\varepsilon}^2 &\leq \frac{(\Delta t)^3}{48} \left\| \frac{\partial^2 (\nabla \cdot (\nabla \nabla \cdot)^2 E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|(\nabla \times \nabla \times)^2 \nabla \times r_{1,E}^n\|_{\varepsilon}^2 &\leq \frac{(\Delta t)^3}{48} \left\| \frac{\partial^2 ((\nabla \times \nabla \times)^2 \nabla \times E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2, \\
\sum_{n=0}^{N-1} \|\nabla \times (\nabla \times \nabla \times)^2 r_{1,H}^{n+\frac{1}{2}}\|_{\mu}^2 &\leq \frac{(\Delta t)^3}{48} \left\| \frac{\partial^2 (\nabla \times (\nabla \times \nabla \times)^2 H)}{\partial t^2} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2.
\end{aligned}$$

Finally, using the regularity assumptions for p , E and H , and 1- and 2-norm equivalence, we obtain the required result:

$$\|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon} + \|e_H^N\|_{\mu} \leq C [(\Delta t)^6 + \|e_p^0\|_{\varepsilon^{-1}} + \|e_E^0\|_{\varepsilon} + \|e_H^0\|_{\mu}]. \quad \square$$

5.2.2 Error Estimate for Full Discretization

We now present the error analysis for the full discretization of the three-field formulation of the Maxwell's equations using a compatible sequence of arbitrary-order simplicial de Rham finite elements in conjunction with the LF₆ scheme. To do so, first we let Π_h^0 , Π_h^1 and Π_h^2 denote the respective smoothed L^2 projection operators, that is, let $\Pi_h^0 : \dot{H}_{\varepsilon^{-1}}^1(\Omega) \rightarrow U_h$, $\Pi_h^1 : \dot{H}_{\varepsilon}^1(\text{curl}; \Omega) \rightarrow V_h$ and $\Pi_h^2 : \dot{H}_{\mu}^1(\text{div}; \Omega) \rightarrow W_h$. The details about these operators has become somewhat standardized, and expositions be found in many articles such as [Sch08; Chr07; AFW06; Arn18; EG21a]. We also refer to the earlier works in [AK25; AK26] for more specific details as it pertains to the (p, E, H) Maxwell's system.

Given this brief background about these smoothed L^2 projection operators, we now define the errors for p , E and H at time $(n + 1/2)\Delta t$ or $n\Delta t$ under the full discretization to be the following:

$$e_{p_h}^{n+\frac{1}{2}} := p(t^{n+\frac{1}{2}}) - p_h^{n+\frac{1}{2}} = \eta^{n+\frac{1}{2}} - \eta_h^{n+\frac{1}{2}}, \quad (5.10)$$

$$e_{E_h}^{n+\frac{1}{2}} := E(t^{n+\frac{1}{2}}) - E_h^{n+\frac{1}{2}} = \zeta^{n+\frac{1}{2}} - \zeta_h^{n+\frac{1}{2}}, \quad (5.11)$$

$$e_{H_h}^n := H(t^n) - H_h^n = \xi^n - \xi_h^n, \quad (5.12)$$

and in which we have the following definitions for the newly introduced terms:

$$\eta^{n+\frac{1}{2}} := p(t^{n+\frac{1}{2}}) - \Pi_h^0 p(t^{n+\frac{1}{2}}), \quad \eta_h^{n+\frac{1}{2}} := p_h^{n+\frac{1}{2}} - \Pi_h^0 p(t^{n+\frac{1}{2}}), \quad (5.13)$$

$$\zeta^{n+\frac{1}{2}} := E(t^{n+\frac{1}{2}}) - \Pi_h^1 E(t^{n+\frac{1}{2}}), \quad \zeta_h^{n+\frac{1}{2}} := E_h^{n+\frac{1}{2}} - \Pi_h^1 E(t^{n+\frac{1}{2}}), \quad (5.14)$$

$$\xi^n := H(t^n) - \Pi_h^2 H(t^n), \quad \xi_h^n := H_h^n - \Pi_h^2 H(t^n). \quad (5.15)$$

For the LF₆ scheme as in Equations (5.3a) to (5.3c), using a de Rham sequence of finite dimensional subspaces of the corresponding function spaces for the spatial discretization of $(p^{n+\frac{1}{2}}, E^{n+\frac{1}{2}}, H^{n+1})$, we obtain the following discrete problem: find $(p_h^{n+\frac{1}{2}}, E_h^{n+\frac{1}{2}}, H_h^{n+1}) \in U_h \times V_h \times W_h \subseteq \mathring{H}_{\varepsilon-1}^1 \times \mathring{H}_\varepsilon(\text{curl}; \Omega) \times \mathring{H}_\mu(\text{div}; \Omega)$ such that:

$$\begin{aligned} \left\langle \frac{p_h^{n+\frac{1}{2}} - p_h^{n-\frac{1}{2}}}{\Delta t}, \tilde{p} \right\rangle - \left\langle \frac{\varepsilon}{2} (E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}}), \nabla \tilde{p} \right\rangle + \frac{\Delta t^2}{12} \left\langle \frac{\varepsilon}{2} \nabla \nabla \cdot (E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}}), \nabla \tilde{p} \right\rangle \\ - \frac{\Delta t^4}{120} \left\langle \frac{\varepsilon}{2} (\nabla \nabla \cdot)^2 (E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}}), \nabla \tilde{p} \right\rangle = 0, \quad (5.16a) \end{aligned}$$

$$\begin{aligned} \left\langle \frac{1}{2} \nabla (p_h^{n+\frac{1}{2}} + p_h^{n-\frac{1}{2}}), \tilde{E} \right\rangle - \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \nabla (p_h^{n+\frac{1}{2}} + p_h^{n-\frac{1}{2}}), \nabla \nabla \cdot \tilde{E} \right\rangle + \frac{\Delta t^4}{120} \left\langle \frac{1}{2} \nabla (p_h^{n+\frac{1}{2}} + p_h^{n-\frac{1}{2}}), (\nabla \nabla \cdot)^2 \tilde{E} \right\rangle \\ + \left\langle \varepsilon \frac{E_h^{n+\frac{1}{2}} - E_h^{n-\frac{1}{2}}}{\Delta t}, \tilde{E} \right\rangle - \left\langle \frac{1}{2} (H_h^{n+1} + H_h^n), \nabla \times \tilde{E} \right\rangle - \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times (H_h^{n+1} + H_h^n), \nabla \times \tilde{E} \right\rangle \\ - \frac{\Delta t^4}{120} \left\langle \frac{1}{2} \mu^{-2} \varepsilon^{-2} (\nabla \times \nabla \times)^2 (H_h^{n+1} + H_h^n), \nabla \times \tilde{E} \right\rangle = 0, \quad (5.16b) \end{aligned}$$

$$\begin{aligned} \left\langle \mu \frac{H_h^{n+1} - H_h^n}{\Delta t}, \tilde{H} \right\rangle + \left\langle \frac{1}{2} \nabla \times (E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}}), \tilde{H} \right\rangle + \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \varepsilon^{-1} \mu^{-1} \nabla \times (E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}}), \nabla \times \nabla \times \tilde{H} \right\rangle \\ + \frac{\Delta t^4}{120} \left\langle \frac{1}{2} \varepsilon^{-2} \mu^{-2} \nabla \times (E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}}), (\nabla \times \nabla \times)^2 \tilde{H} \right\rangle = 0, \quad (5.16c) \end{aligned}$$

for all $(\tilde{p}, \tilde{E}, \tilde{H}) \in U_h \times V_h \times W_h$, and for $n \in \{1, \dots, N-1\}$. Using $n = 0$ for bootstrapping as in Equations (5.4a) to (5.4c) leads to the discrete problem: find $(p_h^{\frac{1}{2}}, E_h^{\frac{1}{2}}, H_h^1) \in U_h \times V_h \times W_h \subseteq \mathring{H}_{\varepsilon-1}^1 \times \mathring{H}_\varepsilon(\text{curl}; \Omega) \times \mathring{H}_\mu(\text{div}; \Omega)$ such that:

$$\begin{aligned} \left\langle \frac{p_h^{\frac{1}{2}} - p_h^0}{\Delta t/2}, \tilde{p} \right\rangle - \frac{1}{2} \left\langle \frac{\varepsilon}{2} (E_h^{\frac{1}{2}} + E_h^0), \nabla \tilde{p} \right\rangle + \frac{1}{8} \cdot \frac{\Delta t^2}{12} \left\langle \frac{\varepsilon}{2} \nabla \nabla \cdot (E_h^{\frac{1}{2}} + E_h^0), \nabla \tilde{p} \right\rangle \\ - \frac{1}{32} \cdot \frac{\Delta t^4}{120} \left\langle \frac{\varepsilon}{2} (\nabla \nabla \cdot)^2 (E_h^{\frac{1}{2}} + E_h^0), \nabla \tilde{p} \right\rangle = 0, \quad (5.17a) \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \left\langle \frac{1}{2} \nabla (p_h^{\frac{1}{2}} + p_h^0), \tilde{E} \right\rangle - \frac{1}{8} \cdot \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \nabla (p_h^{\frac{1}{2}} + p_h^0), \nabla \nabla \cdot \tilde{E} \right\rangle + \frac{1}{32} \cdot \frac{\Delta t^4}{120} \left\langle \frac{1}{2} \nabla (p_h^{\frac{1}{2}} + p_h^0), (\nabla \nabla \cdot)^2 \tilde{E} \right\rangle \\ + \left\langle \varepsilon \frac{E_h^{\frac{1}{2}} - E_h^0}{\Delta t/2}, \tilde{E} \right\rangle - \left\langle \frac{1}{2} (H_h^1 + H_h^0), \nabla \times \tilde{E} \right\rangle - \frac{1}{4} \cdot \frac{\Delta t^2}{12} \left\langle \frac{1}{2} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times (H_h^1 + H_h^0), \nabla \times \tilde{E} \right\rangle \\ + \frac{1}{16} \cdot \frac{\Delta t^4}{120} \left\langle \frac{1}{2} \mu^{-2} \varepsilon^{-2} (\nabla \times \nabla \times)^2 (H_h^1 + H_h^0), \nabla \times \tilde{E} \right\rangle = 0, \quad (5.17b) \end{aligned}$$

$$\begin{aligned} \langle \mu \frac{H_h^1 - H_h^0}{\Delta t}, \tilde{H} \rangle + \frac{1}{2} \langle \frac{1}{2} \nabla \times (E_h^{\frac{1}{2}} + E_h^0), \tilde{H} \rangle + \frac{1}{8} \cdot \frac{\Delta t^2}{12} \langle \frac{1}{2} \varepsilon^{-1} \mu^{-1} \nabla \times (E_h^{\frac{1}{2}} + E_h^0), \nabla \times \nabla \times \tilde{H} \rangle \\ + \frac{1}{32} \cdot \frac{\Delta t^4}{120} \langle \frac{1}{2} \varepsilon^{-2} \mu^{-2} \nabla \times (E_h^{\frac{1}{2}} + E_h^0), (\nabla \times \nabla \times)^2 \tilde{H} \rangle = 0. \end{aligned} \quad (5.17c)$$

for all $(\tilde{p}, \tilde{E}, \tilde{H}) \in U_h \times V_h \times W_h$ given $(p_h^0, E_h^0, H_h^0) \in U_h \times V_h \times W_h$. Let $\Pi_h^0 : \dot{H}_{\varepsilon^{-1}}^1(\Omega) \rightarrow U_h$, $\Pi_h^1 : \dot{H}_{\varepsilon}(\text{curl}; \Omega) \rightarrow V_h$ and $\Pi_h^2 : \dot{H}_{\mu}(\text{div}; \Omega) \rightarrow W_h$ denote the respective smoothed L^2 projection operators as in the sense of [Sch08; Chr07] and a detailed discussion is available in [BS08, Lemma 4.3.8] [AFW06, Theorem 5.3] [EG21a, Lemma 11.9, Corollary 11.11, Theorem 16.10, Theorem 17.5]. For the initial conditions, we set

$$p_h^0 := \Pi_h^0 p_0, E_h^0 := \Pi_h^1 E_0 \text{ and } H_h^0 := \Pi_h^2 H_0. \quad (5.17d)$$

With this, we can now state our theorem for convergence of error in the full discretization of the system of Maxwell's equations using LF_6 and R^{th} -order polynomial Whitney basis finite elements for the various function spaces.

Theorem 5.3 (Full Error Estimate). *Let $p \in C^7(0, T; \dot{H}_{\varepsilon^{-1}}^1(\Omega))$, $E \in C^7(0, T; \dot{H}_{\varepsilon}(\text{curl}; \Omega))$, and $H \in C^7(0, T; \dot{H}_{\mu}(\text{div}; \Omega))$ be the solution to the variational formulation of the Maxwell's equations as in Equations (5.2a) to (5.2c), and let $(p_h^{n+\frac{1}{2}}, E_h^{n+\frac{1}{2}}, H_h^{n+1})$ be the solution of the fully discretized Maxwell's equations using the LF_6 scheme as in Equations (5.16a) to (5.16c) and (5.17a) to (5.17c). If the fixed time step $\Delta t > 0$ and the mesh parameter $h > 0$ are sufficiently small, then there exists a positive bounded constant C independent of both Δt and h such that the following error estimate holds:*

$$\|e_{p_h}^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|e_{E_h}^{N-\frac{1}{2}}\|_{\varepsilon} + \|e_{H_h}^N\|_{\mu} \leq C [(\Delta t)^6 + h^r + h^r(\Delta t)^6],$$

where the finite element subspaces U_h , V_h and W_h are each spanned by their respective Whitney form basis of polynomial order $r \geq 1$.

Proof. First, we shall subtract the set of equations for the full discretization as in Equations (5.16a) to (5.16c) from Equations (5.9a) to (5.9c), and then use the error terms in Equations (5.10) to (5.12) and thereby obtain:

$$\begin{aligned} \langle \frac{e_{p_h}^{n+\frac{1}{2}} - e_{p_h}^{n-\frac{1}{2}}}{\Delta t}, \tilde{p} \rangle - \frac{1}{2} \langle \varepsilon (e_{E_h}^{n+\frac{1}{2}} + e_{E_h}^{n-\frac{1}{2}}), \nabla \tilde{p} \rangle + \frac{\Delta t^2}{24} \langle \varepsilon \nabla \nabla \cdot (e_{E_h}^{n+\frac{1}{2}} + e_{E_h}^{n-\frac{1}{2}}), \nabla \tilde{p} \rangle \\ - \frac{\Delta t^4}{120} \langle \varepsilon (\nabla \nabla \cdot)^2 (e_{E_h}^{n+\frac{1}{2}} + e_{E_h}^{n-\frac{1}{2}}), \nabla \tilde{p} \rangle = \langle R_p^n + \varepsilon \nabla \cdot r_{3,E}^n - \frac{\Delta t^2}{12} \varepsilon \nabla \cdot \nabla \nabla \cdot r_{2,E}^n + \frac{\Delta t^4}{120} \varepsilon \nabla \cdot (\nabla \nabla \cdot)^2 r_{1,E}^n, \tilde{p} \rangle, \\ \frac{1}{2} \langle \nabla (e_{p_h}^{n+\frac{1}{2}} + e_{p_h}^{n-\frac{1}{2}}), \tilde{E} \rangle - \frac{\Delta t^2}{24} \langle \nabla (e_{p_h}^{n+\frac{1}{2}} + e_{p_h}^{n-\frac{1}{2}}), \nabla \nabla \cdot \tilde{E} \rangle + \frac{\Delta t^4}{120} \langle \nabla (e_{p_h}^{n+\frac{1}{2}} + e_{p_h}^{n-\frac{1}{2}}), (\nabla \nabla \cdot)^2 \tilde{E} \rangle \\ + \langle \varepsilon \frac{e_{E_h}^{n+\frac{1}{2}} - e_{E_h}^{n-\frac{1}{2}}}{\Delta t}, \tilde{E} \rangle - \frac{1}{2} \langle e_{H_h}^{n+1} + e_{H_h}^n, \nabla \times \tilde{E} \rangle - \frac{\Delta t^2}{24} \langle \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times (e_{H_h}^{n+1} + e_{H_h}^n), \nabla \times \tilde{E} \rangle \\ - \frac{\Delta t^4}{120} \langle \mu^{-2} \varepsilon^{-2} (\nabla \times \nabla \times)^2 (e_{H_h}^{n+1} + e_{H_h}^n), \nabla \times \tilde{E} \rangle = \langle \varepsilon R_E^n + \nabla r_{3,p}^n - \nabla \times r_{3,H}^{n+\frac{1}{2}} - \frac{\Delta t^2}{12} \nabla \nabla \cdot \nabla r_{2,p}^n \\ - \frac{\Delta t^2}{12} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \nabla \times r_{2,H}^{n+\frac{1}{2}} + \frac{\Delta t^4}{120} (\nabla \nabla \cdot)^2 \nabla r_{1,p}^n - \frac{\Delta t^4}{120} \mu^{-2} \varepsilon^{-2} \nabla \times (\nabla \times \nabla \times)^2 r_{1,H}^{n+\frac{1}{2}}, \tilde{E} \rangle, \end{aligned}$$

$$\begin{aligned}
& \left\langle \mu \frac{e_{H_h}^{n+1} - e_{H_h}^n}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \left\langle \nabla \times \left(e_{E_h}^{n+\frac{1}{2}} + e_{E_h}^{n-\frac{1}{2}} \right), \tilde{H} \right\rangle + \frac{\Delta t^2}{24} \left\langle \varepsilon^{-1} \mu^{-1} \nabla \times \left(e_{E_h}^{n+\frac{1}{2}} + e_{E_h}^{n-\frac{1}{2}} \right), \nabla \times \nabla \times \tilde{H} \right\rangle \\
& + \frac{\Delta t^4}{120} \left\langle \varepsilon^{-2} \mu^{-2} \nabla \times \left(e_{E_h}^{n+\frac{1}{2}} + e_{E_h}^{n-\frac{1}{2}} \right), (\nabla \times \nabla \times)^2 \tilde{H} \right\rangle = \left\langle \mu R_H^{n+\frac{1}{2}} + \nabla \times r_{3,E}^n + \frac{\Delta t^2}{12} \varepsilon^{-1} \mu^{-1} \nabla \times \nabla \times \nabla \times r_{2,E}^n \right. \\
& \quad \left. + \frac{\Delta t^4}{120} \varepsilon^{-2} \mu^{-2} (\nabla \times \nabla \times)^2 \nabla \times r_{1,E}^n, \tilde{H} \right\rangle.
\end{aligned}$$

Next, using the values of the error terms $e_{p_h}^n$, $e_{E_h}^n$ and $e_{H_h}^n$ as in Equations (5.10) to (5.12) in the above equations, we get:

$$\begin{aligned}
& \left\langle \frac{\left(\eta^{n+\frac{1}{2}} - \eta^{n-\frac{1}{2}} \right) - \left(\eta_h^{n+\frac{1}{2}} - \eta_h^{n-\frac{1}{2}} \right)}{\Delta t}, \tilde{p} \right\rangle - \frac{1}{2} \left\langle \varepsilon \left(\left(\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}} \right) - \left(\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \right) \right), \nabla \tilde{p} \right\rangle \\
& + \frac{\Delta t^2}{24} \left\langle \varepsilon \nabla \nabla \cdot \left(\left(\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}} \right) - \left(\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \right) \right), \nabla \tilde{p} \right\rangle \\
& - \frac{\Delta t^4}{240} \left\langle \varepsilon (\nabla \nabla \cdot)^2 \left(\left(\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}} \right) - \left(\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \right) \right), \nabla \tilde{p} \right\rangle \\
& = \left\langle R_p^n + \varepsilon \nabla \cdot r_{3,E}^n - \frac{\Delta t^2}{12} \varepsilon \nabla \cdot \nabla \nabla \cdot r_{2,E}^n + \frac{\Delta t^4}{120} \varepsilon \nabla \cdot (\nabla \nabla \cdot)^2 r_{1,E}^n, \tilde{p} \right\rangle,
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2} \left\langle \nabla \left(\left(\eta^{n+\frac{1}{2}} + \eta^{n-\frac{1}{2}} \right) - \left(\eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \right) \right), \tilde{E} \right\rangle - \frac{\Delta t^2}{24} \left\langle \nabla \left(\left(\eta^{n+\frac{1}{2}} + \eta^{n-\frac{1}{2}} \right) - \left(\eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \right) \right), \nabla \nabla \cdot \tilde{E} \right\rangle \\
& + \frac{\Delta t^4}{240} \left\langle \nabla \left(\left(\eta^{n+\frac{1}{2}} + \eta^{n-\frac{1}{2}} \right) - \left(\eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \right) \right), (\nabla \nabla \cdot)^2 \tilde{E} \right\rangle + \left\langle \varepsilon \frac{\left(\zeta^{n+\frac{1}{2}} - \zeta^{n-\frac{1}{2}} \right) - \left(\zeta_h^{n+\frac{1}{2}} - \zeta_h^{n-\frac{1}{2}} \right)}{\Delta t}, \tilde{E} \right\rangle \\
& - \frac{1}{2} \left\langle \left(\left(\xi^{n+1} + \xi^n \right) - \left(\xi_h^{n+1} + \xi_h^n \right) \right), \nabla \times \tilde{E} \right\rangle - \frac{\Delta t^2}{24} \left\langle \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \left(\left(\xi^{n+1} + \xi^n \right) - \left(\xi_h^{n+1} + \xi_h^n \right) \right), \nabla \times \tilde{E} \right\rangle \\
& - \frac{\Delta t^4}{240} \left\langle \mu^{-2} \varepsilon^{-2} (\nabla \times \nabla \times)^2 \left(\left(\xi^{n+1} + \xi^n \right) - \left(\xi_h^{n+1} + \xi_h^n \right) \right), \nabla \times \tilde{E} \right\rangle = \left\langle \varepsilon R_E^n + \nabla r_{3,p}^n - \nabla \times r_{3,H}^{n+\frac{1}{2}} \right. \\
& \quad \left. - \frac{\Delta t^2}{12} \nabla \nabla \cdot \nabla r_{2,p}^n - \frac{\Delta t^2}{12} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \nabla \times r_{2,H}^{n+\frac{1}{2}} + \frac{\Delta t^4}{120} (\nabla \nabla \cdot)^2 \nabla r_{1,p}^n - \frac{\Delta t^4}{120} \mu^{-2} \varepsilon^{-2} \nabla \times (\nabla \times \nabla \times)^2 r_{1,H}^{n+\frac{1}{2}}, \tilde{E} \right\rangle,
\end{aligned}$$

$$\begin{aligned}
& \left\langle \mu \frac{\left(\xi^{n+1} - \xi^n \right) - \left(\xi_h^{n+1} - \xi_h^n \right)}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \left\langle \nabla \times \left(\left(\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}} \right) - \left(\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \right) \right), \tilde{H} \right\rangle \\
& + \frac{\Delta t^2}{24} \left\langle \varepsilon^{-1} \mu^{-1} \nabla \times \left(\left(\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}} \right) - \left(\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \right) \right), \nabla \times \nabla \times \tilde{H} \right\rangle \\
& + \frac{\Delta t^4}{240} \left\langle \varepsilon^{-2} \mu^{-2} \nabla \times \left(\left(\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}} \right) - \left(\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \right) \right), (\nabla \times \nabla \times)^2 \tilde{H} \right\rangle \\
& = \left\langle \mu R_H^{n+\frac{1}{2}} + \nabla \times r_{3,E}^n + \frac{\Delta t^2}{12} \varepsilon^{-1} \mu^{-1} \nabla \times \nabla \times \nabla \times r_{2,E}^n + \frac{\Delta t^4}{120} \varepsilon^{-2} \mu^{-2} (\nabla \times \nabla \times)^2 \nabla \times r_{1,E}^n, \tilde{H} \right\rangle.
\end{aligned}$$

Since these equations are true for all $(\tilde{p}, \tilde{E}, \tilde{H}) \in U_h \times V_h \times W_h$, we choose $\tilde{p} = -2\Delta t \varepsilon^{-1} \left(\eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \right)$, $\tilde{E} = -2\Delta t \left(\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \right)$ and $\tilde{H} = -2\Delta t \left(\xi_h^n + \xi_h^{n-1} \right)$ and using the fact that $\nabla U_h \subseteq V_h$ and $\nabla \times V_h \subseteq W_h$, we obtain:

$$2 \left\langle \varepsilon^{-1} \left(\eta_h^{n+\frac{1}{2}} - \eta_h^{n-\frac{1}{2}} \right), \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \right\rangle + 2 \left\langle \varepsilon \left(\zeta_h^{n+\frac{1}{2}} - \zeta_h^{n-\frac{1}{2}} \right), \zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \right\rangle + 2 \left\langle \mu \left(\xi_h^n - \xi_h^{n-1} \right), \xi_h^n + \xi_h^{n-1} \right\rangle$$

$$\begin{aligned}
&= 2\langle \varepsilon^{-1} (\eta^{n+\frac{1}{2}} - \eta^{n-\frac{1}{2}}), \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle + 2\langle \varepsilon (\zeta^{n+\frac{1}{2}} - \zeta^{n-\frac{1}{2}}), \zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \rangle + 2\langle \mu (\xi^n - \xi^{n-1}), \xi_h^n + \xi_h^{n-1} \rangle \\
&+ 2\Delta t \langle -R_p^n - \varepsilon \nabla \cdot r_{3,E}^n + \frac{\Delta t^2}{12} \varepsilon \nabla \cdot \nabla \nabla \cdot r_{2,E}^n - \frac{\Delta t^4}{120} \varepsilon \nabla \cdot (\nabla \nabla \cdot)^2 r_{1,E}^n, \varepsilon^{-1} (\eta_h^{n+\frac{1}{2}} - \eta_h^{n-\frac{1}{2}}) \rangle + 2\Delta t \langle -\varepsilon R_E^n - \nabla r_{3,p}^n \\
&\quad + \nabla \times r_{3,H}^{n+\frac{1}{2}} + \frac{\Delta t^2}{12} \nabla \nabla \cdot \nabla r_{2,p}^n + \frac{\Delta t^2}{12} \mu^{-1} \varepsilon^{-1} \nabla \times \nabla \times \nabla \times r_{2,H}^{n+\frac{1}{2}} - \frac{\Delta t^4}{120} (\nabla \nabla \cdot)^2 \nabla r_{1,p}^n \\
&\quad + \frac{\Delta t^4}{120} \mu^{-2} \varepsilon^{-2} \nabla \times (\nabla \times \nabla \times)^2 r_{1,H}^{n+\frac{1}{2}}, \zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \rangle + 2\Delta t \langle -\mu R_H^{n+\frac{1}{2}} - \nabla \times r_{3,E}^n \\
&\quad - \frac{\Delta t^2}{12} \varepsilon^{-1} \mu^{-1} \nabla \times \nabla \times \nabla \times r_{2,E}^n - \frac{\Delta t^4}{120} \varepsilon^{-2} \mu^{-2} (\nabla \times \nabla \times)^2 \nabla \times r_{1,E}^n, \xi_h^n + \xi_h^{n-1} \rangle, \quad (5.18)
\end{aligned}$$

Consider that $\varepsilon^{-1} (\eta^{n+\frac{1}{2}} - \eta^{n-\frac{1}{2}}) = \varepsilon^{-1} (I - \Pi_h^0) (p(t^{n+\frac{1}{2}}) - p(t^{n-\frac{1}{2}}))$ by Equation (5.13). Using the Taylor theorem with remainder as in Theorem 5.2, applying the Cauchy-Schwarz, AM-GM, and Triangle inequalities, we have the following resulting inequality:

$$\begin{aligned}
2\langle \varepsilon^{-1} (\eta^{n+\frac{1}{2}} - \eta^{n-\frac{1}{2}}), \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle &= 2\Delta t \langle \varepsilon^{-1} (I - \Pi_h^0) \\
&\quad \left(\frac{\partial p}{\partial t}(t^n) + \frac{\Delta t^2}{24} \frac{\partial^3 p}{\partial t^3}(t^n) + \frac{\Delta t^4}{1920} \frac{\partial^5 p}{\partial t^5}(t^n) \right), \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle \\
&+ 2\Delta t \langle \varepsilon^{-1} (I - \Pi_h^0) R_p^n, \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle \\
&\leq \Delta t \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 \right. \\
&\quad \left. + \frac{\Delta t^8}{(1920)^2} \left\| (I - \Pi_h^0) \frac{\partial^5 p}{\partial t^5}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 \right] \\
&+ 8\Delta t \left[\|\eta_h^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\eta_h^{n-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 \right].
\end{aligned}$$

Similarly, using Equations (5.14) and (5.15) for the error terms for E and H, we obtain:

$$\begin{aligned}
2\langle \varepsilon (\zeta^{n+\frac{1}{2}} - \zeta^{n-\frac{1}{2}}), \zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \rangle &\leq \Delta t \left[\left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 \right. \\
&\quad \left. + \frac{\Delta t^8}{(1920)^2} \left\| (I - \Pi_h^1) \frac{\partial^5 E}{\partial t^5}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 \right] \\
&\quad + 8\Delta t \left[\|\zeta_h^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|\zeta_h^{n-\frac{1}{2}}\|_{\varepsilon}^2 \right], \\
2\langle \mu (\xi^{n+1} - \xi^n), \xi_h^{n+1} + \xi_h^n \rangle &\leq \Delta t \left[\left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right. \\
&\quad \left. + \frac{\Delta t^8}{(1920)^2} \left\| (I - \Pi_h^2) \frac{\partial^5 H}{\partial t^5}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 + \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu}^2 \right] \\
&\quad + 8\Delta t \left[\|\xi_h^{n+1}\|_{\mu}^2 + \|\xi_h^n\|_{\mu}^2 \right].
\end{aligned}$$

Using these inequalities for η , ζ and ξ in Equation (5.18), we thus obtain the following estimate:

$$\begin{aligned}
& 2 \left[\|\eta_h^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|\eta_h^{n-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{n+\frac{1}{2}}\|_{\varepsilon}^2 - \|\zeta_h^{n-\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^{n+1}\|_{\mu}^2 - \|\xi_h^n\|_{\mu}^2 \right] \leq \Delta t \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 \right. \\
& + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \frac{\Delta t^8}{(1920)^2} \left\| (I - \Pi_h^0) \frac{\partial^5 p}{\partial t^5}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 \\
& + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 + \frac{\Delta t^8}{(1920)^2} \left\| (I - \Pi_h^1) \frac{\partial^5 E}{\partial t^5}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \\
& + \frac{\Delta t^4}{(24)^2} \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 + \frac{\Delta t^8}{(1920)^2} \left\| (I - \Pi_h^2) \frac{\partial^5 H}{\partial t^5}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 + \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu}^2 + \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 \\
& + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 + \|\nabla r_{3,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{3,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{3,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{3,H}^{n+\frac{1}{2}}\|_{\mu}^2 + \frac{\Delta t^4}{144} \left(\|\nabla \nabla \cdot \nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 \right. \\
& + \|\nabla \cdot \nabla \nabla \cdot r_{2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{2,H}^{n+\frac{1}{2}}\|_{\mu}^2 \Big) \\
& + \frac{\Delta t^8}{14400} \left(\|(\nabla \nabla \cdot)^2 \nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot (\nabla \nabla \cdot)^2 r_{1,E}^n\|_{\varepsilon}^2 + \varepsilon^{-5} \mu^{-5} \|(\nabla \times \nabla \times)^2 \nabla \times r_{1,E}^n\|_{\varepsilon}^2 \right. \\
& \left. + \varepsilon^{-5} \mu^{-5} \|\nabla \times (\nabla \times \nabla \times)^2 r_{1,H}^{n+\frac{1}{2}}\|_{\mu}^2 \right) + 10 (\|\eta_h^n\|_{\varepsilon^{-1}}^2 + \|\eta_h^{n-1}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^n\|_{\varepsilon}^2 + \|\zeta_h^{n-1}\|_{\varepsilon}^2 + \|\xi_h^n\|_{\mu}^2 + \|\xi_h^{n-1}\|_{\mu}^2) \Big].
\end{aligned}$$

Then following the same steps for the initial time step and summing up from $n = 0$ to $N - 1$ further with the discrete Gronwall inequality with $\Delta t < 1/30$, we get that:

$$\begin{aligned}
& \|\eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^N\|_{\mu}^2 \leq \left[\frac{6\Delta t}{5} \sum_{n=0}^{N-1} \left(\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 \right. \right. \\
& + \left\| (I - \Pi_h^0) \frac{\partial^5 p}{\partial t^5}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 \\
& + \left\| (I - \Pi_h^1) \frac{\partial^5 E}{\partial t^5}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 + \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \\
& + \left\| (I - \Pi_h^2) \frac{\partial^5 H}{\partial t^5}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 + \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu}^2 + \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 + \|\nabla r_{3,p}^n\|_{\varepsilon^{-1}}^2 \\
& + \|\nabla \cdot r_{3,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{3,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{3,H}^{n+\frac{1}{2}}\|_{\mu}^2 + \frac{\Delta t^4}{144} \left(\|\nabla \nabla \cdot \nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot \nabla \nabla \cdot r_{2,E}^n\|_{\varepsilon}^2 \right. \\
& + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{2,H}^{n+\frac{1}{2}}\|_{\mu}^2 \Big) + \frac{\Delta t^8}{14400} \left(\|(\nabla \nabla \cdot)^2 \nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 \right. \\
& + \|\nabla \cdot (\nabla \nabla \cdot)^2 r_{1,E}^n\|_{\varepsilon}^2 + \varepsilon^{-5} \mu^{-5} \|(\nabla \times \nabla \times)^2 \nabla \times r_{1,E}^n\|_{\varepsilon}^2 + \varepsilon^{-5} \mu^{-5} \|\nabla \times (\nabla \times \nabla \times)^2 r_{1,H}^{n+\frac{1}{2}}\|_{\mu}^2 \Big) \Big] + \\
& \left. \frac{7}{5} \left(\|\eta_h^0\|_{\varepsilon^{-1}}^2 + \|\zeta_h^0\|_{\varepsilon}^2 + \|\xi_h^0\|_{\mu}^2 \right) \right] \exp(20T).
\end{aligned}$$

Using our estimates for the Taylor remainders from Theorem 5.2 for these terms on the right hand side of the above inequality, we further get that:

$$\Delta t \sum_{n=0}^N \left[\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^n\|_{\mu}^2 \right] \leq (\Delta t)^{12} \left[\left\| \frac{\partial^7 p}{\partial t^7} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^7 E}{\partial t^7} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial^7 H}{\partial t^7} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right],$$

and that:

$$\begin{aligned}
\Delta t \sum_{n=0}^N & \left[\|\nabla r_{3,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{3,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{3,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{3,H}^{n+\frac{1}{2}}\|_{\mu}^2 + \frac{\Delta t^4}{144} \left(\|\nabla \nabla \cdot \nabla r_{2,p}^n\|_{\varepsilon^{-1}}^2 \right. \right. \\
& + \|\nabla \cdot \nabla \nabla \cdot r_{2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-3} \mu^{-3} \|\nabla \times \nabla \times \nabla \times r_{2,H}^{n+\frac{1}{2}}\|_{\mu}^2 \Big) \\
& + \frac{\Delta t^8}{14400} \left(\|(\nabla \nabla \cdot)^2 \nabla r_{1,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot (\nabla \nabla \cdot)^2 r_{1,E}^n\|_{\varepsilon}^2 + \varepsilon^{-5} \mu^{-5} \|(\nabla \times \nabla \times)^2 \nabla \times r_{1,E}^n\|_{\varepsilon}^2 \right. \\
& + \varepsilon^{-5} \mu^{-5} \|\nabla \times (\nabla \times \nabla \times)^2 r_{1,H}^{n+\frac{1}{2}}\|_{\mu}^2 \Big) \Big] \leq (\Delta t)^{12} \left[\left\| \frac{\partial^6 (\nabla p)}{\partial t^6} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^6 (\nabla \cdot E)}{\partial t^6} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 \right. \\
& + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^6 (\nabla \times E)}{\partial t^6} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^6 (\nabla \times H)}{\partial t^6} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 + \left\| \frac{\partial^4 (\nabla \nabla \cdot \nabla p)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 \\
& + \left\| \frac{\partial^4 (\nabla \cdot \nabla \nabla \cdot E)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \varepsilon^{-3} \mu^{-3} \left\| \frac{\partial^4 (\nabla \times \nabla \times \nabla \times E)}{\partial t^4} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 \\
& + \varepsilon^{-3} \mu^{-3} \left\| \frac{\partial^4 (\nabla \times \nabla \times \nabla \times H)}{\partial t^4} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 + \left\| \frac{\partial^2 ((\nabla \nabla \cdot)^2 \nabla p)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 \\
& + \left\| \frac{\partial^2 (\nabla \cdot (\nabla \nabla \cdot)^2 E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \varepsilon^{-5} \mu^{-5} \left\| \frac{\partial^2 ((\nabla \times \nabla \times)^2 \nabla \times E)}{\partial t^2} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 \\
& + \varepsilon^{-5} \mu^{-5} \left\| \frac{\partial^2 (\nabla \times (\nabla \times \nabla \times)^2 H)}{\partial t^2} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \Big].
\end{aligned}$$

Next, for $p \in \dot{H}_{\varepsilon^{-1}}^1(\Omega)$, $E \in \dot{H}_{\varepsilon}(\text{curl}; \Omega)$ and $H \in \dot{H}_{\mu}(\text{div}; \Omega)$, there exists positive bounded constants $C_{1,p}$, $C_{2,p}$, $C_{3,p}$, $C_{4,p}$, $C_{1,E}$, $C_{2,E}$, $C_{3,E}$, $C_{4,E}$, $C_{1,H}$, $C_{2,H}$, $C_{3,H}$, and $C_{4,H}$ such that we have the following error bounds for the L^2 projections:

$$\begin{aligned}
\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}} & \leq C_{1,p} h^r \left\| \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}, \quad \left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}} \leq C_{2,p} h^r \left\| \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}, \\
\left\| (I - \Pi_h^0) \frac{\partial^5 p}{\partial t^5}(t^n) \right\|_{\varepsilon^{-1}} & \leq C_{3,p} h^r \left\| \frac{\partial^5 p}{\partial t^5}(t^n) \right\|_{\varepsilon^{-1}}, \quad \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}} \leq C_{4,p} h^r \left\| R_p^n \right\|_{\varepsilon^{-1}}, \\
\left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon} & \leq C_{1,E} h^r \left\| \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}, \quad \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon} \leq C_{2,E} h^r \left\| \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}, \\
\left\| (I - \Pi_h^1) \frac{\partial^5 E}{\partial t^5}(t^n) \right\|_{\varepsilon} & \leq C_{3,E} h^r \left\| \frac{\partial^5 E}{\partial t^5}(t^n) \right\|_{\varepsilon}, \quad \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon} \leq C_{4,E} h^r \left\| R_E^n \right\|_{\varepsilon}, \\
\left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}) \right\|_{\mu} & \leq C_{1,H} h^r \left\| \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}) \right\|_{\mu}, \quad \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}) \right\|_{\mu} \leq C_{2,H} h^r \left\| \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}) \right\|_{\mu}, \\
\left\| (I - \Pi_h^2) \frac{\partial^5 H}{\partial t^5}(t^{n+\frac{1}{2}}) \right\|_{\mu} & \leq C_{3,H} h^r \left\| \frac{\partial^5 H}{\partial t^5}(t^{n+\frac{1}{2}}) \right\|_{\mu}, \quad \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu} \leq C_{4,H} h^r \left\| R_H^{n+\frac{1}{2}} \right\|_{\mu}.
\end{aligned}$$

Set $C_0 := \max\{C_{1,p}, C_{2,p}, C_{3,p}, C_{4,p}, C_{1,E}, C_{2,E}, C_{3,E}, C_{4,E}, C_{1,H}, C_{2,H}, C_{3,H}, C_{4,H}\}$. We therefore have that:

$$\begin{aligned}
& \Delta t \sum_{n=0}^{N-1} \left[\left\| (I - \Pi_h^0) \frac{\partial p}{\partial t}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial E}{\partial t}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial H}{\partial t}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right] \\
& \leq C_0 h^{2r} \sum_{n=0}^{N-1} \Delta t \left[\left\| \frac{\partial p}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial E}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\varepsilon}^2 + \left\| \frac{\partial H}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\mu}^2 \right] \\
& \leq C_0 h^{2r} \int_0^T \left[\left\| \frac{\partial p}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial E}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\varepsilon}^2 + \left\| \frac{\partial H}{\partial t}(t^{n-\frac{1}{2}}) \right\|_{\mu}^2 \right] dt \\
& = C_0 h^{2r} \left[\left\| \frac{\partial p}{\partial t} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial E}{\partial t} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial H}{\partial t} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right],
\end{aligned}$$

$$\begin{aligned}
& \Delta t \sum_{n=0}^{N-1} \left[\left\| (I - \Pi_h^0) \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right] \\
& \leq C_0 h^{2r} \sum_{n=0}^{N-1} \Delta t \left[\left\| \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 + \left\| \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right] \\
& \leq C_0 h^{2r} \int_0^T \left[\left\| \frac{\partial^3 p}{\partial t^3}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial^3 E}{\partial t^3}(t^n) \right\|_{\varepsilon}^2 + \left\| \frac{\partial^3 H}{\partial t^3}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right] dt \\
& = C_0 h^{2r} \left[\left\| \frac{\partial^3 p}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^3 E}{\partial t^3} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial^3 H}{\partial t^3} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right],
\end{aligned}$$

$$\begin{aligned}
& \Delta t \sum_{n=0}^{N-1} \left[\left\| (I - \Pi_h^0) \frac{\partial^5 p}{\partial t^5}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial^5 E}{\partial t^5}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial^5 H}{\partial t^5}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right] \\
& \leq C_0 h^{2r} \sum_{n=0}^{N-1} \Delta t \left[\left\| \frac{\partial^5 p}{\partial t^5}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial^5 E}{\partial t^5}(t^n) \right\|_{\varepsilon}^2 + \left\| \frac{\partial^5 H}{\partial t^5}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right] \\
& \leq C_0 h^{2r} \int_0^T \left[\left\| \frac{\partial^5 p}{\partial t^5}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial^5 E}{\partial t^5}(t^n) \right\|_{\varepsilon}^2 + \left\| \frac{\partial^5 H}{\partial t^5}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right] dt \\
& = C_0 h^{2r} \left[\left\| \frac{\partial^5 p}{\partial t^5} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^5 E}{\partial t^5} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 + \left\| \frac{\partial^5 H}{\partial t^5} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right],
\end{aligned}$$

and likewise for the L^2 projection of the remainder terms:

$$\begin{aligned}
& \Delta t \sum_{n=0}^N \left[\left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}} + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon} + \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu} \right], \\
& \leq C_0 h^{2r} \sum_{n=0}^N \Delta t \left[\|R_p^n\|_{L_{\varepsilon^{-1}}^2(\Omega)} + \|R_E^n\|_{L_{\varepsilon}^2(\Omega)} + \|R_H^{n+\frac{1}{2}}\|_{L_{\mu}^2(\Omega)} \right],
\end{aligned}$$

$$\leq C_0 h^{2r} (\Delta t)^8 \left[\left\| \frac{\partial^7 p}{\partial t^7} \right\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + \left\| \frac{\partial^7 E}{\partial t^7} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + \left\| \frac{\partial^7 H}{\partial t^7} \right\|_{L^2(0,T;L^2_{\mu}(\Omega))}^2 \right].$$

Now, we take $p_h^0 = \Pi_h^0 p_0$, $E_h^0 = \Pi_h^1 E_0$, and $H_h^0 = \Pi_h^2 H_0$, and note that there exists positive bounded constants M_1, M_2, M_3, M_4 and M_5 such that:

$$\begin{aligned} & \left\| \frac{\partial p}{\partial t} \right\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + \left\| \frac{\partial E}{\partial t} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + \left\| \frac{\partial H}{\partial t} \right\|_{L^2(0,T;L^2_{\mu}(\Omega))}^2 \leq M_1, \\ & \left\| \frac{\partial^3 p}{\partial t^3} \right\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + \left\| \frac{\partial^3 E}{\partial t^3} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + \left\| \frac{\partial^3 H}{\partial t^3} \right\|_{L^2(0,T;L^2_{\mu}(\Omega))}^2 \leq M_2, \\ & \left\| \frac{\partial^5 p}{\partial t^5} \right\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + \left\| \frac{\partial^5 E}{\partial t^5} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + \left\| \frac{\partial^5 H}{\partial t^5} \right\|_{L^2(0,T;L^2_{\mu}(\Omega))}^2 \leq M_3, \\ & \left\| \frac{\partial^7 p}{\partial t^7} \right\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + \left\| \frac{\partial^7 E}{\partial t^7} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + \left\| \frac{\partial^7 H}{\partial t^7} \right\|_{L^2(0,T;L^2_{\mu}(\Omega))}^2 \leq M_4, \\ & \left\| \frac{\partial^4 (\nabla p)}{\partial t^4} \right\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + \left\| \frac{\partial^4 (\nabla \cdot E)}{\partial t^4} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^4 (\nabla \times E)}{\partial t^4} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 \\ & + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^4 (\nabla \times H)}{\partial t^4} \right\|_{L^2(0,T;L^2_{\mu}(\Omega))}^2 + \left\| \frac{\partial^2 (\nabla \nabla \cdot \nabla p)}{\partial t^2} \right\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + \left\| \frac{\partial^2 (\nabla \cdot \nabla \nabla \cdot E)}{\partial t^2} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 \\ & + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^2 (\nabla \times \nabla \times \nabla \times E)}{\partial t^2} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + \varepsilon^{-1} \mu^{-1} \left\| \frac{\partial^2 (\nabla \times \nabla \times \nabla \times H)}{\partial t^2} \right\|_{L^2(0,T;L^2_{\mu}(\Omega))}^2 \leq M_5. \end{aligned}$$

Consequently, we have the following estimate:

$$\|\eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^N\|_{\mu}^2 \leq \widetilde{C} [h^{2r} + h^{2r}(\Delta t)^{12} + (\Delta t)^{12}],$$

where $\widetilde{C} = 2 \max\{C_0 M_1, C_0 M_2, C_0 M_3, C_0 M_4, M_4, M_5\} \exp(20T)$, which then with the equivalence between 1- and 2-norms gives us that:

$$\|\eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|\zeta_h^{N-\frac{1}{2}}\|_{\varepsilon} + \|\xi_h^N\|_{\mu} \leq C_1 [h^r + h^r(\Delta t)^6 + (\Delta t)^6],$$

where $C_1 = 2\sqrt{\widetilde{C}}$. Also, there are positive bounded constants $\widetilde{C}_2, \widetilde{C}_3$ and $\widetilde{C}_4$ such that:

$$\begin{aligned} \|\eta^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} &= \|(I - \Pi_h^0)p(t^{N-\frac{1}{2}})\|_{\varepsilon^{-1}} \leq \widetilde{C}_2 h^r \|p(t^{N-\frac{1}{2}})\|_{L^2_{\varepsilon^{-1}}(\Omega)} \implies \|\eta^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} \leq C_2 h^r, \\ \|\zeta^{N-\frac{1}{2}}\|_{\varepsilon} &= \|(I - \Pi_h^1)E(t^{N-\frac{1}{2}})\|_{\varepsilon} \leq \widetilde{C}_3 h^r \|E(t^{N-\frac{1}{2}})\|_{L^2_{\varepsilon}(\Omega)} \implies \|\zeta^{N-\frac{1}{2}}\|_{\varepsilon} \leq C_3 h^r, \\ \|\xi^N\|_{\mu} &= \|(I - \Pi_h^2)H(t^N)\|_{\mu} \leq \widetilde{C}_4 h^r \|H(t^N)\|_{L^2_{\mu}(\Omega)} \implies \|\xi^N\|_{\mu} \leq C_4 h^r, \end{aligned}$$

in which $C_2 = \widetilde{C}_2 \|p(t^{N-\frac{1}{2}})\|_{L^2_{\varepsilon^{-1}}(\Omega)}$, $C_3 = \widetilde{C}_3 \|E(t^{N-\frac{1}{2}})\|_{L^2_{\varepsilon}(\Omega)}$ and $C_4 = \widetilde{C}_4 \|H(t^N)\|_{L^2_{\mu}(\Omega)}$ are all bounded positive constants due to Theorem 5.1. Finally, this provides us with our desired result by choosing $C = C_1 + C_2 + C_3 + C_4$:

$$\|e_{p_h}^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|e_{E_h}^{N-\frac{1}{2}}\|_{\varepsilon} + \|e_{H_h}^N\|_{\mu} = \|\eta^{N-\frac{1}{2}} - \eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|\zeta^{N-\frac{1}{2}} - \zeta_h^{N-\frac{1}{2}}\|_{\varepsilon} + \|\xi^N - \xi_h^N\|_{\mu}$$

$$\leq C [(\Delta t)^6 + h^r + h^r(\Delta t)^6].$$

□

5.3 Characterization of Implicit LF_R Scheme

5.3.1 Time Discretization Stability

Theorem 5.4 (Discrete Energy Estimate). *For the semidiscretization using the LF_R scheme as given in Equations (5.7a) to (5.7c) and (5.8a) to (5.8c), and for any fixed time step $\Delta t > 0$ sufficiently small, we have that:*

$$\mathcal{E}^N := \|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 = \|p_0\|_{\varepsilon^{-1}}^2 + \|E_0\|_{\varepsilon}^2 + \|H_0\|_{\mu}^2 =: \mathcal{E}^0.$$

Proof. Since Equations (5.7a) to (5.7c) are true for all $\tilde{p} \in \dot{H}_{\varepsilon^{-1}}^1(\Omega)$, $\tilde{E} \in \dot{H}_{\varepsilon}(\text{curl}; \Omega)$, $\tilde{H} \in \dot{H}_{\mu}(\text{div}; \Omega)$, using $\tilde{p} = 2\Delta t \varepsilon^{-1} (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}})$, $\tilde{E} = 2\Delta t (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}})$ and $\tilde{H} = 2\Delta t (H^{n+1} + H^n)$, we obtain:

$$\begin{aligned} & 2\langle p^{n+\frac{1}{2}} - p^{n-\frac{1}{2}}, \varepsilon^{-1} (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}) \rangle - \Delta t \langle E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}, \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}) \rangle \\ & + \Delta t \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \langle (\nabla \nabla \cdot) \sum_{m=1}^a k_m (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}) \rangle = 0, \\ & \Delta t \langle \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \rangle \\ & - \Delta t \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \langle \nabla (p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}), (\nabla \nabla \cdot) \sum_{m=1}^a k_m (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}) \rangle \\ & + 2\langle \varepsilon (E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}), E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \rangle - \Delta t \langle (H^{n+1} + H^n), \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}) \rangle \\ & + \Delta t \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1) \sum_{m=1}^a k_m \langle (\mu \varepsilon)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times) \sum_{m=1}^a k_m (H^{n+1} + H^n), \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}) \rangle \\ & = 0, \end{aligned}$$

$$\begin{aligned} & 2\langle \mu (H^{n+1} - H^n), H^{n+1} + H^n \rangle + \Delta t \langle \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), H^{n+1} + H^n \rangle \\ & - \Delta t \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1) \sum_{m=1}^a k_m \langle (\varepsilon \mu)^{-\sum_{m=1}^a k_m} \nabla \times (E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}}), (\nabla \times \nabla \times) \sum_{m=1}^a k_m (H^{n+1} + H^n) \rangle \\ & = 0. \end{aligned}$$

Using the symmetry of the inner product and adding the above equations, we next obtain:

$$\begin{aligned} 2 \left[\left\langle \varepsilon^{-1} \left(p^{n+\frac{1}{2}} - p^{n-\frac{1}{2}} \right), p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}} \right\rangle + \left\langle \varepsilon \left(E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}} \right), E^{n+\frac{1}{2}} + E^{n-\frac{1}{2}} \right\rangle \right. \\ \left. + \left\langle \mu \left(H^{n+1} - H^n \right), H^{n+1} + H^n \right\rangle \right] = 0. \\ \implies \|p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|p^{n-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{n+\frac{1}{2}}\|_{\varepsilon}^2 - \|E^{n-\frac{1}{2}}\|_{\varepsilon}^2 + \|H^{n+1}\|_{\mu}^2 - \|H^n\|_{\mu}^2 = 0. \end{aligned}$$

Summing over $n = 1$ to $N - 1$ results into:

$$\|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 - \|E^{\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 - \|H^1\|_{\mu}^2 = 0.$$

Now, $p^{\frac{1}{2}}$, $E^{\frac{1}{2}}$ and H^1 satisfy Equations (5.8a) to (5.8c), and so using $\tilde{p} = 2\Delta t \varepsilon^{-1} (p^{\frac{1}{2}} + p^0)$, $\tilde{E} = 2\Delta t (E^{\frac{1}{2}} + E^0)$ and $\tilde{H} = 4\Delta t (H^1 + H^0)$ and repeating the arguments leads us to the following estimate:

$$\|p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|p^0\|_{\varepsilon^{-1}}^2 + \|E^{\frac{1}{2}}\|_{\varepsilon}^2 - \|E^0\|_{\varepsilon}^2 + \|H^1\|_{\mu}^2 - \|H^0\|_{\mu}^2 = 0,$$

from which we then get that:

$$\|p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|H^N\|_{\mu}^2 = \|p^0\|_{\varepsilon^{-1}}^2 + \|E^0\|_{\varepsilon}^2 + \|H^0\|_{\mu}^2.$$

□

Theorem 5.5 (Discrete Error Estimate). *For the semidiscretization using the LF_R scheme as in Equations (5.7a) to (5.7c), and (5.8a) to (5.8c), for the solution (p, E, H) of Equations (5.2a) to (5.2c) with initial conditions as in Equation (5.1c), assuming sufficient regularity with $p \in C^{R+1}(0, T; \dot{H}_{\varepsilon^{-1}}^1(\Omega))$, $E \in C^{R+1}(0, T; \dot{H}_{\varepsilon}(\text{curl}; \Omega))$, and $H \in C^{R+1}(0, T; \dot{H}_{\mu}(\text{div}; \Omega))$, and for a time step $\Delta t > 0$ sufficiently small, there exists a positive bounded constant C independent of Δt such that:*

$$\|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon} + \|e_H^N\|_{\mu} \leq C \left[(\Delta t)^R + \|e_p^0\|_{\varepsilon^{-1}} + \|e_E^0\|_{\varepsilon} + \|e_H^0\|_{\mu} \right],$$

where $e_p^{n+\frac{1}{2}} := p(t^{n+\frac{1}{2}}) - p^{n+\frac{1}{2}}$, $e_E^{n+\frac{1}{2}} := E(t^{n+\frac{1}{2}}) - E^{n+\frac{1}{2}}$ and $e_H^n := H(t^n) - H^n$ are the errors in the time semidiscretization of p , E and H , respectively.

Proof. Using the Taylor remainder theorem upto $(R + 1)^{\text{th}}$ -order, and expressing $p(t)$ about $t = t^n$, we have that:

$$\begin{aligned} p(t) = p(t^n) + \frac{\partial p}{\partial t}(t^n)(t - t^n) + \frac{\partial^2 p}{\partial t^2}(t^n) \frac{(t - t^n)^2}{2!} + \dots + \frac{\partial^{R-1} p}{\partial t^{R-1}}(t^n) \frac{(t - t^n)^{R-1}}{(R-1)!} + \frac{\partial^R p}{\partial t^R}(t^n) \frac{(t - t^n)^R}{R!} \\ + \int_{t^n}^t \frac{(t-s)^R}{R!} \frac{\partial^{R+1} p}{\partial t^{R+1}}(s) ds, \end{aligned}$$

which when evaluated at $t = t^{n+\frac{1}{2}}$ and $t = t^{n-\frac{1}{2}}$ yields:

$$p(t^{n+\frac{1}{2}}) = p(t^n) + \frac{\Delta t}{2} \frac{\partial p}{\partial t}(t^n) + \frac{1}{2!} \frac{\Delta t^2}{2^2} \frac{\partial^2 p}{\partial t^2}(t^n) + \dots + \frac{1}{(R-1)!} \frac{\Delta t^{R-1}}{2^{R-1}} \frac{\partial^{R-1} p}{\partial t^{R-1}}(t^n)$$

$$+ \frac{1}{R!} \frac{\Delta t^R}{2^R} \frac{\partial^R p}{\partial t^R}(t^n) + \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^R}{R!} \frac{\partial^{R+1} p}{\partial t^{R+1}}(s) ds,$$

$$\begin{aligned} p(t^{n-\frac{1}{2}}) &= p(t^n) - \frac{\Delta t}{2} \frac{\partial p}{\partial t}(t^n) + \frac{1}{2!} \frac{\Delta t^2}{2^2} \frac{\partial^2 p}{\partial t^2}(t^n) + \dots - \frac{1}{(R-1)!} \frac{\Delta t^{R-1}}{2^{R-1}} \frac{\partial^{R-1} p}{\partial t^{R-1}}(t^n) \\ &\quad + \frac{1}{R!} \frac{\Delta t^R}{2^R} \frac{\partial^R p}{\partial t^R}(t^n) + \int_{t^n}^{t^{n-\frac{1}{2}}} \frac{(t^{n-\frac{1}{2}} - s)^R}{R!} \frac{\partial^{R+1} p}{\partial t^{R+1}}(s) ds. \end{aligned}$$

Subtracting these two equations, and using the result in the inner product term from the semidiscretization of the variational formulation leads to:

$$\left\langle \frac{p(t^{n+\frac{1}{2}}) - p(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{p} \right\rangle = \sum_{k=0}^{\frac{R}{2}-1} \frac{1}{(2k+1)!} \frac{\Delta t^{2k}}{2^{2k}} \left\langle \frac{\partial^{2k+1} p}{\partial t^{2k+1}}(t^n), \tilde{p} \right\rangle + \langle R_p^n, \tilde{p} \rangle,$$

in which we have defined that:

$$R_p^n := \frac{1}{\Delta t} \left[\int_{t^{n-\frac{1}{2}}}^{t^n} \frac{(t^{n-\frac{1}{2}} - s)^R}{R!} \frac{\partial^{R+1} p}{\partial t^{R+1}}(s) ds + \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^R}{R!} \frac{\partial^{R+1} p}{\partial t^{R+1}}(s) ds \right].$$

Similarly, for E and H, we have the following:

$$\begin{aligned} \left\langle \varepsilon \frac{E(t^{n+\frac{1}{2}}) - E(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{E} \right\rangle &= \sum_{k=0}^{\frac{R}{2}-1} \frac{1}{(2k+1)!} \frac{\Delta t^{2k}}{2^{2k}} \left\langle \varepsilon \frac{\partial^{2k+1} E}{\partial t^{2k+1}}(t^n), \tilde{E} \right\rangle + \langle \varepsilon R_E^n, \tilde{E} \rangle, \\ \left\langle \mu \frac{H(t^{n+1}) - H(t^n)}{\Delta t}, \tilde{H} \right\rangle &= \sum_{k=0}^{\frac{R}{2}-1} \frac{1}{(2k+1)!} \frac{\Delta t^{2k}}{2^{2k}} \left\langle \mu \frac{\partial^{2k+1} H}{\partial t^{2k+1}}(t^{n+\frac{1}{2}}), \tilde{H} \right\rangle + \langle \mu R_H^{n+\frac{1}{2}}, \tilde{H} \rangle, \end{aligned}$$

and in each of which we have defined that:

$$\begin{aligned} R_E^n &:= \frac{1}{\Delta t} \left[\int_{t^{n-\frac{1}{2}}}^{t^n} \frac{(t^{n-\frac{1}{2}} - s)^R}{R!} \frac{\partial^{R+1} E}{\partial t^{R+1}}(s) ds + \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^R}{R!} \frac{\partial^{R+1} E}{\partial t^{R+1}}(s) ds \right], \\ R_H^{n+\frac{1}{2}} &:= \frac{1}{\Delta t} \left[\int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^n - s)^R}{R!} \frac{\partial^{R+1} H}{\partial t^{R+1}}(s) ds + \int_{t^{n+\frac{1}{2}}}^{t^{n+1}} \frac{(t^{n+1} - s)^R}{R!} \frac{\partial^{R+1} H}{\partial t^{R+1}}(s) ds \right]. \end{aligned}$$

Again using the Taylor remainder theorem upto S^{th} -order for $2 \leq S \leq R$, for S taken to be even, we have that:

$$\left\langle \frac{p(t^{n+\frac{1}{2}}) - p(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{p} \right\rangle = \sum_{k=0}^{\frac{S}{2}-1} \frac{1}{(2k)!} \frac{\Delta t^{2k}}{2^{2k}} \left\langle \frac{\partial^{2k} p}{\partial t^{2k}}(t^n), \tilde{p} \right\rangle + \langle r_{S/2,p}^n, \tilde{p} \rangle,$$

$$\begin{aligned}\langle \varepsilon \frac{E(t^{n+\frac{1}{2}}) - E(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{E} \rangle &= \sum_{k=0}^{\frac{S}{2}-1} \frac{1}{(2k)!} \frac{\Delta t^{2k}}{2^{2k}} \langle \varepsilon \frac{\partial^{2k} E}{\partial t^{2k}}(t^n), \tilde{E} \rangle + \langle \varepsilon r_{S/2,E}^n, \tilde{E} \rangle, \\ \langle \mu \frac{H(t^{n+1}) - H(t^n)}{\Delta t}, \tilde{H} \rangle &= \sum_{k=0}^{\frac{S}{2}-1} \frac{1}{(2k)!} \frac{\Delta t^{2k}}{2^{2k}} \langle \mu \frac{\partial^{2k} H}{\partial t^{2k}}(t^{n+\frac{1}{2}}), \tilde{H} \rangle + \langle \mu r_{S/2,H}^{n+\frac{1}{2}}, \tilde{H} \rangle,\end{aligned}$$

and in each of which we have defined that:

$$\begin{aligned}r_{S/2,p}^n &:= \frac{1}{2} \left[- \int_{t^{n-\frac{1}{2}}}^{t^n} \frac{(t^{n-\frac{1}{2}} - s)^{S-1}}{(S-1)!} \frac{\partial^S p}{\partial t^S}(s) ds + \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^{S-1}}{(S-1)!} \frac{\partial^S p}{\partial t^S}(s) ds \right], \\ r_{S/2,E}^n &:= \frac{1}{2} \left[- \int_{t^{n-\frac{1}{2}}}^{t^n} \frac{(t^{n-\frac{1}{2}} - s)^{S-1}}{(S-1)!} \frac{\partial^S E}{\partial t^S}(s) ds + \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^{n+\frac{1}{2}} - s)^{S-1}}{(S-1)!} \frac{\partial^S E}{\partial t^S}(s) ds \right], \\ r_{S/2,H}^{n+\frac{1}{2}} &:= \frac{1}{2} \left[- \int_{t^n}^{t^{n+\frac{1}{2}}} \frac{(t^n - s)^{S-1}}{(S-1)!} \frac{\partial^S H}{\partial t^S}(s) ds + \int_{t^{n+\frac{1}{2}}}^{t^{n+1}} \frac{(t^{n+1} - s)^{S-1}}{(S-1)!} \frac{\partial^S H}{\partial t^S}(s) ds \right].\end{aligned}$$

Using these terms in the weak formulation as in Equations (5.2a) to (5.2c) at time $t = t^n$ for p and E terms, and at time $t = t^{n+\frac{1}{2}}$ for H , we obtain:

$$\begin{aligned}\langle \frac{p(t^{n+\frac{1}{2}}) - p(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{p} \rangle &- \langle \frac{\varepsilon}{2} (E(t^{n+\frac{1}{2}}) + E(t^{n-\frac{1}{2}})), \nabla \tilde{p} \rangle \\ &+ \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \langle \frac{\varepsilon}{2} (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} (E(t^{n+\frac{1}{2}}) + E(t^{n-\frac{1}{2}})), \nabla \tilde{p} \rangle \\ &= \langle R_p^n + \varepsilon \nabla \cdot r_{R/2,E}^n - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \varepsilon \nabla \cdot (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n, \tilde{p} \rangle, \quad (5.19a)\end{aligned}$$

$$\begin{aligned}\langle \frac{1}{2} \nabla (p(t^{n+\frac{1}{2}}) + p(t^{n-\frac{1}{2}})), \tilde{E} \rangle &- \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \langle \frac{1}{2} \nabla (p(t^{n+\frac{1}{2}}) + p(t^{n-\frac{1}{2}})), (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \tilde{E} \rangle \\ &+ \langle \varepsilon \frac{E(t^{n+\frac{1}{2}}) - E(t^{n-\frac{1}{2}})}{\Delta t}, \tilde{E} \rangle - \langle \frac{1}{2} (H(t^{n+1}) + H(t^n)), \nabla \times \tilde{E} \rangle \\ &+ \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} \langle \frac{1}{2} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} (H(t^{n+1}) + H(t^n)), \nabla \times \tilde{E} \rangle\end{aligned}$$

$$\begin{aligned}
&= \left\langle \varepsilon R_E^n + \nabla r_{R/2,p}^n - \nabla \times r_{R/2,H}^{n+\frac{1}{2}} - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \left(\varepsilon (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \nabla r_{\left(R/2-\sum_{m=1}^a k_m\right),p}^n \right. \right. \\
&\quad \left. \left. - (-1)^{\sum_{m=1}^a k_m} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} \nabla \times (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),H}^{n+\frac{1}{2}} \right), \tilde{E} \right\rangle, \quad (5.19b)
\end{aligned}$$

$$\begin{aligned}
&\left\langle \mu \frac{H(t^{n+1}) - H(t^n)}{\Delta t}, \tilde{H} \right\rangle + \left\langle \frac{1}{2} \nabla \times \left(E(t^{n+\frac{1}{2}}) + E(t^{n-\frac{1}{2}}) \right), \tilde{H} \right\rangle \\
&- \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} \left\langle \frac{1}{2} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} \nabla \times \left(E(t^{n+\frac{1}{2}}) + E(t^{n-\frac{1}{2}}) \right), (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \tilde{H} \right\rangle \\
&= \left\langle \mu R_H^{n+\frac{1}{2}} + \nabla \times r_{R/2,E}^n \right. \\
&\quad \left. + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \nabla \times r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n, \tilde{H} \right\rangle. \quad (5.19c)
\end{aligned}$$

Then subtracting the LFR scheme expressions as in Equations (5.7a) to (5.7c) leads us to the following set of equations:

$$\begin{aligned}
&\left\langle \frac{e_p^{n+\frac{1}{2}} - e_p^{n-\frac{1}{2}}}{\Delta t}, \tilde{p} \right\rangle - \left\langle \frac{\varepsilon}{2} \left(e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}} \right), \nabla \tilde{p} \right\rangle \\
&+ \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \left\langle \frac{\varepsilon}{2} (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \left(e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}} \right), \nabla \tilde{p} \right\rangle \\
&= \left\langle R_p^n + \varepsilon \nabla \cdot r_{R/2,E}^n - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \varepsilon \nabla \cdot (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n, \tilde{p} \right\rangle, \\
&\left\langle \frac{1}{2} \nabla \left(e_p^{n+\frac{1}{2}} + e_p^{n-\frac{1}{2}} \right), \tilde{E} \right\rangle - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \left\langle \frac{1}{2} \nabla \left(e_p^{n+\frac{1}{2}} + e_p^{n-\frac{1}{2}} \right), (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \tilde{E} \right\rangle \\
&\quad + \left\langle \varepsilon \frac{e_E^{n+\frac{1}{2}} - e_E^{n-\frac{1}{2}}}{\Delta t}, \tilde{E} \right\rangle - \left\langle \frac{1}{2} \left(e_H^{n+1} + e_H^n \right), \nabla \times \tilde{E} \right\rangle \\
&+ \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} \left\langle \frac{1}{2} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \left(e_H^{n+1} + e_H^n \right), \nabla \times \tilde{E} \right\rangle
\end{aligned}$$

$$\begin{aligned}
&= \langle \varepsilon R_E^n + \nabla r_{R/2,p}^n - \nabla \times r_{R/2,H}^{n+\frac{1}{2}} - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \left(\varepsilon (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \nabla r_{\left(R/2-\sum_{m=1}^a k_m\right),p}^n \right. \\
&\quad \left. - (-1)^{\sum_{m=1}^a k_m} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} \nabla \times (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),H}^{n+\frac{1}{2}} \right), \tilde{E} \rangle,
\end{aligned}$$

$$\begin{aligned}
&\langle \mu \frac{e_H^{n+1} - e_H^n}{\Delta t}, \tilde{H} \rangle + \langle \frac{1}{2} \nabla \times (e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}}), \tilde{H} \rangle \\
&- \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} \langle \frac{1}{2} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} \nabla \times (e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}}), (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \tilde{H} \rangle \\
&= \langle \mu R_H^{n+\frac{1}{2}} + \nabla \times r_{R/2,E}^n \\
&\quad + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \nabla \times r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n, \tilde{H} \rangle.
\end{aligned}$$

Likewise, for the semidiscrete approximation of the initial system as in Equations (5.8a) to (5.8c), we obtain for their errors the following system of equations:

$$\begin{aligned}
&\langle \frac{e_p^{\frac{1}{2}} - e_{p_0}}{\Delta t/2}, \tilde{p} \rangle - \frac{1}{2} \langle \frac{\varepsilon}{2} (e_E^{\frac{1}{2}} + e_{E_0}), \nabla \tilde{p} \rangle \\
&\quad + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m+1}} f(a) \langle \frac{\varepsilon}{2} (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} (e_E^{\frac{1}{2}} + e_{E_0}), \nabla \tilde{p} \rangle \\
&= \langle \frac{R_p^0}{2} + \frac{\varepsilon}{2} \nabla \cdot r_{R/2,E}^0 - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m+1}} f(a) \varepsilon \nabla \cdot (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^0, \tilde{p} \rangle, \\
&\frac{1}{2} \langle \frac{1}{2} \nabla (e_p^{\frac{1}{2}} + e_{p_0}), \tilde{E} \rangle - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m+1}} f(a) \langle \frac{1}{2} \nabla (e_p^{\frac{1}{2}} + e_{p_0}), (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \tilde{E} \rangle \\
&\quad + \langle \varepsilon \frac{e_E^{\frac{1}{2}} - e_{E_0}}{\Delta t/2}, \tilde{E} \rangle - \langle \frac{1}{2} (e_H^1 + e_{H_0}), \nabla \times \tilde{E} \rangle \\
&\quad + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m}} f(a) (-1)^{\sum_{m=1}^a k_m} \langle \frac{1}{2} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} (e_H^1 + e_{H_0}), \nabla \times \tilde{E} \rangle
\end{aligned}$$

$$= \left\langle \frac{\varepsilon}{2} R_E^0 + \frac{1}{2} \nabla r_{R/2,p}^0 - \nabla \times r_{R/2,H}^{\frac{1}{2}} - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m+1}} f(a) \left(\varepsilon (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \nabla r_{\left(R/2-\sum_{m=1}^a k_m\right),p}^0 \right. \right. \\ \left. \left. - (-1)^{\sum_{m=1}^a k_m} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} \nabla \times (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),H}^{\frac{1}{2}} \right), \tilde{E} \right\rangle,$$

$$\left\langle \mu \frac{e_H^1 - e_{H_0}}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \left\langle \frac{1}{2} \nabla \times \left(e_E^{\frac{1}{2}} + e_{E_0} \right), \tilde{H} \right\rangle \\ - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m+1}} f(a) (-1)^{\sum_{m=1}^a k_m} \left\langle \frac{1}{2} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} \nabla \times \left(e_E^{\frac{1}{2}} + e_{E_0} \right), (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \tilde{H} \right\rangle \\ = \left\langle \mu R_H^{\frac{1}{2}} + \frac{1}{2} \nabla \times r_{R/2,E}^0 \right. \\ \left. + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m+1}} f(a) (-1)^{\sum_{m=1}^a k_m} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \nabla \times r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^0, \tilde{H} \right\rangle,$$

and for which we define the initial remainder terms as follows:

$$R_p^0 := \frac{1}{\Delta t} \left[\int_{t^0}^{t^{\frac{1}{4}}} \frac{(t^0 - s)^R}{R!} \frac{\partial^{R+1} p}{\partial t^{R+1}}(s) ds + \int_{t^{\frac{1}{4}}}^{t^{\frac{1}{2}}} \frac{(t^{\frac{1}{2}} - s)^R}{R!} \frac{\partial^{R+1} p}{\partial t^{R+1}}(s) ds \right], \\ R_E^0 := \frac{1}{\Delta t} \left[\int_{t^0}^{t^{\frac{1}{4}}} \frac{(t^0 - s)^R}{R!} \frac{\partial^{R+1} E}{\partial t^{R+1}}(s) ds + \int_{t^{\frac{1}{4}}}^{t^{\frac{1}{2}}} \frac{(t^{\frac{1}{2}} - s)^R}{R!} \frac{\partial^{R+1} E}{\partial t^{R+1}}(s) ds \right], \\ R_H^{\frac{1}{2}} := \frac{1}{\Delta t} \left[\int_{t^0}^{t^{\frac{1}{2}}} \frac{(t^0 - s)^R}{R!} \frac{\partial^{R+1} H}{\partial t^{R+1}}(s) ds + \int_{t^{\frac{1}{2}}}^{t^1} \frac{(t^1 - s)^R}{R!} \frac{\partial^{R+1} H}{\partial t^{R+1}}(s) ds \right], \\ r_{S/2,p}^0 := \frac{1}{2} \left[- \int_{t^0}^{t^{\frac{1}{4}}} \frac{(t^0 - s)^{S-1}}{(S-1)!} \frac{\partial^S p}{\partial t^S}(s) ds + \int_{t^{\frac{1}{4}}}^{t^{\frac{1}{2}}} \frac{(t^{\frac{1}{2}} - s)^{S-1}}{(S-1)!} \frac{\partial^S p}{\partial t^S}(s) ds \right], \\ r_{S/2,E}^0 := \frac{1}{2} \left[- \int_{t^0}^{t^{\frac{1}{4}}} \frac{(t^0 - s)^{S-1}}{(S-1)!} \frac{\partial^S E}{\partial t^S}(s) ds + \int_{t^{\frac{1}{4}}}^{t^{\frac{1}{2}}} \frac{(t^{\frac{1}{2}} - s)^{S-1}}{(S-1)!} \frac{\partial^S E}{\partial t^S}(s) ds \right], \\ r_{S/2,H}^{\frac{1}{2}} := \frac{1}{2} \left[- \int_{t^0}^{t^{\frac{1}{2}}} \frac{(t^0 - s)^{S-1}}{(S-1)!} \frac{\partial^S H}{\partial t^S}(s) ds + \int_{t^{\frac{1}{2}}}^{t^1} \frac{(t^1 - s)^{S-1}}{(S-1)!} \frac{\partial^S H}{\partial t^S}(s) ds \right],$$

for $2 \leq S \leq R$ with S even. Now, in this set of weak formulation equations for the errors, we choose the test functions to be $\tilde{p} = 2\Delta t \varepsilon^{-1} (e_p^{n+\frac{1}{2}} + e_p^{n-\frac{1}{2}})$, $\tilde{E} = 2\Delta t (e_E^{n+\frac{1}{2}} + e_E^{n-\frac{1}{2}})$ and $\tilde{H} = 2\Delta t (e_H^n + e_H^{n-1})$. Next, by following essentially the same sequence of steps as in Theorem 5.4, we obtain the estimate for these error terms to be:

$$\begin{aligned} & \|e_p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|e_p^{n-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{n+\frac{1}{2}}\|_{\varepsilon}^2 - \|e_E^{n-\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^{n+1}\|_{\mu}^2 - \|e_H^n\|_{\mu}^2 \leq \Delta t \left[\|e_p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_p^{n-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 \right. \\ & \quad + \|e_E^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|e_E^{n-\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^{n+1}\|_{\mu}^2 + \|e_H^n\|_{\mu}^2 \left. \right] + \Delta t \left[\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 + \|\nabla r_{R/2,p}^n\|_{\varepsilon^{-1}}^2 \right. \\ & \quad + \|\nabla \cdot r_{R/2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{R/2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{R/2,H}^{n+\frac{1}{2}}\|_{\mu}^2 \\ & \quad + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1}^2 f(a)^2 \left(\left\| (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \nabla r_{\left(R/2-\sum_{m=1}^a k_m\right),p}^n \right\|_{\varepsilon^{-1}}^2 + \left\| \nabla \cdot (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n \right\|_{\varepsilon}^2 \right. \\ & \quad + (\varepsilon \mu)^{-(2\sum_{m=1}^a k_m+1)} \left\| (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \nabla \times r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n \right\|_{\varepsilon}^2 \\ & \quad \left. \left. + (\mu \varepsilon)^{-(2\sum_{m=1}^a k_m+1)} \left\| \nabla \times (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),H}^{n+\frac{1}{2}} \right\|_{\mu}^2 \right) \right]. \end{aligned}$$

Now, summing over $n = 1$ to $N - 1$, we get that:

$$\begin{aligned} & \|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|e_p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon}^2 - \|e_E^{\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^N\|_{\mu}^2 - \|e_H^1\|_{\mu}^2 \leq \Delta t \left[\|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon}^2 \right. \\ & \quad + \|e_E^{\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^N\|_{\mu}^2 + \|e_H^1\|_{\mu}^2 \left. \right] + 2\Delta t \sum_{n=1}^{N-2} \left[\|e_p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_p^{n-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{n+1}\|_{\varepsilon}^2 \right] + \Delta t \sum_{n=1}^{N-1} \left[\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 \right. \\ & \quad + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 + \|\nabla r_{R/2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{R/2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{R/2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{R/2,H}^{n+\frac{1}{2}}\|_{\mu}^2 \\ & \quad + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1}^2 f(a)^2 \left(\left\| (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \nabla r_{\left(R/2-\sum_{m=1}^a k_m\right),p}^n \right\|_{\varepsilon^{-1}}^2 + \left\| \nabla \cdot (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n \right\|_{\varepsilon}^2 \right. \\ & \quad + (\varepsilon \mu)^{-(2\sum_{m=1}^a k_m+1)} \left\| (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \nabla \times r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n \right\|_{\varepsilon}^2 \\ & \quad \left. \left. + (\mu \varepsilon)^{-(2\sum_{m=1}^a k_m+1)} \left\| \nabla \times (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),H}^{n+\frac{1}{2}} \right\|_{\mu}^2 \right) \right]. \end{aligned}$$

Similarly, for the case of the initial equations, choosing test functions to be $\tilde{p} = 2\Delta t \varepsilon^{-1} (e_p^{\frac{1}{2}} + e_p^0)$, $\tilde{E} = 2\Delta t (e_E^{\frac{1}{2}} + e_E^0)$ and $\tilde{H} = 4\Delta t (e_H^1 + e_H^0)$, we get that:

$$\begin{aligned} & \|e_p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 - \|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^{\frac{1}{2}}\|_{\varepsilon}^2 - \|e_E^0\|_{\varepsilon}^2 + \|e_H^1\|_{\mu}^2 - \|e_H^0\|_{\mu}^2 \\ & \leq \Delta t \left[\|e_p^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^{\frac{1}{2}}\|_{\varepsilon}^2 + \|e_E^0\|_{\varepsilon}^2 + \|e_H^1\|_{\mu}^2 + \|e_H^0\|_{\mu}^2 \right] + \Delta t \left[\|R_p^0\|_{\varepsilon^{-1}}^2 + \|R_E^0\|_{\varepsilon}^2 + \|R_H^{\frac{1}{2}}\|_{\mu}^2 \right. \end{aligned}$$

$$\begin{aligned}
& + \|\nabla r_{R/2,p}^0\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{R/2,E}^0\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{R/2,E}^0\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{R/2,H}^{\frac{1}{2}}\|_{\mu}^2 \\
& + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}^2}{4^{2\sum_{m=1}^a k_m+1}} f(a)^2 \left(\left\| (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \nabla r_{\left(R/2-\sum_{m=1}^a k_m\right),p}^0 \right\|_{\varepsilon^{-1}}^2 \right. \\
& + \left\| \nabla \cdot (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^0 \right\|_{\varepsilon}^2 + (\varepsilon \mu)^{-(2\sum_{m=1}^a k_m+1)} \left\| (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \nabla \times r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^0 \right\|_{\varepsilon}^2 \\
& \left. + (\mu \varepsilon)^{-(2\sum_{m=1}^a k_m+1)} \left\| \nabla \times (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),H}^{\frac{1}{2}} \right\|_{\mu}^2 \right).
\end{aligned}$$

Adding the previous two equations and using the initial conditions as in Equation (5.1c) and positivity of all the right-hand side terms, we have that:

$$\begin{aligned}
& \|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^N\|_{\mu}^2 \leq \frac{1+\Delta t}{1-\Delta t} \left[\|e_p^0\|_{\varepsilon^{-1}}^2 + \|e_E^0\|_{\varepsilon}^2 + \|e_H^0\|_{\mu}^2 \right] + \\
& \frac{\Delta t}{1-\Delta t} \sum_{n=0}^{N-1} \left[2 \left(\|e_p^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^{n+1}\|_{\mu}^2 \right) + \left(\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 \right. \right. \\
& + \|\nabla r_{R/2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{R/2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{R/2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{R/2,H}^{n+\frac{1}{2}}\|_{\mu}^2 \\
& + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1}^2 f(a)^2 \left(\left\| (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \nabla r_{\left(R/2-\sum_{m=1}^a k_m\right),p}^n \right\|_{\varepsilon^{-1}}^2 + \left\| \nabla \cdot (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n \right\|_{\varepsilon}^2 \right. \\
& + (\varepsilon \mu)^{-(2\sum_{m=1}^a k_m+1)} \left\| (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \nabla \times r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n \right\|_{\varepsilon}^2 \\
& \left. \left. + (\mu \varepsilon)^{-(2\sum_{m=1}^a k_m+1)} \left\| \nabla \times (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),H}^{n+\frac{1}{2}} \right\|_{\mu}^2 \right) \right].
\end{aligned}$$

Applying the discrete Gronwall inequality similar to Theorem 5.4, we obtain the estimate:

$$\begin{aligned}
& \|e_p^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|e_E^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|e_H^N\|_{\mu}^2 \leq \left[\frac{6\Delta t}{5} \sum_{n=0}^{N-1} \left(\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 \right. \right. \\
& + \|\nabla r_{R/2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{R/2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{R/2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{R/2,H}^{n+\frac{1}{2}}\|_{\mu}^2 \\
& + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1}^2 f(a)^2 \left(\left\| (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \nabla r_{\left(R/2-\sum_{m=1}^a k_m\right),p}^n \right\|_{\varepsilon^{-1}}^2 + \left\| \nabla \cdot (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n \right\|_{\varepsilon}^2 \right. \\
& \left. + (\varepsilon \mu)^{-(2\sum_{m=1}^a k_m+1)} \left\| (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \nabla \times r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n \right\|_{\varepsilon}^2 \right. \\
& \left. \left. + (\mu \varepsilon)^{-(2\sum_{m=1}^a k_m+1)} \left\| \nabla \times (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),H}^{n+\frac{1}{2}} \right\|_{\mu}^2 \right) \right].
\end{aligned}$$

$$+(\mu\epsilon)^{-(2\sum_{m=1}^a k_m+1)} \left\| \nabla \times (\nabla \times \nabla \times)_{m=1}^{\sum_{m=1}^a k_m} r^{n+\frac{1}{2}} \left(\frac{R}{2} - \sum_{m=1}^a k_m \right)_{,H} \right\|_{\mu}^2 \right) + \frac{7}{5} \left(\|e_p^0\|_{\epsilon^{-1}}^2 + \|e_E^0\|_{\epsilon}^2 + \|e_H^0\|_{\mu}^2 \right) \exp(4T).$$

Now, we need to obtain bounding estimates for each of the Taylor remainder terms and to do so, we first consider the first remainder term corresponding to p and argue as follows:

$$\begin{aligned} \|R_p^n\|_{\epsilon^{-1}}^2 &= \frac{1}{(R!\Delta t)^2} \left\| \int_{t^{n-1}}^{t^{n-\frac{1}{2}}} (t^{n-1} - s)^R \frac{\partial^{R+1}p}{\partial t^{R+1}}(s) ds + \int_{t^{n-\frac{1}{2}}}^{t^n} (t^n - s)^R \frac{\partial^{R+1}p}{\partial t^{R+1}}(s) ds \right\|_{\epsilon^{-1}}^2, \\ &\leq \frac{1}{(R!\Delta t)^2} \left\| \int_{t^{n-1}}^{t^n} (t^n - s)^R \frac{\partial^{R+1}p}{\partial t^{R+1}}(s) ds \right\|_{\epsilon^{-1}}^2, \quad (\text{using } t^{n-1} < t^n) \\ &\leq \frac{1}{(R!\Delta t)^2} \int_{t^{n-1}}^{t^n} (s - t^n)^{2R} ds \int_{t^{n-1}}^{t^n} \left\| \frac{\partial^{R+1}p}{\partial t^{R+1}}(s) \right\|_{\epsilon^{-1}}^2 ds, \quad (\text{by Cauchy-Schwarz}) \\ &= \frac{(\Delta t)^{2R-1}}{(R!)^2(2R+1)} \int_{t^{n-1}}^{t^n} \left\| \frac{\partial^{R+1}p}{\partial t^{R+1}}(s) \right\|_{\epsilon^{-1}}^2 ds, \end{aligned}$$

and now summing both sides over $n = 0$ to N , we have that:

$$\sum_{n=0}^N \|R_p^n\|_{\epsilon^{-1}}^2 \leq \frac{(\Delta t)^{2R-1}}{(R!)^2(2R+1)} \int_0^T \left\| \frac{\partial^{R+1}p}{\partial t^{R+1}}(s) \right\|_{\epsilon^{-1}}^2 ds = \frac{(\Delta t)^{2R-1}}{(R!)^2(2R+1)} \left\| \frac{\partial^{R+1}p}{\partial t^{R+1}} \right\|_{L^2(0,T;L_{\epsilon^{-1}}^2(\Omega))}^2.$$

Similarly, for the remaining Taylor remainder terms, we have that:

$$\begin{aligned} \|R_E^n\|_{\epsilon}^2 &\leq \frac{(\Delta t)^{2R-1}}{(R!)^2(2R+1)} \left\| \frac{\partial^{R+1}E}{\partial t^{R+1}} \right\|_{L^2(0,T;L_{\epsilon}^2(\Omega))}^2, \\ \sum_{n=0}^{N-1} \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 &\leq \frac{(\Delta t)^{2R-1}}{(R!)^2(2R+1)} \left\| \frac{\partial^{R+1}H}{\partial t^{R+1}} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2, \\ \sum_{n=0}^{N-1} \|\Phi(r_{S/2,p}^n)\|_{\epsilon^{-1}}^2 &\leq \frac{(\Delta t)^{2S-1}}{2^{2S-1}(2S-1)!} \left\| \frac{\partial^S(\Phi(p))}{\partial t^S} \right\|_{L^2(0,T;L_{\epsilon^{-1}}^2(\Omega))}^2, \\ \sum_{n=0}^{N-1} \|\Phi(r_{S/2,E}^n)\|_{\epsilon}^2 &\leq \frac{(\Delta t)^{2S-1}}{2^{2S-1}(2S-1)!} \left\| \frac{\partial^S(\Phi(E))}{\partial t^S} \right\|_{L^2(0,T;L_{\epsilon}^2(\Omega))}^2, \\ \sum_{n=0}^{N-1} \|\Phi(r_{S/2,H}^{n+\frac{1}{2}})\|_{\mu}^2 &\leq \frac{(\Delta t)^{2S-1}}{2^{2S-1}(2S-1)!} \left\| \frac{\partial^S(\Phi(H))}{\partial t^S} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2, \end{aligned}$$

for $2 \leq S \leq R$, S even, and in which Φ is taken to be an operator (for example, $\nabla \times \nabla \times$ or $\nabla \nabla \cdot$) acting on objects taken to be from a correspondingly appropriate function space on Ω . Finally, using the regularity assumptions for p , E and H , and 1- and 2-norm equivalence, we obtain the required result:

$$\|e_p^{N-\frac{1}{2}}\|_{\epsilon^{-1}} + \|e_E^{N-\frac{1}{2}}\|_{\epsilon} + \|e_H^N\|_{\mu} \leq C \left[(\Delta t)^R + \|e_p^0\|_{\epsilon^{-1}} + \|e_E^0\|_{\epsilon} + \|e_H^0\|_{\mu} \right]. \quad \square$$

5.3.2 Error Estimate for Full Discretization

We now present the error analysis for the full discretization of the three-field formulation of the Maxwell's equations using a compatible sequence of arbitrary-order simplicial de Rham finite elements in conjunction with the LF_R scheme. To do so, first we let Π_h^0 , Π_h^1 and Π_h^2 denote the respective smoothed L^2 projection operators, that is, let $\Pi_h^0 : \dot{H}_{\varepsilon^{-1}}^1(\Omega) \rightarrow U_h$, $\Pi_h^1 : \dot{H}_\varepsilon(\text{curl}; \Omega) \rightarrow V_h$ and $\Pi_h^2 : \dot{H}_\mu(\text{div}; \Omega) \rightarrow W_h$ as in 5.2.2 for LF_6 scheme. Given this brief background about these smoothed L^2 projection operators, we now define the errors for p , E and H at time $(n + 1/2)\Delta t$ or $n\Delta t$ under the full discretization to be the following and this is essentially a generalization of the analysis for the full error as in [AK25, Section 5.2] for LF_2 , [AK26, Section 2.2] for LF_4 , and 5.2.2 for LF_6 . So, we have that:

$$e_{p_h}^{n+\frac{1}{2}} := p(t^{n+\frac{1}{2}}) - p_h^{n+\frac{1}{2}} = \eta^{n+\frac{1}{2}} - \eta_h^{n+\frac{1}{2}}, \quad (5.20)$$

$$e_{E_h}^{n+\frac{1}{2}} := E(t^{n+\frac{1}{2}}) - E_h^{n+\frac{1}{2}} = \zeta^{n+\frac{1}{2}} - \zeta_h^{n+\frac{1}{2}}, \quad (5.21)$$

$$e_{H_h}^n := H(t^n) - H_h^n = \xi^n - \xi_h^n, \quad (5.22)$$

and in which we now have the following definitions for the newly introduced terms:

$$\eta^{n+\frac{1}{2}} := p(t^{n+\frac{1}{2}}) - \Pi_h^0 p(t^{n+\frac{1}{2}}), \quad \eta_h^{n+\frac{1}{2}} := p_h^{n+\frac{1}{2}} - \Pi_h^0 p(t^{n+\frac{1}{2}}), \quad (5.23)$$

$$\zeta^{n+\frac{1}{2}} := E(t^{n+\frac{1}{2}}) - \Pi_h^1 E(t^{n+\frac{1}{2}}), \quad \zeta_h^{n+\frac{1}{2}} := E_h^{n+\frac{1}{2}} - \Pi_h^1 E(t^{n+\frac{1}{2}}), \quad (5.24)$$

$$\xi^n := H(t^n) - \Pi_h^2 H(t^n), \quad \xi_h^n := H_h^n - \Pi_h^2 H(t^n). \quad (5.25)$$

For the LF_R scheme as in Equations (5.7a) to (5.7c), using a de Rham sequence of finite dimensional subspaces of the corresponding function spaces for the spatial discretization of $(p^{n+\frac{1}{2}}, E^{n+\frac{1}{2}}, H^{n+1})$, we obtain the following discrete problem: find $(p_h^{n+\frac{1}{2}}, E_h^{n+\frac{1}{2}}, H_h^{n+1}) \in U_h \times V_h \times W_h \subseteq \dot{H}_{\varepsilon^{-1}}^1 \times \dot{H}_\varepsilon(\text{curl}; \Omega) \times \dot{H}_\mu(\text{div}; \Omega)$ such that:

$$\begin{aligned} & \left\langle \frac{p_h^{n+\frac{1}{2}} - p_h^{n-\frac{1}{2}}}{\Delta t}, \tilde{p} \right\rangle - \left\langle \frac{\varepsilon}{2} (E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}}), \nabla \tilde{p} \right\rangle \\ & + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \left\langle \frac{\varepsilon}{2} (\nabla \nabla \cdot) \sum_{m=1}^a k_m (E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}}), \nabla \tilde{p} \right\rangle = 0, \end{aligned} \quad (5.26a)$$

$$\begin{aligned} & \left\langle \frac{1}{2} \nabla (p_h^{n+\frac{1}{2}} + p_h^{n-\frac{1}{2}}), \tilde{E} \right\rangle - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \left\langle \frac{1}{2} \nabla (p_h^{n+\frac{1}{2}} + p_h^{n-\frac{1}{2}}), (\nabla \nabla \cdot) \sum_{m=1}^a k_m \tilde{E} \right\rangle \\ & + \left\langle \frac{E_h^{n+\frac{1}{2}} - E_h^{n-\frac{1}{2}}}{\Delta t}, \tilde{E} \right\rangle - \left\langle \frac{1}{2} (H_h^{n+1} + H_h^n), \nabla \times \tilde{E} \right\rangle \\ & + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} \left\langle \frac{1}{2} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times) \sum_{m=1}^a k_m (H_h^{n+1} + H_h^n), \nabla \times \tilde{E} \right\rangle = 0, \end{aligned} \quad (5.26b)$$

$$\begin{aligned}
& \left\langle \mu \frac{H_h^{n+1} - H_h^n}{\Delta t}, \tilde{H} \right\rangle + \left\langle \frac{1}{2} \nabla \times \left(E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}} \right), \tilde{H} \right\rangle \\
& - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} \left\langle \frac{1}{2} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} \nabla \times \left(E_h^{n+\frac{1}{2}} + E_h^{n-\frac{1}{2}} \right), (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \tilde{H} \right\rangle = 0, \quad (5.26c)
\end{aligned}$$

for all $(\tilde{p}, \tilde{E}, \tilde{H}) \in U_h \times V_h \times W_h$, and for $n \in \{1, \dots, N-1\}$. Using $n = 0$ for bootstrapping as in Equations (5.8a) to (5.8c) leads to the discrete problem: find $(p_h^{\frac{1}{2}}, E_h^{\frac{1}{2}}, H_h^1) \in U_h \times V_h \times W_h \subseteq \dot{H}_{\varepsilon^{-1}}^1 \times \dot{H}_{\varepsilon}(\text{curl}; \Omega) \times \dot{H}_{\mu}(\text{div}; \Omega)$ such that:

$$\begin{aligned}
& \left\langle \frac{p_h^{\frac{1}{2}} - p_h^0}{\Delta t/2}, \tilde{p} \right\rangle - \frac{1}{2} \left\langle \frac{\varepsilon}{2} \left(E_h^{\frac{1}{2}} + E_h^0 \right), \nabla \tilde{p} \right\rangle \\
& + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m+1}} f(a) \left\langle \frac{\varepsilon}{2} (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \left(E_h^{\frac{1}{2}} + E_h^0 \right), \nabla \tilde{p} \right\rangle = 0, \quad (5.27a)
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2} \left\langle \frac{1}{2} \nabla \left(p_h^{\frac{1}{2}} + p_h^0 \right), \tilde{E} \right\rangle - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m+1}} f(a) \left\langle \frac{1}{2} \nabla \left(p_h^{\frac{1}{2}} + p_h^0 \right), (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \tilde{E} \right\rangle \\
& + \left\langle \varepsilon \frac{E_h^{\frac{1}{2}} - E_h^0}{\Delta t/2}, \tilde{E} \right\rangle - \left\langle \frac{1}{2} \left(H_h^1 + H_h^0 \right), \nabla \times \tilde{E} \right\rangle \\
& + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m}} f(a) (-1)^{\sum_{m=1}^a k_m} \left\langle \frac{1}{2} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \left(H_h^1 + H_h^0 \right), \nabla \times \tilde{E} \right\rangle = 0, \quad (5.27b)
\end{aligned}$$

$$\begin{aligned}
& \left\langle \mu \frac{H_h^1 - H_h^0}{\Delta t}, \tilde{H} \right\rangle + \frac{1}{2} \left\langle \frac{1}{2} \nabla \times \left(E_h^{\frac{1}{2}} + E_h^0 \right), \tilde{H} \right\rangle \\
& - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}}{2^{\sum_{m=1}^a k_m+1}} f(a) (-1)^{\sum_{m=1}^a k_m} \left\langle \frac{1}{2} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} \nabla \times \left(E_h^{\frac{1}{2}} + E_h^0 \right), (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \tilde{H} \right\rangle = 0, \quad (5.27c)
\end{aligned}$$

for all $(\tilde{p}, \tilde{E}, \tilde{H}) \in U_h \times V_h \times W_h$ given $(p_h^0, E_h^0, H_h^0) \in U_h \times V_h \times W_h$. Let $\Pi_h^0 : \dot{H}_{\varepsilon^{-1}}^1(\Omega) \longrightarrow U_h$, $\Pi_h^1 : \dot{H}_{\varepsilon}(\text{curl}; \Omega) \longrightarrow V_h$ and $\Pi_h^2 : \dot{H}_{\mu}(\text{div}; \Omega) \longrightarrow W_h$ denote the respective smoothed L^2 projection operators as in the sense of [Sch08; Chr07] and a detailed discussion is available in [BS08, Lemma 4.3.8] [AFW06, Theorem 5.3] [EG21a, Lemma 11.9, Corollary 11.11, Theorem 16.10, Theorem 17.5]. For the initial conditions,

we set:

$$\mathbf{p}_h^0 := \Pi_h^0 \mathbf{p}_0, \quad \mathbf{E}_h^0 := \Pi_h^1 \mathbf{E}_0, \quad \text{and} \quad \mathbf{H}_h^0 := \Pi_h^2 \mathbf{H}_0. \quad (5.27d)$$

With this, we can now state our theorem for convergence of error in the full discretization of the system of Maxwell's equations using LF_R and R^{th} -order polynomial Whitney basis finite elements for the various function spaces.

Theorem 5.6 (Full Error Estimate). *Let $\mathbf{p} \in C^{R+1}(0, T; \dot{H}_{\varepsilon^{-1}}^1(\Omega))$, $\mathbf{E} \in C^{R+1}(0, T; \dot{H}_{\varepsilon}(\text{curl}; \Omega))$, and $\mathbf{H} \in C^{R+1}(0, T; \dot{H}_{\mu}(\text{div}; \Omega))$ be the solution to the variational formulation of the Maxwell's equations as in Equations (5.2a) to (5.2c), and let $(\mathbf{p}_h^{n+\frac{1}{2}}, \mathbf{E}_h^{n+\frac{1}{2}}, \mathbf{H}_h^{n+1})$ be the solution of the fully discretized Maxwell's equations using the LF_R scheme as in Equations (5.26a) to (5.26c) and (5.27a) to (5.27d). If the time step $\Delta t > 0$ and the mesh parameter $h > 0$ are both sufficiently small, then there exists a positive bounded constant C independent of both Δt and h such that the following error estimate holds:*

$$\|\mathbf{e}_{\mathbf{p}_h}^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|\mathbf{e}_{\mathbf{E}_h}^{N-\frac{1}{2}}\|_{\varepsilon} + \|\mathbf{e}_{\mathbf{H}_h}^N\|_{\mu} \leq C [(\Delta t)^R + h^r + h^r(\Delta t)^R],$$

where the finite element subspaces $\mathbf{U}_h$, $\mathbf{V}_h$ and $\mathbf{W}_h$ are each spanned by their respective Whitney form basis of polynomial order $r \geq 1$.

Proof. First, we shall subtract the set of equations for the full discretization as in Equations (5.26a) to (5.26c) from Equations (5.19a) to (5.19c), and then use the error terms in Equations (5.20) to (5.22) and thereby obtain:

$$\begin{aligned} & \left\langle \frac{\mathbf{e}_{\mathbf{p}_h}^{n+\frac{1}{2}} - \mathbf{e}_{\mathbf{p}_h}^{n-\frac{1}{2}}}{\Delta t}, \tilde{\mathbf{p}} \right\rangle - \left\langle \frac{\varepsilon}{2} \left(\mathbf{e}_{\mathbf{E}_h}^{n+\frac{1}{2}} + \mathbf{e}_{\mathbf{E}_h}^{n-\frac{1}{2}} \right), \nabla \tilde{\mathbf{p}} \right\rangle \\ & + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \left\langle \frac{\varepsilon}{2} (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \left(\mathbf{e}_{\mathbf{E}_h}^{n+\frac{1}{2}} + \mathbf{e}_{\mathbf{E}_h}^{n-\frac{1}{2}} \right), \nabla \tilde{\mathbf{p}} \right\rangle \\ & = \left\langle \mathbf{R}_p^n + \varepsilon \nabla \cdot \mathbf{r}_{R/2, E}^n - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \varepsilon \nabla \cdot (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \mathbf{r}_{\left(\frac{R}{2}-\sum_{m=1}^a k_m\right), E}^n, \tilde{\mathbf{p}} \right\rangle, \\ & \left\langle \frac{1}{2} \nabla \left(\mathbf{e}_{\mathbf{p}_h}^{n+\frac{1}{2}} + \mathbf{e}_{\mathbf{p}_h}^{n-\frac{1}{2}} \right), \tilde{\mathbf{E}} \right\rangle - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \left\langle \frac{1}{2} \nabla \left(\mathbf{e}_{\mathbf{p}_h}^{n+\frac{1}{2}} + \mathbf{e}_{\mathbf{p}_h}^{n-\frac{1}{2}} \right), (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \tilde{\mathbf{E}} \right\rangle \\ & + \left\langle \varepsilon \frac{\mathbf{e}_{\mathbf{E}_h}^{n+\frac{1}{2}} - \mathbf{e}_{\mathbf{E}_h}^{n-\frac{1}{2}}}{\Delta t}, \tilde{\mathbf{E}} \right\rangle - \left\langle \frac{1}{2} \left(\mathbf{e}_{\mathbf{H}_h}^{n+1} + \mathbf{e}_{\mathbf{H}_h}^n \right), \nabla \times \tilde{\mathbf{E}} \right\rangle \\ & + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} \left\langle \frac{1}{2} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \left(\mathbf{e}_{\mathbf{H}_h}^{n+1} + \mathbf{e}_{\mathbf{H}_h}^n \right), \nabla \times \tilde{\mathbf{E}} \right\rangle \end{aligned}$$

$$\begin{aligned}
&= \langle \varepsilon R_E^n + \nabla r_{R/2,p}^n - \nabla \times r_{R/2,H}^{n+\frac{1}{2}} - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \left(\varepsilon (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \nabla r_{\left(R/2-\sum_{m=1}^a k_m\right),p}^n \right. \\
&\quad \left. - (-1)^{\sum_{m=1}^a k_m} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} \nabla \times (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),H}^{n+\frac{1}{2}} \right), \tilde{E} \rangle,
\end{aligned}$$

$$\begin{aligned}
&\langle \mu \frac{e_{H_h}^{n+1} - e_{H_h}^n}{\Delta t}, \tilde{H} \rangle + \langle \frac{1}{2} \nabla \times (e_{E_h}^{n+\frac{1}{2}} + e_{E_h}^{n-\frac{1}{2}}), \tilde{H} \rangle \\
&- \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} \langle \frac{1}{2} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} \nabla \times (e_{E_h}^{n+\frac{1}{2}} + e_{E_h}^{n-\frac{1}{2}}), (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \tilde{H} \rangle \\
&= \langle \mu R_H^{n+\frac{1}{2}} + \nabla \times r_{R/2,E}^n \\
&\quad + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \nabla \times r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n, \tilde{H} \rangle.
\end{aligned}$$

Next, using the values of the error terms $e_{p_h}^n$, $e_{E_h}^n$ and $e_{H_h}^n$ as in Equations (5.20) to (5.22) in the above equations, we get:

$$\begin{aligned}
&\langle \frac{(\eta^{n+\frac{1}{2}} - \eta^{n-\frac{1}{2}}) - (\eta_h^{n+\frac{1}{2}} - \eta_h^{n-\frac{1}{2}})}{\Delta t}, \tilde{p} \rangle - \langle \frac{\varepsilon}{2} \left((\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}}) - (\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}}) \right), \nabla \tilde{p} \rangle \\
&+ \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \langle \frac{\varepsilon}{2} (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \left((\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}}) - (\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}}) \right), \nabla \tilde{p} \rangle \\
&= \langle R_p^n + \varepsilon \nabla \cdot r_{R/2,E}^n - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \varepsilon \nabla \cdot (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n, \tilde{p} \rangle,
\end{aligned}$$

$$\begin{aligned}
&\langle \frac{1}{2} \nabla \left((\eta^{n+\frac{1}{2}} + \eta^{n-\frac{1}{2}}) - (\eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}}) \right), \tilde{E} \rangle \\
&- \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \langle \frac{1}{2} \nabla \left((\eta^{n+\frac{1}{2}} + \eta^{n-\frac{1}{2}}) - (\eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}}) \right), (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \tilde{E} \rangle \\
&+ \langle \varepsilon \frac{(\zeta^{n+\frac{1}{2}} - \zeta^{n-\frac{1}{2}}) - (\zeta_h^{n+\frac{1}{2}} - \zeta_h^{n-\frac{1}{2}})}{\Delta t}, \tilde{E} \rangle - \langle \frac{1}{2} \left((\xi^{n+1} + \xi^n) - (\xi_h^{n+1} - \xi_h^n) \right), \nabla \times \tilde{E} \rangle
\end{aligned}$$

$$\begin{aligned}
& + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} \left\langle \frac{1}{2} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)_{m=1}^{\sum_{m=1}^a k_m} \left((\xi^{n+1} + \xi^n) - (\xi_h^{n+1} - \xi_h^n) \right), \nabla \times \tilde{E} \right\rangle \\
& = \left\langle \varepsilon R_E^n + \nabla r_{R/2,p}^n - \nabla \times r_{R/2,H}^{n+\frac{1}{2}} - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \left(\varepsilon (\nabla \nabla \cdot)_{m=1}^{\sum_{m=1}^a k_m} \nabla r_{\left(R/2-\sum_{m=1}^a k_m\right),p}^n \right. \right. \\
& \quad \left. \left. - (-1)^{\sum_{m=1}^a k_m} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} \nabla \times (\nabla \times \nabla \times)_{m=1}^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),H}^{n+\frac{1}{2}} \right), \tilde{E} \right\rangle, \\
& \left\langle \mu \frac{(\xi^{n+1} - \xi^n) - (\xi_h^{n+1} - \xi_h^n)}{\Delta t}, \tilde{H} \right\rangle + \left\langle \frac{1}{2} \nabla \times \left((\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}}) - (\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}}) \right), \tilde{H} \right\rangle \\
& \quad - \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} \\
& \quad \left(\left\langle \frac{1}{2} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} \nabla \times \left((\zeta^{n+\frac{1}{2}} + \zeta^{n-\frac{1}{2}}) - (\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}}) \right), (\nabla \times \nabla \times)_{m=1}^{\sum_{m=1}^a k_m} \tilde{H} \right\rangle = \left\langle \mu R_H^{n+\frac{1}{2}} + \nabla \times r_{R/2,E}^n \right. \right. \\
& \quad \left. \left. + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)_{m=1}^{\sum_{m=1}^a k_m} \nabla \times r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n, \tilde{H} \right\rangle.
\end{aligned}$$

Since these equations are true for all $(\tilde{p}, \tilde{E}, \tilde{H}) \in U_h \times V_h \times W_h$, we choose $\tilde{p} = -2\Delta t \varepsilon^{-1} \left(\eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \right)$, $\tilde{E} = -2\Delta t \left(\zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \right)$ and $\tilde{H} = -2\Delta t \left(\xi_h^n + \xi_h^{n-1} \right)$ and using the fact that $\nabla U_h \subseteq V_h$ and $\nabla \times V_h \subseteq W_h$ and so on, we obtain that:

$$\begin{aligned}
& 2\langle \varepsilon^{-1} \left(\eta_h^{n+\frac{1}{2}} - \eta_h^{n-\frac{1}{2}} \right), \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle + 2\langle \varepsilon \left(\zeta_h^{n+\frac{1}{2}} - \zeta_h^{n-\frac{1}{2}} \right), \zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \rangle + 2\langle \mu \left(\xi_h^n - \xi_h^{n-1} \right), \xi_h^n + \xi_h^{n-1} \rangle \\
& = 2\langle \varepsilon^{-1} \left(\eta_h^{n+\frac{1}{2}} - \eta_h^{n-\frac{1}{2}} \right), \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle + 2\langle \varepsilon \left(\zeta_h^{n+\frac{1}{2}} - \zeta_h^{n-\frac{1}{2}} \right), \zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \rangle + 2\langle \mu \left(\xi_h^n - \xi_h^{n-1} \right), \xi_h^n + \xi_h^{n-1} \rangle \\
& \quad + 2\Delta t \langle -R_p^n - \varepsilon \nabla \cdot r_{R/2,E}^n + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \varepsilon \nabla \cdot (\nabla \nabla \cdot)_{m=1}^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),E}^n, \varepsilon^{-1} \left(\eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \right) \rangle \\
& \quad + 2\Delta t \langle -\varepsilon R_E^n - \nabla r_{R/2,p}^n + \nabla \times r_{R/2,H}^{n+\frac{1}{2}} + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) \left(\varepsilon (\nabla \nabla \cdot)_{m=1}^{\sum_{m=1}^a k_m} \nabla r_{\left(R/2-\sum_{m=1}^a k_m\right),p}^n \right. \right. \\
& \quad \left. \left. - (-1)^{\sum_{m=1}^a k_m} (\mu \varepsilon)^{-\sum_{m=1}^a k_m} \nabla \times (\nabla \times \nabla \times)_{m=1}^{\sum_{m=1}^a k_m} r_{\left(R/2-\sum_{m=1}^a k_m\right),H}^{n+\frac{1}{2}} \right), \zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \right\rangle \\
& \quad + 2\Delta t \langle -\mu R_H^{n+\frac{1}{2}} - \nabla \times r_{R/2,E}^n
\end{aligned}$$

$$- \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1} f(a) (-1)^{\sum_{m=1}^a k_m} (\varepsilon \mu)^{-\sum_{m=1}^a k_m} (\nabla \times \nabla \times)_{m=1}^{\sum_{m=1}^a k_m} \nabla \times r_{\left(\frac{R}{2}-\sum_{m=1}^a k_m\right), E}^n, \xi_h^n + \xi_h^{n-1}. \quad (5.28)$$

Consider that $\varepsilon^{-1} \left(\eta^{n+\frac{1}{2}} - \eta^{n-\frac{1}{2}} \right) = \varepsilon^{-1} (I - \Pi_h^0) \left(p(t^{n+\frac{1}{2}}) - p(t^{n-\frac{1}{2}}) \right)$ by Equation (5.23). Using the Taylor theorem with remainder as in Theorem 5.5, applying the Cauchy-Schwarz, AM-GM, and triangle inequalities, we have the following inequality:

$$\begin{aligned} 2 \langle \varepsilon^{-1} \left(\eta^{n+\frac{1}{2}} - \eta^{n-\frac{1}{2}} \right), \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle &= 2 \Delta t \langle \varepsilon^{-1} (I - \Pi_h^0) \sum_{k=0}^{\frac{R}{2}-1} \frac{1}{(2k+1)!} \frac{\Delta t^{2k}}{2^{2k}} \frac{\partial^{2k+1} p}{\partial t^{2k+1}}(t^n), \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle \\ &\quad + 2 \Delta t \langle \varepsilon^{-1} (I - \Pi_h^0) R_p^n, \eta_h^{n+\frac{1}{2}} + \eta_h^{n-\frac{1}{2}} \rangle \\ &\leq \Delta t \left[\sum_{k=0}^{\frac{R}{2}-1} \frac{1}{(2k+1)!} \frac{\Delta t^{2k}}{2^{2k}} \left\| (I - \Pi_h^0) \frac{\partial^{2k+1} p}{\partial t^{2k+1}}(t^n) \right\|_{\varepsilon^{-1}}^2 \right. \\ &\quad \left. + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 \right] + (R+2) \Delta t \left[\left\| \eta_h^{n+\frac{1}{2}} \right\|_{\varepsilon^{-1}}^2 + \left\| \eta_h^{n-\frac{1}{2}} \right\|_{\varepsilon^{-1}}^2 \right]. \end{aligned}$$

Similarly, using Equations (5.24) and (5.25) for the error terms for E and H, we obtain that:

$$\begin{aligned} 2 \langle \varepsilon \left(\zeta^{n+\frac{1}{2}} - \zeta^{n-\frac{1}{2}} \right), \zeta_h^{n+\frac{1}{2}} + \zeta_h^{n-\frac{1}{2}} \rangle &\leq \Delta t \left[\sum_{k=0}^{\frac{R}{2}-1} \frac{1}{(2k+1)!} \frac{\Delta t^{2k}}{2^{2k}} \left\| (I - \Pi_h^1) \frac{\partial^{2k+1} E}{\partial t^{2k+1}}(t^n) \right\|_{\varepsilon}^2 \right. \\ &\quad \left. + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 \right] + (R+2) \Delta t \left[\left\| \zeta_h^{n+\frac{1}{2}} \right\|_{\varepsilon}^2 + \left\| \zeta_h^{n-\frac{1}{2}} \right\|_{\varepsilon}^2 \right], \\ 2 \langle \mu \left(\xi^{n+1} - \xi^n \right), \xi_h^{n+1} + \xi_h^n \rangle &\leq \Delta t \left[\sum_{k=0}^{\frac{R}{2}-1} \frac{1}{(2k+1)!} \frac{\Delta t^{2k}}{2^{2k}} \left\| (I - \Pi_h^2) \frac{\partial^{2k+1} H}{\partial t^{2k+1}}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right. \\ &\quad \left. + \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu}^2 \right] + (R+2) \Delta t \left[\left\| \xi_h^{n+1} \right\|_{\mu}^2 + \left\| \xi_h^n \right\|_{\mu}^2 \right]. \end{aligned}$$

Using these inequalities for η , ζ and ξ in Equation (5.28), and summing from $n = 1$ to $N - 1$, we get that:

$$\begin{aligned} \left\| \eta_h^{N-\frac{1}{2}} \right\|_{\varepsilon^{-1}}^2 + \left\| \zeta_h^{N-\frac{1}{2}} \right\|_{\varepsilon}^2 + \left\| \xi_h^N \right\|_{\mu}^2 &\leq (R+4) \Delta t \sum_{n=0}^{N-1} \left[\left\| \eta_h^{n+\frac{1}{2}} \right\|_{\varepsilon^{-1}}^2 + \left\| \zeta_h^{n+\frac{1}{2}} \right\|_{\varepsilon}^2 + \left\| \xi_h^n \right\|_{\mu}^2 \right] \\ &\quad + \Delta t \sum_{n=0}^N \left[\sum_{k=0}^{\frac{R}{2}-1} \left(\frac{1}{(2k+1)!} \right)^2 \frac{\Delta t^{4k}}{2^{4k}} \left(\left\| (I - \Pi_h^0) \frac{\partial^{2k+1} p}{\partial t^{2k+1}}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial^{2k+1} E}{\partial t^{2k+1}}(t^{n-\frac{1}{2}}) \right\|_{\varepsilon}^2 \right. \right. \\ &\quad \left. \left. + \left\| (I - \Pi_h^2) \frac{\partial^{2k+1} H}{\partial t^{2k+1}}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right) + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) R_H^n \right\|_{\mu}^2 \right] \\ &\quad + \left\| R_p^n \right\|_{\varepsilon^{-1}}^2 + \left\| R_E^n \right\|_{\varepsilon}^2 + \left\| R_H^{n+\frac{1}{2}} \right\|_{\mu}^2 \left\| \nabla r_{R/2,p}^n \right\|_{\varepsilon^{-1}}^2 + \left\| \nabla \cdot r_{R/2,E}^n \right\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \left\| \nabla \times r_{R/2,E}^n \right\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \left\| \nabla \times r_{R/2,H}^{n+\frac{1}{2}} \right\|_{\mu}^2 \end{aligned}$$

$$\begin{aligned}
& + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1}^2 f(a)^2 \left(\|(\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \nabla r^n\|_{\left(R/2-\sum_{m=1}^a k_m\right),p}^2 + \|\nabla \cdot (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} r^n\|_{\left(R/2-\sum_{m=1}^a k_m\right),E}^2 \right. \\
& \quad \left. + (\varepsilon \mu)^{-(2\sum_{m=1}^a k_m+1)} \|(\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \nabla \times r^n\|_{\left(R/2-\sum_{m=1}^a k_m\right),E}^2 \right. \\
& \quad \left. + (\mu \varepsilon)^{-(2\sum_{m=1}^a k_m+1)} \|\nabla \times (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} r^{n+\frac{1}{2}}\|_{\left(R/2-\sum_{m=1}^a k_m\right),H}^2 \right) \Big] + \left[\|\eta_h^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^1\|_{\mu}^2 \right].
\end{aligned}$$

Likewise, for the semidiscrete approximation of the initial system as in Equations (5.8a) to (5.8c), following the same steps as for general one, we obtain for their errors the following inequality:

$$\begin{aligned}
& \|\eta_h^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^1\|_{\mu}^2 \leq \frac{(R+4)}{2} \Delta t \left[\|\eta_h^{\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\eta_h^0\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{\frac{1}{2}}\|_{\varepsilon}^2 + \|\zeta_h^0\|_{\varepsilon}^2 + \|\xi_h^1\|_{\mu}^2 + \|\xi_h^0\|_{\mu}^2 \right] \\
& \quad + \Delta t \left[\|\mathbf{R}_p^0\|_{\varepsilon^{-1}}^2 + \|\mathbf{R}_E^0\|_{\varepsilon}^2 + \|\mathbf{R}_H^{\frac{1}{2}}\|_{\mu}^2 + \|\nabla r_{R/2,p}^0\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{R/2,E}^0\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{R/2,E}^0\|_{\varepsilon}^2 \right. \\
& \quad + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{R/2,H}^{\frac{1}{2}}\|_{\mu}^2 + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} \frac{C_{k_1}^2}{4^{\sum_{m=1}^a k_m+1}} f(a)^2 \left(\|(\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \nabla r^0\|_{\left(R/2-\sum_{m=1}^a k_m\right),p}^2 \right. \\
& \quad + \|\nabla \cdot (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} r^0\|_{\left(R/2-\sum_{m=1}^a k_m\right),E}^2 + (\varepsilon \mu)^{-(2\sum_{m=1}^a k_m+1)} \|(\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} \nabla \times r^0\|_{\left(R/2-\sum_{m=1}^a k_m\right),E}^2 \\
& \quad \left. \left. + (\mu \varepsilon)^{-(2\sum_{m=1}^a k_m+1)} \|\nabla \times (\nabla \times \nabla \times)^{\sum_{m=1}^a k_m} r^{\frac{1}{2}}\|_{\left(R/2-\sum_{m=1}^a k_m\right),H}^2 \right) \right] + \left[\|\eta_h^0\|_{\varepsilon^{-1}}^2 + \|\zeta_h^0\|_{\varepsilon}^2 + \|\xi_h^0\|_{\mu}^2 \right].
\end{aligned}$$

Adding the last two inequalities, we arrive at the following resulting inequality:

$$\begin{aligned}
& \|\eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^N\|_{\mu}^2 \leq \\
& \left(\frac{1 + \frac{R+4}{2} \Delta t}{1 - \frac{R+4}{2} \Delta t} \right) \left[\|\eta_h^0\|_{\varepsilon^{-1}}^2 + \|\zeta_h^0\|_{\varepsilon}^2 + \|\xi_h^0\|_{\mu}^2 \right] + \left(\frac{(R+4) \Delta t}{1 - \frac{R+4}{2} \Delta t} \right) \sum_{n=0}^{N-1} \left[\|\eta_h^{n+\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{n+\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^{n+1}\|_{\mu}^2 \right] + \\
& \left(\frac{\Delta t}{1 - \frac{R+4}{2} \Delta t} \right) \sum_{n=0}^{N-1} \left[\sum_{k=0}^{\frac{R}{2}-1} \left(\frac{1}{(2k+1)!} \right)^2 \frac{\Delta t^{4k}}{2^{4k}} \left(\left\| (I - \Pi_h^0) \frac{\partial^{2k+1} p}{\partial t^{2k+1}}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial^{2k+1} E}{\partial t^{2k+1}}(t^n) \right\|_{\varepsilon}^2 \right. \right. \\
& \quad \left. \left. + \left\| (I - \Pi_h^2) \frac{\partial^{2k+1} H}{\partial t^{2k+1}}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right) + \left\| (I - \Pi_h^0) \mathbf{R}_p^n \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \mathbf{R}_E^n \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \mathbf{R}_H^{n+\frac{1}{2}} \right\|_{\mu}^2 + \|\mathbf{R}_p^n\|_{\varepsilon^{-1}}^2 + \|\mathbf{R}_E^n\|_{\varepsilon}^2 \right. \\
& \quad \left. + \|\mathbf{R}_H^{n+\frac{1}{2}}\|_{\mu}^2 + \|\nabla r_{R/2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{R/2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{R/2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1} \mu^{-1} \|\nabla \times r_{R/2,H}^{n+\frac{1}{2}}\|_{\mu}^2 \right. \\
& \quad \left. + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1}^2 f(a)^2 \left(\|(\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} \nabla r^n\|_{\left(R/2-\sum_{m=1}^a k_m\right),p}^2 + \|\nabla \cdot (\nabla \nabla \cdot)^{\sum_{m=1}^a k_m} r^n\|_{\left(R/2-\sum_{m=1}^a k_m\right),E}^2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& +(\varepsilon\mu)^{-(2\sum_{m=1}^a k_m+1)} \left\| (\nabla \times \nabla \times)_{m=1}^{\sum_{m=1}^a k_m} \nabla \times r^n \right\|_{\left(R/2-\sum_{m=1}^a k_m\right),E}^2 \\
& +(\mu\varepsilon)^{-(2\sum_{m=1}^a k_m+1)} \left\| \nabla \times (\nabla \times \nabla \times)_{m=1}^{\sum_{m=1}^a k_m} r^{n+\frac{1}{2}} \right\|_{\left(R/2-\sum_{m=1}^a k_m\right),H}^2 \Bigg].
\end{aligned}$$

We now apply the discrete Gronwall inequality with $\Delta t < \frac{1}{3(R+4)}$ to get that:

$$\begin{aligned}
& \|\eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^N\|_{\mu}^2 \leq \left[\frac{6\Delta t}{5} \sum_{n=0}^{N-1} \left(\sum_{k=0}^{\frac{R}{2}-1} \left(\frac{1}{(2k+1)!} \right)^2 \frac{\Delta t^{4k}}{2^{4k}} \left(\left\| (I - \Pi_h^0) \frac{\partial^{2k+1} p}{\partial t^{2k+1}}(t^n) \right\|_{\varepsilon^{-1}}^2 \right. \right. \right. \\
& \quad \left. \left. + \left\| (I - \Pi_h^1) \frac{\partial^{2k+1} E}{\partial t^{2k+1}}(t^n) \right\|_{\varepsilon}^2 + \left\| (I - \Pi_h^2) \frac{\partial^{2k+1} H}{\partial t^{2k+1}}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right) + \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon}^2 \\
& \quad + \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu}^2 + \|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 + \|\nabla r_{R/2,p}^n\|_{\varepsilon^{-1}}^2 + \|\nabla \cdot r_{R/2,E}^n\|_{\varepsilon}^2 + \varepsilon^{-1}\mu^{-1} \|\nabla \times r_{R/2,E}^n\|_{\varepsilon}^2 \\
& \quad + \varepsilon^{-1}\mu^{-1} \|\nabla \times r_{R/2,H}^{n+\frac{1}{2}}\|_{\mu}^2 + \sum_{a=1}^{\frac{R}{2}-1} \sum_{k_a=1}^{\frac{R}{2}-1-\sum_{b=1}^{a-1} k_b} C_{k_1}^2 f(a)^2 \left(\left\| (\nabla \nabla \cdot)_{m=1}^{\sum_{m=1}^a k_m} \nabla r^n \right\|_{\left(R/2-\sum_{m=1}^a k_m\right),p}^2 \right. \\
& \quad \left. + \left\| \nabla \cdot (\nabla \nabla \cdot)_{m=1}^{\sum_{m=1}^a k_m} r^n \right\|_{\left(R/2-\sum_{m=1}^a k_m\right),E}^2 + (\varepsilon\mu)^{-(2\sum_{m=1}^a k_m+1)} \left\| (\nabla \times \nabla \times)_{m=1}^{\sum_{m=1}^a k_m} \nabla \times r^n \right\|_{\left(R/2-\sum_{m=1}^a k_m\right),E}^2 \right. \\
& \quad \left. + (\mu\varepsilon)^{-(2\sum_{m=1}^a k_m+1)} \left\| \nabla \times (\nabla \times \nabla \times)_{m=1}^{\sum_{m=1}^a k_m} r^{n+\frac{1}{2}} \right\|_{\left(R/2-\sum_{m=1}^a k_m\right),H}^2 \right) \Bigg] \\
& \quad + \frac{7}{5} \left(\|\eta_h^0\|_{\varepsilon^{-1}}^2 + \|\zeta_h^0\|_{\varepsilon}^2 + \|\xi_h^0\|_{\mu}^2 \right) \exp(2(R+4)T).
\end{aligned}$$

Using our estimates for the Taylor remainders from Theorem 5.5 for these terms on the right-hand side of the above inequality, we further get that:

$$\begin{aligned}
\Delta t \sum_{n=0}^N \left[\|R_p^n\|_{\varepsilon^{-1}}^2 + \|R_E^n\|_{\varepsilon}^2 + \|R_H^{n+\frac{1}{2}}\|_{\mu}^2 \right] & \leq (\Delta t)^{2R} \left[\left\| \frac{\partial^{R+1} p}{\partial t^{R+1}} \right\|_{L^2(0,T;L_{\varepsilon^{-1}}^2(\Omega))}^2 + \left\| \frac{\partial^{R+1} E}{\partial t^{R+1}} \right\|_{L^2(0,T;L_{\varepsilon}^2(\Omega))}^2 \right. \\
& \quad \left. + \left\| \frac{\partial^{R+1} H}{\partial t^{R+1}} \right\|_{L^2(0,T;L_{\mu}^2(\Omega))}^2 \right].
\end{aligned}$$

Now, for $2 \leq S \leq R$, S even, $\ell > 0$, and Φ to be an appropriate operator acting on functions from correspondingly appropriate function spaces on Ω (as in the proof of Theorem 5.5), we have that:

$$\Delta t \sum_{n=0}^{N-1} (\Delta t)^{2(R-S)} \left[\|\Phi(r_{S/2,p}^n)\|_{\varepsilon^{-1}}^2 + (\mu\varepsilon)^{-\ell} \|\Phi(r_{S/2,E}^n)\|_{\varepsilon}^2 + (\mu\varepsilon)^{-\ell} \|\Phi(r_{S/2,H}^{n+\frac{1}{2}})\|_{\mu}^2 \right]$$

$$\leq (\Delta t)^{2R} \left[\left\| \frac{\partial^S(\Phi(p))}{\partial t^S} \right\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + (\mu\varepsilon)^{-\ell} \left\| \frac{\partial^S(\Phi(E))}{\partial t^S} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + (\mu\varepsilon)^{-\ell} \left\| \frac{\partial^S(\Phi(H))}{\partial t^S} \right\|_{L^2(0,T;L^2_{\mu}(\Omega))}^2 \right].$$

Next, for $p \in \dot{H}_{\varepsilon^{-1}}^1(\Omega)$, $E \in \dot{H}_{\varepsilon}(\text{curl}; \Omega)$ and $H \in \dot{H}_{\mu}(\text{div}; \Omega)$, there exists positive bounded constants $C_{0,p}$, $C_{0,E}$, $C_{0,H}$, $C_{m,p}$, $C_{m,E}$, $C_{m,H}$, for $0 \leq m \leq \frac{R}{2} - 1$ such that we have the following error bounds for the L^2 projections:

$$\begin{aligned} \left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}} &\leq C_{0,p} h^r \left\| R_p^n \right\|_{\varepsilon^{-1}}, & \left\| (I - \Pi_h^0) \frac{\partial^{2m+1} p}{\partial t^{2m+1}}(t^n) \right\|_{\varepsilon^{-1}} &\leq C_{m,p} h^r \left\| \frac{\partial^{2m+1} p}{\partial t^{2m+1}}(t^n) \right\|_{\varepsilon^{-1}}, \\ \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon} &\leq C_{0,E} h^r \left\| R_E^n \right\|_{\varepsilon}, & \left\| (I - \Pi_h^1) \frac{\partial^{2m+1} E}{\partial t^{2m+1}}(t^n) \right\|_{\varepsilon} &\leq C_{m,E} h^r \left\| \frac{\partial^{2m+1} E}{\partial t^{2m+1}}(t^n) \right\|_{\varepsilon}, \\ \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu} &\leq C_{0,H} h^r \left\| R_H^{n+\frac{1}{2}} \right\|_{\mu}, & \left\| (I - \Pi_h^2) \frac{\partial^{2m+1} H}{\partial t^{2m+1}}(t^{n+\frac{1}{2}}) \right\|_{\mu} &\leq C_{m,H} h^r \left\| \frac{\partial^{2m+1} H}{\partial t^{2m+1}}(t^{n+\frac{1}{2}}) \right\|_{\mu}. \end{aligned}$$

Set $C_0 := \max\{C_{0,p}, C_{0,E}, C_{0,H}, \max_{0 \leq m \leq R/2} \{C_{m,p}, C_{m,E}, C_{m,H}\}\}$. We therefore have that:

$$\begin{aligned} \Delta t \sum_{n=0}^N \left[\sum_{k=0}^{\frac{R}{2}-1} \left(\frac{1}{(2k+1)!} \right)^2 \frac{\Delta t^{4k}}{2^{4k}} \left(\left\| (I - \Pi_h^0) \frac{\partial^{2k+1} p}{\partial t^{2k+1}}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| (I - \Pi_h^1) \frac{\partial^{2k+1} E}{\partial t^{2k+1}}(t^n) \right\|_{\varepsilon}^2 \right. \right. \\ \left. \left. + \left\| (I - \Pi_h^2) \frac{\partial^{2k+1} H}{\partial t^{2k+1}}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right) \right] \\ \leq C_0 h^{2r} \sum_{n=0}^{N-1} \Delta t \left[\sum_{k=0}^{\frac{R}{2}-1} \left(\frac{1}{(2k+1)!} \right)^2 \frac{\Delta t^{4k}}{2^{4k}} \left(\left\| \frac{\partial^{2k+1} p}{\partial t^{2k+1}}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial^{2k+1} E}{\partial t^{2k+1}}(t^n) \right\|_{\varepsilon}^2 + \left\| \frac{\partial^{2k+1} H}{\partial t^{2k+1}}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right) \right] \\ \leq C_0 h^{2r} \sum_{k=0}^{\frac{R}{2}-1} \int_0^T \left[\left\| \frac{\partial^{2k+1} p}{\partial t^{2k+1}}(t^n) \right\|_{\varepsilon^{-1}}^2 + \left\| \frac{\partial^{2k+1} E}{\partial t^{2k+1}}(t^n) \right\|_{\varepsilon}^2 + \left\| \frac{\partial^{2k+1} H}{\partial t^{2k+1}}(t^{n+\frac{1}{2}}) \right\|_{\mu}^2 \right] dt \\ = C_0 h^{2r} \sum_{k=0}^{\frac{R}{2}-1} \left[\left\| \frac{\partial^{2k+1} p}{\partial t^{2k+1}} \right\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + \left\| \frac{\partial^{2k+1} E}{\partial t^{2k+1}} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + \left\| \frac{\partial^{2k+1} H}{\partial t^{2k+1}} \right\|_{L^2(0,T;L^2_{\mu}(\Omega))}^2 \right], \end{aligned}$$

and likewise for the L^2 projection of the remainder terms:

$$\begin{aligned} \Delta t \sum_{n=0}^N \left[\left\| (I - \Pi_h^0) R_p^n \right\|_{\varepsilon^{-1}} + \left\| (I - \Pi_h^1) R_E^n \right\|_{\varepsilon} + \left\| (I - \Pi_h^2) R_H^{n+\frac{1}{2}} \right\|_{\mu} \right] \\ \leq C_0 h^{2r} \sum_{n=0}^N \Delta t \left[\left\| R_p^n \right\|_{L^2_{\varepsilon^{-1}}(\Omega)} + \left\| R_E^n \right\|_{L^2_{\varepsilon}(\Omega)} + \left\| R_H^{n+\frac{1}{2}} \right\|_{L^2_{\mu}(\Omega)} \right] \\ \leq C_0 h^{2r} (\Delta t)^{2R} \left[\left\| \frac{\partial^{R+1} p}{\partial t^{R+1}} \right\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + \left\| \frac{\partial^{R+1} E}{\partial t^{R+1}} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + \left\| \frac{\partial^{R+1} H}{\partial t^{R+1}} \right\|_{L^2(0,T;L^2_{\mu}(\Omega))}^2 \right]. \end{aligned}$$

We next set $p_h^0 := \Pi_h^0 p_0$, $E_h^0 := \Pi_h^1 E_0$, and $H_h^0 := \Pi_h^2 H_0$, and note that for $0 \leq m \leq \frac{R}{2}$, $2 \leq S \leq R$, S even, $\ell > 0$, and for Φ to be an appropriate operator (as before), there exists positive bounded constants M_m and

M_S such that:

$$\begin{aligned} & \left\| \frac{\partial^{2m+1} p}{\partial t^{2m+1}} \right\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + \left\| \frac{\partial^{2m+1} E}{\partial t^{2m+1}} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + \left\| \frac{\partial^{2m+1} H^{2m+1}}{\partial t} \right\|_{L^2(0,T;L^2_{\mu}(\Omega))}^2 \leq M_m, \\ & \left\| \frac{\partial^S (\Phi(p))}{\partial t^S} \right\|_{L^2(0,T;L^2_{\varepsilon^{-1}}(\Omega))}^2 + (\mu\varepsilon)^{-\ell} \left\| \frac{\partial^S (\Phi(E))}{\partial t^S} \right\|_{L^2(0,T;L^2_{\varepsilon}(\Omega))}^2 + (\mu\varepsilon)^{-\ell} \left\| \frac{\partial^S (\Phi(H))}{\partial t^S} \right\|_{L^2(0,T;L^2_{\mu}(\Omega))}^2 \leq M_S, \end{aligned}$$

Consequently, we have the following estimate:

$$\|\eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}}^2 + \|\zeta_h^{N-\frac{1}{2}}\|_{\varepsilon}^2 + \|\xi_h^N\|_{\mu}^2 \leq \tilde{C} [h^{2r} + h^{2r}(\Delta t)^{2R} + (\Delta t)^{2R}],$$

where $\tilde{C} = 2 \max\{C_0 M_m, M_m, M_S\} \exp(2(R+4)T)$ for $0 \leq m \leq \frac{R}{2}$, $2 \leq S \leq R$, S even and then with the equivalence between 1- and 2-norms we get that:

$$\|\eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|\zeta_h^{N-\frac{1}{2}}\|_{\varepsilon} + \|\xi_h^N\|_{\mu} \leq C_1 [h^r + h^r(\Delta t)^R + (\Delta t)^R],$$

where $C_1 = 2\sqrt{\tilde{C}}$. Also, there are positive bounded constants $\tilde{C}_2$, $\tilde{C}_3$ and $\tilde{C}_4$ such that:

$$\begin{aligned} \|\eta^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} &= \|(I - \Pi_h^0)p(t^{N-\frac{1}{2}})\|_{\varepsilon^{-1}} \leq \tilde{C}_2 h^r \|p(t^{N-\frac{1}{2}})\|_{L^2_{\varepsilon^{-1}}(\Omega)} \implies \|\eta^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} \leq C_2 h^r, \\ \|\zeta^{N-\frac{1}{2}}\|_{\varepsilon} &= \|(I - \Pi_h^1)E(t^{N-\frac{1}{2}})\|_{\varepsilon} \leq \tilde{C}_3 h^r \|E(t^{N-\frac{1}{2}})\|_{L^2_{\varepsilon}(\Omega)} \implies \|\zeta^{N-\frac{1}{2}}\|_{\varepsilon} \leq C_3 h^r, \\ \|\xi^N\|_{\mu} &= \|(I - \Pi_h^2)H(t^N)\|_{\mu} \leq \tilde{C}_4 h^r \|H(t^N)\|_{L^2_{\mu}(\Omega)} \implies \|\xi^N\|_{\mu} \leq C_4 h^r, \end{aligned}$$

in which $C_2 = \tilde{C}_2 \|p(t^{N-\frac{1}{2}})\|_{L^2_{\varepsilon^{-1}}(\Omega)}$, $C_3 = \tilde{C}_3 \|E(t^{N-\frac{1}{2}})\|_{L^2_{\varepsilon}(\Omega)}$ and $C_4 = \tilde{C}_4 \|H(t^N)\|_{L^2_{\mu}(\Omega)}$ are all bounded positive constants due to Theorem 5.4. Finally, this provides us with our desired result by choosing $C = C_1 + C_2 + C_3 + C_4$:

$$\begin{aligned} \|e_{p_h}^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|e_{E_h}^{N-\frac{1}{2}}\|_{\varepsilon} + \|e_{H_h}^N\|_{\mu} &= \|\eta^{N-\frac{1}{2}} - \eta_h^{N-\frac{1}{2}}\|_{\varepsilon^{-1}} + \|\zeta^{N-\frac{1}{2}} - \zeta_h^{N-\frac{1}{2}}\|_{\varepsilon} + \|\xi^N - \xi_h^N\|_{\mu} \\ &\leq C [(\Delta t)^R + h^r + h^r(\Delta t)^R]. \end{aligned} \quad \square$$

5.4 Numerical Results

We present two examples in $\mathbb{R}^2$ as stated below, and demonstrate their numerical solution using LF_6 in conjunction with linear and quadratic Whitney simplicial finite elements on a triangulation of Ω . Our examples are both specified on the domain Ω which is a unit square in $\mathbb{R}^2$ and we compute the solution for time $t \in [0, 1]$.

Example 5.1: We consider the problem on Ω with analytical solutions, and material parameters as below.

$$p = 0, \quad E = \begin{bmatrix} \sin \pi y \cos \pi t \\ \sin \pi x \cos \pi t \end{bmatrix}, \quad H = (\cos \pi y - \cos \pi x) \sin \pi t,$$

$$\varepsilon = 1, \quad \mu = 1.$$

The results of solving this problem with LF_6 , and with linear and quadratic Whitney basis are shown in Figures 5.1 and 5.2, respectively.

Example 5.2: This problem is also on Ω but with $p \neq 0$. The boundary conditions for p and E are nonhomogeneous, and they are homogeneous for H . The analytical solutions and the material parameters are as below.

$$p = (\cos \pi x + \cos \pi y) \sin \pi t,$$

$$E = \begin{bmatrix} \sin \pi(\sqrt{2}t - x - y) - \sin \pi x \cos \pi t \\ -\sin \pi(\sqrt{2}t - x - y) - \sin \pi y \cos \pi t \end{bmatrix}, \quad H = -\sqrt{2} \sin \pi(\sqrt{2}t - x - y),$$

$$\epsilon = 1, \quad \mu = 1.$$

The results of solving this problem with LF_6 , and with linear and quadratic Whitney basis are shown in Figures 5.3 and 5.4, respectively.

We wish to conclude by stating that we have delineated an arbitrarily higher (even) order time discretization scheme LF_R for the (p, E, H) Maxwell's systems in $\mathbb{R}^2$ and $\mathbb{R}^3$. The LF_R scheme is implicit and uses two staggered time grids, and therefore is in the category of leapfrog schemes. We have analysed LF_R in the full generality in conjunction with compatible and arbitrary-order de Rham finite elements spanned by Whitney basis. In doing so, we have also demonstrated its energy conservation and the rate of convergence of its error in terms of the time step Δt , mesh parameter h , time order R , and spatial polynomial order r . Our chapter is thus comprehensive in both defining a generalized scheme and performing its error analysis. We hope that as future work our method can be extended to other instances of this Maxwell's system as it applies to, for example, scattering and reflection problems, and other similar interesting real-world problems in computational electromagnetics. We also hope that our work would be beneficial for the paradigm of scientific machine learning especially Physics-informed neural networks for Maxwell's equations.

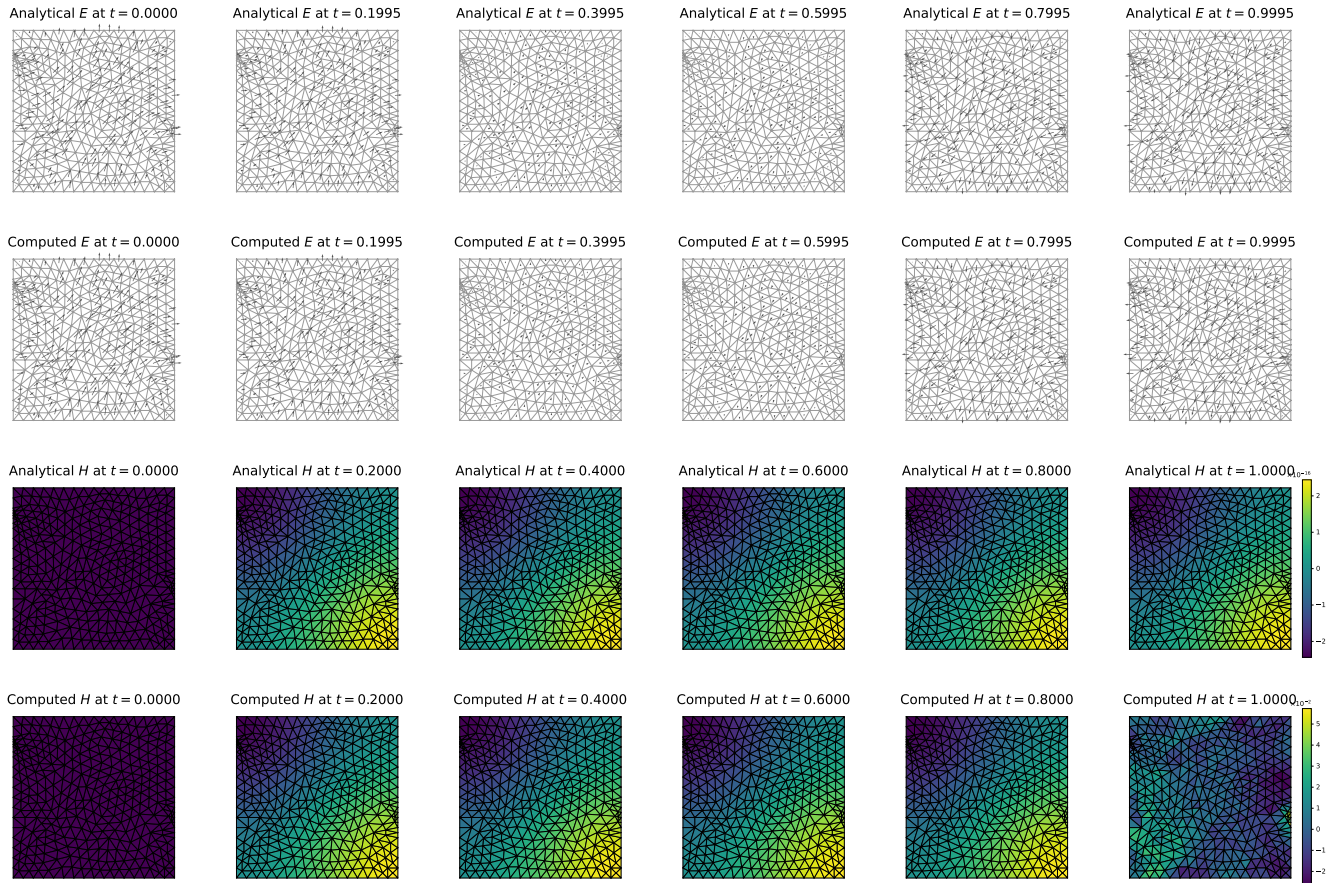


FIGURE 5.1: **Linear finite elements with LF_6** : Plots of solutions $(E_h^{n-1/2}, H_h^n)$ at different times for **Example 5.1** of Section 5.4 using LF_6 and linear Whitney forms for the FEEC spaces. The solutions for p are not shown due to them being identically 0. We also show projections of the analytical solutions to the linear Whitney forms spaces at these time steps for a visual comparison.

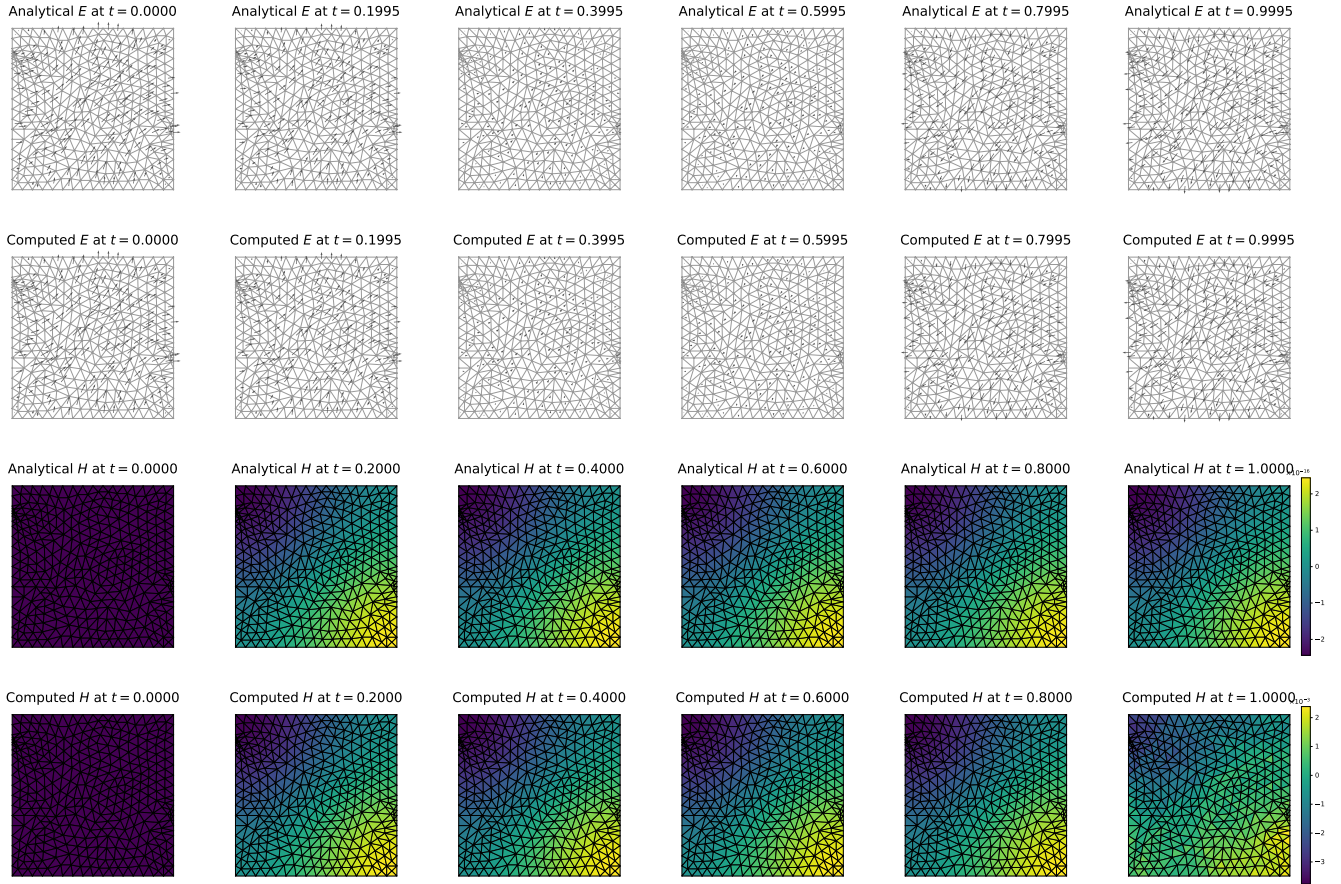


FIGURE 5.2: **Quadratic finite elements with LF_6 :** Plots of solutions $(E_h^{n-1/2}, H_h^n)$ at different times for **Example 5.1** of Section 5.4 using LF_6 and quadratic Whitney forms in FEEC. The solutions for p are again not shown due to them being identically 0. We also show projections of the analytical solutions to the quadratic Whitney forms spaces at these time steps for a visual comparison.

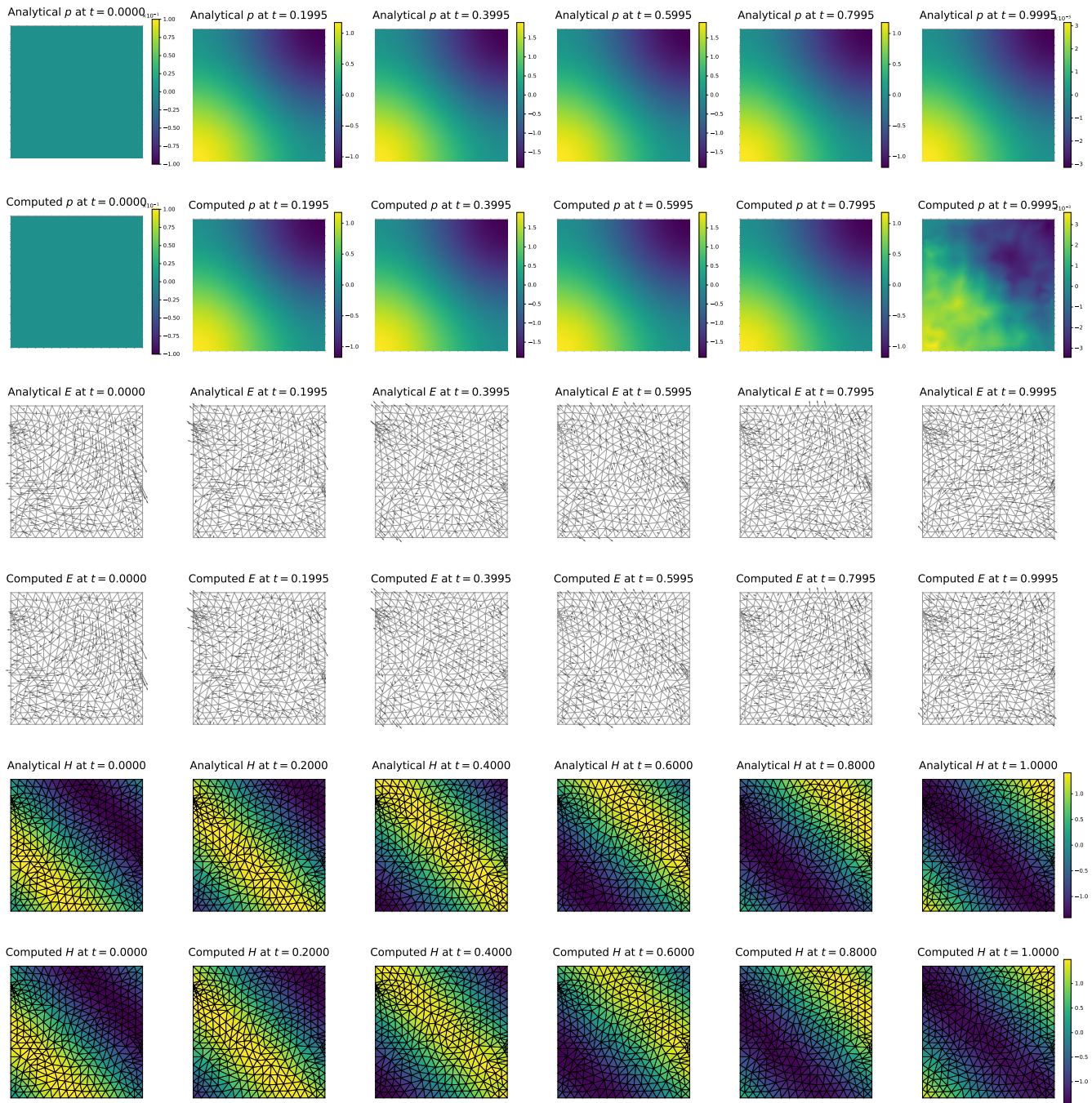


FIGURE 5.3: **Linear finite elements with LF₆:** Plots of solutions $(p_h^{n+1/2}, E_h^{n-1/2}, H_h^n)$ at different times for **Example 5.2** of Section 5.4 using the LF₆ and linear Whitney forms in FEEC. We also show projections of the analytical solutions to the linear Whitney forms spaces at these time steps for a visual comparison.

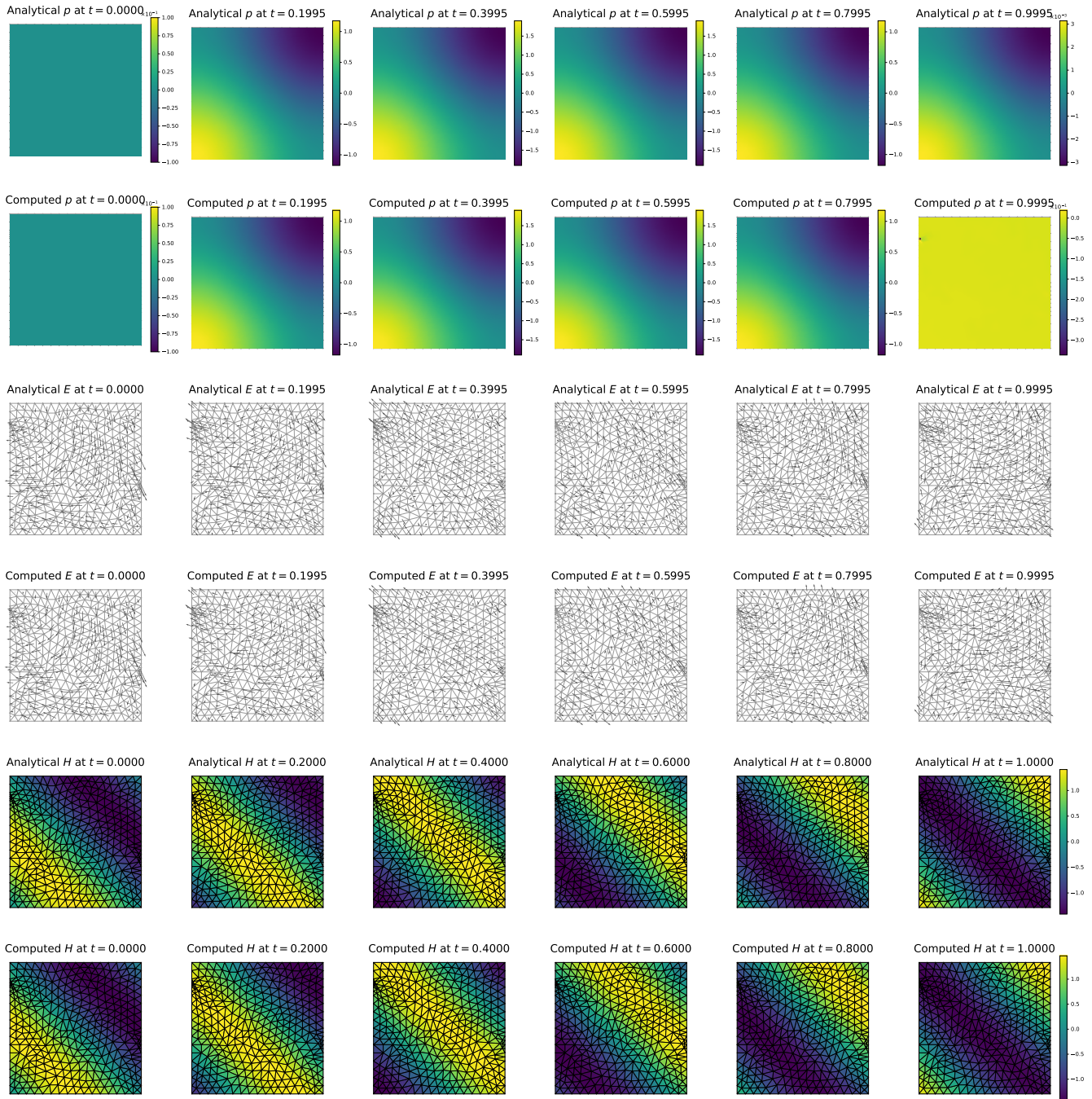


FIGURE 5.4: Quadratic finite elements with LF₆: Plots of solutions $(p_h^{n-1/2}, E_h^{n-1/2}, H_h^n)$ at different times for **Example 5.2** of Section 5.4 using the LF₆ and quadratic Whitney forms in FEEC. We also show projections of the analytical solutions to the quadratic Whitney forms spaces at these time steps for a visual comparison.

Chapter 6

Summary and Future Directions

6.1 Summary

1. Our work in this thesis focuses on higher-order discretizations of Maxwell's equations in both space and time. We begin with a brief introduction including an overview of the recent literature following which we present the finite element method for linear PDEs with their stability analysis. We also introduce Maxwell's equations with basic time discretization techniques, de Rham finite element spaces and several fundamental inequalities that are used throughout this work.
2. Subsequently, we discuss in detail the implementation of higher-order finite element discretization using the de Rham sequence of finite element spaces for two- and three-dimensional Poisson's equations and for various boundary conditions. Our discussion here includes also some notes on the interpolation of solutions and computation of L^2 errors. We also have a brief discussion on the non-trivial kernel problems and present computational results on Poisson's problems, eigenvalue problem, and basic discretizations of Maxwell's equations.
3. We then study two second-order in time discretizations for Maxwell's equations: the classical Crank-Nicolson and our proposed implicit leapfrog scheme. For both schemes, we establish stability results and estimates for discrete energy, semi-discrete in time, and full errors.
4. Following this, we introduce two fourth-order in time discretizations for Maxwell's equations: LF_4 scheme or the fourth-order leapfrog scheme, and TS_4 scheme or the fourth-order temporal scheme. These schemes are developed using two different strategies which we term as the spatial strategy and the temporal strategy, respectively. These strategies also provide a generalization framework for the classical second-order Crank-Nicolson scheme and our proposed second-order implicit leapfrog scheme. We provide a comprehensive analysis of stability, discrete energy conservation, and error estimates for each of these schemes.
5. Motivated by these developments, we generalize the LF_4 scheme to an arbitrary, even order in time, method leading to the LF_R scheme (an R^{th} order leapfrog scheme), where R is an arbitrary even integer. We establish again the stability of this general scheme and its energy conservation, and also provide error estimates for this scheme. For clarity and illustration, we explicitly present in this context the various proofs for the sixth-order version of the scheme to make the generalized proofs more readily accessible.
6. In the presentation of all our schemes, we complement the various theoretical analysis with ample computational results on model problems with known exact solutions, in both 2d and 3d for the second-order schemes and only in 2d for the remaining higher-order schemes. Thereby, we validate our various error analysis with concrete numerical solutions as well.

6.2 Future Directions

In this work, we have tried to comprehensively address some aspects of energy conserving, higher-order discretizations for a system of Maxwell's equations to be used in conjunction with higher-order de Rham finite elements. Nevertheless, there are many interesting problems stemming from our work which we simply could not get to work on, and we believe that these offshoot problems are not only useful to pursue further but can also help clarify the applicability of our developed methods and their analysis. Some of our proposed problems are related to further development of our methods, in particular, the time strategy that we have proposed, and others pertain to the application of our methods for other kinds of electromagnetics problems. We also list some other numerical aspects that we simply could not concern ourselves with in this work. We hope that this list of questions will provide valuable directions for further research and development of our thesis.

1. **Non-trivial homology and higher-order harmonic fields.** We have already presented some foundational computations on domains with non-trivial homology in Section 2.7. A natural next step, therefore, is to extend the entire framework developed in this thesis to such domains with non-trivial topology. As a minor related work here, we also believe that there can be further computational developments for obtaining a higher-order harmonic vector field using the Hodge isomorphism as in [Hir+23].

Proposed Direction: Perform necessary computations to approximate a harmonic vector field using higher-order polynomial spaces. Extend the analysis for our higher-order time discretization schemes to include the problems on domains with non-trivial homology.

2. **Solution of Maxwell's equations with electric circuit coupling.** A field-circuit coupled problem consists of Maxwell's equations being driven by an electrical circuit and is an example of multiphysics coupling. It brings together many of our theoretical and computational underpinnings to such a multiphysics setup. An abstract representation for such an electromagnetic fields-electric circuit coupled problem is shown in Figure 6.1. As can be seen, this problem has electric and magnetic boundary conditions in different parts of the boundary of the domain. The circuit elements introduce holes and voids, and any computational solution will have to account for the harmonic vector fields being in the kernel of the vector Laplacian. As an example, [HO21], such computations are discussed for the lowest-order finite element spatial approximation, and in a descriptive manner without precise error analysis. We can now do better and consider higher-order time and spatial discretizations for this problem.

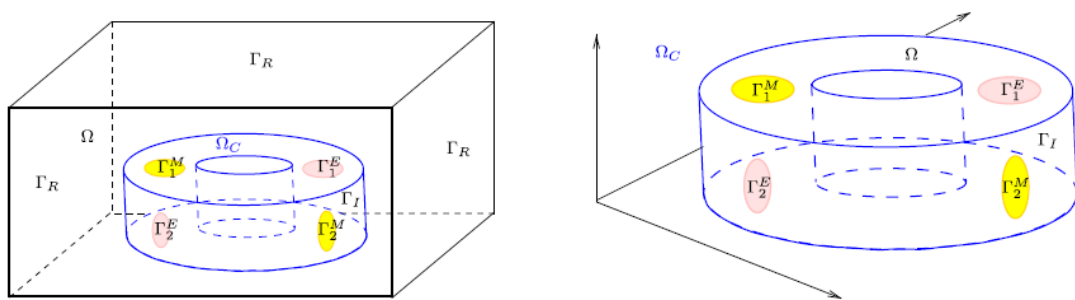


FIGURE 6.1: Figure taken from [HO21]

Proposed Direction: Obtain computational results for the solution of the Maxwell's equation using higher-order finite element and time discretizations for the circuit coupled problems by also considering that the harmonic vector fields are also approximated using higher-order polynomial spaces. Provide necessary error analysis for these higher-order computations, and also demonstrate on some standard electrical engineering benchmark problems, for example, as in [Duf+14].

3. **Scattering and other similar problems.** This proposal is a relatively straightforward application of our developments to a different class of electromagnetic problems, namely, scattering or reflection of incident fields. The following set of equations models a simplified scattering problem and in these equations, the subscript “inc” is used for incident fields and the subscript “sca” for scattered fields where a source is usually emitting such electromagnetic radiation.

$$\begin{aligned}\frac{\partial}{\partial t}(p_{\text{inc}} + p_{\text{sca}}) + \nabla \cdot (E_{\text{inc}} + E_{\text{sca}}) &= f_p, \\ \nabla(p_{\text{inc}} + p_{\text{sca}}) + \epsilon \frac{\partial(E_{\text{inc}} + E_{\text{sca}})}{\partial t} - \mu \nabla \times (H_{\text{inc}} + H_{\text{sca}}) &= f_E, \\ \mu \frac{\partial}{\partial t}(H_{\text{inc}} + H_{\text{sca}}) + \nabla \times (E_{\text{inc}} + E_{\text{sca}}) &= f_H.\end{aligned}$$

$$\begin{aligned}\frac{\partial p_{\text{inc}}}{\partial t} + \nabla \cdot E_{\text{inc}} &= 0, & \frac{\partial p_{\text{sca}}}{\partial t} + \nabla \cdot E_{\text{sca}} &= f_p, \\ \nabla p_{\text{inc}} + \epsilon \frac{\partial E_{\text{inc}}}{\partial t} - \mu \nabla \times H_{\text{inc}} &= 0, & \nabla p_{\text{sca}} + \epsilon \frac{\partial E_{\text{sca}}}{\partial t} - \mu \nabla \times H_{\text{sca}} &= f_E, \\ \mu \frac{\partial H_{\text{inc}}}{\partial t} + \nabla \times E_{\text{inc}} &= 0. & \mu \frac{\partial H_{\text{sca}}}{\partial t} + \nabla \times E_{\text{sca}} &= f_H.\end{aligned}$$

The scattering problem models many real-world usage of engineering electromagnetics, for example, mobile phones communicating with a mobile tower, or radar tracking of objects such as flights, rockets, or even cars and humans on roads. In principles, the domains for such problems are *open* with the scattered fields allowed to radiate to infinity. In practice, however, one uses absorbing boundary conditions (ABCs) on some suitable restriction to a compact domain. The governing equation for ABCs is the following:

$$L(u) = \beta(x)\nabla \times \nabla \times u + \alpha(x)u_t - \beta(x)\nabla \nabla \cdot u = 0.$$

Consequently, this problem readily lends itself to application of our developments.

Proposed Direction: Compute a higher-order solution of the radiating electromagnetic fields due to a source excitation in conjunction with higher-order energy conserving temporal discretizations. Also, provide arbitrarily higher-order model for the ABCs which can then be consistently used with our discretizations.

4. **Arbitrarily higher-order temporal discretization TS_R .** In Chapter 4, we have proposed the temporal strategy as one of our frameworks for obtaining higher-order time discretizations. However, this strategy allows for a multitude of schemes for larger orders, and in particular, it is not easy to write down a generalized analysis for an arbitrary order R . We leave this to a future study to explore as well as clearly elucidate the analysis for the combinatorial explosion of such schemes of order R .

Proposed Direction: Organize and perform the analysis for an arbitrary temporal discretization strategy scheme TS_R .

5. **Linear solvers** Our final proposal is a related problem to the higher order discretizations in this work. In all our validating numerical computations, we have “black boxed” the linear solver using direct and iterative solvers as and when we needed them. For example, LF_4 computations in this work required us to switch using iterative solvers from a sparse direct solver. We leave this structured study of the problem of devising appropriate linear solvers including preconditioners, and multigrid-like methods to a future work.

Proposed Direction: Formulate sparse direct and iterative solvers and investigate appropriate preconditioners including both of the algebraic and geometric multigrid variety as necessary. Furthermore, devise such solvers to be amenable to parallel computations on general-purpose parallel processors.

Appendix A

Curl and Divergence of Arbitrary Whitney basis

Suppose, for instance, we need to compute the curl and divergence for quadratic Whitney basis corresponding to the edges of a triangle which are given by:

$$W_{01_0} = \lambda_0(\lambda_0 \nabla \lambda_1 - \lambda_1 \nabla \lambda_0) = \lambda_0^2 \nabla \lambda_1 - \lambda_0 \lambda_1 \nabla \lambda_0.$$

We start with the computation of curl of this Whitney basis, so consider

$$\begin{aligned} \Rightarrow \nabla \times W_{01_0} &= \nabla \times (\lambda_0^2 \nabla \lambda_1 - \lambda_0 \lambda_1 \nabla \lambda_0), \\ &= \nabla \lambda_0^2 \times \nabla \lambda_1 - \nabla(\lambda_0 \lambda_1) \times \nabla \lambda_0, \quad (\text{Using } \nabla(\lambda_i \nabla \lambda_j) = \nabla \lambda_i \times \nabla \lambda_j) \\ &= 2\lambda_0 \nabla \lambda_0 \times \nabla \lambda_1 - (\lambda_0 \nabla \lambda_1 + \lambda_1 \nabla \lambda_0) \times \nabla \lambda_0. \end{aligned}$$

To simplify, in general, we can compute the value of $\nabla \times (\lambda_0^a \lambda_1^b \lambda_2^c \nabla \lambda_t)$ and use it for particular values. So, consider

$$\begin{aligned} \nabla(\lambda_0^a \lambda_1^b \lambda_2^c) &= \nabla(\lambda_0^a) \lambda_1^b \lambda_2^c + \lambda_0^a \nabla(\lambda_1^b \lambda_2^c), \\ &= (a\lambda_0^{a-1} \nabla \lambda_0)(\lambda_1^b \lambda_2^c) + \lambda_0^a (\lambda_1^b \nabla(\lambda_2^c) + \lambda_2^c \nabla(\lambda_1^b)), \\ &= a\lambda_0^{a-1} \lambda_1^b \lambda_2^c \nabla \lambda_0 + b\lambda_0^a \lambda_1^{b-1} \lambda_2^c \nabla \lambda_1 + c\lambda_0^a \lambda_1^b \lambda_2^{c-1} \nabla \lambda_2. \end{aligned}$$

Thus, we have:

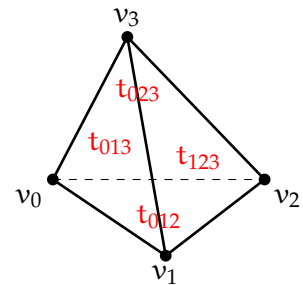
$$\begin{aligned} \nabla \times (\lambda_0^a \lambda_1^b \lambda_2^c \nabla \lambda_t) &= \nabla(\lambda_0^a \lambda_1^b \lambda_2^c) \times \nabla \lambda_t, \\ &= (a\lambda_0^{a-1} \lambda_1^b \lambda_2^c \nabla \lambda_0 + b\lambda_0^a \lambda_1^{b-1} \lambda_2^c \nabla \lambda_1 + c\lambda_0^a \lambda_1^b \lambda_2^{c-1} \nabla \lambda_2) \times \nabla \lambda_t. \end{aligned}$$

Next, consider the linear Whitney basis on faces(triangles) of tetrahedron which are given by:

$$\phi_{ijk} = \lambda_i d\lambda_j \wedge d\lambda_k - \lambda_j d\lambda_i \wedge d\lambda_k + \lambda_k d\lambda_i \wedge d\lambda_j,$$

where $\wedge = \times$ (cross product). So, we have the basis as:

$$\begin{aligned} W_{012} &= \lambda_0 \nabla \lambda_1 \times \nabla \lambda_2 - \lambda_1 \nabla \lambda_0 \times \nabla \lambda_2 + \lambda_2 \nabla \lambda_0 \times \nabla \lambda_1, \\ W_{013} &= \lambda_0 \nabla \lambda_1 \times \nabla \lambda_3 - \lambda_1 \nabla \lambda_0 \times \nabla \lambda_3 + \lambda_3 \nabla \lambda_0 \times \nabla \lambda_1, \\ W_{023} &= \lambda_0 \nabla \lambda_2 \times \nabla \lambda_3 - \lambda_2 \nabla \lambda_0 \times \nabla \lambda_3 + \lambda_3 \nabla \lambda_0 \times \nabla \lambda_2, \\ W_{123} &= \lambda_1 \nabla \lambda_2 \times \nabla \lambda_3 - \lambda_2 \nabla \lambda_1 \times \nabla \lambda_3 + \lambda_3 \nabla \lambda_1 \times \nabla \lambda_2. \end{aligned}$$



We will now compute the curl of these Whitney basis W_{ijk} , where

$$W_{ijk} = \lambda_i \nabla \lambda_j \times \nabla \lambda_k - \lambda_j \nabla \lambda_i \times \nabla \lambda_k + \lambda_k \nabla \lambda_i \times \nabla \lambda_j.$$

In general, we compute $\nabla \times \underbrace{(\lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d \nabla \lambda_s)}_A \times \underbrace{(\nabla \lambda_t)}_B$. We will use the following identity to compute the value of this curl:

$$\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B.$$

First consider $A(\nabla \cdot B) = (\lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d)(\nabla \cdot \nabla \lambda_t) = 0$, because

$$\nabla \cdot \nabla \lambda_t = \nabla \cdot \begin{bmatrix} dl_{0,t} \\ dl_{1,t} \\ dl_{2,t} \end{bmatrix} = \frac{\partial}{\partial x} dl_{0,t} + \frac{\partial}{\partial y} dl_{1,t} + \frac{\partial}{\partial z} dl_{2,t} = 0.$$

Similarly, $(B \cdot \nabla)A = 0$. Then $B(\nabla \cdot A) = \nabla \lambda_t (\nabla \cdot (\lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d \nabla \lambda_s))$, where

$$\begin{aligned} \nabla \cdot (\lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d \nabla \lambda_s) &= \nabla \cdot \begin{bmatrix} \lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d dl_{0,s} \\ \lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d dl_{1,s} \\ \lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d dl_{2,s} \end{bmatrix}, \\ &= dl_{0,s} \frac{\partial}{\partial x} (\lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d) + dl_{1,s} \frac{\partial}{\partial x} (\lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d) + dl_{2,s} \frac{\partial}{\partial x} (\lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d), \\ &= dl_{0,s} \left[a \lambda_0^{a-1} \lambda_1^b \lambda_2^c \lambda_3^d \frac{\partial \lambda_0}{\partial x} + b \lambda_0^a \lambda_1^{b-1} \lambda_2^c \lambda_3^d \frac{\partial \lambda_1}{\partial x} + c \lambda_0^a \lambda_1^b \lambda_2^{c-1} \lambda_3^d \frac{\partial \lambda_2}{\partial x} \right. \\ &\quad \left. + d \lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^{d-1} \frac{\partial \lambda_3}{\partial x} \right] + dl_{1,s} \left[a \lambda_0^{a-1} \lambda_1^b \lambda_2^c \lambda_3^d \frac{\partial \lambda_0}{\partial y} + b \lambda_0^a \lambda_1^{b-1} \lambda_2^c \lambda_3^d \frac{\partial \lambda_1}{\partial y} \right. \\ &\quad \left. + c \lambda_0^a \lambda_1^b \lambda_2^{c-1} \lambda_3^d \frac{\partial \lambda_2}{\partial y} + d \lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^{d-1} \frac{\partial \lambda_3}{\partial y} \right] + dl_{2,s} \left[a \lambda_0^{a-1} \lambda_1^b \lambda_2^c \lambda_3^d \frac{\partial \lambda_0}{\partial z} \right. \\ &\quad \left. + b \lambda_0^a \lambda_1^{b-1} \lambda_2^c \lambda_3^d \frac{\partial \lambda_1}{\partial z} + c \lambda_0^a \lambda_1^b \lambda_2^{c-1} \lambda_3^d \frac{\partial \lambda_2}{\partial z} + d \lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^{d-1} \frac{\partial \lambda_3}{\partial z} \right], \end{aligned}$$

and $(A \cdot \nabla)B = \left((\lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d \nabla \lambda_s) \cdot \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \right) \nabla \lambda_t = (\nabla \cdot (\lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d)) \nabla \lambda_t$. So, $\nabla \times (\lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d \nabla \lambda_s \times \nabla \lambda_t) = -\nabla \lambda_t (\nabla \cdot (\lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d \nabla \lambda_s)) - (\nabla \cdot (\lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d)) \nabla \lambda_t$. Now we compute divergence of above Whitney basis, i.e.,

$$\nabla \cdot W_{ijk} = \nabla \cdot (\lambda_i \nabla \lambda_j \times \nabla \lambda_k - \lambda_j \nabla \lambda_i \times \nabla \lambda_k + \lambda_k \nabla \lambda_i \times \nabla \lambda_j).$$

Consider $\nabla \cdot (\lambda_i \nabla \lambda_j \times \nabla \lambda_k) = (\nabla \times \lambda_i \nabla \lambda_j) \cdot \nabla \lambda_k - \underbrace{(\nabla \times \nabla \lambda_k)}^0 \cdot \lambda_i \nabla \lambda_j = (\nabla \times \lambda_i \nabla \lambda_j) \cdot \nabla \lambda_k$

More generally,

$$\begin{aligned} \nabla \cdot (\lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d \nabla \lambda_s \times \nabla \lambda_t) &= \nabla \cdot ((\lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d \nabla \lambda_s) \cdot \nabla \lambda_t) - \underbrace{(\nabla \times \nabla \lambda_t)}^0 \cdot \lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d \nabla \lambda_s, \\ &= \nabla \cdot ((\lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^d \nabla \lambda_s) \cdot \nabla \lambda_t), \\ &= ((a \lambda_0^{(a-1)} \lambda_1^b \lambda_2^c \lambda_3^d \nabla \lambda_0 + b \lambda_0^a \lambda_1^{(b-1)} \lambda_2^c \lambda_3^d \nabla \lambda_1 \\ &\quad + c \lambda_0^a \lambda_1^b \lambda_2^{(c-1)} \lambda_3^d \nabla \lambda_2 + d \lambda_0^a \lambda_1^b \lambda_2^c \lambda_3^{(d-1)} \nabla \lambda_3) \times \nabla \lambda_s) \cdot \nabla \lambda_t. \end{aligned}$$

Appendix B

Curl and Divergence of Arbitrary Raviart-Thomas basis

Similar to Whitney basis, we can compute curl and divergence for Raviart-Thomas basis which are defined as:

$$RT_{ij} = (\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i)_{\text{rotated}} = (\lambda_i \nabla \lambda_j)_{\text{rotated}} - (\lambda_j \nabla \lambda_i)_{\text{rotated}}.$$

Again starting with the computation of curl of RT_i s, consider as follows:

$$\begin{aligned} \text{curl}((\lambda_0^a \lambda_1^b \lambda_2^c \nabla \lambda_t)_{\text{rotated}}) &= \nabla \cdot \begin{bmatrix} -\lambda_0^a \lambda_1^b \lambda_2^c dl_{1,t} \\ \lambda_0^a \lambda_1^b \lambda_2^c dl_{0,t} \end{bmatrix}, \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\lambda_0^a \lambda_1^b \lambda_2^c dl_{1,t} & \lambda_0^a \lambda_1^b \lambda_2^c dl_{0,t} & 0 \end{vmatrix}, \\ &= \begin{bmatrix} dl_{0,t} \left(a \lambda_0^{a-1} \lambda_1^b \lambda_2^c \frac{\partial \lambda_0}{\partial x} + b \lambda_0^a \lambda_1^{b-1} \lambda_2^c \frac{\partial \lambda_1}{\partial x} + c \lambda_0^a \lambda_1^b \lambda_2^{c-1} \frac{\partial \lambda_2}{\partial x} \right) \\ + dl_{1,t} \left(a \lambda_0^{a-1} \lambda_1^b \lambda_2^c \frac{\partial \lambda_0}{\partial y} + b \lambda_0^a \lambda_1^{b-1} \lambda_2^c \frac{\partial \lambda_1}{\partial y} + c \lambda_0^a \lambda_1^b \lambda_2^{c-1} \frac{\partial \lambda_2}{\partial y} \right) \end{bmatrix} \hat{z}. \end{aligned}$$

or $RT_{ij} = (\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i)_{\text{rotated}}$.

$\implies \nabla \times RT_{ij} = \nabla \times (\lambda_i \nabla \lambda_j)_{\text{rotated}} - \nabla \times (\lambda_j \nabla \lambda_i)_{\text{rotated}}$, where

$$\begin{aligned} \nabla \times (\lambda_i \nabla \lambda_j)_{\text{rotated}} &= \nabla \times \left(\lambda_i \begin{bmatrix} dl_{0,j} \\ dl_{1,j} \end{bmatrix}_{\text{rotated}} \right), \\ &= \nabla \times \left(\lambda_i \begin{bmatrix} -dl_{1,j} \\ dl_{0,j} \end{bmatrix} \right), \\ &= \underbrace{\lambda_i \nabla \times \begin{bmatrix} -dl_{1,j} \\ dl_{0,j} \end{bmatrix}}_{=0} + \nabla \lambda_i \times \begin{bmatrix} -dl_{1,j} \\ dl_{0,j} \end{bmatrix}, \\ &= \nabla \lambda_i \times \begin{bmatrix} -dl_{1,j} \\ dl_{0,j} \end{bmatrix}. \end{aligned}$$

Therefore, $\nabla \times RT_{ij} = \nabla \lambda_i \times \begin{bmatrix} -dl_{1,j} \\ dl_{0,j} \end{bmatrix} - \nabla \lambda_j \times \begin{bmatrix} -dl_{1,i} \\ dl_{0,i} \end{bmatrix}$.

Similarly, we can compute the divergence of Whitney basis as follows:

$$\begin{aligned}
 \operatorname{div} RT_{ij} &= \operatorname{div}((\lambda_i \nabla \lambda_j)_{\text{rotated}}) - \operatorname{div}((\lambda_j \nabla \lambda_i)_{\text{rotated}}), \\
 &= \operatorname{div} \begin{bmatrix} -\lambda_i b_j \\ \lambda_i a_j \end{bmatrix} - \operatorname{div} \begin{bmatrix} -\lambda_j b_i \\ \lambda_j a_i \end{bmatrix}, \\
 &= \left(-b_j \frac{\partial \lambda_i}{\partial x} + a_j \frac{\partial \lambda_i}{\partial y} \right) - \left(-b_i \frac{\partial \lambda_j}{\partial x} + a_i \frac{\partial \lambda_j}{\partial y} \right).
 \end{aligned}$$

So, in general we can compute:

$$\begin{aligned}
 \operatorname{div}((\lambda_0^a \lambda_1^b \lambda_2^c \nabla \lambda_t)_{\text{rotated}}) &= \nabla \cdot \left(\left(\lambda_0^a \lambda_1^b \lambda_2^c \begin{bmatrix} dl_{0,t} \\ dl_{1,t} \end{bmatrix} \right)_{\text{rotated}} \right), \\
 &= \nabla \cdot \begin{bmatrix} -\lambda_0^a \lambda_1^b \lambda_2^c dl_{1,t} \\ \lambda_0^a \lambda_1^b \lambda_2^c dl_{0,t} \end{bmatrix}, \\
 &= -dl_{1,t} \left(a \lambda_0^{a-1} \lambda_1^b \lambda_2^c \frac{\partial \lambda_0}{\partial x} + b \lambda_0^a \lambda_1^{b-1} \lambda_2^c \frac{\partial \lambda_1}{\partial x} + c \lambda_0^a \lambda_1^b \lambda_2^{c-1} \frac{\partial \lambda_2}{\partial x} \right) \\
 &\quad + dl_{0,t} \left(a \lambda_0^{a-1} \lambda_1^b \lambda_2^c \frac{\partial \lambda_0}{\partial y} + b \lambda_0^a \lambda_1^{b-1} \lambda_2^c \frac{\partial \lambda_1}{\partial y} + c \lambda_0^a \lambda_1^b \lambda_2^{c-1} \frac{\partial \lambda_2}{\partial y} \right).
 \end{aligned}$$

Appendix C

Example Symbolic Computations

We discuss some example computations on 2d torus, 2d and 3d annulus.

C.1 Solid Torus

We try explaining the conversion of coordinates with the help of an example. For this, let analytical solution to be $u = (r - r_0)^2 r \hat{r} = u_r \hat{r}$ where $r = 0.25, R = 1, h = \hat{\phi} \implies u \cdot h = 0$. And

$$\begin{aligned} \sigma = \operatorname{div} u &= \nabla \cdot (u_r \hat{r} + u_\theta \hat{\theta} + u_\phi \hat{\phi}), \\ &= \frac{\partial u_r}{\partial r} + \left(\frac{1}{r} + \frac{\cos \theta}{R + r \cos \theta} \right) u_r + \underbrace{\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} - \frac{\sin \theta}{R + r \cos \theta} u_\theta + \frac{1}{R + r \cos \theta} \frac{\partial u_\phi}{\partial \phi}}_{\substack{\parallel \\ 0}}, \quad (\text{because } u_\theta = 0 = u_\phi) \\ &= (r - r_0) \left[4r - 2r_0 + \frac{r(r - r_0) \cos \theta}{R + r \cos \theta} \right]. \end{aligned}$$

$$\begin{aligned} \implies \operatorname{grad} \operatorname{div} u &= \operatorname{grad} \sigma = \frac{\partial \sigma}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \sigma}{\partial \theta} r \hat{\theta} + \frac{1}{R + r \cos \theta} \frac{\partial \sigma}{\partial \phi} (R + r \cos \theta) \hat{\phi}, \\ &= \left[\left(4r - 2r_0 + \frac{r(r - r_0) \cos \theta}{R + r \cos \theta} \right) \right. \\ &\quad \left. + (r - r_0) \left(4 + \frac{(2r - r_0)(R + r \cos \theta) \cos \theta - r(r - r_0) \cos \theta}{(R + r \cos \theta)^2} \right) \right] \hat{r} \\ &\quad + \frac{1}{r} \left[(r - r_0) \frac{r(r - r_0)(-\sin \theta)(R + r \cos \theta) - r(r - r_0) \cos \theta(-r \sin \theta)}{(R + r \cos \theta)^2} \right] r \hat{\theta}. \end{aligned}$$

$$\operatorname{curl} u = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \frac{1}{r(R + r \cos \theta)} \frac{\partial}{\partial r} & \frac{1}{R + r \cos \theta} \frac{\partial}{\partial \theta} & \frac{1}{r} \frac{\partial}{\partial \phi} \\ u_r & r u_\theta & (R + r \cos \theta) u_\phi \end{vmatrix} = 0, \quad (\text{because } u_\theta = 0 = u_\phi)$$

$$\implies \operatorname{curl} \operatorname{curl} u = 0.$$

$$\implies f = \operatorname{curl} \operatorname{curl} u - \operatorname{grad} \operatorname{div} u = -\operatorname{grad} \operatorname{div} u,$$

$$\begin{aligned}
&= - \left[\underbrace{\left(4r - 2r_0 + \frac{r(r-r_0)\cos\theta}{R+r\cos\theta} \right) + (r-r_0) \left(4 + \frac{(2r-r_0)(R+r\cos\theta)\cos\theta - r(r-r_0)\cos\theta}{(R+r\cos\theta)^2} \right)}_{f_r} \right] \hat{r} \\
&\quad - \underbrace{\frac{1}{r} \left[(r-r_0) \frac{r(r-r_0)(-\sin\theta)(R+r\cos\theta) - r(r-r_0)\cos\theta(-r\sin\theta)}{(R+r\cos\theta)^2} \right]}_{f_\theta} r \hat{\theta}.
\end{aligned}$$

In Cartesian coordinates,

$$\mathbf{u} = (r-r_0)^2 r (\cos\theta \cos\phi \hat{x} + \cos\theta \sin\phi \hat{y} + \sin\theta \hat{z}),$$

$$\mathbf{h} = \hat{\phi} = \frac{1}{(R+r\cos\theta)} (-\sin\phi \hat{x} + \cos\phi \hat{y},$$

$$\sigma = (r-r_0) \left[4r - 2r_0 + \frac{r(r-r_0)\cos\theta}{R+r\cos\theta} \right],$$

$$\mathbf{f} = f_r (\cos\theta \cos\phi \hat{x} + \cos\theta \sin\phi \hat{y} + \sin\theta \hat{z}) + f_\theta (-\sin\theta \cos\phi \hat{x} - \sin\theta \sin\phi \hat{y} + \cos\theta \hat{z}).$$

C.2 2d Annulus

We solve this problem using Polar coordinates instead of Cartesian ones.

Take $\mathbf{u} = \underbrace{(r-r_1)^2(r-r_0)^2}_{u_r} \hat{r} + \underbrace{0}_{u_\theta} \hat{\theta}$, where r_0 and r_1 are radii of inner and outer circles respectively.

For computing $\mathbf{f} = \text{curl curl } \mathbf{u} - \text{grad div } \mathbf{u}$:

$$\text{div } \mathbf{u} = \frac{1}{r} \frac{\partial}{\partial r} (r \cdot u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (u_\theta) = \frac{1}{r} \frac{\partial}{\partial r} (r \cdot u_r). \quad (\text{because } u_\theta = 0)$$

$$\Rightarrow \sigma = \frac{1}{r} \frac{\partial}{\partial r} (r(r-r_1)^2(r-r_0)^2) = \frac{(r-r_1)(r-r_0)}{r} [5r^2 - 3(r_1+r_0)r + r_1r_0].$$

$$\begin{aligned}
\text{More generally, } \text{grad}(\text{div } \mathbf{u}) &= \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \cdot u_r) \right) \hat{r} + \underbrace{\frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \cdot u_r) \right)}_{\substack{u_r \text{ is a} \\ \text{function of } r}} \hat{\theta}, \\
&\quad \parallel \\
&\quad 0
\end{aligned}$$

$$\begin{aligned}
&= \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \cdot u_r) \right) \hat{r}, \\
&= - \left(\frac{u_r}{r^2} - \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{\partial^2 u_r}{\partial r^2} \right) \hat{r}.
\end{aligned}$$

$$\text{and } \text{curl } \mathbf{u} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ u_r & 0 & 0 \end{vmatrix} = 0 \Rightarrow \text{curl curl } \mathbf{u} = 0.$$

$$\Rightarrow \mathbf{f} = -\text{grad div } \mathbf{u} = \left(\frac{u_r}{r^2} - \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{\partial^2 u_r}{\partial r^2} \right) \hat{r}.$$

Here, $u_r = (r-r_1)^2(r-r_0)^2$,

$$\Rightarrow \frac{\partial u_r}{\partial r} = 2(r - r_1)(r - r_0)[2r - (r_1 + r_0)],$$

$$\Rightarrow \frac{\partial^2 u_r}{\partial r^2} = 8(r - r_1)(r - r_0) + 2(r - r_1)^2 + 2(r - r_0)^2,$$

$$\Rightarrow f = \left[\underbrace{\frac{(r - r_1)^2(r - r_0)^2}{r^2} - \frac{2}{r}(r - r_1)(r - r_0)(2r - (r_1 + r_0)) - (8(r - r_1)(r - r_0) + 2(r - r_1)^2 + 2(r - r_0)^2)}_{f_r} \right] \hat{r},$$

And $h = \hat{\theta}$.

Now in terms of Cartesian coordinates,

$$h = \hat{\theta} = -\frac{\sin \theta}{r} \hat{x} + \frac{\cos \theta}{r} \hat{y},$$

$$u = (r - r_1)^2(r - r_0)^2 \hat{r} = ((r - r_1)^2(r - r_0)^2)(\cos \theta \hat{x} + \sin \theta \hat{y}),$$

$$f = f_r \hat{r} = f_r (\cos \theta \hat{x} + \sin \theta \hat{y}).$$

$$\left[\begin{array}{l} \text{Check : } u \cdot h = 0. \\ r_0 = 1, r_1 = 2 \Rightarrow u \cdot h = \int_0^{2\pi} \int_1^2 u \cdot h \, dr \, (r \, d\theta) = 0. \end{array} \right]$$

C.3 3d Annulus

We discuss conversion of coordinates with the help of an example, so take analytical solution to be $u = (r - R_1)^2(r - R_2)^2 \hat{r} = u_r \hat{r}$, where $R_1 = 1, R_2 = 2$.

$$\Rightarrow \frac{\partial u_r}{\partial r} = 2(r - R_1)(r - R_2)^2 + 2(r - R_1)^2(r - R_2).$$

We will use:

$$\nabla F = \frac{\partial F}{\partial r} \hat{r} + \frac{1}{r \sin \phi} \frac{\partial F}{\partial \theta} \hat{\theta} + \frac{1}{r} \frac{\partial F}{\partial \phi} \hat{\phi},$$

$$\nabla \cdot f = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 f_r) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \theta} (f_\theta) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} (f_\phi \sin \phi),$$

$$\nabla \times f = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial r} & \frac{1}{r} \frac{\partial}{\partial \theta} & \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} \\ f_r & r \sin \phi f_\theta & r f_\phi \end{vmatrix}.$$

$$\text{Therefore, } \operatorname{div} u = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) = 2(r - R_1)(r - R_2) \left[\frac{(r - R_1)(r - R_2)}{r} + 2r - (R_1 + R_2) \right],$$

$$\Rightarrow \operatorname{grad} \operatorname{div} u = 2 \left[9r^2 - 10(R_1 + R_2 - 2)r + 4R_1 R_2 + 2(R_1 + R_2)^2 - \left(\frac{R_1 R_2}{r} \right)^2 \right] \hat{r},$$

$$\text{and } \sigma = \operatorname{curl} u = 0,$$

$$\Rightarrow \operatorname{curl} \operatorname{curl} u = 0,$$

$$\Rightarrow f = \operatorname{curl} \operatorname{curl} u - \operatorname{grad} \operatorname{div} u = -\operatorname{grad} \operatorname{div} u,$$

$$= 2 \left[\underbrace{9r^2 - 10(R_1 + R_2 - 2)r + 4R_1 R_2 + 2(R_1 + R_2)^2 - \left(\frac{R_1 R_2}{r} \right)^2}_{f_r} \right] \hat{r}.$$

In Cartesian coordinates,

$$\mathbf{u} = (r - R_1)^2(r - R_2)^2 (\cos \theta \sin \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \phi \hat{z}),$$

$$\mathbf{h} = \sin \phi \hat{r} = \sin \phi (\cos \theta \sin \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \phi \hat{z}),$$

$$\sigma = 0,$$

$$f = f_r (\cos \theta \sin \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \phi \hat{z}).$$

Note that $\operatorname{div} \mathbf{h} = \frac{2 \sin \phi}{r} \neq 0$ and $\operatorname{curl} \mathbf{h} = \frac{\cos \phi}{r} \hat{\theta} \neq 0$ but both should be zero, so we can't take this $\mathbf{h}$ and not $\mathbf{h} = \frac{\sin \phi}{r^2} \hat{r}$ as well as it has non-zero curl. Similarly, we can't take $\mathbf{h} = \hat{r}$ as $\operatorname{curl} \mathbf{h} = 0$ but $\operatorname{div} \mathbf{h} = \frac{2}{r} \neq 0$. Therefore, we take $\mathbf{h} = \frac{\hat{r}}{r^2}$ as its both curl and divergence are zero. Also, note that $\mathbf{u} \cdot \mathbf{n} \neq 0$, so we need to multiply by $\cos \phi$ in $\mathbf{u}_r$.

Appendix D

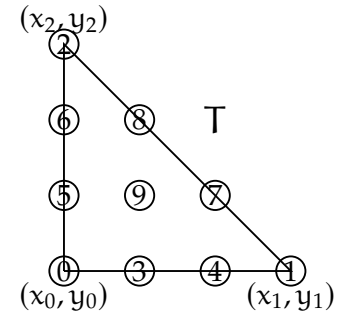
Arrangement of Mass Matrix using Cubic Lagrange basis

We demonstrate the arrangement of mass matrix using cubic Lagrange basis. Let $\{\lambda_0, \lambda_1, \lambda_2\}$ be the barycentric functions of the triangle T represented on the right h, then we have:

$$\lambda_0 = 1 - x - y, \lambda_1 = x, \lambda_2 = y.$$

And the cubic Lagrange basis are given by:

- Corresponding to vertices: $\frac{1}{2}\lambda_i(3\lambda_i - 1)(3\lambda_i - 2)$, for $i = 0, 1, 2$.
- Corresponding to edges: $\frac{9}{2}\lambda_i(3\lambda_i - 1)\lambda_j$, and $\frac{9}{2}\lambda_i(3\lambda_j - 1)\lambda_j$, for $i, j = 0, 1, 2, i < j$.
- Corresponding to triangle: $27\lambda_i\lambda_j\lambda_k$, for $i, j, k = 0, 1, 2, i < j < k$.



Using the barycentric coordinates values we can convert these cubic Lagrange basis in terms of cartesian coordinates as follows:

$$W_0 = \frac{1}{2}\lambda_0(3\lambda_0 - 1)(3\lambda_0 - 2) = -\frac{9}{2}x^3 - \frac{27}{2}x^2y + 9x^2 - \frac{27}{2}xy^2 + 18xy - \frac{11}{2}x - \frac{9}{2}y^3 + 9y^2 - \frac{11}{2}y + 1,$$

$$W_1 = \frac{1}{2}\lambda_1(3\lambda_1 - 1)(3\lambda_1 - 2) = \frac{1}{2}x(9x^2 - 9x + 2),$$

$$W_2 = \frac{1}{2}\lambda_2(3\lambda_2 - 1)(3\lambda_2 - 2) = \frac{1}{2}y(9y^2 - 9y + 2),$$

$$W_3 = \frac{9}{2}\lambda_0(3\lambda_0 - 1)\lambda_1 = \frac{9}{2}x(3x^2 + 6xy - 5x + 3y^2 - 5y + 2),$$

$$W_4 = \frac{9}{2}\lambda_0(3\lambda_1 - 1)\lambda_1 = \frac{9}{2}x(-3x^2 - 3xy + 4x + y - 1),$$

$$W_5 = \frac{9}{2}\lambda_0(3\lambda_0 - 1)\lambda_2 = \frac{9}{2}y(3x^2 + 6xy - 5x + 3y^2 - 5y + 2),$$

$$W_6 = \frac{9}{2}\lambda_0(3\lambda_2 - 1)\lambda_2 = \frac{9}{2}y(-3xy + x - 3y^2 + 4y - 1),$$

$$W_7 = \frac{9}{2}\lambda_1(3\lambda_1 - 1)\lambda_2 = \frac{9}{2}xy(3x - 1),$$

$$W_8 = \frac{9}{2}\lambda_1(3\lambda_2 - 1)\lambda_2 = \frac{9}{2}xy(3y - 1),$$

$$W_9 = 27\lambda_0\lambda_1\lambda_2 = 27xy(-x - y + 1),$$

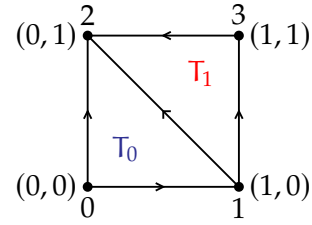
Mass matrix corresponding to T can be represented as:

$$M_T = \begin{bmatrix} \langle W_0, W_0 \rangle & \langle W_0, W_1 \rangle & \langle W_0, W_2 \rangle & \dots & \langle W_0, W_9 \rangle \\ \langle W_1, W_0 \rangle & \langle W_1, W_1 \rangle & \langle W_1, W_2 \rangle & \dots & \langle W_1, W_9 \rangle \\ \langle W_2, W_0 \rangle & \langle W_2, W_1 \rangle & \langle W_2, W_2 \rangle & \dots & \langle W_2, W_9 \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle W_9, W_0 \rangle & \langle W_9, W_1 \rangle & \langle W_9, W_2 \rangle & \dots & \langle W_9, W_9 \rangle \end{bmatrix},$$

where $\langle W_i, W_j \rangle = \int_T W_i \cdot W_j \, ds$. Since $\langle W_i, W_j \rangle = \langle W_j, W_i \rangle$, thus M_T is symmetric.

Now we will discuss the global computation of mass matrix on a unit square divided into two triangles as shown in the right.

$$\begin{array}{ll} \text{Global} & \rightarrow \text{Local} \\ \{0, 1, 2\} & \rightarrow \{0, 1, 2\} \\ \{3, 2, 1\} & \rightarrow \{0, 1, 2\} \end{array}$$



We will compute all the entries of M_{T_0} and M_{T_1} , first we will start with T_0 , and

$$\langle W_i, W_j \rangle = \int_{T_0} W_i \cdot W_j \, ds = \int_0^1 \int_0^{1-y} W_i(x, y) \cdot W_j(x, y) \, dx dy.$$

Thus, considering symmetricity of M_{T_0} , we have the following values of the entries of M_{T_0} :

- $\langle W_0, W_0 \rangle = \frac{19}{3360} \approx 0.0056548,$
- $\langle W_1, W_1 \rangle = \frac{19}{3360} \approx 0.0056548,$
- $\langle W_2, W_2 \rangle = \frac{19}{3360} \approx 0.0056548,$
- $\langle W_3, W_3 \rangle = \frac{9}{224} \approx 0.040179,$
- $\langle W_4, W_4 \rangle = \frac{9}{224} \approx 0.040179,$
- $\langle W_5, W_5 \rangle = \frac{9}{224} \approx 0.040179,$
- $\langle W_6, W_6 \rangle = \frac{9}{224} \approx 0.040179,$
- $\langle W_7, W_7 \rangle = \frac{9}{224} \approx 0.040179,$

- $\langle W_8, W_8 \rangle = \frac{9}{224} \approx 0.040179,$
- $\langle W_9, W_9 \rangle = \frac{81}{560} \approx 0.14464,$
- $\langle W_0, W_1 \rangle = \langle W_1, W_0 \rangle = \frac{11}{13440} \approx 0.000818452,$
- $\langle W_0, W_2 \rangle = \langle W_2, W_0 \rangle = \frac{11}{13440} \approx 0.000818452,$
- $\langle W_0, W_3 \rangle = \langle W_3, W_0 \rangle = \frac{3}{2240} \approx 0.00133929,$
- $\langle W_0, W_4 \rangle = \langle W_4, W_0 \rangle \approx 0,$
- $\langle W_0, W_5 \rangle = \langle W_5, W_0 \rangle = \frac{3}{2240} \approx 0.00133929,$
- $\langle W_0, W_6 \rangle = \langle W_6, W_0 \rangle \approx 0,$
- $\langle W_0, W_7 \rangle = \langle W_7, W_0 \rangle = \frac{9}{4480} \approx 0.00200893,$
- $\langle W_0, W_8 \rangle = \langle W_8, W_0 \rangle = \frac{9}{4480} \approx 0.00200893,$
- $\langle W_0, W_9 \rangle = \langle W_9, W_0 \rangle = \frac{3}{1120} \approx 0.00267857,$
- $\langle W_1, W_2 \rangle = \langle W_2, W_1 \rangle = \frac{11}{13440} \approx 0.000818452,$
- $\langle W_1, W_3 \rangle = \langle W_3, W_1 \rangle \approx 0,$
- $\langle W_1, W_4 \rangle = \langle W_4, W_1 \rangle = \frac{3}{2240} \approx 0.00133929,$
- $\langle W_1, W_5 \rangle = \langle W_5, W_1 \rangle = \frac{9}{4480} \approx 0.00200893,$
- $\langle W_1, W_6 \rangle = \langle W_6, W_1 \rangle = \frac{9}{4480} \approx 0.00200893,$
- $\langle W_1, W_7 \rangle = \langle W_7, W_1 \rangle = \frac{3}{2240} \approx 0.00133929,$
- $\langle W_1, W_8 \rangle = \langle W_8, W_1 \rangle \approx 0,$
- $\langle W_1, W_9 \rangle = \langle W_9, W_1 \rangle = \frac{3}{1120} \approx 0.00267857,$
- $\langle W_2, W_3 \rangle = \langle W_3, W_2 \rangle = \frac{9}{4480} \approx 0.00200893,$
- $\langle W_2, W_4 \rangle = \langle W_4, W_2 \rangle = \frac{9}{4480} \approx 0.00200893,$
- $\langle W_2, W_5 \rangle = \langle W_5, W_2 \rangle \approx 0,$
- $\langle W_2, W_6 \rangle = \langle W_6, W_2 \rangle = \frac{3}{2240} \approx 0.00133929,$

- $\langle W_2, W_7 \rangle = \langle W_7, W_2 \rangle \approx 0,$
- $\langle W_2, W_8 \rangle = \langle W_8, W_2 \rangle = \frac{3}{2240} \approx 0.00133929,$
- $\langle W_2, W_9 \rangle = \langle W_9, W_2 \rangle = \frac{3}{1120} \approx 0.00267857,$
- $\langle W_3, W_4 \rangle = \langle W_4, W_3 \rangle = -\frac{9}{640} \approx -0.0140625,$
- $\langle W_3, W_5 \rangle = \langle W_5, W_3 \rangle = \frac{9}{448} \approx 0.020089,$
- $\langle W_3, W_6 \rangle = \langle W_6, W_3 \rangle = -\frac{9}{896} \approx -0.010045,$
- $\langle W_3, W_7 \rangle = \langle W_7, W_3 \rangle = -\frac{9}{896} \approx -0.010045,$
- $\langle W_3, W_8 \rangle = \langle W_8, W_3 \rangle = -\frac{9}{2240} \approx -0.004017861,$
- $\langle W_3, W_9 \rangle = \langle W_9, W_3 \rangle = \frac{27}{2240} \approx 0.012054,$
- $\langle W_4, W_5 \rangle = \langle W_5, W_4 \rangle = -\frac{9}{896} \approx -0.010045,$
- $\langle W_4, W_6 \rangle = \langle W_6, W_4 \rangle = -\frac{9}{2240} \approx -0.004017861,$
- $\langle W_4, W_7 \rangle = \langle W_7, W_4 \rangle = \frac{9}{448} \approx 0.020089,$
- $\langle W_4, W_8 \rangle = \langle W_8, W_4 \rangle = -\frac{9}{896} \approx -0.010045,$
- $\langle W_4, W_9 \rangle = \langle W_9, W_4 \rangle = \frac{27}{2240} \approx 0.012054,$
- $\langle W_5, W_6 \rangle = \langle W_6, W_5 \rangle = -\frac{9}{640} \approx -0.0140625,$
- $\langle W_5, W_7 \rangle = \langle W_7, W_5 \rangle = -\frac{9}{2240} \approx -0.004017861,$
- $\langle W_5, W_8 \rangle = \langle W_8, W_5 \rangle = -\frac{9}{896} \approx -0.010045,$
- $\langle W_5, W_9 \rangle = \langle W_9, W_5 \rangle = \frac{27}{2240} \approx 0.012054,$
- $\langle W_6, W_7 \rangle = \langle W_7, W_6 \rangle = -\frac{9}{896} \approx -0.010045,$
- $\langle W_6, W_8 \rangle = \langle W_8, W_6 \rangle = \frac{9}{448} \approx 0.020089,$
- $\langle W_6, W_9 \rangle = \langle W_9, W_6 \rangle = \frac{27}{2240} \approx 0.012054,$
- $\langle W_7, W_8 \rangle = \langle W_8, W_7 \rangle = -\frac{9}{640} \approx -0.0140625,$

- $\langle W_7, W_9 \rangle = \langle W_9, W_7 \rangle = \frac{27}{2240} \approx 0.012054,$
- $\langle W_8, W_9 \rangle = \langle W_9, W_8 \rangle = \frac{27}{2240} \approx 0.012054.$

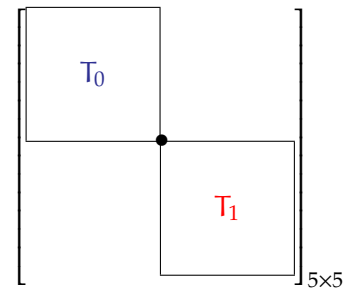
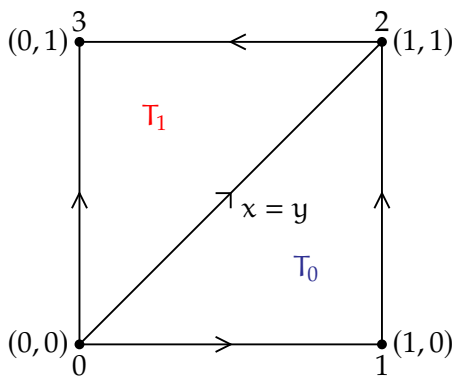
Similarly, we can compute the entries of M_{T_1} . Now we will arrange the local entries into the global matrix M_g as follows:

T_0	T_1	$w_0 0$	$w_1 1 w_2$	$w_2 2 w_1$	$3 w_0$	$w_3 0 1$	$w_4 0 1$	$w_5 0 2$	$w_6 0 2$	$w_7 1 2 w_7$	$w_8 1 2 w_8$	$13 w_5$	$13 w_6$	$23 w_3$	$23 w_4$	$w_9 0 1 2$	$3 2 1 w_9$	Index
$w_0 0$		00	01	02	0	03	04	05	06	07	08	0	0	0	0	09	0	0
$w_1 1 w_2$		10	11 + 22	12 + 21	20	13	14	15	16	17 + 27	18 + 28	25	26	23	24	19	29	1
$w_2 2 w_1$		20	21 + 12	22 + 11	10	23	24	25	26	27 + 17	28 + 18	15	16	13	14	29	19	2
$3 w_0$		0	02	01	00	0	0	0	0	07	08	05	06	03	04	0	09	3
$w_3 0 1$		30	31	32	0	33	34	35	36	37	38	0	0	0	0	39	0	4
$w_4 0 1$		40	41	42	0	43	44	45	46	47	48	0	0	0	0	49	0	5
$w_5 0 2$		50	51	52	0	53	54	55	56	57	58	0	0	0	0	59	0	6
$w_6 0 2$		60	61	62	0	63	64	65	66	67	68	0	0	0	0	69	0	7
$M_g = w_7 1 2 w_7$		70	71 + 72	72 + 71	70	73	74	75	76	77 + 77	78 + 78	75	76	73	74	79	79	8
$w_8 1 2 w_8$		80	81 + 82	82 + 81	80	83	84	85	86	87 + 87	88 + 88	85	86	83	84	89	89	9
$13 w_5$		0	52	51	50	0	0	0	0	57	58	55	56	53	54	0	59	10
$13 w_6$		0	62	61	60	0	0	0	0	67	68	65	66	63	64	0	69	11
$23 w_3$		0	32	31	30	0	0	0	0	37	38	35	36	33	34	0	39	12
$23 w_4$		0	42	41	40	0	0	0	0	47	48	45	46	43	44	0	49	13
$w_9 0 1 2$		90	91	92	0	93	94	95	96	97	98	0	0	0	0	99	0	14
$3 2 1 w_9$		0	92	91	90	0	0	0	0	97	98	95	96	93	94	0	99	15
Index		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	

Appendix E

Approximating a Function using Raviart-Thomas Basis to Check Requirement of Edge Parities in Integrals

We discuss the approximation of a function using Raviart-Thomas basis in order to check the requirement of edge parities in the integral computations.

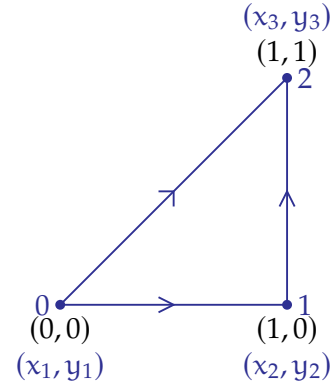


Our goal is to find the coefficients on all edges so that f can be represented using the 5 RT basis for which we follow the following steps:

1. Setup basis for RT(Raviart-Thomas) from reference triangle.
2. Obtain RT basis for each triangle in coordinates.
 - There are six in total.
3. Let us compute the mass matrix the 3×3 mass matrices by hand.
4. Obtain the global mass matrix.
5. Obtain the inner product $\langle f, \phi_i \rangle$ for both T_0 and T_1 ; it is 3×1 .

T_0 :

$$\begin{aligned} T &= \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \\ &= \overset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} \overset{x}{\begin{bmatrix} x \\ y \end{bmatrix}} + \overset{b}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}. \\ &\Downarrow \\ \det A &= 1 \end{aligned}$$



$$\begin{aligned} RT_{01} &= \begin{bmatrix} -(x-0)+1 \\ -(y-0)+1 \end{bmatrix} = \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix}, \\ RT_{02} &= \begin{bmatrix} -(x-0)+1 \\ -(y-0)+0 \end{bmatrix} = \begin{bmatrix} -x+1 \\ -y \end{bmatrix}, \\ RT_{12} &= \begin{bmatrix} -(x-0) \\ -(y-0) \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}. \end{aligned}$$

$$M_{T_0} = \begin{bmatrix} \langle RT_{01}, RT_{01} \rangle & \langle RT_{01}, RT_{02} \rangle & \langle RT_{01}, RT_{12} \rangle \\ \langle RT_{02}, RT_{01} \rangle & \langle RT_{02}, RT_{02} \rangle & \langle RT_{02}, RT_{12} \rangle \\ \langle RT_{12}, RT_{01} \rangle & \langle RT_{12}, RT_{02} \rangle & \langle RT_{12}, RT_{12} \rangle \end{bmatrix} \longrightarrow \text{Symmetric}$$

$$\langle RT_{01}, RT_{01} \rangle = \int \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix} = \int_0^1 \int_0^x (1-x)^2 + (1-y)^2 \, dy \, dx = \frac{1}{3},$$

$$\langle RT_{02}, RT_{01} \rangle = \int \begin{bmatrix} -x+1 \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix} = \int_0^1 \int_0^x (1-x)^2 - y(1-y) \, dy \, dx = 0,$$

$$\langle RT_{02}, RT_{02} \rangle = \int \begin{bmatrix} -x+1 \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y \end{bmatrix} = \int_0^1 \int_0^x (1-x)^2 + y^2 \, dy \, dx = \frac{1}{6},$$

$$\langle RT_{12}, RT_{01} \rangle = \int \begin{bmatrix} -x \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix} = \int_0^1 \int_0^x -x(1-x) - y(1-y) \, dy \, dx = -\frac{1}{6},$$

$$\langle RT_{12}, RT_{02} \rangle = \int \begin{bmatrix} -x \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y \end{bmatrix} = \int_0^1 \int_0^x -x(1-x) + y^2 \, dy \, dx = 0,$$

$$\langle RT_{12}, RT_{12} \rangle = \int \begin{bmatrix} -x \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y \end{bmatrix} = \int_0^1 \int_0^x x^2 + y^2 \, dy \, dx = \frac{1}{3}.$$

$$\therefore M_{T_0} = \begin{matrix} & \begin{matrix} 01 & 02 & 12 \end{matrix} \\ \begin{matrix} 01 \\ 02 \\ 12 \end{matrix} & \begin{bmatrix} \frac{1}{3} & 0 & -\frac{1}{6} \\ 0 & \frac{1}{6} & 0 \\ -\frac{1}{6} & 0 & \frac{1}{3} \end{bmatrix} \end{matrix}.$$

Now for $f = \begin{bmatrix} x+1 \\ y+1 \end{bmatrix}$,

$$\langle f, RT_{01} \rangle = \int \begin{bmatrix} x+1 \\ y+1 \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix} = \int_0^1 \int_0^x 1 - x^2 + 1 - y^2 \, dy \, dx = \frac{2}{3},$$

$$\langle f, RT_{02} \rangle = \int \begin{bmatrix} x+1 \\ y+1 \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y \end{bmatrix} = \int_0^1 \int_0^x 1 - x^2 - y(1+y) \, dy \, dx = 0,$$

$$\langle f, RT_{12} \rangle = \int \begin{bmatrix} x+1 \\ y+1 \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y \end{bmatrix} = \int_0^1 \int_0^x -x(1+x) - y(1+y) \, dy \, dx = -\frac{5}{6}.$$

$$\therefore f_{T_0} = \begin{matrix} 01 \\ 02 \\ 12 \end{matrix} \begin{bmatrix} \frac{2}{3} \\ 0 \\ -\frac{5}{6} \end{bmatrix}.$$

$$M_{T_0} u_0 = f_{T_0} \implies u_0 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \implies f_{\text{approx.}} = \begin{bmatrix} 1-x+2x \\ 1-y+2y \end{bmatrix} = \begin{bmatrix} 1+x \\ 1+y \end{bmatrix} = f.$$

T_1 :

$$T = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix},$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\Downarrow \\ \det A = 1$$

$$RT_{01} = \begin{bmatrix} -(x-0)+0 \\ -(y-0)+1 \end{bmatrix} = \begin{bmatrix} -x \\ -y+1 \end{bmatrix},$$

$$RT_{02} = \begin{bmatrix} -(x-0)+1 \\ -(y-0)+1 \end{bmatrix} = \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix},$$

$$RT_{12} = \begin{bmatrix} -(x-0) \\ -(y-0) \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}.$$

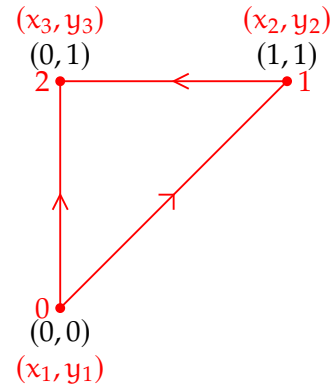
$$M_{T_1} = \begin{bmatrix} \langle RT_{01}, RT_{01} \rangle & \langle RT_{01}, RT_{02} \rangle & \langle RT_{01}, RT_{12} \rangle \\ \langle RT_{02}, RT_{01} \rangle & \langle RT_{02}, RT_{02} \rangle & \langle RT_{02}, RT_{12} \rangle \\ \langle RT_{12}, RT_{01} \rangle & \langle RT_{12}, RT_{02} \rangle & \langle RT_{12}, RT_{12} \rangle \end{bmatrix} \longrightarrow \text{Symmetric}$$

$$\langle RT_{01}, RT_{01} \rangle = \int \begin{bmatrix} -x \\ -y+1 \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y+1 \end{bmatrix} = \int_0^1 \int_x^1 x^2 + (1-y)^2 \, dy \, dx = \frac{1}{6},$$

$$\langle RT_{02}, RT_{01} \rangle = \int \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y+1 \end{bmatrix} = \int_0^1 \int_x^1 -x(1-x) + (1-y)^2 \, dy \, dx = 0,$$

$$\langle RT_{02}, RT_{02} \rangle = \int \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix} = \int_0^1 \int_x^1 (1-x)^2 + (1-y)^2 \, dy \, dx = \frac{1}{3},$$

$$\langle RT_{12}, RT_{01} \rangle = \int \begin{bmatrix} -x \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y+1 \end{bmatrix} = \int_0^1 \int_x^1 x^2 - y(1-y) \, dy \, dx = 0,$$



$$\langle \mathbf{RT}_{12}, \mathbf{RT}_{02} \rangle = \int \begin{bmatrix} -x \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix} = \int_0^1 \int_x^1 -x(1-x) - y(1-y) \, dy \, dx = -\frac{1}{6},$$

$$\langle \mathbf{RT}_{12}, \mathbf{RT}_{12} \rangle = \int \begin{bmatrix} -x \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y \end{bmatrix} = \int_0^1 \int_x^1 x^2 + y^2 \, dy \, dx = \frac{1}{3}.$$

$$\therefore \mathbf{M}_{T_1} = \begin{matrix} & \begin{matrix} 01 & 02 & 12 \end{matrix} \\ \begin{matrix} 01 \\ 02 \\ 12 \end{matrix} & \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{6} \\ 0 & -\frac{1}{6} & \frac{1}{3} \end{bmatrix} \end{matrix}.$$

Now for $f = \begin{bmatrix} x+1 \\ y+1 \end{bmatrix}$,

$$\langle f, \mathbf{RT}_{01} \rangle = \int \begin{bmatrix} x+1 \\ y+1 \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y+1 \end{bmatrix} = \int_0^1 \int_x^1 -x(1+x) + 1 - y^2 \, dy \, dx = 0,$$

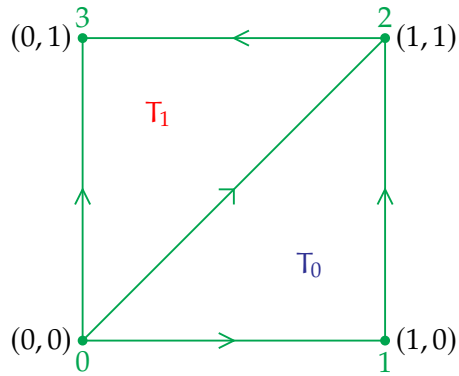
$$\langle f, \mathbf{RT}_{02} \rangle = \int \begin{bmatrix} x+1 \\ y+1 \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix} = \int_0^1 \int_x^1 1 - x^2 + 1 - y^2 \, dy \, dx = \frac{2}{3},$$

$$\langle f, \mathbf{RT}_{12} \rangle = \int \begin{bmatrix} x+1 \\ y+1 \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y \end{bmatrix} = \int_0^1 \int_x^1 -x(1+x) - y(1+y) \, dy \, dx = -\frac{5}{6}.$$

$$\therefore f_{T_1} = \begin{matrix} & \begin{matrix} 01 \\ 02 \\ 12 \end{matrix} \end{matrix} \begin{bmatrix} 0 \\ \frac{2}{3} \\ -\frac{5}{6} \end{bmatrix}.$$

$$\mathbf{M}_{T_1} \mathbf{u}_1 = f_{T_1} \implies \mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \implies f_{\text{approx.}} = \begin{bmatrix} 1-x+2x \\ 1-y+2y \end{bmatrix} = \begin{bmatrix} 1+x \\ 1+y \end{bmatrix} = f.$$

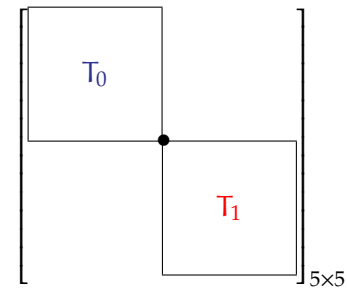
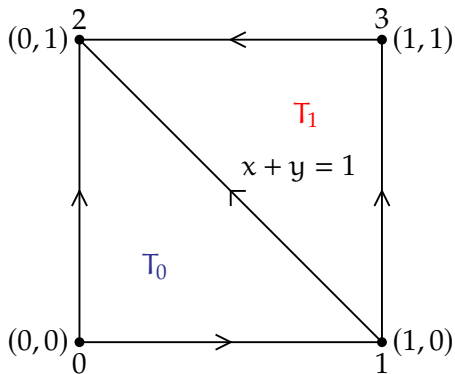
Global:



$$\begin{array}{ll} 01 \rightarrow 01 & 01 \rightarrow 02 \\ 02 \rightarrow 02 & 02 \rightarrow 03 \\ 12 \rightarrow 12 & 12 \rightarrow 23 \end{array}$$

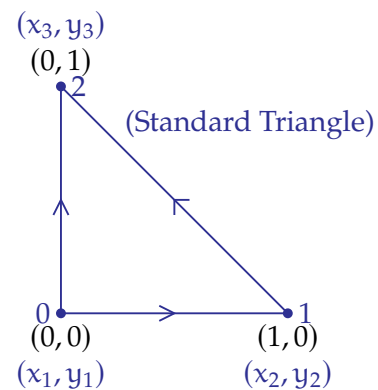
$$M = \begin{matrix} & \begin{matrix} 01^{01} & 02^{02}_{01} & 03^{02}_{02} & 12^{12}_{12} & 23^{12}_{12} \end{matrix} \\ \begin{matrix} 01^{01} \\ 02^{02}_{01} \\ 03^{02}_{02} \\ 12^{12}_{12} \\ 23^{12}_{12} \end{matrix} & \begin{bmatrix} \frac{1}{3} & 0 & 0 & -\frac{1}{6} & 0 \\ 0 & \frac{1}{6} + \frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & -\frac{1}{6} \\ -\frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{6} & 0 & \frac{1}{3} \end{bmatrix} \end{matrix} \text{ and } f = \begin{matrix} & \begin{matrix} 01^{01} & 02^{02}_{01} & 03^{02}_{02} & 12^{12}_{12} & 23^{12}_{12} \end{matrix} \\ \begin{matrix} 01^{01} \\ 02^{02}_{01} \\ 03^{02}_{02} \\ 12^{12}_{12} \\ 23^{12}_{12} \end{matrix} & \begin{bmatrix} \frac{2}{3} \\ 0 + 0 \\ \frac{2}{3} \\ -\frac{5}{6} \\ -\frac{5}{6} \end{bmatrix} \end{matrix}.$$

But this won't work as the common edge has 0 coefficient in local solutions. So, we take following mesh:

T₀:

$$\begin{aligned} T &= \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \\ &\Downarrow \\ \det A &= 1 \end{aligned}$$

$$\begin{aligned} RT_{01} &= \begin{bmatrix} -(x-0)+0 \\ -(y-0)+1 \end{bmatrix} = \begin{bmatrix} -x \\ -y+1 \end{bmatrix}, \\ RT_{02} &= \begin{bmatrix} -(x-0)+1 \\ -(y-0)+0 \end{bmatrix} = \begin{bmatrix} -x+1 \\ -y \end{bmatrix}, \\ RT_{12} &= \begin{bmatrix} -(x-0) \\ -(y-0) \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}. \end{aligned}$$



$$M_{T_0} = \begin{bmatrix} \langle RT_{01}, RT_{01} \rangle & \langle RT_{01}, RT_{02} \rangle & \langle RT_{01}, RT_{12} \rangle \\ \langle RT_{02}, RT_{01} \rangle & \langle RT_{02}, RT_{02} \rangle & \langle RT_{02}, RT_{12} \rangle \\ \langle RT_{12}, RT_{01} \rangle & \langle RT_{12}, RT_{02} \rangle & \langle RT_{12}, RT_{12} \rangle \end{bmatrix} \longrightarrow \text{Symmetric}$$

$$\langle RT_{01}, RT_{01} \rangle = \int \begin{bmatrix} -x \\ -y+1 \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y+1 \end{bmatrix} = \int_0^1 \int_0^{1-x} x^2 + (1-y)^2 dy dx = \frac{1}{3},$$

$$\langle RT_{02}, RT_{01} \rangle = \int \begin{bmatrix} -x+1 \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y+1 \end{bmatrix} = \int_0^1 \int_0^{1-x} -x(1-x) - y(1-y) dy dx = -\frac{1}{6},$$

$$\langle RT_{02}, RT_{02} \rangle = \int \begin{bmatrix} -x+1 \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y \end{bmatrix} = \int_0^1 \int_0^{1-x} (1-x)^2 + y^2 dy dx = \frac{1}{3},$$

$$\langle RT_{12}, RT_{01} \rangle = \int \begin{bmatrix} -x \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y+1 \end{bmatrix} = \int_0^1 \int_0^x x^2 - y(1-y) dy dx = 0,$$

$$\langle RT_{12}, RT_{02} \rangle = \int \begin{bmatrix} -x \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y \end{bmatrix} = \int_0^1 \int_0^{1-x} (1-x)^2 + y^2 dy dx = 0,$$

$$\langle RT_{12}, RT_{12} \rangle = \int \begin{bmatrix} -x \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y \end{bmatrix} = \int_0^1 \int_0^{1-x} x^2 + y^2 dy dx = \frac{1}{6}.$$

$$\therefore M_{T_0} = \begin{matrix} & \begin{matrix} 01 & 02 & 12 \end{matrix} \\ \begin{matrix} 01 \\ 02 \\ 12 \end{matrix} & \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} \end{matrix}.$$

$$\text{Now for } f = \begin{bmatrix} x+1 \\ y+1 \end{bmatrix},$$

$$\langle f, RT_{01} \rangle = \int \begin{bmatrix} x+1 \\ y+1 \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y+1 \end{bmatrix} = \int_0^1 \int_0^{1-x} -x(1+x) + 1 - y^2 dy dx = \frac{1}{6},$$

$$\langle f, RT_{02} \rangle = \int \begin{bmatrix} x+1 \\ y+1 \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y \end{bmatrix} = \int_0^1 \int_0^{1-x} 1 - x^2 - y(1+y) dy dx = \frac{1}{6},$$

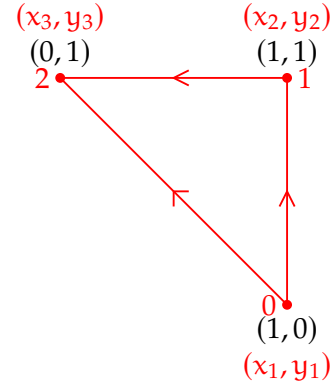
$$\langle f, RT_{12} \rangle = \int \begin{bmatrix} x+1 \\ y+1 \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y \end{bmatrix} = \int_0^1 \int_0^{1-x} -x(1+x) - y(1+y) dy dx = -\frac{1}{2}.$$

$$\therefore f_{T_0} = \begin{matrix} & \begin{matrix} 01 \\ 02 \\ 12 \end{matrix} \\ \begin{matrix} 01 \\ 02 \\ 12 \end{matrix} & \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{2} \end{bmatrix} \end{matrix}.$$

$$M_{T_0} u_0 = f_{T_0} \implies u_0 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \implies f_{\text{approx.}} = \begin{bmatrix} -x+1-x+3x \\ 1-y-y+3y \end{bmatrix} = \begin{bmatrix} 1+x \\ 1+y \end{bmatrix} = f.$$

T_1 :

$$\begin{aligned} T &= \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \\ &= \overset{A}{\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} + \overset{b}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}. \\ &\Downarrow \\ \det A &= 1 \end{aligned}$$



$$RT_{01} = \begin{bmatrix} -(x-1) + (-1) \\ -(y-0) + 1 \end{bmatrix} = \begin{bmatrix} -x \\ -y+1 \end{bmatrix},$$

$$RT_{02} = \begin{bmatrix} -(x-1) + 0 \\ -(y-0) + 1 \end{bmatrix} = \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix},$$

$$RT_{12} = \begin{bmatrix} -(x-1) \\ -(y-0) \end{bmatrix} = \begin{bmatrix} -x+1 \\ -y \end{bmatrix}.$$

$$M_{T_1} = \begin{bmatrix} \langle RT_{01}, RT_{01} \rangle & \langle RT_{01}, RT_{02} \rangle & \langle RT_{01}, RT_{12} \rangle \\ \langle RT_{02}, RT_{01} \rangle & \langle RT_{02}, RT_{02} \rangle & \langle RT_{02}, RT_{12} \rangle \\ \langle RT_{12}, RT_{01} \rangle & \langle RT_{12}, RT_{02} \rangle & \langle RT_{12}, RT_{12} \rangle \end{bmatrix} \longrightarrow \text{Symmetric}$$

$$\langle RT_{01}, RT_{01} \rangle = \int \begin{bmatrix} -x \\ -y+1 \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y+1 \end{bmatrix} = \int_0^1 \int_{1-y}^1 x^2 + (1-y)^2 dy dx = \frac{1}{3},$$

$$\langle RT_{02}, RT_{01} \rangle = \int \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y+1 \end{bmatrix} = \int_0^1 \int_{1-y}^1 -x(1-x) + (1-y)^2 dy dx = 0,$$

$$\langle RT_{02}, RT_{02} \rangle = \int \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix} = \int_0^1 \int_{1-y}^1 (1-x)^2 + (1-y)^2 dy dx = \frac{1}{6},$$

$$\langle RT_{12}, RT_{01} \rangle = \int \begin{bmatrix} -x+1 \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y+1 \end{bmatrix} = \int_0^1 \int_{1-y}^1 -x(1-x) - y(1-y) dy dx = -\frac{1}{6},$$

$$\langle RT_{12}, RT_{02} \rangle = \int \begin{bmatrix} -x+1 \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix} = \int_0^1 \int_{1-y}^1 (1-x)^2 - y(1-y) dy dx = 0,$$

$$\langle RT_{12}, RT_{12} \rangle = \int \begin{bmatrix} -x+1 \\ -y \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y \end{bmatrix} = \int_0^1 \int_{1-y}^1 (1-x)^2 + y^2 dy dx = \frac{1}{3}.$$

$$\therefore M_{T_1} = \begin{matrix} & \begin{matrix} 01 & 02 & 12 \end{matrix} \\ \begin{matrix} 01 \\ 02 \\ 12 \end{matrix} & \begin{bmatrix} \frac{1}{3} & 0 & -\frac{1}{6} \\ 0 & \frac{1}{6} & 0 \\ -\frac{1}{6} & 0 & \frac{1}{3} \end{bmatrix} \end{matrix}.$$

$$\text{Now for } f = \begin{bmatrix} x+1 \\ y+1 \end{bmatrix},$$

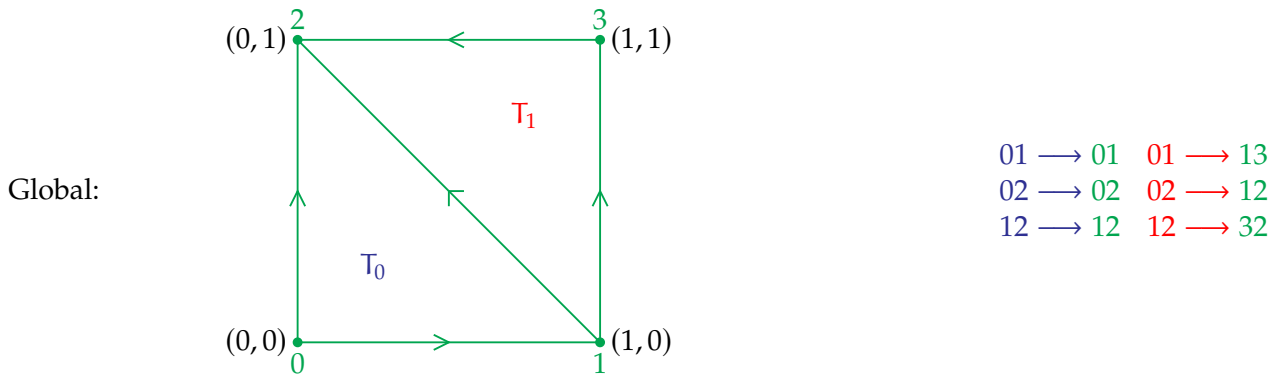
$$\langle f, RT_{01} \rangle = \int \begin{bmatrix} x+1 \\ y+1 \end{bmatrix} \cdot \begin{bmatrix} -x \\ -y+1 \end{bmatrix} = \int_0^1 \int_{1-y}^1 -x(1+x) + 1 - y^2 \, dy \, dx = -\frac{1}{3},$$

$$\langle f, RT_{02} \rangle = \int \begin{bmatrix} x+1 \\ y+1 \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y+1 \end{bmatrix} = \int_0^1 \int_{1-y}^1 1 - x^2 + 1 - y^2 \, dy \, dx = \frac{1}{2},$$

$$\langle f, RT_{12} \rangle = \int \begin{bmatrix} x+1 \\ y+1 \end{bmatrix} \cdot \begin{bmatrix} -x+1 \\ -y \end{bmatrix} = \int_0^1 \int_{1-y}^1 1 - x^2 - y(1+y) \, dy \, dx = -\frac{1}{3}.$$

$$\therefore f_{T_1} = \begin{matrix} 01 \\ 02 \\ 12 \end{matrix} \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{2} \\ -\frac{1}{3} \end{bmatrix}.$$

$$M_{T_1} u_1 = f_{T_1} \implies u_1 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} \implies f_{\text{approx.}} = \begin{bmatrix} 2x+3-3x-2+2x \\ -2+2y+3-3y+2y \end{bmatrix} = \begin{bmatrix} 1+x \\ 1+y \end{bmatrix} = f.$$



$$M = \begin{matrix} & \begin{matrix} 01^{01} & 02^{02} & 12^{12}_{02} & 13_{01} & 23_{12} \end{matrix} \\ \begin{matrix} 01^{01} \\ 02^{02} \\ 12^{12}_{02} \\ 13_{01} \\ 23_{12} \end{matrix} & \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} + \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & 0 & -\frac{1}{6} & \frac{1}{3} \end{bmatrix} \end{matrix} \text{ and } f = \begin{matrix} & \begin{matrix} 01^{01} & 02^{02} & 12^{12}_{02} & 13_{01} & 23_{12} \end{matrix} \\ \begin{matrix} 01^{01} \\ 02^{02} \\ 12^{12}_{02} \\ 13_{01} \\ 23_{12} \end{matrix} & \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{2} - \frac{1}{2} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} \end{matrix}.$$

$$Mu = f \implies u = \begin{bmatrix} 1 \\ 1 \\ -3 \\ -2 \\ -2 \end{bmatrix}.$$

We need to multiply with edge parities to basis which is 1 if edge is oriented according to lexicographic order else is -1 .

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List of Publications

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Preprints:

1. Arya, Archana, and Kaushik Kalyanaraman. An Arbitrarily Higher-Order Time Implicit Scheme for Maxwell's Equations. arXiv PREPRINT: <http://arxiv.org/abs/2404.05004>.
2. Arya, Archana, and Kaushik Kalyanaraman. Two Frameworks and their Fourth Order Implicit Schemes for Time Discretization of Maxwell's Equations. arXiv PREPRINT: <http://arxiv.org/abs/2601.12793>